9 Fourier Analysis

The single most recurring theme of mathematical physics is Fourier analysis. It shows up, for example, in classical mechanics and the analysis of normal modes, in electromagnetic theory and the frequency analysis of waves, in noise considerations and thermal physics, in quantum theory and the transformation between momentum and coordinate representations, and in relativistic quantum field theory and creation and annihilation operation formalism.

9.1 Fourier Series

One way to begin the study of Fourier series and transforms is to invoke a generalization of the Stone-Weierstrass Approximation Theorem (Theorem 7.2.3), which established the completeness of monomials, x^k. The generalization of Theorem 7.2.3 permits us to find another set of orthogonal functions in terms of which we can expand an arbitrary function. This generalization involves polynomials in more than one variable ([Simm 83, pp. 160–161]):

generalized Stone-Weierstrass theorem

Theorem 9.1.1 (Generalized Stone-Weierstrass Theorem) *If the function $f(x_1, x_2, \ldots, x_n)$ is continuous in the domain $\{a_i \le x_i \le b_i\}_{i=1}^n$, then it can be expanded in terms of the monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, where the k_i are nonnegative integers.*

Now let us consider functions that are periodic and investigate their expansion in terms of elementary periodic functions. We use the generalized Stone-Weierstrass theorem with two variables, x and y. A function $g(x, y)$ can be written as $g(x, y) = \sum_{k,m=0}^{\infty} a_{km} x^k y^m$. In this equation, x and y can be considered as coordinates in the xy-plane, which in turn can be written in terms of polar coordinates r and θ. In that case, we obtain

$$f(r, \theta) \equiv g(r\cos\theta, r\sin\theta) = \sum_{k,m=0}^{\infty} a_{km} r^{k+m} \cos^k\theta \sin^m\theta.$$

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In particular, if we let $r = 1$, we obtain a function of θ alone, which upon substitution of complex exponentials for $\sin\theta$ and $\cos\theta$ becomes

$$f(\theta) = \sum_{k,m=0}^{\infty} a_{km} \frac{1}{2^k}\left(e^{i\theta} + e^{-i\theta}\right)^k \frac{1}{(2i)^m}\left(e^{i\theta} - e^{-i\theta}\right)^m = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \tag{9.1}$$

where b_n is a constant that depends on a_{km}. The RHS of (9.1) is periodic with period 2π; thus, it is especially suitable for periodic functions $f(\theta)$ that satisfy the periodicity condition $f(\theta - \pi) = f(\theta + \pi)$.

We can also write Eq. (9.1) as

$$\begin{aligned} f(\theta) &= b_0 + \sum_{n=1}^{\infty}\left(b_n e^{in\theta} + b_{-n} e^{-in\theta}\right) \\ &= b_0 + \sum_{n=1}^{\infty}\Big[\underbrace{(b_n + b_{-n})}_{\equiv A_n}\cos n\theta + \underbrace{i(b_n - b_{-n})}_{\equiv B_n}\sin n\theta)\Big] \end{aligned}$$

or

$$f(\theta) = b_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta). \tag{9.2}$$

If $f(\theta)$ is real, then b_0, A_n, and B_n are also real. Equation (9.1) or (9.2) is called the **Fourier series expansion** of $f(\theta)$.

Let us now concentrate on the elementary periodic functions $e^{in\theta}$. We define the $\{|e_n\rangle\}_{n=1}^{\infty}$ such that their "θth components" are given by

$$\langle\theta|e_n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta}, \quad \text{where } \theta \in (-\pi, \pi).$$

These functions—or ket vectors—which belong to $\mathcal{L}^2(-\pi, \pi)$, are orthonormal, as can be easily verified. It can also be shown that they are complete. In fact, for functions that are *continuous* on $(-\pi, \pi)$, this is a result of the generalized Stone-Weierstrass theorem. It turns out, however, that $\{|e_n\rangle\}_{n=1}^{\infty}$ is also a complete orthonormal sequence for *piecewise continuous* functions on $(-\pi, \pi)$.[1] Therefore, any periodic piecewise continuous function of θ can be expressed as a linear combination of these orthonormal vectors. Thus if $|f\rangle \in \mathcal{L}^2(-\pi, \pi)$, then

piecewise continuous

$$|f\rangle = \sum_{n=-\infty}^{\infty} f_n|e_n\rangle, \quad \text{where } f_n = \langle e_n|f\rangle. \tag{9.3}$$

Fourier series expansion: angular expression

We can write this as a functional relation if we take the θth component of both sides: $\langle\theta|f\rangle = \sum_{n=-\infty}^{\infty} f_n\langle\theta|e_n\rangle$, or

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \tag{9.4}$$

[1] A piecewise continuous function on a finite interval is one that has a finite number of discontinuities in its interval of definition.

with f_n given by

$$f_n = \langle e_n|\mathbf{1}|f\rangle = \langle e_n|\left(\int_{-\pi}^{\pi} |\theta\rangle\langle\theta|\,d\theta\right)|f\rangle$$
$$= \int_{-\pi}^{\pi} \langle e_n|\theta\rangle\langle\theta|f\rangle\,d\theta = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\,d\theta. \tag{9.5}$$

It is important to note that even though $f(\theta)$ may be defined only for $-\pi \le \theta \le \pi$, Eq. (9.4) extends the domain of definition of $f(\theta)$ to all the intervals $(2k-1)\pi \le \theta \le (2k+1)\pi$ for all $k \in \mathbb{Z}$. Thus, if a function is to be represented by Eq. (9.4) without any specification of the interval of definition, it must be periodic in θ. For such functions, the interval of their definition can be translated by a factor of 2π. Thus, $f(\theta)$ with $-\pi \le \theta \le \pi$ is equivalent to $f(\theta - 2m\pi)$ with $2m\pi - \pi \le \theta \le 2m\pi + \pi$; both will give the same Fourier series expansion. We shall define periodic functions in their **fundamental cell** such as $(-\pi, \pi)$.

fundamental cell of a periodic function

Historical Notes

Joseph Fourier (1768–1830) did very well as a young student of mathematics but had set his heart on becoming an army officer. Denied a commission because he was the son of a tailor, he went to a Benedictine school with the hope that he could continue studying mathematics at its seminary in Paris. The French Revolution changed those plans and set the stage for many of the personal circumstances of Fourier's later years, due in part to his courageous defense of some of its victims, an action that led to his arrest in 1794. He was released later that year, and he enrolled as a student in the Ecole Normale, which opened and closed within a year. His performance there, however, was enough to earn him a position as assistant lecturer (under Lagrange and Monge) in the *Ecole Polytechnique*. He was an excellent mathematical physicist, was a friend of Napoleon (so far as such people have friends), and accompanied him in 1798 to Egypt, where Fourier held various diplomatic and administrative posts while also conducting research. Napoleon took note of his accomplishments and, on Fourier's return to France in 1801, appointed him prefect of the district of Isère, in southeastern France, and in this capacity built the first real road from Grenoble to Turin. He also befriended the boy Champollion, who later deciphered the *Rosetta stone* as the first long step toward understanding the hieroglyphic writing of the ancient Egyptians.

"The profound study of nature is the most fruitful source of mathematical discoveries." Joseph Fourier

Like other scientists of his time, Fourier took up the flow of heat. The flow was of interest as a practical problem in the handling of metals in industry and as a scientific problem in attempts to determine the temperature in the interior of the earth, the variation of that temperature with time, and other such questions. He submitted a basic paper on heat conduction to the Academy of Sciences of Paris in 1807. The paper was judged by Lagrange, Laplace, and Legendre, and was not published, mainly due to the objections of Lagrange, who had earlier rejected the use of trigonometric series. But the Academy did wish to encourage Fourier to develop his ideas, and so made the problem of the propagation of heat the subject of a grand prize to be awarded in 1812. Fourier submitted a revised paper in 1811, which was judged by the men already mentioned and others. It won the prize but was criticized for its lack of rigor and so was not published at that time in the *Mémoires* of the Academy.

Joseph Fourier 1768–1830

Fourier developed a mastery of clear notation, some of which is still in use today. (The modern integral sign and the placement of the limits of integration near its top and bottom were introduced by Fourier.) It was also his habit to maintain close association between mathematical relations and physically measurable quantities, especially in limiting or asymptotic cases, even performing some of the experiments himself. He was one of the first to begin full incorporation of physical constants into his equations, and made considerable strides toward the modern ideas of units and dimensional analysis.

Fourier continued to work on the subject of heat and, in 1822, published one of the classics of mathematics, *Théorie Analytique de la Chaleur*, in which he made extensive use of the series that now bear his name and incorporated the first part of his 1811 paper practically

without change. Two years later he became secretary of the Academy and was able to have his 1811 paper published in its original form in the *Mémoires*.
Fourier series were of profound significance in connection with the evolution of the concept of a function, the rigorous theory of definite integrals, and the development of **Hilbert spaces**. Fourier claimed that "arbitrary" graphs can be represented by trigonometric series and should therefore be treated as legitimate functions, and it came as a shock to many that he turned out to be right. The classical definition of the definite integral due to Riemann was first given in his fundamental paper of 1854 on the subject of Fourier series. Hilbert thought of a function as represented by an infinite sequence, the Fourier coefficients of the function.
Fourier himself is one of the fortunate few: his name has become rooted in all civilized languages as an adjective that is well-known to physical scientists and mathematicians in every part of the world.

