Green's Functions in One Dimension 20

Our treatment of differential equations, with the exception of SOLDEs with constant coefficients, did not consider inhomogeneous equations. At this point, however, we can put into use one of the most elegant pieces of machinery in higher mathematics, Green's functions, to solve inhomogeneous differential equations.

This chapter addresses Green's functions in one dimension, that is, Green's functions of ordinary differential equations. Consider the ODE $\mathbf{L}_x[u] = f(x)$ where $\mathbf{L}_x$ is a linear differential operator. In the abstract Dirac notation this can be formally written as $\mathbf{L}|u\rangle = |f\rangle$. If $\mathbf{L}$ has an inverse $\mathbf{L}^{-1} \equiv \mathbf{G}$, the solution can be formally written as $|u\rangle = \mathbf{L}^{-1}|f\rangle = \mathbf{G}|f\rangle$. Multiplying this by $\langle x|$ and inserting $\mathbf{1} = \int dy|y\rangle w(y)\langle y|$ between $\mathbf{G}$ and $|f\rangle$ gives

$$u(x) = \int dy G(x, y) w(y) f(y), \tag{20.1}$$

where the integration is over the range of definition of the functions involved. Once we know $G(x, y)$, Eq. (20.1) gives the solution $u(x)$ in an integral form. But how do we find $G(x, y)$?

Sandwiching both sides of $\mathbf{LG} = \mathbf{1}$ between $\langle x|$ and $|y\rangle$ and using

$$\mathbf{1} = \int dx' |x'\rangle w(x') \langle x'|$$

between $\mathbf{L}$ and $\mathbf{G}$ yields

$$\int dx' L(x, x') w(x') G(x', y) = \langle x|y\rangle = \frac{\delta(x-y)}{w(x)}$$

if we use Eq. (7.19). In particular, if $\mathbf{L}$ is a local differential operator (see Sect. 17.1), then $L(x, x') = [\delta(x - x')/w(x)]\mathbf{L}_x$, and we obtain

differential equation for Green's function

$$\mathbf{L}_x G(x, y) = \frac{\delta(x-y)}{w(x)} \quad \text{or} \quad \mathbf{L}_x G(x, y) = \delta(x - y), \tag{20.2}$$

where the second equation makes the frequently used assumption that $w(x) = 1$. $G(x, y)$ is called the **Green's function** (GF) for the differential operator (DO) $\mathbf{L}_x$.

Green's function

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As discussed in Chaps. 17 and 19, $\mathbf{L}_x$ might not be defined for all functions on $\mathbb{R}$. Moreover, a complete specification of $\mathbf{L}_x$ requires some initial (or boundary) conditions. Therefore, we expect $G(x, y)$ to depend on such initial conditions as well. We note that when $\mathbf{L}_x$ is applied to (20.1), we get

$$\mathbf{L}_x u(x) = \int dy \big[\mathbf{L}_x(G(x, y))\big] w(y) f(y)$$
$$= \int dy \frac{\delta(x-y)}{w(x)} w(y) f(y) = f(x),$$

indicating that $u(x)$ is indeed a solution of the original ODE. Equation (20.2), involving the generalized function $\delta(x-y)$ (or distribution in the language of Sect. 7.3), is meaningful only in the same context. Thus, we treat $G(x, y)$ not as an ordinary function but as a *distribution*. Finally, (20.1) is assumed to hold for an arbitrary (well-behaved) function f.

20.1 Calculation of Some Green's Functions

This section presents some examples of calculating $G(x, y)$ for very simple DOs. Later we will see how to obtain Green's functions for a general second-order linear differential operator. Although the complete specification of GFs requires boundary conditions, we shall introduce unspecified constants in some of the examples below, and calculate some *indefinite* GFs.

indefinite Green's Functions

Example 20.1.1 Let us find the GF for the simplest DO, $\mathbf{L}_x = d/dx$. We need to find a distribution such that its derivative is the Dirac delta function:[1] $G'(x, y) = \delta(x-y)$. In Sect. 7.3, we encountered such a distribution—the step function $\theta(x-y)$. Thus,

$$G(x, y) = \theta(x-y) + \alpha(y),$$

where $\alpha(y)$ is the "constant" of integration.

