34 Fiber Bundles and Connections

The elegance of the geometrical expression of physical ideas has attracted much attention ever since Einstein proposed his geometrical theory of gravity in 1916. Such an expression was, however, confined to the general theory of relativity until the 1970s when the language of geometry was found to be most suitable, not only for gravity, but also for the other three fundamental forces of nature. Geometry, in the form of gauge field theories of electroweak and strong interactions, has been successful not only in creating a model—the so-called **standard model**—that explains all experimental results to remarkable accuracy, but also in providing a common language for describing all fundamental forces of nature, and with that a hope for unifying these forces into a single all-embracing force. This hope is encouraged by the successful unification of electromagnetism with the weak nuclear force through the medium of geometry and gauge field theory.

The word "geometry" is normally used in the mathematics literature for a manifold on which a "machine" is defined with the property that it gives a number when two vectors are fed into it. Symplectic geometry's machine was a nondegenerate 2-form. Riemannian (or pseudo-Riemannian or semi-Riemannian) geometry has a symmetric bilinear form (metric, inner product). Both of these geometries are important: Symplectic geometry is the natural setting for Hamiltonian dynamics, and (pseudo- or semi-) Riemannian geometry is the basis of the general theory of relativity.

The most elegant way of studying geometry, which very naturally encompasses the (pseudo-)Riemannian geometry of general relativity and the gauge theory of the fundamental interactions of physics, is the language of the fiber bundles, which we set out to do in this chapter.

34.1 Principal Fiber Bundles

In Sect. 28.4, we defined the tangent bundle $T(M)$ as the union of the tangent spaces at all points of a manifold M. It can be shown that $T(M)$ is a manifold, and that there is a differentiable surjective map $\pi : T(M) \to M$, sending the tangent space $T_x(M)$ at $x \in M$ to x.[1] The inverse of this map at

[1]For ease of notation, we have changed $\mathcal{T}_P(M)$ of Definition 28.4.1 to $T_x(M)$.

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x, $\pi^{-1}(x)$, is the collection of all vectors at x. The notion of tangent *bundle* and the corresponding map π can be generalized to the extremely fruitful notion of principal fiber bundle.

principal fiber bundle

Definition 34.1.1 A **principal fiber bundle** (PFB) over a manifold M with Lie group G is a manifold P, called the **total space** or the **bundle space**, and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right: $R_g(p) \equiv p \cdot g \equiv pg \in P$.
(2) M is the space P/G of the orbits of G in P and the canonical map $\pi : P \to P/G$ is differentiable.
(3) P is **locally trivial**, i.e., for every point $x \in M$, there is a neighborhood U containing x and a diffeomorphism $T_u : \pi^{-1}(U) \to U \times G$ of the form $T_u(p) = (\pi(p), s_u(p))$ where $s_u : \pi^{-1}(U) \to G$ has the property $s_u(pg) = s_u(p)g$ for all $g \in G$ and $p \in \pi^{-1}(U)$. The map T_u is called a **local trivialization** (LT).

local trivialization

A principal fiber bundle will be denoted by $P(M, G, \pi)$, or $P(M, G)$, or even just P. M is called the **base space**, G the **structure group**, and π the **projection** of the PFB. For each $x \in M$, $\pi^{-1}(x)$ is a submanifold of P, called the **fiber** over x. If $x = \pi(p)$, then $\pi^{-1}(x)$ is just the orbit of G at p. By Theorem 29.1.7, every fiber is diffeomorphic to G. There is no natural group structure on $\pi^{-1}(x)$. So, although fibers can be thought of as copies of G, they are so *only as manifolds*.

base space; structure group; projection; fiber

Remark 34.1.1 Just as a fiber sprouts from a single point of the earth (a spherical 2-manifold), so does a fiber $\pi^{-1}(x)$ sprout out of a single point x of the manifold M. And just as you can collect a bunch of fibers and make a *bundle* out of them, so can you collect a bunch of $\pi^{-1}(x)$'s and make $P = \bigcup_x \pi^{-1}(x)$. Furthermore, fibers sprout vertically from the ground. Similarly, in a sense to be elaborated in our discussion of connections, $\pi^{-1}(x)$ are "vertical" manifolds, while M is "horizontal."

trivial bundle

Example 34.1.2 Let M be any manifold and G any Lie group. Let $P = M \times G$ and let G act on P on the right by the rule: $(x, g)g' = (x, gg')$. We note that the action is free because

$$(x, g)g' = (x, g) \iff \bigl(x, gg'\bigr) = (x, g) \iff gg' = g \iff g' = e.$$

Two points (x, g) and (x', g') belong to the same orbit iff there is $h \in G$ such that $(x, g)h = (x', g')$. This happens iff $x' = x$ and $gh = g'$. It follows that for any g, (x, g) belongs to the orbit at (x, e). Therefore, $[\![(x, g)]\!] = [\![(x, e)]\!]$. This gives a natural identification of P/G with M. For trivialization, let the neighborhood U of any point be M and let $s_u(x, g) = g$. This choice makes P *globally* trivial, thus the name **trivial** for such a bundle.

Definition 34.1.3 A **homomorphism** of a principal fiber bundle $P'(M', G')$ into another $P(M, G)$ is a pair (f, f_G) of maps $f : P' \to P$ and $f_G : G' \to G$ with f_G a group homomorphism such that $f(p'g') = f(p')f_G(g')$ for all $p' \in P'$ and $g' \in G'$. Every bundle homomorphism induces a map $f_M : M' \to M$. If f is bijective and f_G a group isomorphism, then f_M is a diffeomorphism and (f, f_G) is called an **isomorphism** of $P'(M', G')$ onto $P(M, G)$. An isomorphism of $P(M, G)$ onto itself in which $f_G = \mathrm{id}_G$ is called an **automorphism** of P.

homomorphism, isomorphism, and automorphism of PFBs

Requirement (3) in Definition 34.1.1 situates $x \in M$ in the (sub)bundle $\pi^{-1}(U)$ which, through the diffeomorphism T_u, can be identified as the trivial bundle $U \times G$. The natural right action of G on the trivial bundle $U \times G$ should therefore be identified with its action on $\pi^{-1}(U)$. On the one hand,

$$T_u(p) = \big(\pi(p), s_u(p)\big), \qquad T_u(pg) = \big(\pi(pg), s_u(pg)\big) = \big(\pi(p), s_u(pg)\big),$$

where the last equality follows because p and pg both belong to the orbit at p. On the other hand,

$$\big(\pi(p), s_u(p)\big)g = \big(\pi(p), s_u(p)g\big) \quad \text{for the trivial bundle } U \times G.$$

So if the action of G on $U \times G$ is to be identified with its action on $\pi^{-1}(U)$, we must have $s_u(pg) = s_u(p)g$. That is why this equality was demanded in Definition 34.1.1. We summarize this by saying that T_u *respects the action of* G.

A local trivialization respects the action of structure group.

Now let $T_u : \pi^{-1}(U) \to U \times G$ and $T_v : \pi^{-1}(V) \to V \times G$ be two LTs. If $x \in U \cap V$ and $\pi(p) = x$, then $T_u(p) = (\pi(p), s_u(p))$, and $T_v(p) = (\pi(p), s_v(p))$. Since $s_u(p), s_v(p) \in G$, there must exist $g \in G$ such that $gs_v(p) = s_u(p)$. In fact, $g = s_u(p)(s_u(p))^{-1}$. What is interesting about g is that it can be defined on M.

transition functions for a PFB

Definition 34.1.4 Let $T_u : \pi^{-1}(U) \to U \times G$ and $T_v : \pi^{-1}(V) \to V \times G$ be two LTs of a PFB $P(M, G, \pi)$. The **transition function** from T_u to T_v is the map $g_{uv} : U \cap V \to G$, given by $g_{uv}(x) = s_u(p)(s_u(p))^{-1}$.