Functions are not always defined on $(-\pi, \pi)$. Let us consider a function $F(x)$ that is defined on (a, b) and is periodic with period $L = b - a$. We define a new variable,

$$\theta \equiv \frac{2\pi}{L}\left(x - a - \frac{L}{2}\right) \quad \Rightarrow \quad x = \frac{L}{2\pi}\theta + a + \frac{L}{2},$$

and note that $f(\theta) \equiv F((L/2\pi)\theta + a + L/2)$ has period $(-\pi, \pi)$ because

$$f(\theta \pm \pi) = F\left(\frac{L}{2\pi}(\theta \pm \pi) + a + \frac{L}{2}\right) = F\left(x \pm \frac{L}{2}\right)$$

and $F(x + L/2) = F(x - L/2)$. If follows that we can expand the latter as in Eq. (9.4). Using that equation, but writing θ in terms of x, we obtain

Fourier series expansion: general expression

$$F(x) = F\left(\frac{L}{2\pi}\theta + a + \frac{L}{2}\right) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f_n \exp\left[in\frac{2\pi}{L}\left(x - a - \frac{L}{2}\right)\right]$$

$$= \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} F_n e^{2n\pi i x/L}, \tag{9.6}$$

where we have introduced[2] $F_n \equiv \sqrt{L/2\pi} f_n e^{-i(2\pi n/L)(a+L/2)}$. Using Eq. (9.5), we can write

$$F_n = \sqrt{\frac{L}{2\pi}} e^{-i(2\pi n/L)(a+L/2)} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta$$

$$= \frac{\sqrt{L}}{2\pi} e^{-i(2\pi n/L)(a+L/2)} \int_a^{a+L} e^{-i(2\pi n/L)(x-a-L/2)} F(x) \frac{2\pi}{L}\, dx$$

$$= \frac{1}{\sqrt{L}} \int_a^b e^{-i(2\pi n/L)x} F(x)\, dx. \tag{9.7}$$

The functions $\exp(2\pi i n x/L)/\sqrt{L}$ are easily seen to be orthonormal as members of $\mathcal{L}^2(a, b)$. We can introduce $\{|e_n\rangle\}_{n=1}^{\infty}$ with the "xth component" given by $\langle x|e_n\rangle = (1/\sqrt{L})e^{2\pi i n x/L}$. Then the reader may check

[2] The F_n are defined such that what they multiply in the expansion are orthonormal in the interval (a, b).

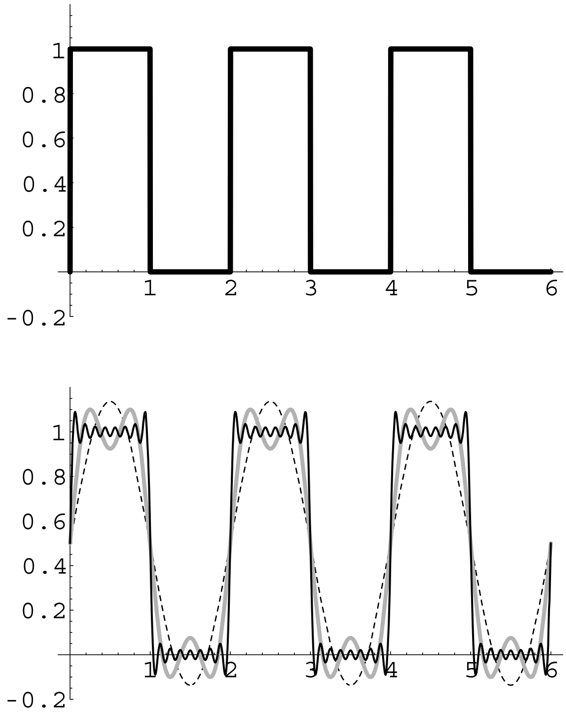

Fig. 9.1 *Top*: The periodic square wave potential with height taken to be 1. *Bottom*: Various approximations to the Fourier series of the square-wave potential. The *dashed plot* is that of the first term of the series, the *thick grey plot* keeps 3 terms, and the *solid plot* 15 terms

that Eqs. (9.6) and (9.7) can be written as $|F\rangle = \sum_{n=-\infty}^{\infty} F_n|e_n\rangle$ with $F_n = \langle n|F\rangle$.

Example 9.1.2 In the study of electrical circuits, periodic voltage signals of different shapes are encountered. An example is a **square wave** voltage of height U_0, "duration" T, and "rest duration" T [see Fig. 9.1(a)]. The potential as a function of time $V(t)$ can be expanded as a Fourier series. The interval is $(0, 2T)$ because that is one whole cycle of the potential variation. We therefore use Eq. (9.6) and write

square wave voltage

$$V(t) = \frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} V_n e^{2n\pi it/2T}, \quad \text{where}$$

$$V_n = \frac{1}{\sqrt{2T}} \int_0^{2T} e^{-2\pi int/2T} V(t)\,dt.$$

The problem is to find V_n. This is easily done by substituting

$$V(t) = \begin{cases} U_0 & \text{if } 0 \le t \le T, \\ 0 & \text{if } T \le t \le 2T \end{cases}$$

in the last integral:

$$V_n = \frac{U_0}{\sqrt{2T}} \int_0^T e^{-in\pi t/T}\, dt = \frac{U_0}{\sqrt{2T}} \left(-\frac{T}{in\pi} \right) \left[(-1)^n - 1 \right] \quad \text{where } n \neq 0$$
$$= \begin{cases} 0 & \text{if } n \text{ is even and } n \neq 0, \\ \frac{\sqrt{2T}\, U_0}{in\pi} & \text{if } n \text{ is odd.} \end{cases}$$

For $n = 0$, we obtain $V_0 = \frac{1}{\sqrt{2T}} \int_0^{2T} V(t)\, dt = \frac{1}{\sqrt{2T}} \int_0^T U_0\, dt = U_0 \sqrt{\frac{T}{2}}$. Therefore, we can write

$$V(t) = \frac{1}{\sqrt{2T}} \left[U_0 \sqrt{\frac{T}{2}} + \frac{\sqrt{2T}\, U_0}{i\pi} \left(\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{-1} \frac{1}{n} e^{in\pi t/T} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} e^{in\pi t/T} \right) \right]$$
$$= U_0 \left\{ \frac{1}{2} + \frac{1}{i\pi} \left[\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{-n} e^{-in\pi t/T} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} e^{in\pi t/T} \right] \right\}$$
$$= U_0 \left\{ \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(\frac{[2k+1]\pi t}{T} \right) \right\}.$$

Figure 9.1(b) shows the graphical representation of the above infinite sum when only a finite number of terms are present.

sawtooth voltage

Example 9.1.3 Another frequently used voltage is the **sawtooth** voltage [see Fig. 9.2(a)]. The equation for $V(t)$ with period T is $V(t) = U_0 t/T$ for $0 \leq t \leq T$, and its Fourier representation is

$$V(t) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} V_n e^{2n\pi i t/T}, \quad \text{where} \quad V_n = \frac{1}{\sqrt{T}} \int_0^T e^{-2\pi i n t/T} V(t)\, dt.$$

Substituting for $V(t)$ in the integral above yields

$$V_n = \frac{1}{\sqrt{T}} \int_0^T e^{-2\pi i n t/T} U_0 \frac{t}{T}\, dt = U_0 T^{-3/2} \int_0^T e^{-2\pi i n t/T}\, t\, dt$$
$$= U_0 T^{-3/2} \left(\frac{Tt}{-i2n\pi} e^{-2\pi i n t/T} \Big|_0^T + \frac{T}{i2n\pi} \underbrace{\int_0^T e^{-2\pi i n t/T} dt}_{=0} \right)$$
$$= U_0 T^{-3/2} \left(\frac{T^2}{-i2n\pi} \right) = -\frac{U_0 \sqrt{T}}{i2n\pi} \quad \text{where } n \neq 0,$$
$$V_0 = \frac{1}{\sqrt{T}} \int_0^T V(t)\, dt = \frac{1}{\sqrt{T}} \int_0^T U_0 \frac{t}{T}\, dt = \frac{1}{2} U_0 \sqrt{T}.$$

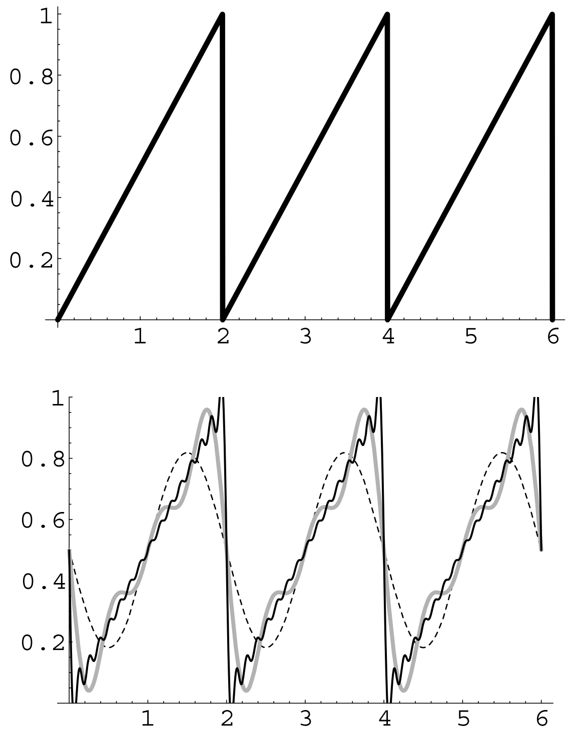

Fig. 9.2 *Top*: The periodic saw-tooth potential with height taken to be 1. *Bottom*: Various approximations to the Fourier series of the sawtooth potential. The *dashed plot* is that of the first term of the series, the *thick grey plot* keeps 3 terms, and the *solid plot* 15 terms

Thus,

$$V(t) = \frac{1}{\sqrt{T}}\left[\frac{1}{2}U_0\sqrt{T} - \frac{U_0\sqrt{T}}{i2\pi}\left(\sum_{n=-\infty}^{-1}\frac{1}{n}e^{i2n\pi t/T} + \sum_{n=1}^{\infty}\frac{1}{n}e^{i2n\pi t/T}\right)\right]$$
$$= U_0\left\{\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi t}{T}\right)\right\}.$$

Figure 9.2(b) shows the graphical representation of the above series keeping the first few terms.