The example above did not include a boundary (or initial) condition. Let us see how boundary conditions affect the resulting GF.

Example 20.1.2 Let us solve $u'(x) = f(x)$ where $x \in [0, \infty)$ and $u(0) = 0$. A general solution of this DE is given by Eq. (20.1) and the preceding example:

$$u(x) = \int_0^\infty \theta(x-y) f(y) dy + \int_0^\infty \alpha(y) f(y) dy.$$

The factor $\theta(x-y)$ in the first term on the RHS chops off the integral at x:

$$u(x) = \int_0^x f(y) dy + \int_0^\infty \alpha(y) f(y) dy.$$

[1] Here and elsewhere in this chapter, a prime over a GF indicates differentiation with respect to its first argument.

The BC gives

$$0 = u(0) = 0 + \int_0^\infty \alpha(y) f(y) dy.$$

The only way that this can be satisfied for arbitrary $f(y)$ is for $\alpha(y)$ to be zero. Thus, $G(x, y) = \theta(x - y)$, and

$$u(x) = \int_0^\infty \theta(x - y) f(y) dy = \int_0^x f(y) dy.$$

This is killing a fly with a sledgehammer! We could have obtained the result by a simple integration. However, the roundabout way outlined here illustrates some important features of GFs that will be discussed later. The BC introduced here is very special. What happens if it is changed to $u(0) = a$? Problem 20.1 answers that.

Example 20.1.3 A more complicated DO is $\mathbf{L}_x = d^2/dx^2$. Let us find its indefinite GF. To do so, we integrate $G''(x, y) = \delta(x - y)$ once with respect to x to obtain

$$\frac{d}{dx} G(x, y) = \theta(x - y) + \alpha(y).$$

A second integration yields

$$G(x, y) = \int dx \theta(x - y) + x\alpha(y) + \eta(y),$$

where α and η are arbitrary functions and the integral is an indefinite integral to be evaluated next.

Let $\Omega(x, y)$ be the primitive of $\theta(x - y)$; that is,

$$\frac{d\Omega}{dx} = \theta(x - y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x < y. \end{cases} \tag{20.3}$$

The solution to this equation is

$$\Omega(x, y) = \begin{cases} x + a(y) & \text{if } x > y, \\ b(y) & \text{if } x < y. \end{cases}$$

Note that we have not defined $\Omega(x, y)$ at $x = y$. It will become clear below that $\Omega(x, y)$ is continuous at $x = y$. It is convenient to write $\Omega(x, y)$ as

$$\Omega(x, y) = \big[x + a(y)\big]\theta(x - y) + b(y)\theta(y - x). \tag{20.4}$$

To specify $a(y)$ and $b(y)$ further, we differentiate (20.4) and compare it with (20.3):

$$\begin{aligned} \frac{d\Omega}{dx} &= \theta(x - y) + \big[x + a(y)\big]\delta(x - y) - b(y)\delta(x - y) \\ &= \theta(x - y) + \big[x - b(y) + a(y)\big]\delta(x - y), \end{aligned} \tag{20.5}$$

where we have used

$$\frac{d}{dx}\theta(x-y)=-\frac{d}{dx}\theta(y-x)=\delta(x-y).$$

For Eq. (20.5) to agree with (20.3), we must have $[x-b(y)+a(y)]\delta(x-y)=0$, which, upon integration over x, yields $a(y)-b(y)=-y$. Substituting this in the expression for $\Omega(x,y)$ gives

$$\Omega(x,y)=(x-y)\theta(x-y)+b(y)\big[\theta(x-y)+\theta(y-x)\big].$$

But $\theta(x)+\theta(-x)=1$; therefore, $\Omega(x,y)=(x-y)\theta(x-y)+b(y)$. It follows, among other things, that $\Omega(x,y)$ is continuous at $x=y$. We can now write

$$G(x,y)=(x-y)\theta(x-y)+x\alpha(y)+\beta(y),$$

where $\beta(y)=\eta(y)+b(y)$.