For this definition to make sense, $g_{uv}(x)$ must be independent of $p \in \pi^{-1}(x)$. Indeed, we have

Proposition 34.1.5 *The transition function g_{uv} from the local trivialization T_u to the local trivialization T_v is independent of the choice of $p \in \pi^{-1}(x)$. Furthermore,*

(1) $g_{uu}(x) = e \quad \forall x \in U;$
(2) $g_{uv}(x) = g_{vu}(x)^{-1} \quad \forall x \in U \cap V;$
(3) $g_{uv}(x) = g_{uw}(x)g_{wv}(x) \quad \forall x \in U \cap V \cap W.$

Proof Let $p' \in \pi^{-1}(x)$ be a different point from p. Since p' is in the same orbit as p, we must have $p' = pg$ for some $g \in G$. Then

$$s_u(p')\big(s_u(p')\big)^{-1} = s_u(pg)\big(s_u(pg)\big)^{-1} = s_u(p)g\big(s_u(p)g\big)^{-1}$$
$$= s_u(p)gg^{-1}\big(s_u(p)\big)^{-1} = s_u(p)\big(s_u(p)\big)^{-1}$$

Thus g_{uv} is well defined. The other parts of the proposition are trivial. □

Consider a manifold M. Let $\{U_\alpha\}$ be an open cover of M, i.e., open sets such that $M = \bigcup_\alpha U_\alpha$. Let G be a Lie group. Construct the set of trivial PFBs $P_\alpha = U_\alpha \times G$. Connect all pairs P_α and P_β by transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying (1)–(3) of Proposition 34.1.5. This process constructs a PFB with transition functions $g_{\alpha\beta}$. Therefore, a PFB is *defined* by its transition functions. In fact, it is only the transition functions that determine the bundle. *Any* PFB can be broken down into a collection $\{U_\alpha \times G\}$ of trivial bundles. It is *how* these trivial bundles are "glued together" via the transition functions that distinguishes between different PFBs.

Given a PFB $P(M, G)$ and a subgroup G' of G, it may be possible to find some covering $\{U_\alpha\}$ of M and transition functions $g_{\alpha\beta}$ which take values in G'. The new covering and transition functions define a new PFB $P'(M, G')$. Then we say that the PFB $P(M, G)$ is **reducible** to $P'(M, G')$. We also say that the structure group G of $P(M, G)$ is reducible to G' if $P(M, G)$ is reducible to $P'(M, G')$.

reducible bundle

Example 34.1.6 Let's reconsider the trivial bundle $M \times G$. What is the most general *local* trivialization of this bundle? Let $x \in U_\alpha$ if $p = (x, g)$, then $T_\alpha(p) = (x, g')$ for some $g' \in G$, and $s_\alpha(p) = s_\alpha(x, g) = g'$. This means that s_α affects only g, and therefore, can be reduced to a function $f_\alpha : G \to G$ having the property that $f_\alpha(gg') = f_\alpha(g)g'$. With $g = e$, this gives $f_\alpha(g') = f_\alpha(e)g'$. Thus, f_α is simply left multiplication by $h_\alpha \equiv f_\alpha(e)$, where h_α may depend on U_α. Hence, the most general LT for the trivial bundle $M \times G$ is

$$T_\alpha(p) = (x, h_\alpha g) \equiv \big(x, s_\alpha(p)\big) \quad \text{or} \quad s_\alpha\big((x, g)\big) = h_\alpha g$$

So, the transition functions are of the form $g_{\alpha\beta}(x) = h_\alpha h_\beta^{-1}$, and can easily be shown to satisfy the three conditions of Proposition 34.1.5.

Can the trivial bundle be reduced? Are there a covering $\{U_\alpha\}$ and transition functions $g_{\alpha\beta}$ which take values in a subgroup of G? In fact, G can be drastically reduced! In the above discussion let $h_\alpha = h$ for all α. Then, $g_{\alpha\beta}(x) = e$ for all $x \in U_\alpha \cap U_\beta$ (and therefore for all $x \in M$).

The converse of the last statement of the example above is also true:

Proposition 34.1.7 *Any PFB whose structure group can be reduced to the identity of the group is isomorphic to the trivial PFB.*

local section and section

Definition 34.1.8 A **local section** (or *local cross section*) of a principal fiber bundle $P(M, G, \pi)$ on an open set $U \subset M$ is a map $\sigma_u : U \to P$ such that $\pi \circ \sigma_u = \mathrm{id}_U$. If $U = M$, then $\sigma_u \equiv \sigma$ is called a **global section** or simply a **section** on M.

Proposition 34.1.9 *There is a natural* 1-1 *correspondence between the set of local trivializations and the set of local sections. In particular, if* $P(M, G, \pi)$ *has a* (*global*) *section, then* $P(M, G, \pi) = M \times G$, *the trivial bundle.*

Proof For each local trivialization T_u let $\sigma_u = T_u^{-1}|_{U\times\{e\}} : U \cong U \times \{e\} \to P$. Conversely, for each σ_u, define $S_u : U \times G \to \pi^{-1}(U)$ by $S_u(x, g) = \sigma_u(x)g$. Then it can be shown that S_u is a bijection and $T_u = S_u^{-1}$ is a local trivialization. □

Let σ_u be a local section on U and σ_v on V. If $x \in U \cap V$, then $\sigma_u(x)$ and $\sigma_v(x)$ both belong to $\pi^{-1}(x)$. Hence, there must be a $g \in G$ such that $\sigma_v(x) = \sigma_u(x)g$. We want to find this g. From the definition of T_u, we have $T_u^{-1}(x, e) = p_0$ for some $p_0 \in P$. Thus, $T_u(p_0) = (x, s_u(p_0)) = (x, e)$ implies that $s_u(p_0) = e$. But $T_u^{-1}(x, e) = \sigma_u(x)$. Therefore, we have $\sigma_u(x) = p_0$ with $s_u(p_0) = e$. Similarly, $\sigma_v(x) = p_1$ with $s_v(p_1) = e$. Let $p_1 = p_0g$. Then $e = s_v(p_1) = s_v(p_0g) = s_v(p_0)g$, or $g = s_v(p_0)^{-1}$. We thus get $\sigma_v(x) = p_0s_v(p_0)^{-1}$ or $\sigma_v(x)s_v(p_0) = p_0$. Multiplying both sides by an arbitrary g, we get

$$\sigma_v(x)s_v(p_0)g = p_0g \quad \text{or} \quad \sigma_v(x)s_v(p_0g) = p_0g \quad \text{or}$$
$$\sigma_v(x)s_v(p) = p \ \forall p \in P.$$

An identical reasoning gives $\sigma_u(x)s_u(p) = p$. Therefore, $\sigma_v(x)s_v(p) = \sigma_u(x)s_u(p)$, or $\sigma_v(x) = \sigma_u(x)s_u(p)s_v(p)^{-1}$. Thus,

$$\sigma_v(x) = \sigma_u(x)g_{uv}(x) \tag{34.1}$$

where g_{uv} is the transition function from T_u to T_v.

Example 34.1.10 (The bundle $G(G/H, H)$) Let G be a Lie group and H one of its Lie subgroups. Let H act on G on the right by right multiplication. Let G/H be the factor group of this action and $\pi : G \to G/H$, the natural projection. It is shown in Lie group theory that such a construction has local trivializations. Then with G as the total space, $M = G/H$ as the base space, and $\pi : G \to G/H$ as the projection, $G(G/H, H, \pi)$ becomes a principal fiber bundle.

Example 34.1.11 (Bundle of linear frames) Let M be an n-manifold. A **linear frame** p at $x \in M$ is an ordered basis $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ of the tangent space $T_x(M)$. Let $L_x(M)$ be the set of all linear frames at x and $L(M)$ the set of all $L_x(M)$ for all $x \in M$. Let $\pi : L(M) \to M$ be the map that sends $L_x(M)$ to x. If $\mathsf{A} \in GL(n, \mathbb{R})$ is a matrix with components a^i_j, then the action of $GL(n, \mathbb{R})$ on $L(M)$ on the right is written in matrix form as

bundle of linear frames

$$(\mathbf{X}_1\ \mathbf{X}_2\ \dots\ \mathbf{X}_n)\begin{pmatrix} a^1_1 & a^1_2 & \dots & a^1_n \\ a^2_1 & a^2_2 & \dots & a^2_n \\ \vdots & \vdots & & \vdots \\ a^n_1 & a^n_2 & \dots & a^n_n \end{pmatrix} \equiv (\mathbf{Y}_1\ \mathbf{Y}_2\ \dots\ \mathbf{Y}_n).$$

In "component" form, this can be written as $\mathbf{Y}_i = a^j_i \mathbf{X}_j$ (with summation convention in place). Since A is invertible, $(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n) \in L(M)$. So, indeed $GL(n, \mathbb{R})$ acts on $L(M)$ on the right. It is easy to show that the action is free (Problem 34.4). Furthermore, if $p, q \in \pi^{-1}(x) \equiv L_x(M)$, i.e., if p and q are two (ordered) bases of $T_x(M)$, then there must exist an invertible matrix A such that $q = p\mathsf{A}$. Therefore, $\pi^{-1}(x)$ is indeed the orbit of $GL(n, \mathbb{R})$ at p. This shows (1) and (2) of Definition 34.1.1. Foregoing the rather technical details of (3), we find that $L(M)(M, GL(n, \mathbb{R}))$ is indeed a principal fiber bundle.