The foregoing examples indicate an important fact about Fourier series. At points of discontinuity (for example, $t = T$ in the preceding two examples), the value of the function is not defined, but the Fourier series expansion assigns it a value—the average of the two values on the right and left of the discontinuity. For instance, when we substitute $t = T$ in the series of Example 9.1.3, all the sine terms vanish and we obtain $V(T) = U_0/2$, the average of U_0 (on the left) and 0 (on the right). We express this as

$$V(T) = \frac{1}{2}\big[V(T-0) + V(T+0)\big] \equiv \frac{1}{2}\lim_{\epsilon\to 0}\big[V(T-\epsilon) + V(T+\epsilon)\big].$$

This is a general property of Fourier series. In fact, the main theorem of Fourier series, which follows, incorporates this property. (For a proof of this theorem, see [Cour 62].)

Theorem 9.1.4 *The Fourier series of a function $f(\theta)$ that is piecewise continuous in the interval $(-\pi,\pi)$ converges to*

$$\begin{aligned} &\frac{1}{2}\big[f(\theta+0)+f(\theta-0)\big] && \text{for } -\pi<\theta<\pi,\\ &\frac{1}{2}\big[f(\pi)+f(-\pi)\big] && \text{for } \theta=\pm\pi. \end{aligned}$$

Although we used exponential functions to find the Fourier expansion of the two examples above, it is more convenient to start with the trigonometric series when the expansion of a real function is sought. Equation (9.2) already gives such an expansion. All we need to do now is find expressions for A_n and B_n. From the definitions of A_n and the relation between b_n and f_n we get

$$\begin{aligned} A_n &= b_n + b_{-n} = \frac{1}{\sqrt{2\pi}}(f_n + f_{-n})\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\,d\theta + \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{in\theta} f(\theta)\,d\theta\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big[e^{-in\theta}+e^{in\theta}\big]f(\theta)\,d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos n\theta f(\theta)\,d\theta. \end{aligned} \tag{9.8}$$

Similarly,

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin n\theta f(\theta)\,d\theta,\\ b_0 &= \frac{1}{\sqrt{2\pi}} f_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \equiv \frac{1}{2}A_0. \end{aligned} \tag{9.9}$$

So, for a function $f(\theta)$ defined in $(-\pi,\pi)$, the Fourier trigonometric series is as in Eq. (9.2) with the coefficients given by Eqs. (9.8) and (9.9). For a function $F(x)$, defined on (a,b), the trigonometric series becomes

$$F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{2n\pi x}{L} + B_n\sin\frac{2n\pi x}{L}\right), \tag{9.10}$$

where

$$\begin{aligned} A_n &= \frac{2}{L}\int_a^b \cos\left(\frac{2n\pi x}{L}\right)F(x)\,dx,\\ B_n &= \frac{2}{L}\int_a^b \sin\left(\frac{2n\pi x}{L}\right)F(x)\,dx. \end{aligned} \tag{9.11}$$

A convenient rule to remember is that for even (odd) functions—which are necessarily defined on a symmetric interval around the origin—only cosine (sine) terms appear in the Fourier expansion.

It is useful to have a representation of the Dirac delta function in terms of the present orthonormal basis of Fourier expansion. First we note that we can represent the delta function in terms of a series in *any* set of orthonormal functions (see Problem 9.23):

$$\delta(x - x') = \sum_n f_n(x) f_n^*(x') w(x). \tag{9.12}$$

Next we use the basis of the Fourier expansion for which $w(x) = 1$. We then obtain

$$\delta(x - x') = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi inx/L}}{\sqrt{L}} \frac{e^{-2\pi inx'/L}}{\sqrt{L}} = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{2\pi in(x-x')/L}.$$

9.1.1 The Gibbs Phenomenon

The plot of the Fourier series expansions in Figs. 9.1(b) and 9.2(b) exhibit a feature that is common to all such expansions: At the discontinuity of the periodic function, the truncated Fourier series overestimates the actual function. This is called the **Gibbs phenomenon**, and is the subject of this subsection. Gibbs phenomenon

Let us approximate the infinite series with a finite sum. Then

$$\begin{aligned} f_N(\theta) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} f_n e^{in\theta} = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} e^{in\theta} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-in\theta'} f(\theta')\, d\theta' \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta' f(\theta') \sum_{n=-N}^{N} e^{in(\theta-\theta')}, \end{aligned}$$

where we substituted Eq. (9.5) in the sum and, without loss of generality, changed the interval of integration from $(-\pi, \pi)$ to $(0, 2\pi)$. Problem 9.2 shows that

$$\sum_{n=-N}^{N} e^{in(\theta-\theta')} = \frac{\sin[(N+\frac{1}{2})(\theta-\theta')]}{\sin[\frac{1}{2}(\theta-\theta')]}.$$

It follows that

$$\begin{aligned} f_N(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta' f(\theta') \frac{\sin[(N+\frac{1}{2})(\theta-\theta')]}{\sin[\frac{1}{2}(\theta-\theta')]} \\ &= \frac{1}{2\pi} \int_{-\theta}^{2\pi-\theta} d\phi f(\phi+\theta) \underbrace{\frac{\sin[(N+\frac{1}{2})\phi]}{\sin(\frac{1}{2}\phi)}}_{\equiv S(\phi)} \\ &\equiv \frac{1}{2\pi} \int_{-\theta}^{2\pi-\theta} d\phi f(\phi+\theta) S(\phi). \end{aligned} \tag{9.13}$$

We want to investigate the behavior of f_N at a discontinuity of f. By translating the limits of integration if necessary, we can assume that the discontinuity of f occurs at a point α such that $0 \neq \alpha \neq 2\pi$. Let us denote the jump at this discontinuity for the function itself by Δf, and for its finite Fourier sum by Δf_N:

$$\Delta f \equiv f(\alpha+\epsilon) - f(\alpha-\epsilon), \qquad \Delta f_N \equiv f_N(\alpha+\epsilon) - f_N(\alpha-\epsilon).$$

Then, we have

$$\begin{aligned}
\Delta f_N &= \frac{1}{2\pi}\int_{-\alpha-\epsilon}^{2\pi-\alpha-\epsilon} d\phi f(\phi+\alpha+\epsilon)S(\phi) \\
&\quad - \frac{1}{2\pi}\int_{-\alpha+\epsilon}^{2\pi-\alpha+\epsilon} d\phi f(\phi+\alpha-\epsilon)S(\phi) \\
&= \frac{1}{2\pi}\left\{\int_{-\alpha-\epsilon}^{-\alpha+\epsilon} d\phi f(\phi+\alpha+\epsilon)S(\phi)\right. \\
&\quad \left. + \int_{-\alpha+\epsilon}^{2\pi-\alpha-\epsilon} d\phi f(\phi+\alpha+\epsilon)S(\phi)\right\} \\
&\quad - \frac{1}{2\pi}\left\{\int_{-\alpha+\epsilon}^{2\pi-\alpha-\epsilon} d\phi f(\phi+\alpha-\epsilon)S(\phi)\right. \\
&\quad \left. + \int_{2\pi-\alpha-\epsilon}^{2\pi-\alpha+\epsilon} d\phi f(\phi+\alpha-\epsilon)S(\phi)\right\} \\
&= \frac{1}{2\pi}\left\{\int_{-\alpha-\epsilon}^{-\alpha+\epsilon} d\phi f(\phi+\alpha+\epsilon)S(\phi)\right. \\
&\quad \left. - \int_{2\pi-\alpha-\epsilon}^{2\pi-\alpha+\epsilon} d\phi f(\phi+\alpha-\epsilon)S(\phi)\right\} \\
&\quad + \frac{1}{2\pi}\int_{-\alpha+\epsilon}^{2\pi-\alpha-\epsilon} d\phi \big[f(\phi+\alpha+\epsilon) - f(\phi+\alpha-\epsilon)\big]S(\phi).
\end{aligned}$$

The first two integrals give zero because of the small ranges of integration and the continuity of the integrands in those intervals. The integrand of the third integral is almost zero for all values of the range of integration except when $\phi \approx 0$. Hence, we can confine the integration to the small interval $(-\delta, +\delta)$ for which the difference in the square brackets is simply Δf. It now follows that

$$\Delta f_N(\delta) \approx \frac{\Delta f}{2\pi}\int_{-\delta}^{\delta} \frac{\sin[(N+\frac{1}{2})\phi]}{\sin(\frac{1}{2}\phi)}\, d\phi \approx \frac{\Delta f}{\pi}\int_0^{\delta} \frac{\sin[(N+\frac{1}{2})\phi]}{\frac{1}{2}\phi}\, d\phi,$$

where we have emphasized the dependence of f_N on δ and approximated the sine in the denominator by its argument, a good approximation due to the smallness of ϕ. The reader may find the plot of the integrand in Fig. 7.3, where it is shown that the major contribution to the integral comes from the interval $[0, \pi/(N+\frac{1}{2})]$, where $\pi/(N+\frac{1}{2})$ is the first zero of the integrand. Furthermore, it is clear that if the upper limit is larger than $\pi/(N+\frac{1}{2})$, the result of the integral will decrease, because in each interval of length 2π, the area below the horizontal axis is larger than that above. Therefore, if we are interested in the *maximum* overshoot of the finite sum, we must set the upper limit equal to $\pi/(N+\frac{1}{2})$. It follows firstly that the maximum overshoot of the finite sum occurs at $\pi/(N+\frac{1}{2}) \approx \pi/N$ to the right of the discontinuity. Secondly, the amount of the maximum overshoot is

maximum overshoot in Gibbs phenomenon calculated

$$\begin{aligned}(\Delta f_N)_{\max} &\approx \frac{2\Delta f}{\pi}\int_0^{\pi/(N+\frac{1}{2})}\frac{\sin[\overbrace{(N+\frac{1}{2})\phi}^{\equiv x}]}{\phi}\,d\phi \\ &= \frac{2}{\pi}\Delta f\int_0^{\pi}\frac{\sin x}{x}\,dx \approx 1.179\Delta f.\end{aligned} \tag{9.14}$$

Thus

> **Box 9.1.5** (The Gibbs Phenomenon) *The finite (large-N) sum approximation of the discontinuous function overshoots the function itself at a discontinuity by about* 18 *percent*.