The GF in the example above has two arbitrary functions, $\alpha(y)$ and $\beta(y)$, which are the result of underspecification of $\mathbf{L}_x$: A full specification of $\mathbf{L}_x$ requires BCs, as the following example shows.

Example 20.1.4 Let us calculate the GF of $\mathbf{L}_x[u]=u''(x)=f(x)$ subject to the BC $u(a)=u(b)=0$ where $[a,b]$ is the interval on which $\mathbf{L}_x$ is defined. Example 20.1.3 gives us the (indefinite) GF for $\mathbf{L}_x$. Using that, we can write

$$\begin{aligned}u(x)&=\int_a^b(x-y)\theta(x-y)f(y)\,dy+x\int_a^b\alpha(y)f(y)\,dy\\&\quad+\int_a^b\beta(y)f(y)\,dy\\&=\int_a^x(x-y)f(y)\,dy+x\int_a^b\alpha(y)f(y)\,dy+\int_a^b\beta(y)f(y)\,dy.\end{aligned}$$

Applying the BCs yields

$$\begin{aligned}0=u(a)&=a\int_a^b\alpha(y)f(y)\,dy+\int_a^b\beta(y)f(y)\,dy,\\0=u(b)&=\int_a^b(b-y)f(y)\,dy+b\int_a^b\alpha(y)f(y)\,dy\\&\quad+\int_a^b\beta(y)f(y)\,dy.\end{aligned}\tag{20.6}$$

From these two relations it is possible to determine $\alpha(y)$ and $\beta(y)$: Substitute for the last integral on the RHS of the second equation of (20.6) from the first equation and get

$$0=\int_a^b\big[b-y+b\alpha(y)-a\alpha(y)\big]f(y)\,dy.$$

Since this must hold for arbitrary $f(y)$, we conclude that

$$b - y + (b-a)\alpha(y) = 0 \quad \Rightarrow \quad \alpha(y) = -\frac{b-y}{b-a}.$$

Substituting for $\alpha(y)$ in the first equation of (20.6) and noting that the result holds for arbitrary f, we obtain $\beta(y) = a(b-y)/(b-a)$. Insertion of $\alpha(y)$ and $\beta(y)$ in the expression for $G(x,y)$ obtained in Example 20.1.3 gives

$$G(x,y) = (x-y)\theta(x-y) + (x-a)\frac{y-b}{b-a} \quad \text{where } a \le x \text{ and } y \le b.$$

It is striking that $G(a,y) = (a-y)\theta(a-y) = 0$ (because $a - y \le 0$), and

$$G(b,y) = (b-y)\theta(b-y) + (b-a)\frac{y-b}{b-a} = 0$$

because $\theta(b-y) = 1$ for all $y \le b$ [recall that x and y lie in the interval (a,b)]. These two equations reveal the important fact that as a function of x, $G(x,y)$ satisfies the same (homogeneous) BC as the solution of the DE. This is a general property that will be discussed later.

In all the preceding examples, the BCs were very simple. Specifically, the value of the solution and/or its derivative at the boundary points was zero. What if the BCs are not so simple? In particular, how can we handle a case where $u(a)$ [or $u'(a)$] and $u(b)$ [or $u'(b)$] are nonzero?

Consider a general (second-order) differential operator $\mathbf{L}_x$ and the differential equation $\mathbf{L}_x[u] = f(x)$ subject to the BCs $u(a) = a_1$ and $u(b) = b_1$. We claim that we can reduce this system to the case where $u(a) = u(b) = 0$. Recall from Chap. 14 that the most general solution to such a DE is of the form $u = u_h + u_i$ where u_h, the solution to the homogeneous equation, satisfies $\mathbf{L}_x[u_h] = 0$ and contains the arbitrary parameters inherent in solutions of differential equations. For instance, if the linearly independent solutions are v and w, then $u_h(x) = C_1 v(x) + C_2 w(x)$ and u_i is any solution of the inhomogeneous DE.