Definition 34.1.12 The PFB described in Example 34.1.11 is called the **bundle of linear frames** and denoted by $L(M)(M, GL(n, \mathbb{R}))$, or simply $L(M)$.

34.1.1 Associated Bundles

Let $P(M, G)$ be a PFB and F a manifold on which G acts on the left: $G \times F \ni (g, \xi) \mapsto g\xi \in F$. On the product manifold $P \times F$ let G act on the right by the rule $R_g(p, \xi) \equiv (p, \xi)g \equiv (pg, g^{-1}\xi)$. Denote the quotient space of this action by $P \times_G F$, and let $E \equiv P \times_G F$. For $[\![p, \xi]\!] \in E$ let $\pi_E([\![p, \xi]\!]) = \pi(p) = x \in M$. Then π_E is a projection of E onto M. Define $\phi_u : \pi_E^{-1}(U) \to U \times F$ by $\phi_u([\![p, \xi]\!]) = (\pi(p), s_u(p)\xi)$, where $s_u : P \to G$ is as defined in the local trivialization of $P(M, G)$. One can show that ϕ_u is a diffeomorphism (Problem 34.5) and that E is a fiber bundle.

associated bundle and its standard fiber and structure group

Definition 34.1.13 The fiber bundle constructed above is called the fiber bundle over the base M with **standard fiber** F and **structure group** G which is **associated** with the principal fiber bundle $P(M, G)$. E is more elaborately denoted by $E(M, F, G, P)$. The **fiber over** x in E, $\pi_E^{-1}(x)$ is denoted by F_x.

The diffeomorphism ϕ_u, when restricted to the fiber over x, is denoted by ϕ_x. It is a diffeomorphism: $\phi_x : \pi_E^{-1}(x) \to \{x\} \times F \cong F$ by $\phi_x([\![p, \xi]\!]) = s_u(p)\xi$. Note that this map is determined entirely by p. The inverse of this mapping, also determined entirely by p, can be thought of as a map $p : F \to F_x$, given by $p(\xi) \equiv p\xi = [\![p, \xi]\!]$. It can easily be shown that this map satisfies

$$(pg)\xi = p(g\xi) \quad \text{for } p \in P,\ g \in G,\ \xi \in F. \tag{34.2}$$

Theorem 34.1.14 *Let $P(M, G)$ be a principal fiber bundle with the associated bundle $E(M, F, G, P)$. Then each $p \in P$ can be considered as a diffeomorphic map $p : F \to F_x$ satisfying (34.2).*

Consider two fibers F_x and F_y. They are diffeomorphic, because each is diffeomorphic to F. In fact if $p : E \to F_x$ and $q : E \to F_y$, then $q \circ p^{-1} : F_x \to F_y$ is called an **isomorphism** of F_y and F_x. For $x = y$, $q \circ p^{-1}$ becomes an **automorphism** of F_x. Moreover, since $\pi(q) = x = \pi(p)$, we must have $q = pg$ for some $g \in G$. Therefore, any automorphism of F_x is of the form $p \circ g \circ p^{-1}$.

Proposition 34.1.15 *The group of automorphisms of F_x is isomorphic with the structure group G.*

Example 34.1.16 The bundle of linear frames consists of fibers which include all ordered bases of $T_x(M)$ and a right action by $GL(n, \mathbb{R})$, which is the group of invertible linear transformation of $\mathbb{R}^n$. For every ordered basis $p = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \in L(M)$, let $p(\hat{\mathbf{e}}_i) = \mathbf{X}_i$, where $\{\hat{\mathbf{e}}_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. This then defines a map $p : \mathbb{R}^n \to T_x(M)$. All this, in conjunction with Theorem 34.1.14, leads to the conclusion that $E(M, \mathbb{R}^n, GL(n, \mathbb{R}), L(M))$ is the bundle associated with the bundle of linear frames with standard fiber $\mathbb{R}^n$, and that $\pi_E^{-1}(x) = T_x(M)$. But Definition 28.4.1, and the discussion at the very beginning of this chapter, indicate that $T_x(M)$ is the fiber over x of the tangent bundle $T(M)$. We also note that the right action $p \mapsto p\mathsf{A}$ of $GL(n, \mathbb{R})$ on $L(M)$ can be interpreted as the composite map $p \circ \mathsf{A}$:

$$\mathbb{R}^n \xrightarrow{\mathsf{A}} \mathbb{R}^n \xrightarrow{p} T_x(M).$$

All this discussion is summarized in

Box 34.1.17 *The tangent bundle $T(M)$ is associated with the bundle of linear frames $L(M)$ with standard fiber $\mathbb{R}^n$. The right action $p \mapsto p\mathsf{A}$ of $GL(n, \mathbb{R})$ on $L(M)$ can be interpreted as $\mathbb{R}^n \xrightarrow{\mathsf{A}} \mathbb{R}^n \xrightarrow{p} T_x(M)$, the composite map $p \circ \mathsf{A}$.*

Example 34.1.18 Let $\mathcal{T}_s^r(\mathbb{R}^n)$ be the set of tensors of type (r, s) over the vector space $\mathbb{R}^n$. The action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$ can be extended to an action on $\mathcal{T}_s^r(\mathbb{R}^n)$ by acting on each of the r vectors and s dual vectors separately.

Tensor fields as sections of bundles associated to $L(M)$

With $\mathcal{T}^r_s(\mathbb{R}^n)$ as the standard fiber, we obtain the tensor bundle $T^r_s(M)$ of type (r,s) over M which is associated with $L(M)$. A tensor field of type (r,s) is a section of this bundle: $\mathbf{T}^r_s : M \to T^r_s(M)$.

34.2 Connections in a PFB

A principal fiber bundle can be thought of (locally, at least) as a continuous collection of fibers, each fiber located at $x \in U \subset M$. The points of each fiber are naturally connected through the action of G. In fact, given a point of the fiber, we can construct the entire fiber by applying all $g \in G$ to that point. This is by construction and the fact that G acts freely on each fiber. Because each fiber is an orbit of G, and because G acts freely on the fiber, each fiber is diffeomorphic to G. However, there is no natural diffeomorphism connecting one fiber to its neighbor. Such a connection requires an extra structure on the principal fiber bundle, not surprisingly called connection.

fundamental vector field

Given a principal fiber bundle $P(M,G)$, the action of G on P induces a vector field on P for each $A \in \mathfrak{g}$ (see Definition 29.1.30). In fiber bundle theory, it is common to denote this vector field by A^* and call it the **fundamental vector field** corresponding to A. With $\gamma_A(t) \equiv \exp(At)$ the integral curve of A, the fundamental vector field at any point $p \in P$ is defined as

$$A^*_p = \frac{d}{dt}\big(p\gamma_A(t)\big)\bigg|_{t=0} = \frac{d}{dt}\big(p\exp(At)\big)\bigg|_{t=0}. \tag{34.3}$$

Note that $\gamma_A(0) = e$, i.e., the curve passes through the identity of the Lie group G. This is required because only $\mathcal{T}_e(G)$ is identified as the Lie algebra of G. Thus, any G-curve that passes through the identity induces a fundamental vector field in P. Since the action on P is right multiplication, Proposition 29.1.34 gives $(Ad_g A)^* = R^{-1}_{g*}A^* \equiv R_{g^{-1}*}A^*$ or equivalently, $R_{g*}A^* = (Ad_{g^{-1}}A)^* \equiv (Ad^{-1}_g A)^*$.

The diffeomorphism of each fiber $\pi^{-1}(x)$ with G leads to the isomorphism of the Lie algebra $\mathfrak{g}$ of G with the tangent space $T_p(\pi^{-1}(x))$ at each point p of the fiber; and since the action of G is confined to a fiber, the fundamental field A^* must also be confined to the tangent spaces of the fibers. To "connect" one fiber to its neighbor, we use the fundamental vector fields defined on them. This is a natural thing to do since each A^* originates from the same $A \in \mathfrak{g}$.

Is there anyway that we can make an association of A^* with its origin A in $\mathfrak{g}$? Is there a "machine" that spits out A when A^* is fed into it? The most obvious answer is a $\mathfrak{g}$-valued one form! So define a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega}$ by $\boldsymbol{\omega}(A^*) = A$. How would $\boldsymbol{\omega}$ change on $T_p(\pi^{-1}(x))$? The right action of G on $T_p(\pi^{-1}(x))$ induces a right transformation $R^*_g\boldsymbol{\omega}$. What would this give when acting on A^*?

$$R^*_g\boldsymbol{\omega}\big(A^*\big) \equiv \boldsymbol{\omega}\big(R_{g*}A^*\big) = \boldsymbol{\omega}\big(\big(Ad^{-1}_g A\big)^*\big) = Ad^{-1}_g A = Ad^{-1}_g\boldsymbol{\omega}\big(A^*\big).$$

Therefore, if a 1-form is to associate A^* with A, it must satisfy $R_g^*\boldsymbol{\omega} = Ad_g^{-1}\boldsymbol{\omega}$. When extended to the entire P (not just $\pi^{-1}(x)$), $\boldsymbol{\omega}$ defines a connection on P.