9.1.2 Fourier Series in Higher Dimensions

It is instructive to generalize the Fourier series to more than one dimension. This generalization is especially useful in crystallography and solid-state physics, which deal with three-dimensional periodic structures. To generalize to N dimensions, we first consider a special case in which an N-dimensional periodic function is a product of N one-dimensional periodic functions. That is, we take the N functions

$$f^{(j)}(x) = \frac{1}{\sqrt{L_j}}\sum_{k=-\infty}^{\infty} f_k^{(j)} e^{2i\pi kx/L_j}, \qquad j = 1, 2, \dots, N,$$

and multiply them on both sides to obtain

$$F(\mathbf{r}) = f^{(1)}(x_1)f^{(2)}(x_2)\cdots f^{(N)}(x_N) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} F_{\mathbf{k}} e^{i\mathbf{g_k}\cdot\mathbf{r}}, \tag{9.15}$$

where we have used the following new notations:

$$\begin{aligned} F(\mathbf{r}) &\equiv f^{(1)}(x_1)f^{(2)}(x_2)\cdots f^{(N)}(x_N), & V &= L_1L_2\cdots L_N, \\ \mathbf{k} &\equiv (k_1, k_2, \dots, k_N), & F_{\mathbf{k}} &\equiv f_{k_1}\cdots f_{k_N}, \\ \mathbf{g_k} &= 2\pi(k_1/L_1, \dots, k_N/L_N), & \mathbf{r} &= (x_1, x_2, \dots, x_N). \end{aligned}$$

We take Eq. (9.15) as the definition of the Fourier series for *any* periodic function of N variables (not just the product of N functions of a single variable). However, application of (9.15) requires some clarification. In one dimension, the shape of the smallest region of periodicity is unique. It is simply a line segment of length L, for example. In two and more dimensions, however, such regions may have a variety of shapes. For instance, in two dimensions, they can be rectangles, pentagons, hexagons, and so forth. Thus, we let V in Eq. (9.15) stand for a primitive cell of the N-dimensional lattice. This cell is important in solid-state physics, and (in three dimensions) is called the **Wigner-Seitz cell**.

Wigner-Seitz cell

It is customary to absorb the factor $1/\sqrt{V}$ into $F_{\mathbf{k}}$, and write

$$F(\mathbf{r}) = \sum_{\mathbf{k}} F_{\mathbf{k}} e^{i\mathbf{g}_{\mathbf{k}}\cdot\mathbf{r}} \quad \Leftrightarrow \quad F_{\mathbf{k}} = \frac{1}{V}\int_V F(\mathbf{r}) e^{-i\mathbf{g}_{\mathbf{k}}\cdot\mathbf{r}}\, d^N x, \tag{9.16}$$

where the sum is a multiple sum over $(k_1, \ldots, k_N)$ and the integral is a multiple integral over a single Wigner-Seitz cell.

Recall that $F(\mathbf{r})$ is a periodic function of $\mathbf{r}$. This means that when $\mathbf{r}$ is changed by $\mathbf{R}$, where $\mathbf{R}$ is a vector describing the boundaries of a cell, then we should get the same function: $F(\mathbf{r}+\mathbf{R}) = F(\mathbf{r})$. When substituted in (9.16), this yields

$$F(\mathbf{r}+\mathbf{R}) = \sum_{\mathbf{k}} F_{\mathbf{k}} e^{i\mathbf{g}_{\mathbf{k}}\cdot(\mathbf{r}+\mathbf{R})} = \sum_{\mathbf{k}} e^{i\mathbf{g}_{\mathbf{k}}\cdot\mathbf{R}} F_{\mathbf{k}} e^{i\mathbf{g}_{\mathbf{k}}\cdot\mathbf{r}},$$

which is equal to $F(\mathbf{r})$ if

$$e^{i\mathbf{g}_{\mathbf{k}}\cdot\mathbf{R}} = 1, \tag{9.17}$$

i.e., if $\mathbf{g}_{\mathbf{k}}\cdot\mathbf{R}$ is an integral multiple of 2π.

In three dimensions $\mathbf{R} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$, where m_1, m_2, and m_3 are integers and $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are *crystal axes*, which are not generally orthogonal. On the other hand, $\mathbf{g}_{\mathbf{k}} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2 + n_3\mathbf{b}_3$, where n_1, n_2, and n_3 are integers, and $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$ are the **reciprocal lattice vectors** defined by

reciprocal lattice vectors

$$\mathbf{b}_1 = \frac{2\pi(\mathbf{a}_2\times\mathbf{a}_3)}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)}, \qquad \mathbf{b}_2 = \frac{2\pi(\mathbf{a}_3\times\mathbf{a}_1)}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)}, \qquad \mathbf{b}_3 = \frac{2\pi(\mathbf{a}_1\times\mathbf{a}_2)}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)}.$$

The reader may verify that $\mathbf{b}_i\cdot\mathbf{a}_j = 2\pi\delta_{ij}$. Thus

$$\begin{aligned}\mathbf{g}_{\mathbf{k}}\cdot\mathbf{R} &= \left(\sum_{i=1}^{3} n_i\mathbf{b}_i\right)\cdot\left(\sum_{j=1}^{3} m_j\mathbf{a}_j\right) = \sum_{i,j} n_i m_j \mathbf{b}_i\cdot\mathbf{a}_j \\ &= 2\pi\sum_{j=1}^{3} m_j n_j = 2\pi(\text{integer}),\end{aligned}$$

and Eq. (9.17) is satisfied.

9.2 Fourier Transform

The Fourier series representation of $F(x)$ is valid for the entire real line as long as $F(x)$ is periodic. However, most functions encountered in physical applications are defined in some interval (a, b) without repetition beyond

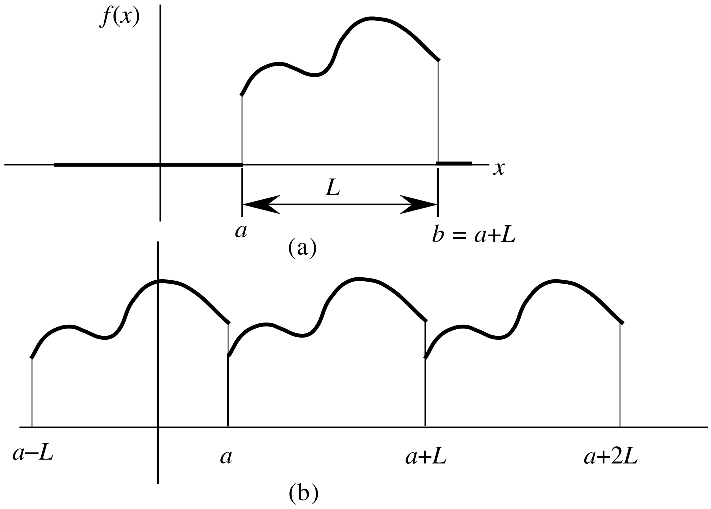

Fig. 9.3 (**a**) The function we want to represent. (**b**) The Fourier series representation of the function

that interval. It would be useful if we could also expand such functions in some form of Fourier "series".

One way to do this is to start with the periodic series and then let the period go to infinity while extending the domain of the definition of the function. As a specific case, suppose we are interested in representing a function $f(x)$ that is defined only for the interval (a, b) and is assigned the value zero everywhere else [see Fig. 9.3(a)]. To begin with, we might try the Fourier series representation, but this will produce a repetition of our function. This situation is depicted in Fig. 9.3(b).