If we demand that $u_h(a) = a_1$ and $u_h(b) = b_1$, then u_i satisfies the system

$$\mathbf{L}_x[u_i] = f(x), \qquad u_i(a) = u_i(b) = 0,$$

which is of the type discussed in the preceding examples. Since $\mathbf{L}_x$ is a SOLDO, we can put all the machinery of Chap. 14 to work to obtain $v(x)$, $w(x)$, and therefore $u_h(x)$. The problem then reduces to a DE for which the BCs are homogeneous; that is, the value of the solution and/or its derivative is zero at the boundary points.

Example 20.1.5 Let us assume that $\mathbf{L}_x = d^2/dx^2$. Calculation of u_h is trivial:

$$\mathbf{L}_x[u_h] = 0 \quad \Rightarrow \quad \frac{d^2 u_h}{dx^2} = 0 \quad \Rightarrow \quad u_h(x) = C_1 x + C_2.$$

To evaluate C_1 and C_2, we impose the BCs $u_h(a) = a_1$ and $u_h(b) = b_1$:

$$C_1 a + C_2 = a_1,$$
$$C_1 b + C_2 = b_1.$$

This gives $C_1 = (b_1 - a_1)/(b - a)$ and $C_2 = (a_1 b - a b_1)/(b - a)$.

The inhomogeneous equation defines a problem identical to that of Example 20.1.4. Thus, we can immediately write $u_i(x) = \int_a^b G(x, y) f(y)\, dy$, where $G(x, y)$ is as given in that example. Thus, the general solution is

$$u(x) = \frac{b_1 - a_1}{b - a} x + \frac{a_1 b - a b_1}{b - a} + \int_a^x (x - y) f(y)\, dy + \frac{x - a}{b - a} \int_a^b (y - b) f(y)\, dy.$$

Example 20.1.5 shows that an inhomogeneous DE with inhomogeneous BCs can be separated into two DEs, one homogeneous with inhomogeneous BCs and the other inhomogeneous with homogeneous BCs, the latter being appropriate for the GF. Furthermore, all the preceding examples indicate that solutions of DEs can be succinctly written in terms of GFs that automatically incorporate the BCs as long as the BCs are homogeneous. Can a GF also give the solution to a homogeneous DE with inhomogeneous BCs?

20.2 Formal Considerations

The discussion and examples of the preceding section hint at the power of Green's functions. The elegance of such a function becomes apparent from the realization that it contains all the information about the solutions of a DE for any type of BCs, as we are about to show. Since GFs are inverses of DOs, let us briefly reexamine the inverse of an operator, which is closely tied to its spectrum.

The question as to whether or not an operator $\mathbf{A}$ in a finite-dimensional vector space is invertible is succinctly answered by the value of its determinant: $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$. In fact, as we saw at the beginning of Chap. 17, one translates the abstract operator equation $\mathbf{A}|u\rangle = |v\rangle$ into a matrix equation $\mathsf{A}\mathsf{u} = \mathsf{v}$ and reduces the question to that of the inverse of a matrix. This matrix takes on an especially simple form when $\mathbf{A}$ is diagonal, that is, when $A_{ij} = \lambda_i \delta_{ij}$. For this special situation we have

$$\lambda_i u_i = v_i \quad \text{for } i = 1, 2, \ldots, N \text{ (no sum over } i). \tag{20.7}$$

This equation has a unique solution (for arbitrary v_i) if and only if $\lambda_i \neq 0$ for all i. In that case $u_i = v_i/\lambda_i$ for $i = 1, 2, \ldots, N$. In particular, if $v_i = 0$ for all i, that is, when Eq. (20.7) is homogeneous, the unique solution is the trivial solution. On the other hand, when some of the λ_i are zero, there may be no solution to (20.7), but the homogeneous equation has a nontrivial solution (u_i need not be zero). Proposition 6.2.6 applies to vector spaces of finite as well as infinite dimensions. Therefore, we restate it here:

Theorem 20.2.1 *An operator* $\mathbf{A}$ *on a Hilbert space has an inverse if and only if* $\lambda = 0$ *is not an eigenvalue of* $\mathbf{A}$. *Equivalently,* $\mathbf{A}$ *is invertible if and only if the homogeneous equation* $\mathbf{A}|u\rangle = 0$ *has no nontrivial solutions.*

Green's functions are inverses of differential operators. Therefore, it is important to have a clear understanding of the DOs. An nth-order linear differential operator (NOLDO) satisfies the following theorem (for a proof, see [Birk 78, Chap. 6]).