Definition 34.2.1 A **connection** Γ is a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega}$ on P such that for any vector field $\mathbf{X} \in T(P)$, $\boldsymbol{\omega}(\mathbf{X})$ is the unique $A \in \mathfrak{g}$ related to A^* that passes through that point. We demand that $\boldsymbol{\omega}$ satisfy the following conditions:

connection 1-form

(a) $\boldsymbol{\omega}(A^*) = A$.
(b) $R_g^*\boldsymbol{\omega} = Ad_g^{-1}\boldsymbol{\omega}$ on P, i.e., for any vector field $\mathbf{X} \in T(P)$,

$$\boldsymbol{\omega}(R_{g*}\mathbf{X}) = Ad_g^{-1}\boldsymbol{\omega}(\mathbf{X}).$$

We call $\boldsymbol{\omega}$ a **connection 1-form**.

Any vector field $\mathbf{Y}$ in $T(P)$ can be written as $\mathbf{Y} = \mathbf{Y}_h + A^*$ with A^* in the tangent space of a fiber. Then, since $\boldsymbol{\omega}(\mathbf{Y}) = \boldsymbol{\omega}(A^*)$, we get $\boldsymbol{\omega}(\mathbf{Y}_h) = 0$. A vector field $\mathbf{X}$ in P satisfying $\boldsymbol{\omega}(\mathbf{X}) = 0$ is called a **horizontal** vector field. A vector $\mathbf{Z}$ in a tangent space of a fiber has the property that $\pi_*(\mathbf{Z}) = 0$, because π is the constant map on any fiber: $\pi : \pi^{-1}(x) \to x$ (see Box 28.3.3). Any vector field $\mathbf{Y}$ in P satisfying $\pi_*(\mathbf{Z}) = 0$ is called a **vertical** vector field.[2] If we define the horizontal and vertical subspaces as

horizontal and vertical vector fields

$$H = \{\mathbf{X} \in T(P) \,|\, \boldsymbol{\omega}(\mathbf{X}) = 0\}, \qquad V = \{\mathbf{Z} \in T(P) | \pi_*(\mathbf{Z})\} = 0,$$

then $T(P) = H \oplus V$. Furthermore, because of (b) in the definition above, $H_{pg} = R_{g*}H_p$. Thus, at every point of P, $T(P)$ can be written as the direct sum of a horizontal and a vertical subspace and these subspaces are smoothly connected to each other. From local trivialization, we conclude that $\dim P = \dim M + \dim G$, and since $\dim V = \dim G$, we obtain $\dim H = \dim M$. Hence, $\pi_* : H_p \to T_x(M)$ is a linear isomorphism.

π_* is a linear isomorphism of H_p and T_xM.

34.2.1 Local Expression for a Connection

The local trivialization of a bundle with its corresponding sections could be used to define a $\mathfrak{g}$-valued 1-form on the base manifold M. The pull-back of $\boldsymbol{\omega}$ by a local section σ_u is indeed a $\mathfrak{g}$-valued 1-form on $U \subset M$: For $\mathbf{Y} \in T(U)$, we have (by definition) $\sigma_u^*\boldsymbol{\omega}(\mathbf{Y}) = \boldsymbol{\omega}(\sigma_{u*}\mathbf{Y})$. Since local sections depend on the subsets chosen, and since they are connected via transition functions as in Eq. (34.1), we have to know how $\boldsymbol{\omega}_u \equiv \sigma_u^*\boldsymbol{\omega}$ is related to $\boldsymbol{\omega}_v \equiv \sigma_v^*\boldsymbol{\omega}$. To take full advantage of formalism, let us write Eq. (34.1) as a composite map

$$\sigma_v : U \cap V \xrightarrow{\alpha} P \times G \xrightarrow{\Phi} P,$$

[2] See the remark after Definition 34.1.1.

where $\alpha(x) = (\sigma_u(x), g_{uv}(x))$ and $\Phi(\sigma_u(x), g_{uv}(x)) = \sigma_u(x)g_{uv}(x)$. Then $\sigma_{v*} = \Phi_* \circ \alpha_*$, and using Proposition 28.3.7, we obtain

$$\sigma_{v*}(\mathbf{Y}) = \Phi_*(\alpha_*\mathbf{Y}) = \Phi_{\sigma_u(x)*}\big(g_{uv*}(\mathbf{Y})\big) + \Phi_{g_{uv}(x)*}\big(\sigma_{u*}(\mathbf{Y})\big). \tag{34.4}$$

In the second term on the right, $\Phi_{g_{uv}(x)*} = R_{g_{uv}(x)*}$, the right multiplication. For the first term, we note that

$$\Phi_{\sigma_u(x)*}\big(g_{uv*}(\mathbf{Y})\big) = \frac{d}{dt}\big(\sigma_u(x)g_{uv}\big(\gamma_Y(t)\big)\big)\bigg|_{t=0}.$$

Now, we note that $\sigma_u(x)$ is a point in P and $g_{uv}(\gamma_Y(t))$ is a curve in G. Therefore, $\sigma_u(x)g_{uv}(\gamma_Y(t))$ is a curve in P. It is not the curve of a fundamental vector field, because $g_{uv}(\gamma_Y(0)) = g_{uv}(x) \neq e$. However, if we rewrite it as

$$\sigma_u(x)g_{uv}(x)g_{uv}(x)^{-1}g_{uv}\big(\gamma_Y(t)\big) = \sigma_v(x)\underbrace{g_{uv}(x)^{-1}g_{uv}\big(\gamma_Y(t)\big)}_{\equiv\gamma_{A^Y}(t)}$$
$$\equiv \sigma_v(x)\gamma_{A^Y}(t),$$

then $\gamma_{A^Y}(0) = e$ and the vector field associated with $\gamma_{A^Y}(t)$ is indeed a fundamental vector field. It is clear that $A^Y = L_{g_{uv}(x)^{-1}*}g_{uv*}(\mathbf{Y}) = L^{-1}_{g_{uv}(x)*}g_{uv*}(\mathbf{Y})$. We therefore, write Eq. (34.4) as

$$\sigma_{v*}(\mathbf{Y}) = A^{Y*}_{\sigma_v(x)} + R_{g_{uv}(x)*}\big(\sigma_{u*}(\mathbf{Y})\big). \tag{34.5}$$

Applying $\boldsymbol{\omega}$ on both sides, we get

$$\boldsymbol{\omega}\big(\sigma_{v*}(\mathbf{Y})\big) \equiv \boldsymbol{\omega}_v(\mathbf{Y}) = \boldsymbol{\omega}\big(A^{Y*}_{\sigma_v(x)}\big) + \boldsymbol{\omega}\big(R_{g_{uv}(x)*}\big(\sigma_{u*}(\mathbf{Y})\big)\big),$$

or

local connection forms defined on M

$$\boldsymbol{\omega}_v(\mathbf{Y}) = L^{-1}_{g_{uv}(x)*}g_{uv*}(\mathbf{Y}) + Ad^{-1}_{g_{uv}(x)}\boldsymbol{\omega}_u(\mathbf{Y}), \tag{34.6}$$

where for the first term on the right-hand side, we used (a) and for the second term we used (b) of Definition 34.2.1.

Let $\boldsymbol{\theta}$ be the left-invariant canonical 1-form of Definition 29.1.29. For each $U \cap V$ define a $\mathfrak{g}$-valued 1-form by $\boldsymbol{\theta}_{uv} = g^*_{uv}\boldsymbol{\theta}$. Then it can be shown that Eq. (34.6) can be written succinctly as

$$\boldsymbol{\omega}_v = \boldsymbol{\theta}_{uv} + Ad^{-1}_{g_{uv}}\boldsymbol{\omega}_u. \tag{34.7}$$

We defined the connection Γ on a principal fiber bundle as a $\mathfrak{g}$-valued 1-form having properties (a) and (b) of Definition 34.2.1. Then we showed two important consequences of the definition: that the 1-form splits $T(P)$ into horizontal and vertical subspaces at each point of the bundle; and that it defines a $\mathfrak{g}$-valued 1-form on each domain of the local trivializations and these 1-forms are connected by (34.6). It turns out that the two consequences are actually equivalent to the definition, i.e., that if $T_p(P) = H_p \oplus V_p$ at each $p \in P$ such that $H_{pg} = R_{g*}H_p$, then there exists a 1-form satisfying (a) and (b) of Definition 34.2.1. Similarly, the existence of a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega}_u$

on the domain U of each trivialization T_u leads to the $\mathfrak{g}$-valued 1-form of Definition 34.2.1.