Next we may try a function $g_\Lambda(x)$ defined in the interval $(a - \Lambda/2, b + \Lambda/2)$, where Λ is an arbitrary positive number:

$$g_\Lambda(x) = \begin{cases} 0 & \text{if } a - \Lambda/2 < x < a, \\ f(x) & \text{if } a < x < b, \\ 0 & \text{if } b < x < b + \Lambda/2. \end{cases}$$

This function, which is depicted in Fig. 9.4, has the Fourier series representation

$$g_\Lambda(x) = \frac{1}{\sqrt{L+\Lambda}} \sum_{n=-\infty}^{\infty} g_{\Lambda,n} e^{2i\pi nx/(L+\Lambda)}, \tag{9.18}$$

where

$$g_{\Lambda,n} = \frac{1}{\sqrt{L+\Lambda}} \int_{a-\Lambda/2}^{b+\Lambda/2} e^{-2i\pi nx/(L+\Lambda)} g_\Lambda(x)\, dx. \tag{9.19}$$

We have managed to separate various copies of the original periodic function by Λ. It should be clear that if $\Lambda \to \infty$, we can completely isolate the function and stop the repetition. Let us investigate the behavior of Eqs. (9.18) and (9.19) as Λ grows without bound. First, we notice that the

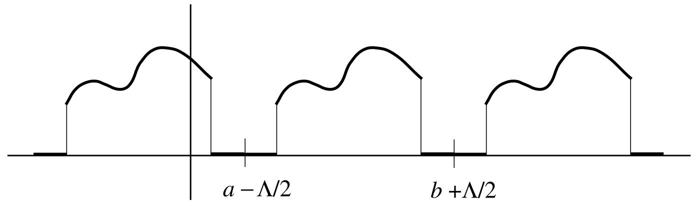

Fig. 9.4 By introducing the parameter Λ, we have managed to separate the copies of the function

quantity k_n defined by $k_n \equiv 2n\pi/(L+\Lambda)$ and appearing in the exponent becomes almost continuous. In other words, as n changes by one unit, k_n changes only slightly. This suggests that the terms in the sum in Eq. (9.18) can be lumped together in j intervals of width Δn_j, giving

$$g_\Lambda(x) \approx \sum_{j=-\infty}^{\infty} \frac{g_\Lambda(k_j)}{\sqrt{L+\Lambda}} e^{ik_jx} \Delta n_j,$$

where $k_j \equiv 2n_j\pi/(L+\Lambda)$, and $g_\Lambda(k_j) \equiv g_{\Lambda,n_j}$. Substituting $\Delta n_j = [(L+\Lambda)/2\pi]\Delta k_j$ in the above sum, we obtain

$$g_\Lambda(x) \approx \sum_{j=-\infty}^{\infty} \frac{g^\Lambda(k_j)}{\sqrt{L+\Lambda}} e^{ik_jx} \frac{L+\Lambda}{2\pi} \Delta k_j = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \tilde{g}_\Lambda(k_j) e^{ik_jx} \Delta k_j,$$

where we introduced $\tilde{g}_\Lambda(k_j)$ defined by $\tilde{g}_\Lambda(k_j) \equiv \sqrt{(L+\Lambda)/2\pi}\, g_\Lambda(k_j)$. It is now clear that the preceding sum approaches an integral in the limit that $\Lambda \to \infty$. In the same limit, $g_\Lambda(x) \to f(x)$, and we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \tag{9.20}$$

where

$$\begin{aligned} \tilde{f}(k) &\equiv \lim_{\Lambda\to\infty} \tilde{g}_\Lambda(k_j) = \lim_{\Lambda\to\infty} \sqrt{\frac{L+\Lambda}{2\pi}}\, g_\Lambda(k_j) \\ &= \lim_{\Lambda\to\infty} \sqrt{\frac{L+\Lambda}{2\pi}} \frac{1}{\sqrt{L+\Lambda}} \int_{a-\Lambda/2}^{b+\Lambda/2} e^{-ik_jx} g_\Lambda(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \end{aligned} \tag{9.21}$$

Fourier integral transforms

Equations (9.20) and (9.21) are called the **Fourier integral transforms** of $\tilde{f}(k)$ and $f(x)$, respectively.

Example 9.2.1 Let us evaluate the Fourier transform of the function defined by

$$f(x) = \begin{cases} b & \text{if } |x| < a, \\ 0 & \text{if } |x| > a \end{cases}$$

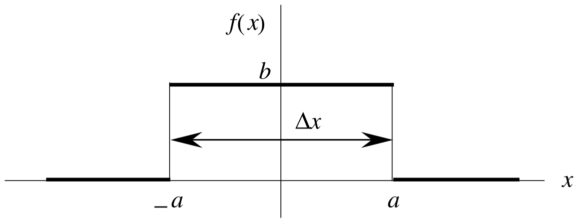

Fig. 9.5 The square "bump" function

(see Fig. 9.5). From (9.21) we have

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{b}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} dx = \frac{2ab}{\sqrt{2\pi}} \left(\frac{\sin ka}{ka} \right),$$

which is the function encountered (and depicted) in Example 7.3.2.

Let us discuss this result in detail. First, note that if $a \to \infty$, the function $f(x)$ becomes a constant function over the entire real line, and we get

$$\tilde{f}(k) = \frac{2b}{\sqrt{2\pi}} \lim_{a\to\infty} \frac{\sin ka}{k} = \frac{2b}{\sqrt{2\pi}} \pi \delta(k)$$

by the result of Example 7.3.2. This is the Fourier transform of an everywhere-constant function (see Problem 9.12). Next, let $b \to \infty$ and $a \to 0$ in such a way that $2ab$, which is the area under $f(x)$, is 1. Then $f(x)$ will approach the delta function, and $\tilde{f}(k)$ becomes

$$\tilde{f}(k) = \lim_{\substack{b\to\infty \\ a\to 0}} \frac{2ab}{\sqrt{2\pi}} \frac{\sin ka}{ka} = \frac{1}{\sqrt{2\pi}} \lim_{a\to 0} \frac{\sin ka}{ka} = \frac{1}{\sqrt{2\pi}}.$$

So the Fourier transform of the delta function is the constant $1/\sqrt{2\pi}$.

Finally, we note that the width of $f(x)$ is $\Delta x = 2a$, and the width of $\tilde{f}(k)$ is roughly the distance, on the k-axis, between its first two roots, k_+ and k_-, on either side of $k = 0$: $\Delta k = k_+ - k_- = 2\pi/a$. Thus increasing the width of $f(x)$ results in a decrease in the width of $\tilde{f}(k)$. In other words, when the function is wide, its Fourier transform is narrow. In the limit of infinite width (a constant function), we get infinite sharpness (the delta function). The last two statements are very general. In fact, it can be shown that $\Delta x \Delta k \geq 1$ for any function $f(x)$. When both sides of this inequality are multiplied by the (reduced) Planck constant $\hbar \equiv h/(2\pi)$, the result is the celebrated **Heisenberg uncertainty relation**:[3]

Heisenberg uncertainty relation

$$\Delta x \Delta p \geq \hbar,$$

where $p = \hbar k$ is the momentum of the particle.

[3] In the context of the uncertainty relation, the width of the function—the so-called wave packet—measures the uncertainty in the position x of a quantum mechanical particle. Similarly, the width of the Fourier transform measures the uncertainty in k, which is related to momentum p via $p = \hbar k$.

Having obtained the transform of $f(x)$, we can write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2b}{\sqrt{2\pi}} \frac{\sin ka}{k} e^{ikx} dk = \frac{b}{\pi} \int_{-\infty}^{\infty} \frac{\sin ka}{k} e^{ikx} dk.$$

Example 9.2.2 Let us evaluate the Fourier transform of a Gaussian $g(x) = ae^{-bx^2}$ with $a, b > 0$:

$$\tilde{g}(k) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-b(x^2+ikx/b)} dx = \frac{ae^{-k^2/4b}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-b(x+ik/2b)^2} dx.$$

To evaluate this integral rigorously, we would have to use techniques developed in complex analysis, which are not introduced until Chap. 11 (see Example 11.3.8). However, we can ignore the fact that the exponent is complex, substitute $y = x + ik/(2b)$, and write

$$\int_{-\infty}^{\infty} e^{-b[x+ik/(2b)]^2} dx = \int_{-\infty}^{\infty} e^{-by^2} dy = \sqrt{\frac{\pi}{b}}.$$

Thus, we have $\tilde{g}(k) = \frac{a}{\sqrt{2b}} e^{-k^2/(4b)}$, which is also a Gaussian.

We note again that the width of $g(x)$, which is proportional to $1/\sqrt{b}$, is in inverse relation to the width of $\tilde{g}(k)$, which is proportional to $\sqrt{b}$. We thus have $\Delta x \Delta k \sim 1$.

Equations (9.20) and (9.21) are reciprocals of one another. However, it is not obvious that they are consistent. In other words, if we substitute (9.20) in the RHS of (9.21), do we get an identity? Let's try this:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k') e^{ik'x} dk' \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \tilde{f}(k') e^{i(k'-k)x} dk'. \end{aligned}$$

We now change the order of the two integrations:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dk'\, \tilde{f}(k') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k'-k)x} \right].$$

But the expression in the square brackets is the delta function (see Example 7.3.2). Thus, we have $\tilde{f}(k) = \int_{-\infty}^{\infty} dk'\, \tilde{f}(k')\delta(k'-k)$, which is an identity.

As in the case of Fourier series, Eqs. (9.20) and (9.21) are valid even if f and $\tilde{f}$ are piecewise continuous. In that case the Fourier transforms are written as

$$\begin{aligned} \frac{1}{2}\big[f(x+0) + f(x-0)\big] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \\ \frac{1}{2}\big[\tilde{f}(k+0) + \tilde{f}(k-0)\big] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \end{aligned} \qquad (9.22)$$

where each zero on the LHS is an ϵ that has gone to its limit.