Example 34.2.2 Suppose that G is a matrix group, i.e., a subgroup of $GL(n, \mathbb{R})$. Then g_{uv} is a matrix-valued *function* or 0-form. Hence, $g_{uv*} = dg_{uv}$ (see the discussion on page 874), and $L^{-1}_{g_{uv}(x)*}$ is just left multiplication by $g^{-1}_{uv}(x)$. Thus the first term on the right-hand side of Eq. (34.6) is $g^{-1}_{uv}(x)dg_{uv}(\mathbf{Y})$. For the second term, we note that for matrices A and B, we have

$$Ad_{\mathsf{A}}\mathsf{B} = \frac{d}{dt}Ad_{\mathsf{A}}\big(\gamma_{\mathsf{B}}(t)\big)\Big|_{t=0} = \frac{d}{dt}\big(\mathsf{A}\gamma_{\mathsf{B}}(t)\mathsf{A}^{-1}\big)\Big|_{t=0} = \mathsf{A}\mathsf{B}\mathsf{A}^{-1}.$$

Therefore, $Ad^{-1}_{g_{uv}(x)}\boldsymbol{\omega}_u(\mathbf{Y}) = g_{uv}(x)^{-1}\boldsymbol{\omega}_u(\mathbf{Y})g_{uv}(x)$. Consequently, the transformation rule (34.6) can be expressed as

local connection forms when G is a matrix group

$$\boldsymbol{\omega}_v = g^{-1}_{uv}dg_{uv} + g^{-1}_{uv}\boldsymbol{\omega}_u g_{uv}, \tag{34.8}$$

where it is understood that the vector field will be evaluated by dg_{uv} and $\boldsymbol{\omega}_u$ on the right-hand side.

34.2.2 Parallelism

The diffeomorphism of $\pi^{-1}(U)$ with $U \times G$ given by a local trivialization gives rise to an isomorphism of $T_p(P)$ and $T_x(U) \times T_g(G)$. A connection splits $T_p(P)$ into a horizontal subspace and a vertical subspace, of which the vertical subspace is isomorphic to $T_g(G)$. Therefore, the horizontal subspace must be isomorphic to $T_x(U) = T_x(M)$. In fact since $\pi_*(V_p) = 0$, π_* maps the horizontal subspace isomorphically to $T_x(M)$, $\pi_* : H_p \xrightarrow{\cong} T_x(M)$.

horizontal lift of vector fields

Definition 34.2.3 The **horizontal lift** (or simply the **lift**) of a vector field $\mathbf{X}$ on M is the unique vector field $\mathbf{X}^*$ on P, which is horizontal and $\pi_*(\mathbf{X}^*_p) = \mathbf{X}_{\pi(p)}$ for every $p \in P$.

From the lift of a vector field, we can move on to the lift of a curve in M. Given a curve $\gamma(t) \equiv x_t$ in M, we can lift each point of it into P and get a curve $\gamma^*(t) \equiv p_t$ in P in such a way that the tangent vector to γ^* is horizontal and maps to the tangent vector to γ at its corresponding point. More precisely,

horizontal lift of curves

Definition 34.2.4 Let $\gamma(t) \equiv x_t$ be a curve in M. A **horizontal lift** (or simply the **lift**) of γ is a horizontal curve $\gamma^*(t) \equiv p_t$ in P such that $\pi(\gamma^*(t)) = \gamma(t)$. By a horizontal curve is meant one whose tangent vectors are horizontal.

By local triviality, a curve $\gamma(t)$ in $U \subset M$ maps to a curve $\alpha(t)$ in P, which may not be horizontal. If there is a horizontal curve $\gamma^*(t)$, each of its points can be obtained from $\alpha(t)$ by right multiplication by an element of G, because $\alpha(t)$ and $\gamma^*(t)$ both belong to the same fiber for each given t. So, we have $\gamma^*(t) = \alpha(t)g(t)$. The question is if this construction actually works. The answer is yes, and we have the following proposition, whose proof can be found in [Koba 63, pp. 69–70]:

Proposition 34.2.5 *Let $\gamma(t)$, $0 \le t \le 1$, be a curve in M. For an arbitrary point p_0 of P with $\pi(p_0) = \gamma(0)$, there exists a unique horizontal lift γ^* of γ starting at p_0 (i.e., with $\gamma^*(0) = p_0$). Furthermore, the unique lift that starts at $p = p_0 g$ is $\gamma^*(t)g$.*

parallel displacement of fibers

Let $\gamma = x_t$, $0 \le t \le 1$ be a curve in M. Let p_0 be an arbitrary point in P with $\pi(p_0) = \gamma(0) = x_0$. The unique lift γ^* of γ through p_0 has the end point $p_1 = \gamma^*(1)$ such that $\pi(p_1) = \gamma(1) = x_1$. By varying p_0 in the fiber $\pi^{-1}(x_0)$, we obtain a bijection for the two fibers $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$. Denote this mapping by the same letter γ and call it the **parallel displacement** along the curve γ.

The notion of parallelism can be extended to the associated bundles as well. For this, we need to split the tangent spaces of E into horizontal and vertical at all points $w \in E$. If $\pi_E(w) = x \in M$, then the tangent space to $\pi_E^{-1}(x)$ at w, denoted by V_w, is by definition the *vertical* subspace. Let $\pi_G : P \times F \to E$ be the natural projection, so that $\pi_G(p, \xi) = [\![p, \xi]\!] \equiv w$. Choose a pair $(p, \xi) \in \pi_G^{-1}(w)$. If you fix p and let ξ vary over the entire F, by Theorem 34.1.14, you get a diffeomorphic image of F, namely $\pi_E^{-1}(x)$ if $\pi_E(w) = x$. More precisely, the diffeomorphism $p : F \to \pi_E^{-1}(x)$ has the differential map $p_* : T(F) \to T(\pi_E^{-1}(x))$ and $V_w = p_*(T(F))$.

horizontal and vertical subspaces of the associated bundle

The procedure described above for obtaining the vertical subspace gives us a hint for defining the horizontal subspace H_w as follows. Instead of fixing p, now fix ξ and let p vary. More precisely, define the map $f_\xi : P \to E$ by $f_\xi(p) = p\xi$ with differential $f_{\xi *} : T_p(P) \to T_w(E)$. Define the horizontal subspace of $T_w(E)$ to be the image of the horizontal subspace H_p of $T_p(P)$. So, $H_w \equiv f_{\xi *}(H_p)$. For this assignment to be meaningful (well-defined), it must be independent of the choice (p, ξ).

Proposition 34.2.6 *If $\pi_G(p_1, \xi_1) = w = \pi_G(p_2, \xi_2)$, then $f_{\xi_1 *}(H_{p_1}) = f_{\xi_2 *}(H_{p_2})$.*

Proof First note that $f_{g\xi} = f_\xi \circ R_g$. Next note that if $\pi_G(p_1, \xi_1) = \pi_G(p_2, \xi_2)$, then there must exist a $g \in G$ such that $p_2 = p_1 g^{-1}$ and $\xi_2 = g\xi_1$. Now use these two facts plus the invariance of the horizontal space H_p under right translation to prove the statement. □

From the diffeomorphism of $\pi_E^{-1}(U)$ and $U \times F$, and the fact that U is an *open* submanifold of M, we conclude that

$$\dim T_w(E) = \dim T_x(M) + \dim T_\xi(F) = \dim T_x(M) + \dim V_w.$$

From the diffeomorphism of $\pi^{-1}(U)$ and $U \times G$, and the split of $T_p(P)$ into horizontal and vertical subspaces, we conclude that

$$\begin{aligned}\dim H_p + \dim V_p &= \dim T_p(P) = \dim T_x(M) + \dim T_g(G)\\ &= \dim T_x(M) + \dim V_p\end{aligned}$$

so that $\dim H_p = \dim T_x(M)$. Furthermore, one can show that f_ξ is an injection. Hence, $\dim H_w = \dim H_p$, and therefore, $\dim H_w = \dim T_x(M)$. This, plus the first equation above yields $T_w(E) = H_w \oplus V_w$.

Definition 34.2.7 A vector field $\mathbf{Z}$ in E is **horizontal** if $\mathbf{Z} = f_{\xi *}(\mathbf{X}^*)$ for some horizontal vector field $\mathbf{X}^*$ in P. A curve in E is horizontal if its tangent vector is horizontal at each point of the curve. Given a curve γ in M, a (**horizontal**) **lift** is a horizontal curve γ_E^* in E such that $\pi_E(\gamma^*) = \gamma$.

horizontal lift in the associated bundle

Just as there was a unique horizontal lift for every curve γ in M starting at a given point of the principal fiber bundle P, so is there a unique horizontal lift of every curve γ in M starting at a given point of the associated bundle E. In fact, let $\gamma(t) = x_t$ be a curve in M. Let $w_0 \in E$ be such that $\pi_E(w_0) = x_0$. Then there is a $p_0 \in P$ such that $p_0\xi = w_0$. Let $\gamma^*(t) = p_t$ be the lift of x_t starting at p_0. Let $w_t = p_t\xi$. Then w_t is a horizontal curve starting at w_0. The fact that it is unique follows from the uniqueness of the solution of differential equations with give initial conditions. We thus have

Theorem 34.2.8 *Given a curve $\gamma(t) = x_t$, $0 \le t \le 1$ in M and a point $w_0 \in E$ such that $\pi_E(w_0) = x_0$, there is a unique lift $\gamma_E^*(t) = w_t$ starting at w_0. In fact, if $w_0 = [\![p_0, \xi]\!]$, then $w_t = p_t\xi$, where p_t is the lift of x_t in P starting at p_0.*

Recall that a (cross) section σ of E is a map $\sigma : U \to E$ such that $\pi_E\sigma(x) = x$. Let x_t, $0 \le t \le 1$ be a curve in U. Let $w_0 = \sigma(x_0)$. Then, clearly $\sigma(x_t)$ is a curve in E starting at w_0. Let w_t be the horizontal lift of x_t starting at w_0. In general, of course, $w_t \ne \sigma(x_t)$.

parallel section

Definition 34.2.9 We say the section σ of E is **parallel** if $\sigma(x_t)$, $0 \le t \le 1$ is the horizontal lift of x_t.

34.3 Curvature Form

A connection is a 1-form on P which allows a parallel displacement of sections of its associated bundles. Infinitesimal displacements carry the notion of differentiation which is important in differential geometry. As a 1-form, the connection accepts another kind of differentiation, namely exterior derivative. But this differentiation ought to be generalized so that it is compatible with the action of the structure group.

pseudotensorial and tensorial forms

Definition 34.3.1 Let $P(M,G)$ be a principal fiber bundle and $\rho : G \to GL(\mathcal{V})$ a representation of the structure group on a vector space $\mathcal{V}$. A **pseudotensorial form** of degree r on P of type $(\rho, \mathcal{V})$ is a $\mathcal{V}$-valued r-form $\boldsymbol{\phi}$ on P such that

$$R_g^* \boldsymbol{\phi} = \rho\big(g^{-1}\big) \cdot \boldsymbol{\phi}.$$

$\boldsymbol{\phi}$ is called a **tensorial form** if $\boldsymbol{\phi}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r) = 0$ when any of the $\mathbf{X}_i \in T(P)$ is vertical. In this case we say that $\boldsymbol{\phi}$ is horizontal.

Example 34.3.2 (a) Let ρ be the adjoint representation $Ad : g \to GL(\mathfrak{g})$ given in Definition 29.1.26. Then (b) of Definition 34.2.1 shows that the connection form $\boldsymbol{\omega}$ is a pseudotensorial form of degree 1 of type $(Ad, \mathfrak{g})$.

(b) Let ρ_0 be the trivial representation, sending all elements of G to the identity of $GL(\mathcal{V})$. Then a tensorial form of degree r of type $(\rho_0, \mathcal{V})$ is simply an r-form on P which can be written as $\boldsymbol{\phi} = \pi^* \boldsymbol{\phi}_M$ where $\boldsymbol{\phi}_M$ is a $\mathcal{V}$-valued r-form on the base manifold M. In particular, if $\mathcal{V} = \mathbb{R}$, then $\boldsymbol{\phi}$ is the pull-back by π of an ordinary r-form on M.

Remark 34.3.1 Let $E(M, \mathcal{V}, G, P)$ be the bundle associated with the principal fiber bundle P with standard fiber $\mathcal{V}$ on which G acts through a representation ρ. A tensorial form $\boldsymbol{\phi}$ of degree r of type $(\rho, \mathcal{V})$ can be considered as an assignment to each x a multilinear skewsymmetric mapping $\tilde{\boldsymbol{\phi}}_x$ of $T_x(M) \times T_x(M) \times \cdots \times T_x(M)$ (r times) into the vector space $\pi_E^{-1}(x)$ which is the fiber of E over x. Here is how:

$$\tilde{\boldsymbol{\phi}}_x(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r) \equiv p\big(\boldsymbol{\phi}\big(\mathbf{X}_1^*, \mathbf{X}_2^*, \ldots, \mathbf{X}_r^*\big)\big), \quad \mathbf{X}_i \in T_x(M). \tag{34.9}$$

On the right-hand side, $\mathbf{X}_i^*$ is *any* vector field at p such that $\pi_*(\mathbf{X}_i^*) = \mathbf{X}_i$, and p is any point of P with $\pi(p) = x$. Since $\boldsymbol{\phi}$ is $\mathcal{V}$-valued, $\boldsymbol{\phi}(\mathbf{X}_1^*, \mathbf{X}_2^*, \ldots, \mathbf{X}_r^*)$ is in $\mathcal{V}$, the standard fiber of E, on which p acts according to Theorem 34.1.14 to give a vector in $\pi_E^{-1}(x)$. As Problem 34.12 shows, the left-hand side of Eq. (34.9) is independent of the choice of p and $\mathbf{X}_i^*$ on the right-hand side. Conversely, given a skewsymmetric multilinear mapping $\tilde{\boldsymbol{\phi}}_x$ of $T_x(M) \times T_x(M) \times \cdots \times T_x(M)$ to $\pi_E^{-1}(x)$ for each $x \in M$, $p^{-1} \circ \tilde{\boldsymbol{\phi}}_x \circ \pi_*$ is a tensorial r-form of type $(\rho, \mathcal{V})$, with p chosen such that $\pi(p) = x$. In particular, a cross section $\tilde{f} : M \to E$ can be identified with $f = p^{-1} \circ \tilde{f} \circ \pi$, which is a $\mathcal{V}$-valued function on P satisfying $f(pg) = \rho(g^{-1}) f(p)$.

In the special case where ρ is the identity representation and $\mathcal{V} = \mathbb{R}$, $\tilde{\boldsymbol{\phi}}$ is just an ordinary r-form, i.e., $\tilde{\boldsymbol{\phi}} \in \Lambda^r(M)$.

Let $P(M,G)$ be a principal fiber bundle with a connection, giving rise to the split $T_p(P) = H_p \oplus V_p$ into horizontal and vertical subspaces at each point $p \in P$. Define $h : T_p(P) \to H_p$ to be the projection onto the horizontal subspace. For a pseudotensorial r-form $\boldsymbol{\phi}$ on P, define $\boldsymbol{\phi}h$ by

$$(\boldsymbol{\phi}h)(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r) = \boldsymbol{\phi}(h\mathbf{X}_1, h\mathbf{X}_2, \ldots, h\mathbf{X}_r), \quad \mathbf{X}_i \in T_p(P). \tag{34.10}$$

Definition 34.3.3 Let $P(M, G)$ be a principal fiber bundle with a connection 1-form $\boldsymbol{\omega}$ that induces the split $T_p(P) = H_p \oplus V_p$. Let $h : T_p(P) \to H_p$ be the projection onto the horizontal subspace. The **exterior covariant derivative** (associated with $\boldsymbol{\omega}$) of a (pseudo)tensorial r-form $\boldsymbol{\phi}$ is defined as $D^\omega \boldsymbol{\phi} = (d\boldsymbol{\phi})h$ and D^ω is called the **exterior covariant differentiation**.

exterior covariant differentiation

The proof of the following proposition is straightforward:

Proposition 34.3.4 *Let $\boldsymbol{\phi}$ be a pseudotensorial r-form on P of type $(\rho, \mathcal{V})$. Then*

(a) *the form $\boldsymbol{\phi}h$ is a tensorial r-form of type $(\rho, \mathcal{V})$;*
(b) *$d\boldsymbol{\phi}$ is a pseudotensorial $(r+1)$-form of type $(\rho, \mathcal{V})$;*
(c) *$D^\omega \boldsymbol{\phi}$ is a tensorial $(r+1)$-form of type $(\rho, \mathcal{V})$.*

Definition 34.3.5 The tensorial 2-form $\boldsymbol{\Omega}^\omega \equiv D^\omega \boldsymbol{\omega}$ of type $(Ad, \mathfrak{g})$ is called the **curvature form** of $\boldsymbol{\omega}$.

curvature form

The proof of the following **structure equation** can be found in [Koba 63, pp. 77–78]

Theorem 34.3.6 *The curvature form $\boldsymbol{\Omega}^\omega$ of the connection form $\boldsymbol{\omega}$ satisfies the following equation*:

$$\boldsymbol{\Omega}^\omega(\mathbf{X}, \mathbf{Y}) = d\boldsymbol{\omega}(\mathbf{X}, \mathbf{Y}) + \frac{1}{2}\big[\boldsymbol{\omega}(\mathbf{X}), \boldsymbol{\omega}(\mathbf{Y})\big], \quad \mathbf{X}, \mathbf{Y} \in T_p(P).$$

The equation of this theorem is abbreviated as

structure equation

$$\boldsymbol{\Omega}^\omega = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]. \tag{34.11}$$

The commutator in (34.11) is a Lie algebra commutator (in particular, it is *not* zero). In fact, Eq. (34.13) below captures the meaning of (34.11).