It is useful to generalize Fourier transform equations to more than one dimension. The generalization is straightforward:

$$\begin{aligned} f(\mathbf{r}) &= \frac{1}{(2\pi)^{n/2}} \int d^n k e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}), \\ \tilde{f}(\mathbf{k}) &= \frac{1}{(2\pi)^{n/2}} \int d^n x f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{9.23}$$

Let us now use the abstract notation of Sect. 7.3 to get more insight into the preceding results. In the language of Sect. 7.3, Eq. (9.20) can be written as

$$\langle x|f\rangle = \int_{-\infty}^{\infty} \langle k|\tilde{f}\rangle\langle x|k\rangle dk = \langle x|\left(\int_{-\infty}^{\infty} |k\rangle\langle k|\, dk\right)|\tilde{f}\rangle, \tag{9.24}$$

where we have defined

$$\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{9.25}$$

Equation (9.24) suggests the identification $|\tilde{f}\rangle \equiv |f\rangle$ as well as the identity

$$\mathbf{1} = \int_{-\infty}^{\infty} |k\rangle\langle k|\, dk, \tag{9.26}$$

which is the same as (7.17). Equation (7.19) yields

$$\langle k|k'\rangle = \delta(k - k'), \tag{9.27}$$

which upon the insertion of a unit operator gives an integral representation of the delta function:

$$\begin{aligned} \delta(k - k') &= \langle k|\mathbf{1}|k'\rangle = \langle k|\left(\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx\right)|k'\rangle \\ &= \int_{-\infty}^{\infty} \langle k|x\rangle\langle x|k'\rangle\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{i(k'-k)x}. \end{aligned}$$

Obviously, we can also write

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{i(x-x')k}.$$

If more than one dimension is involved, we use

$$\begin{aligned} \delta(\mathbf{k} - \mathbf{k}') &= \frac{1}{(2\pi)^n}\int d^n x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}, \\ \delta(\mathbf{r} - \mathbf{r}') &= \frac{1}{(2\pi)^n}\int d^n k e^{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{k}}, \end{aligned} \tag{9.28}$$

with the inner product relations

$$\langle \mathbf{r}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{n/2}} e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad \langle \mathbf{k}|\mathbf{r}\rangle = \frac{1}{(2\pi)^{n/2}} e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{9.29}$$

Equations (9.28) and (9.29) and the identification $|\tilde{f}\rangle \equiv |f\rangle$ exhibit a striking resemblance between $|\mathbf{r}\rangle$ and $|\mathbf{k}\rangle$. In fact, any given abstract vector $|f\rangle$ can be expressed either in terms of its r representation, $\langle\mathbf{r}|f\rangle = f(\mathbf{r})$, or in terms of its k representation, $\langle\mathbf{k}|f\rangle \equiv \tilde{f}(\mathbf{k})$. These two representations are completely equivalent, and there is a one-to-one correspondence between the two, given by Eq. (9.23). The representation that is used in practice is dictated by the physical application. In quantum mechanics, for instance, most of the time the r representation, corresponding to the position, is used, because then the operator equations turn into differential equations that are (in many cases) linear and easier to solve than the corresponding equations in the k representation, which is related to the momentum.

Fourier transform of the Coulomb potential

Example 9.2.3 In this example we evaluate the Fourier transform of the Coulomb potential $V(r)$ of a point charge q: $V(r) = q/r$. The Fourier transform is important in scattering experiments with atoms, molecules, and solids. As we shall see in the following, the Fourier transform of $V(r)$ is not defined. However, if we work with the **Yukawa potential**,

Yukawa potential

$$V_\alpha(r) = \frac{qe^{-\alpha r}}{r}, \quad \alpha > 0,$$

the Fourier transform will be well-defined, and we can take the limit $\alpha \to 0$ to recover the Coulomb potential. Thus, we seek the Fourier transform of $V_\alpha(r)$.

We are working in three dimensions and therefore may write

$$\tilde{V}_\alpha(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \iiint d^3x\, e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{qe^{-\alpha r}}{r}.$$

It is clear from the presence of r that spherical coordinates are appropriate. We are free to pick any direction as the z-axis. A simplifying choice in this case is the direction of $\mathbf{k}$. So, we let $\mathbf{k} = |\mathbf{k}|\hat{\mathbf{e}}_z = k\hat{\mathbf{e}}_z$, or $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$, where θ is the polar angle in spherical coordinates. Now we have

$$\tilde{V}_\alpha(\mathbf{k}) = \frac{q}{(2\pi)^{3/2}} \int_0^\infty r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\varphi\, e^{-ikr\cos\theta} \frac{e^{-\alpha r}}{r}.$$

The φ integration is trivial and gives 2π. The θ integration is done next:

$$\int_0^\pi \sin\theta\, e^{-ikr\cos\theta}\,d\theta = \int_{-1}^1 e^{-ikru}\,du = \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right).$$

We thus have

$$\begin{aligned}\tilde{V}_\alpha(\mathbf{k}) &= \frac{q(2\pi)}{(2\pi)^{3/2}} \int_0^\infty dr\, r^2 \frac{e^{-\alpha r}}{r} \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right) \\ &= \frac{q}{(2\pi)^{1/2}} \frac{1}{ik} \int_0^\infty dr\left[e^{(-\alpha+ik)r} - e^{-(\alpha+ik)r}\right] \\ &= \frac{q}{(2\pi)^{1/2}} \frac{1}{ik} \left(\frac{e^{(-\alpha+ik)r}}{-\alpha+ik}\bigg|_0^\infty + \frac{e^{-(\alpha+ik)r}}{\alpha+ik}\bigg|_0^\infty\right).\end{aligned}$$

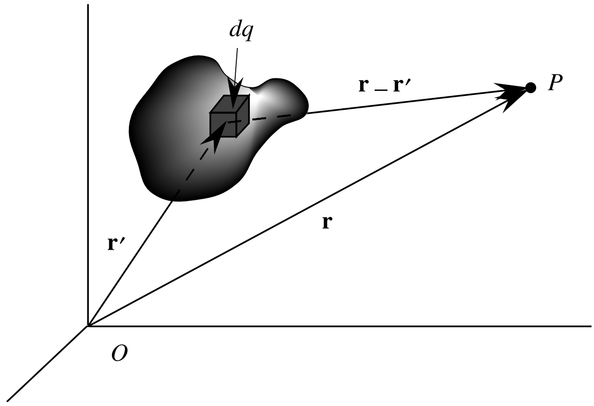

Fig. 9.6 The Fourier transform of the potential of a continuous charge distribution at P is calculated using this geometry

Note how the factor $e^{-\alpha r}$ has tamed the divergent behavior of the exponential at $r \to \infty$. This was the reason for introducing it in the first place. Simplifying the last expression yields

$$\tilde{V}_\alpha(\mathbf{k}) = \frac{2q}{\sqrt{2\pi}} \frac{1}{k^2 + \alpha^2}.$$

The parameter α is a measure of the range of the potential. It is clear that the larger α is, the smaller the range. In fact, it was in response to the short range of nuclear forces that Yukawa introduced α. For electromagnetism, where the range is infinite, α becomes zero and $V_\alpha(r)$ reduces to $V(r)$. Thus, the Fourier transform of the Coulomb potential is

$$\tilde{V}_{\text{Coul}}(\mathbf{k}) = \frac{2q}{\sqrt{2\pi}} \frac{1}{k^2}.$$

If a charge *distribution* is involved, the Fourier transform will be different.

Example 9.2.4 The example above deals with the electrostatic potential of a point charge. Let us now consider the case where the charge is distributed over a finite volume. Then the potential is

$$V(\mathbf{r}) = \iiint \frac{q\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3x' \equiv q \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3x',$$

where $q\rho(\mathbf{r}')$ is the charge density at $\mathbf{r}'$, and we have used a single integral because d^3x' already indicates the number of integrations to be performed. Note that we have normalized $\rho(\mathbf{r}')$ so that its integral over the volume is 1, which is equivalent to assuming that the total charge is q. Figure 9.6 shows the geometry of the situation.

Making a change of variables, $\mathbf{R} \equiv \mathbf{r}' - \mathbf{r}$, or $\mathbf{r}' = \mathbf{R} + \mathbf{r}$, and $d^3x' = d^3X$, with $\mathbf{R} \equiv (X, Y, Z)$, we get

$$\tilde{V}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{r}} q \int \frac{\rho(\mathbf{R} + \mathbf{r})}{R} d^3X. \tag{9.30}$$

To evaluate Eq. (9.30), we substitute for $\rho(\mathbf{R}+\mathbf{r})$ in terms of its Fourier transform,

$$\rho(\mathbf{R}+\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k' \tilde{\rho}(\mathbf{k}') e^{i\mathbf{k}'\cdot(\mathbf{R}+\mathbf{r})}. \tag{9.31}$$

Combining (9.30) and (9.31), we obtain

$$\begin{aligned}\tilde{V}(\mathbf{k}) &= \frac{q}{(2\pi)^3} \int d^3x\, d^3X\, d^3k' \frac{e^{i\mathbf{k}'\cdot\mathbf{R}}}{R} \tilde{\rho}(\mathbf{k}') e^{i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})} \\ &= q \int d^3X\, d^3k' \frac{e^{i\mathbf{k}'\cdot\mathbf{R}}}{R} \tilde{\rho}(\mathbf{k}') \underbrace{\left(\frac{1}{(2\pi)^3} \int d^3x\, e^{i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})}\right)}_{\delta(\mathbf{k}'-\mathbf{k})} \\ &= q\tilde{\rho}(\mathbf{k}) \int d^3X \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{R}. \end{aligned} \tag{9.32}$$

What is nice about this result is that the contribution of the charge distribution, $\tilde{\rho}(\mathbf{k})$, has been completely factored out. The integral, aside from a constant and a change in the sign of $\mathbf{k}$, is simply the Fourier transform of the Coulomb potential of a point charge obtained in the previous example. We can therefore write Eq. (9.32) as

$$\tilde{V}(\mathbf{k}) = (2\pi)^{3/2} \tilde{\rho}(\mathbf{k}) \tilde{V}_{\text{Coul}}(-\mathbf{k}) = \frac{4\pi q \tilde{\rho}(\mathbf{k})}{|\mathbf{k}|^2}.$$

This equation is important in analyzing the structure of atomic particles. The Fourier transform $\tilde{V}(\mathbf{k})$ is directly measurable in scattering experiments. In a typical experiment a (charged) target is probed with a charged point particle (electron). If the analysis of the scattering data shows a deviation from $1/k^2$ in the behavior of $\tilde{V}(\mathbf{k})$, then it can be concluded that the target particle has a charge distribution. More specifically, a plot of $k^2\tilde{V}(\mathbf{k})$ versus k gives the variation of $\tilde{\rho}(\mathbf{k})$, the **form factor**, with k. If the resulting graph is a constant, then $\tilde{\rho}(\mathbf{k})$ is a constant, and the target is a point particle [$\tilde{\rho}(\mathbf{k})$ is a constant for point particles, where $\tilde{\rho}(\mathbf{r}') \propto \delta(\mathbf{r}-\mathbf{r}')$]. If there is any deviation from a constant function, $\tilde{\rho}(\mathbf{k})$ must have a dependence on k, and correspondingly, the target particle must have a charge distribution.

form factor

Fourier transform and the discovery of quarks

The above discussion, when generalized to four-dimensional relativistic space-time, was the basis for a strong argument in favor of the existence of point-like particles—quarks—inside a proton in 1968, when the results of the scattering of high-energy electrons off protons at the Stanford Linear Accelerator Center revealed deviation from a constant for the proton form factor.