If $\mathbf{X}$ and $\mathbf{Y}$ are both horizontal vector fields on P, then Theorem 28.5.11 yields

$$\boldsymbol{\Omega}^\omega(\mathbf{X}, \mathbf{Y}) = -\boldsymbol{\omega}\big([\mathbf{X}, \mathbf{Y}]\big). \tag{34.12}$$

Note that the right-hand side is not zero, because the Lie bracket of two horizontal vector fields is not necessarily horizontal.

It is convenient to have an expression for the structure equation in terms of real-valued forms. So, let $\{\mathbf{E}_i\}_{i=1}^m$ be a basis for the Lie algebra $\mathfrak{g}$ with structure constants c^i_{jk}, so that

$$[\mathbf{E}_j, \mathbf{E}_k] = c^i_{jk}\mathbf{E}_i, \quad j, k = 1, 2, \ldots, m.$$

As $\mathfrak{g}$-valued forms, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}^\omega$ can be expressed as $\boldsymbol{\omega} = \omega^i \mathbf{E}_i$ and $\boldsymbol{\Omega}^\omega = \Omega^i \mathbf{E}_i$, where ω^i and Ω^i are ordinary real-valued forms. It is straightforward to show that the structure equation can be expressed as

$$\Omega^i = d\omega^i + \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k, \quad i = 1, 2, \dots, m. \tag{34.13}$$

Taking the exterior derivative of both sides of this equation, one can show that

$$d\Omega^i = c^i_{jk} \Omega^j \wedge \omega^k, \quad \text{or} \quad d\boldsymbol{\Omega}^\omega = [\boldsymbol{\Omega}^\omega, \boldsymbol{\omega}]. \tag{34.14}$$

The proof of the following theorem, which follows easily from the last two equations, is left as an exercise for the reader (see Problem 34.16):

Bianchi's identity

Theorem 34.3.7 (Bianchi's identity) $D^\omega \boldsymbol{\Omega}^\omega = 0$.

In Sect. 34.2.1, we expressed a connection Γ in terms of its 1-form defined on the base manifold M. It is instructive to obtain similar expressions for the curvature form as well. In fact, since the local connection form was simply the pull-back by the local sections, and since the exterior derivative and exterior product both commute with the pullback, we define $\boldsymbol{\Omega}^\omega_u \equiv \sigma^*_u \boldsymbol{\Omega}^\omega$ and easily prove

Theorem 34.3.8 $\boldsymbol{\Omega}^\omega_u = d\boldsymbol{\omega}_u + \frac{1}{2}[\boldsymbol{\omega}_u, \boldsymbol{\omega}_u]$.

We found how different pieces of the connection 1-form, defined on different subsets of M, were related to each other (see (34.6)). We can do the same with the curvature 2-form. Using Eq. (34.5) and the definition of pullback, we find

$$\begin{aligned}
\boldsymbol{\Omega}^\omega_v(\mathbf{X}, \mathbf{Y}) &\equiv \sigma^{v*} \boldsymbol{\Omega}^\omega(\mathbf{X}, \mathbf{Y}) = \boldsymbol{\Omega}^\omega(\sigma_{v*}\mathbf{X}, \sigma_{v*}\mathbf{Y}) \\
&= \boldsymbol{\Omega}^\omega\big(A^{X*}_{\sigma_v(x)} + R_{g_{uv}(x)*}(\sigma_{u*}\mathbf{X}), A^{Y*}_{\sigma_v(x)} + R_{g_{uv}(x)*}(\sigma_{u*}\mathbf{Y})\big) \\
&= \boldsymbol{\Omega}^\omega\big(R_{g_{uv}(x)*}(\sigma_{u*}\mathbf{X}), R_{g_{uv}(x)*}(\sigma_{u*}\mathbf{Y})\big),
\end{aligned}$$

because $\boldsymbol{\Omega}^\omega$ is a *tensorial* form, and hence, gives zero for its vertical arguments. By Proposition 34.3.4, $\boldsymbol{\Omega}^\omega$ is of the same type as $\boldsymbol{\omega}$, i.e., of type $(Ad, \mathfrak{g})$. Therefore, we have

$$\boldsymbol{\Omega}^\omega_v(\mathbf{X}, \mathbf{Y}) = Ad_{g^{-1}_{uv}} \boldsymbol{\Omega}^\omega(\sigma_{u*}\mathbf{X}, \sigma_{u*}\mathbf{Y}) = Ad_{g^{-1}_{uv}} \sigma^{u*} \boldsymbol{\Omega}^\omega(\mathbf{X}, \mathbf{Y}),$$

or

$$\boldsymbol{\Omega}^\omega_v = Ad_{g^{-1}_{uv}} \boldsymbol{\Omega}^\omega_u \tag{34.15}$$

Using Eq. (34.14), a similar derivation leads to the local version of the Bianchi's identity:

$$d\boldsymbol{\Omega}^\omega_u = [\boldsymbol{\Omega}^\omega_u, \boldsymbol{\omega}_u]. \tag{34.16}$$

All the foregoing discussion simplifies considerably if the structure group is abelian. In this case all the structure constants c^i_{jk} vanish and Equations (34.13) and (34.14) become $\Omega^i = d\omega^i$ and $d\Omega^i = 0$, respectively. Furthermore, it can be shown (Problem 34.17) that $Ad_g = \mathrm{id}_{\mathfrak{g}}$, the identity of the Lie algebra of G. We summarize all this in

The abelian case

Proposition 34.3.9 *If the structure group is abelian, then*

$$\boldsymbol{\omega}_v = \boldsymbol{\theta}_{uv} + \boldsymbol{\omega}_u, \qquad \boldsymbol{\Omega}^\omega = d\boldsymbol{\omega}, \qquad d\boldsymbol{\Omega}^\omega = 0,$$
$$\boldsymbol{\Omega}^\omega_u = d\boldsymbol{\omega}_u, \qquad \boldsymbol{\Omega}^\omega_v = \boldsymbol{\Omega}^\omega_u$$

where $\boldsymbol{\theta}_{uv}$ is as in Eq. (34.7) *and represents the first term on the right-hand side of Eq.* (34.6).

34.3.1 Flat Connections

Let $P = M \times G$ be a trivial principal fiber bundle. Let $\pi_2 : M \times G \to G$ be the projection onto the second factor. The differential of this map, $\pi_{2*} : T(M) \times T(G) \to T(G)$, has the property that $\pi_{2*}(\mathbf{X}) = 0$ if $\mathbf{X} \in T(M)$ (see Box 28.3.3). With $\boldsymbol{\theta}$ the canonical left-invariant 1-form on G, let $\boldsymbol{\omega} = \pi_2^*\boldsymbol{\theta}$. Then $\boldsymbol{\omega}$ is a 1-form on P, and one can show that it satisfies the two conditions of Definition 34.2.1. Hence, $\boldsymbol{\omega}$ is a connection on P. The horizontal space of this connection is clearly $T(M)$. The connection associated with this $\boldsymbol{\omega}$ is called the **canonical flat connection** of P. The Maurer-Cartan equation (29.17) yields

canonical flat connection

$$d\boldsymbol{\omega} = d\left(\pi_2^*\boldsymbol{\theta}\right) = \pi_2^*(d\boldsymbol{\theta}) = \pi_2^*\left(-\frac{1}{2}[\boldsymbol{\theta}, \boldsymbol{\theta}]\right)$$
$$= -\frac{1}{2}\left[\pi_2^*(\boldsymbol{\theta}), \pi_2^*(\boldsymbol{\theta})\right] = -\frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}].$$

Comparison with Eq. (34.11) implies that the curvature of the canonical flat connection is zero.