9.2.1 Fourier Transforms and Derivatives

The Fourier transform is very useful for solving differential equations. This is because the derivative operator in $\mathbf{r}$ space turns into ordinary multiplication in $\mathbf{k}$ space. For example, if we differentiate $f(\mathbf{r})$ in Eq. (9.23) with respect to x_j, we obtain

$$\frac{\partial}{\partial x_j} f(\mathbf{r}) = \frac{1}{(2\pi)^{n/2}} \int d^n k \frac{\partial}{\partial x_j} e^{i(k_1 x_1 + \cdots + k_j x_j + \cdots + k_n x_n)} \tilde{f}(\mathbf{k})$$
$$= \frac{1}{(2\pi)^{n/2}} \int d^n k (ik_j) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}).$$

That is, every time we differentiate with respect to any component of $\mathbf{r}$, the corresponding component of $\mathbf{k}$ "comes down". Thus, the n-dimensional gradient is

$$\boldsymbol{\nabla} f(\mathbf{r}) = \frac{1}{(2\pi)^{n/2}} \int d^n k (i\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}),$$

and the n-dimensional Laplacian is

$$\nabla^2 f(\mathbf{r}) = \frac{1}{(2\pi)^{n/2}} \int d^n k \big(-k^2\big) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}).$$

We shall use Fourier transforms extensively in solving differential equations later in the book. Here, we can illustrate the above points with a simple example. Consider the ordinary second-order differential equation

$$C_2 \frac{d^2 y}{dx^2} + C_1 \frac{dy}{dx} + C_0 y = f(x),$$

where C_0, C_1, and C_2 are constants. We can "solve" this equation by simply substituting the following in it:

$$y(x) = \frac{1}{\sqrt{2\pi}} \int dk \tilde{y}(k) e^{ikx}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{2\pi}} \int dk \tilde{y}(k)(ik) e^{ikx},$$
$$\frac{d^2 y}{dx^2} = -\frac{1}{\sqrt{2\pi}} \int dk \tilde{y}(k) k^2 e^{ikx}, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int dk \tilde{f}(k) e^{ikx}.$$

This gives

$$\frac{1}{\sqrt{2\pi}} \int dk \tilde{y}(k) \big(-C_2 k^2 + iC_1 k + C_0\big) e^{ikx} = \frac{1}{\sqrt{2\pi}} \int dk \tilde{f}(k) e^{ikx}.$$

Equating the coefficients of e^{ikx} on both sides, we obtain[4]

$$\tilde{y}(k) = \frac{\tilde{f}(k)}{-C_2 k^2 + iC_1 k + C_0}.$$

If we know $\tilde{f}(k)$ [which can be obtained from $f(x)$], we can calculate $y(x)$ by Fourier-transforming $\tilde{y}(k)$. The resulting integrals are not generally easy to evaluate. In some cases the methods of complex analysis may be helpful; in others numerical integration may be the last resort. However, the real power of the Fourier transform lies in the *formal analysis* of differential equations.

[4] Alternatively, we can multiply both sides by $e^{-ik'x}$ and integrate over x. The result of this integration yields $\delta(k - k')$, which collapses the k-integrations and yields the equality of the integrands.

9.2.2 The Discrete Fourier Transform

The preceding remarks alluded to the power of the Fourier transform in solving certain differential equations. If such a solution is combined with numerical techniques, the integrals must be replaced by sums. This is particularly true if our function is given by a table rather than a mathematical relation, a common feature of numerical analysis. So suppose that we are given a set of measurements performed in equal time intervals of Δt. Suppose that the overall period in which these measurements are done is T. We are seeking a Fourier transform of this finite set of data. First we write

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f(t_n) e^{-i\omega t_n} \Delta t,$$

or, discretizing the frequency as well and writing $\omega_m = m\Delta\omega$, with $\Delta\omega$ to be determined later, we have

$$\tilde{f}(m\Delta\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f(n\Delta t) e^{-i(m\Delta\omega)n\Delta t} \left(\frac{T}{N}\right). \tag{9.33}$$

Since the Fourier transform is given in terms of a finite sum, let us explore the idea of writing the inverse transform also as a sum. So, multiply both sides of the above equation by $[e^{i(m\Delta\omega)k\Delta t}/(\sqrt{2\pi})]\Delta\omega$ and sum over m:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \sum_{m=0}^{N-1} \tilde{f}(m\Delta\omega) e^{i(m\Delta\omega)k\Delta t} \Delta\omega \\
&\quad = \frac{T\Delta\omega}{2\pi N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(n\Delta t) e^{im\Delta\omega\Delta t(k-n)} \\
&\quad = \frac{T\Delta\omega}{2\pi N} \sum_{n=0}^{N-1} f(n\Delta t) \sum_{m=0}^{N-1} e^{im\Delta\omega\Delta t(k-n)}.
\end{aligned}$$

Problem 9.2 shows that

$$\sum_{m=0}^{N-1} e^{im\Delta\omega\Delta t(k-n)} = \begin{cases} N & \text{if } k = n, \\ \frac{e^{iN\Delta\omega\Delta t(k-n)} - 1}{e^{i\Delta\omega\Delta t(k-n)} - 1} & \text{if } k \neq n. \end{cases}$$

We want the sum to vanish when $k \neq n$. This suggests demanding that $N\Delta\omega\Delta t(k-n)$ be an integer multiple of 2π. Since $\Delta\omega$ and Δt are to be independent of this (arbitrary) integer (as well as k and n), we must write

$$N\Delta\omega\Delta t(k-n) = 2\pi(k-n) \quad \Rightarrow \quad N\Delta\omega\frac{T}{N} = 2\pi \quad \Rightarrow \quad \Delta\omega = \frac{2\pi}{T}.$$

discrete Fourier transforms

With this choice, we have the following **discrete Fourier transforms**:

$$\tilde{f}(\omega_j) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(t_n) e^{-i\omega_j t_n}, \quad \omega_j = \frac{2\pi j}{T},$$
$$f(t_n) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \tilde{f}(\omega_j) e^{i\omega_j t_n}, \quad t_n = n\Delta t, \tag{9.34}$$

where we have redefined the new $\tilde{f}$ to be $\sqrt{2\pi N/T}$ times the old $\tilde{f}$.

Discrete Fourier transforms are used extensively in numerical calculation of problems in which ordinary Fourier transforms are used. For instance, if a differential equation lends itself to a solution via the Fourier transform as discussed before, then discrete Fourier transforms will give a procedure for finding the solution numerically. Similarly, the frequency analysis of signals is nicely handled by discrete Fourier transforms.

It turns out that discrete Fourier analysis is very intensive computationally. Its status as a popular tool in computational physics is due primarily to a very efficient method of calculation known as the **fast Fourier transform**. In a typical Fourier transform, one has to perform a sum of N terms for every point. Since there are N points to transform, the total computational time will be of order N^2. In the fast Fourier transform, one takes N to be even and divides the sum into two other sums, one over the even terms and one over the odd terms. Then the computation time will be of order $2 \times (N/2)^2$, or half the original calculation. Similarly, if $N/2$ is even, one can further divide the odd and even sums by two and obtain a computation time of $4 \times (N/4)^2$, or a quarter of the original calculation. In general, if $N = 2^k$, then by dividing the sums consecutively, we end up with N transforms to be performed after k steps. So, the computation time will be $kN = N \log_2 N$. For $N = 128$, the computation time will be $100 \log_2 128 = 700$ as opposed to $128^2 \approx 16{,}400$, a reduction by a factor of over 20. The fast Fourier transform is indeed fast!

fast Fourier transform

9.2.3 Fourier Transform of a Distribution

Although one can define the Fourier transform of a distribution in exact analogy to an ordinary function, sometimes it is convenient to define the Fourier transform of the distribution as a linear functional.

Let us ignore the distinction between the two variables x and k, and simply define the Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{iut} dt.$$

Now we consider two functions, f and g, and note that

$$\langle f, \tilde{g} \rangle \equiv \int_{-\infty}^{\infty} f(u) \tilde{g}(u)\, du = \int_{-\infty}^{\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-iut} dt \right] du$$

$$= \int_{-\infty}^{\infty} g(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-iut}\,du\right]dt$$
$$= \int_{-\infty}^{\infty} g(t)\tilde{f}(t)\,dt = \langle \tilde{f}, g\rangle.$$

The following definition is motivated by the last equation.

Definition 9.2.5 Let φ be a distribution and let f be a $\mathcal{C}_F^\infty$ function whose Fourier transform $\tilde{f}$ exists and is also a $\mathcal{C}_F^\infty$ function. Then we define the Fourier transform $\tilde{\varphi}$ of φ to be the distribution given by

$$\langle \tilde{\varphi}, f\rangle = \langle \varphi, \tilde{f}\rangle.$$

Example 9.2.6 The Fourier transform of $\delta(x)$ is given by

$$\langle \tilde{\delta}, f\rangle = \langle \delta, \tilde{f}\rangle = \tilde{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,dt$$
$$= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\right) f(t)\,dt = \left\langle \frac{1}{\sqrt{2\pi}}, f\right\rangle.$$

Thus, $\tilde{\delta} = 1/\sqrt{2\pi}$, as expected.

The Fourier transform of $\delta(x - x') \equiv \delta_{x'}(x)$ is given by

$$\langle \tilde{\delta}_{x'}, f\rangle = \langle \delta_{x'}, \tilde{f}\rangle = \tilde{f}\bigl(x'\bigr) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ix't}\,dt$$
$$= \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}e^{-ix't}\right) f(t)\,dt.$$

Thus, if $\varphi(x) = \delta(x - x')$, then $\tilde{\varphi}(t) = (1/\sqrt{2\pi})e^{-ix't}$.

9.3 Problems

9.1 Consider the function $f(\theta) = \sum_{m=-\infty}^{\infty} \delta(\theta - 2m\pi)$.

(a) Show that f is periodic of period 2π.
(b) What is the Fourier series expansion for $f(\theta)$.

9.2 Break the sum $\sum_{n=-N}^{N} e^{in(\theta-\theta')}$ into $\sum_{n=-N}^{-1} + 1 + \sum_{n=1}^{N}$. Use the geometric sum formula

$$\sum_{n=0}^{N} ar^n = a\frac{r^{N+1} - 1}{r - 1}$$

to obtain

$$\sum_{n=1}^{N} e^{in(\theta-\theta')} = e^{i(\theta-\theta')}\frac{e^{iN(\theta-\theta')} - 1}{e^{i(\theta-\theta')} - 1} = e^{i\frac{1}{2}(N+1)(\theta-\theta')}\frac{\sin[\frac{1}{2}N(\theta-\theta')]}{\sin[\frac{1}{2}(\theta-\theta')]}.$$

By changing n to $-n$ or equivalently, $(\theta - \theta')$ to $-(\theta - \theta')$ find a similar sum from $-N$ to -1. Now put everything together and use the trigonometric identity

$$2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

to show that

$$\sum_{n=-N}^{N} e^{in(\theta-\theta')} = \frac{\sin[(N+\frac{1}{2})(\theta-\theta')]}{\sin[\frac{1}{2}(\theta-\theta')]}.$$

9.3 Find the Fourier series expansion of the periodic function defined on its fundamental cell as

$$f(\theta) = \begin{cases} -\frac{1}{2}(\pi + \theta) & \text{if } -\pi \le \theta < 0, \\ \frac{1}{2}(\pi - \theta) & \text{if } 0 < \theta \le \pi. \end{cases}$$

9.4 Show that A_n and B_n in Eq. (9.2) are real when $f(\theta)$ is real.

9.5 Find the Fourier series expansion of the periodic function $f(\theta)$ defined as $f(\theta) = \cos\alpha\theta$ on its fundamental cell, $(-\pi, \pi)$

(a) when α is an integer;
(b) when α is not an integer.

9.6 Find the Fourier series expansion of the periodic function defined on its fundamental cell, $(-\pi, \pi)$, as $f(\theta) = \theta$.

9.7 Consider the periodic function that is defined on its fundamental cell, $(-a, a)$, as $f(x) = |x|$.

(a) Find its Fourier series expansion.
(b) Show that the infinite series gives the same result as the function when both are evaluated at $x = a$.
(c) Evaluate both sides of the expansion at $x = 0$, and show that

$$\pi^2 = 8\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

9.8 Let $f(x) = x$ be a periodic function defined over the interval $(0, 2a)$. Find the Fourier series expansion of f.

9.9 Show that the piecewise parabolic "approximation" to $a^2 \sin(\pi x/a)$ in the interval $(-a, a)$ given by the function

$$f(x) = \begin{cases} 4x(a + x) & \text{if } -a \le x \le 0 \\ 4x(a - x) & \text{if } 0 \le x \le a \end{cases}$$

has the Fourier series expansion

$$f(x) = \frac{32a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\frac{(2n+1)\pi x}{a}.$$

Plot $f(x)$, $a^2 \sin(\pi x/a)$, and the series expansion (up to 20 terms) for $a = 1$ between -1 and $+1$ on the same graph.

9.10 Find the Fourier series expansion of $f(\theta) = \theta^2$ for $|\theta| < \pi$. Then show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

9.11 Find the Fourier series expansion of

$$f(t) = \begin{cases} \sin \omega t & \text{if } 0 \le t \le \pi/\omega, \\ 0 & \text{if } -\pi/\omega \le t \le 0. \end{cases}$$

9.12 What is the Fourier transform of

(a) the constant function $f(x) = C$, and
(b) the Dirac delta function $\delta(x)$?

9.13 Show that

(a) if $g(x)$ is real, then $\tilde{g}^*(k) = \tilde{g}(-k)$, and
(b) if $g(x)$ is even (odd), then $\tilde{g}(k)$ is also even (odd).

9.14 Let $g_c(x)$ stand for the single function that is nonzero only on a subinterval of the fundamental cell $(a, a + L)$. Define the function $g(x)$ as

$$g(x) = \sum_{j=-\infty}^{\infty} g_c(x - jL).$$

(a) Show that $g(x)$ is periodic with period L.
(b) Find its Fourier transform $\tilde{g}(k)$, and verify that

$$\tilde{g}(k) = L\tilde{g}_c(k) \sum_{m=-\infty}^{\infty} \delta(kL - 2m\pi).$$

(c) Find the (inverse) *transform* of $\tilde{g}(k)$, and show that it is the Fourier *series* of $g_c(x)$.

9.15 Evaluate the Fourier transform of

$$g(x) = \begin{cases} b - b|x|/a & \text{if } |x| < a, \\ 0 & \text{if } |x| > a. \end{cases}$$

9.16 Let $f(\theta)$ be a periodic function given by $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Find its Fourier transform $\tilde{f}(t)$.

9.17 Let

$$f(t)=\begin{cases}\sin\omega_0 t & \text{if } |t|<T,\\ 0 & \text{if } |t|>T.\end{cases}$$

Show that

$$\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\left\{\frac{\sin[(\omega-\omega_0)T]}{\omega-\omega_0}-\frac{\sin[(\omega+\omega_0)T]}{\omega+\omega_0}\right\}.$$

Verify the uncertainty relation $\Delta\omega\Delta t\approx 4\pi$.

9.18 If $f(x)=g(x+a)$, show that $\tilde{f}(k)=e^{-iak}\tilde{g}(k)$.

9.19 For $a>0$ find the Fourier transform of $f(x)=e^{-a|x|}$. Is $\tilde{f}(k)$ symmetric? Is it real? Verify the uncertainty relations.

9.20 The displacement of a damped harmonic oscillator is given by

$$f(t)=\begin{cases}Ae^{-\alpha t}e^{i\omega_0 t} & \text{if } t>0,\\ 0 & \text{if } t<0.\end{cases}$$

Find $\tilde{f}(\omega)$ and show that the frequency distribution $|\tilde{f}(\omega)|^2$ is given by

$$\left|\tilde{f}(\omega)\right|^2=\frac{A^2}{2\pi}\frac{1}{(\omega-\omega_0)^2+\alpha^2}.$$

9.21 Prove the **convolution theorem**:

convolution theorem

$$\int_{-\infty}^{\infty}f(x)g(y-x)\,dx=\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}(k)e^{iky}\,dk.$$

What will this give when $y=0$?

9.22 Prove **Parseval's relation** for Fourier transforms:

Parseval's relation

$$\int_{-\infty}^{\infty}f(x)g^*(x)\,dx=\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}^*(k)\,dk.$$

In particular, the norm of a function—with weight function equal to 1—is invariant under Fourier transform.

9.23 Use the completeness relation $\mathbf{1}=\sum_n|n\rangle\langle n|$ and sandwich it between $|x\rangle$ and $\langle x'|$ to find an expression for the Dirac delta function in terms of an infinite series of orthonormal functions.

9.24 Use a Fourier transform in three dimensions to find a solution of the Poisson equation: $\nabla^2\Phi(\mathbf{r})=-4\pi\rho(\mathbf{r})$.

9.25 For $\varphi(x)=\delta(x-x')$, find $\tilde{\varphi}(y)$.

9.26 Show that $\tilde{\tilde{f}}(t)=f(-t)$.

9.27 The Fourier transform of a distribution φ is given by

$$\tilde{\varphi}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta'(t-n).$$

What is $\varphi(x)$? Hint: Use $\tilde{\tilde{\varphi}}(x) = \varphi(-x)$

9.28 For $f(x) = \sum_{k=0}^{n} a_k x^k$, show that

$$\tilde{f}(u) = \sqrt{2\pi} \sum_{k=0}^{n} i^k a_k \delta^{(k)}(u), \quad \text{where} \quad \delta^{(k)}(u) \equiv \frac{d^k}{du^k} \delta(u).$$