Definition 34.3.10 A connection Γ in *any* principal fiber bundle $P(M, G)$ is called **flat** if every $x \in M$ has a neighborhood U such that the connection in $\pi^{-1}(U)$ is isomorphic to the canonical flat connection in $U \times G$.

flat connection

The vanishing of the curvature is a necessary condition for a connection to be flat. It turns out that it is also sufficient (see [Koba 63, p. 92] for a proof):

Theorem 34.3.11 *A connection in a principal fiber bundle $P(M, G)$ is flat if and only if the curvature form vanishes identically.*

The existence of a flat connection in a principal fiber bundle determines the nature of the bundle:

Corollary 34.3.12 *If $P(M, G)$ has a connection whose curvature form vanishes identically, then $P(M, G)$ is (isomorphic to) the trivial bundle $M \times G$ and the connection is the canonical flat connection.*

34.3.2 Matrix Structure Group

The structure groups encountered in physics are almost exclusively matrix groups, or subgroups of $GL(n, \mathbb{R})$. For these groups, the equations derived above take a simpler form [see, for example, Eq. (34.8)]. Furthermore, it is a good idea to have these special formulas, so we can use them when need arises.

Proposition 34.3.13 *Let N be a manifold and G a matrix Lie group with Lie algebra $\mathfrak{g}$. For $\boldsymbol{\phi} \in \Lambda^k(N, \mathfrak{g})$ and $\boldsymbol{\psi} \in \Lambda^j(N, \mathfrak{g})$, we have*

$$[\boldsymbol{\phi}, \boldsymbol{\psi}] = \boldsymbol{\phi} \wedge \boldsymbol{\psi} - (-1)^{kj} \boldsymbol{\psi} \wedge \boldsymbol{\phi}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are regarded as matrices of $\mathbb{R}$-valued forms and $\boldsymbol{\phi} \wedge \boldsymbol{\psi}$ is matrix multiplication with elements multiplied via wedge product.

Proof For matrix algebras, the commutator is just the difference in products of matrices. Therefore,

$$\begin{aligned}
&[\boldsymbol{\phi}, \boldsymbol{\psi}](\mathbf{X}_1, \ldots, \mathbf{X}_{k+j}) \\
&\quad \equiv \underbrace{\frac{1}{k!j!} \sum_{\pi} \epsilon_\pi \boldsymbol{\phi}(\mathbf{X}_{\pi(1)}, \ldots, \mathbf{X}_{\pi(k)}) \boldsymbol{\psi}(\mathbf{X}_{\pi(k+1)}, \ldots, \mathbf{X}_{\pi(k+j)})}_{=(\boldsymbol{\phi} \wedge \boldsymbol{\psi})(\mathbf{X}_1, \ldots, \mathbf{X}_{k+j})} \\
&\qquad - \frac{1}{k!j!} \sum_{\pi} \epsilon_\pi \boldsymbol{\psi}(\mathbf{X}_{\pi(k+1)}, \ldots, \mathbf{X}_{\pi(k+j)}) \boldsymbol{\phi}(\mathbf{X}_{\pi(1)}, \ldots, \mathbf{X}_{\pi(k)}) \\
&\quad = (\boldsymbol{\phi} \wedge \boldsymbol{\psi})(\mathbf{X}_1, \ldots, \mathbf{X}_{k+j}) \\
&\qquad - \frac{1}{k!j!} \sum_{\pi} \epsilon_\pi (-1)^{kj} \boldsymbol{\psi}(\mathbf{X}_{\pi(1)}, \ldots, \mathbf{X}_{\pi(j)}) \boldsymbol{\phi}(\mathbf{X}_{\pi(j+1)}, \ldots, \mathbf{X}_{\pi(j+k)}).
\end{aligned}$$

Noting that the last sum is $(-1)^{kj}(\boldsymbol{\psi} \wedge \boldsymbol{\phi})(\mathbf{X}_1, \ldots, \mathbf{X}_{k+j})$, we obtain the result we are after. □

Corollary 34.3.14 *If G is a matrix group, then*

$$\boldsymbol{\Omega}^\omega = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} \quad \text{and} \quad \boldsymbol{\Omega}_u^\omega = d\boldsymbol{\omega}_u + \boldsymbol{\omega}_u \wedge \boldsymbol{\omega}_u.$$

Proof The proof follows immediately from Eq. (34.11) and Proposition 34.3.13 with $k = j = 1$. □

The following theorem can also be easily proved:

Theorem 34.3.15 *Let T_u and T_v be two local trivializations with transition function $g_{uv} : U \cap V \to G$, where G is a matrix group. Then*

(a) $\mathbf{\Omega}_v^\omega = g_{uv}^{-1}\mathbf{\Omega}_u^\omega g_{uv}$;
(b) $d\mathbf{\Omega}_u^\omega = \mathbf{\Omega}_u^\omega \wedge \boldsymbol{\omega}_u - \boldsymbol{\omega}_u \wedge \mathbf{\Omega}_u^\omega$.

34.4 Problems

34.1 Show that

(a) a fiber bundle homomorphism preserves the fibers, i.e., two points belonging to the same fiber of P' get mapped to the same fiber of P;
(b) if (f, f_G) of Definition 34.1.3 is an isomorphism, then the induced map $f_M : M' \to M$ is a bijection.

34.2 Finish the proof of Proposition 34.1.5.

34.3 Complete the proof of Proposition 34.1.9.

34.4 Using the linear independence of the vectors in a basis, show that the action of $GL(n, \mathbb{R})$ on $L(M)$ is free.

34.5 Show that $\phi_u : \pi^{-1}(U) \to U \times F$ defined by $\phi_u([\![p, \xi]\!]) = (\pi(p), s_u(p)\xi)$ for the associated fiber bundle is well defined (i.e., if $[\![p', \xi']\!] = [\![p, \xi]\!]$ then $\phi_u([\![p', \xi']\!]) = \phi_u([\![p, \xi]\!])$) and bijective.

34.6 Show that the map $p : E \to F_x$, given by $p(\xi) \equiv p\xi = [\![p, \xi]\!]$ satisfies

$$(pg)\xi = p(g\xi) \quad \text{for } p \in P,\ g \in G,\ \xi \in E.$$

34.7 Show that the map $S_u : U \times G \to \pi^{-1}(U)$, given by $S_u(x, g) = \sigma_u(x)g$ for a local cross section σ_u is a bijection, and that $T_u = S_u^{-1}$ is a local trivialization satisfying condition (3) of Definition 34.1.1.

34.8 Provide the details of the proof of Proposition 34.2.6.

34.9 Show that the map $f_\xi : P \to E$ defined by $f_\xi(p) = p\xi$ is injective. Hint: Show that $[\![p_1, \xi]\!] = [\![p_2, \xi]\!]$ implies $p_1 = p_2$.

34.10 Show that Eq. (34.7) follows from Eq. (34.6).

34.11 Show that the canonical flat connection on $P = M \times G$ given by $\boldsymbol{\omega} = \pi_2^*\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is the canonical 1-form on G satisfies both conditions of Definition 34.2.1.

34.12 Show that Eq. (34.9) is independent of the choice of p and $\mathbf{X}_i^*$ on the right-hand side.

34.13 Prove Proposition 34.3.4.

34.14 Derive Eq. (34.12).

34.15 Derive Eq. (34.13).

34.16 Taking the exterior derivative of both sides of Eq. (34.13), show that

$$d\Omega^i = c^i_{jk}\Omega^j \wedge \omega^k - \frac{1}{2}c^i_{jk}c^j_{lm}\omega^l \wedge \omega^m \wedge \omega^k.$$

Using Lie's third theorem (29.13), show that the second term on the right-hand side vanishes. Now prove Bianchi's identity of Theorem 34.3.7. Hint: $\boldsymbol{\omega}(\mathbf{X}) = 0$ if $\mathbf{X}$ is horizontal.

34.17 Let $\mathrm{id}_M : M \to M$ be the identity map on M. Prove that id_{M*} is the identity map on $T(M)$. Let $I_g = R_{g^{-1}} \circ L_g$ be the inner automorphism of a Lie group G. Show that if G is abelian, then $I_g = \mathrm{id}_G$ for all $g \in G$. Now show that $Ad_g = \mathrm{id}_{\mathfrak{g}}$.

34.18 Prove Theorem 34.3.8.

34.19 Provide the details of Corollary 34.3.14.

34.20 Provide the details of Theorem 34.3.15.