Theorem 20.2.2 *Let*

$$\mathbf{L}_x = p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x) \tag{20.8}$$

where $p_n(x) \not\equiv 0$ *in* $[a, b]$. *Let* $x_0 \in [a, b]$ *and let* $\{\gamma_k\}_{k=1}^n$ *be given numbers and* $f(x)$ *a given piecewise continuous function on* $[a, b]$. *Then the initial value problem* (*IVP*)

$$\begin{aligned} &\mathbf{L}_x[u] = f \quad \text{for } x \in [a, b], \\ &u(x_0) = \gamma_1, \qquad u'(x_0) = \gamma_2, \ldots, \; u^{(n-1)}(x_0) = \gamma_n \end{aligned} \tag{20.9}$$

has one and only one solution.

This is simply the existence and uniqueness theorem for a NOLDE. Equation (20.9) is referred to as the **IVP with data** $\{f(x); \gamma_1, \ldots, \gamma_n\}$. This theorem is used to define $\mathbf{L}_x$. Part of that definition are the BCs that the solutions to $\mathbf{L}_x$ must satisfy.

initial value problem

A particularly important BC is the homogeneous one in which $\gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$. In such a case it can be shown (see Problem 20.3) that the only nontrivial solution of the homogeneous DE $\mathbf{L}_x[u] = 0$ is $u \equiv 0$. Theorem 20.2.1 then tells us that $\mathbf{L}_x$ is invertible; that is, there is a unique operator $\mathbf{G}$ such that $\mathbf{LG} = \mathbf{1}$. The "components" version of this last relation is part of the content of the next theorem.

Theorem 20.2.3 *The DO* $\mathbf{L}_x$ *of Eq.* (20.8) *associated with the IVP with data* $\{f(x); 0, 0, \ldots, 0\}$ *is invertible*; *that is, there exists a function* $G(x, y)$ *such that*

$$\mathbf{L}_x G(x, y) = \frac{\delta(x - y)}{w(x)}.$$

The importance of homogeneous BCs can now be appreciated. Theorem 20.2.3 is the reason why we had to impose homogeneous BCs to obtain the GF in all the examples of the previous section.

The BCs in (20.9) clearly are not the only ones that can be used. The most general linear BCs encountered in differential operator theory are

$$\mathbf{R}_i[u] \equiv \sum_{j=1}^{n} \big(\alpha_{ij} u^{(j-1)}(a) + \beta_{ij} u^{(j-1)}(b)\big) = \gamma_i, \quad i = 1, \ldots, n. \quad (20.10)$$

The n row vectors $\{(\alpha_{i1}, \ldots, \alpha_{in}, \beta_{i1}, \ldots, \beta_{in})\}_{i=1}^{n}$ are assumed to be independent (in particular, no row is identical to zero). We refer to $\mathbf{R}_i$ as **boundary functionals** because for each (sufficiently smooth) function u, they give a number γ_i. The DO of (20.8) and the BCs of (20.10) together form a **boundary value problem** (BVP). The DE $\mathbf{L}_x[u] = f$ subject to the BCs of (20.10) is a BVP with data $\{f(x); \gamma_1, \ldots, \gamma_n\}$.

boundary functionals and boundary value problem

We note that the $\mathbf{R}_i$ are linear; that is,

$$\mathbf{R}_i[u_1 + u_2] = \mathbf{R}_i[u_1] + \mathbf{R}_i[u_2] \quad \text{and} \quad \mathbf{R}_i[\alpha u] = \alpha \mathbf{R}_i[u].$$

Since $\mathbf{L}_x$ is also linear, we conclude that the superposition principle applies to the system consisting of $\mathbf{L}_x[u] = f$ and the BCs of (20.10), which is sometimes denoted by $(\mathbf{L}; \mathbf{R}_1, \ldots, \mathbf{R}_n)$. If u satisfies the BVP with data $\{f; \gamma_1, \ldots, \gamma_n\}$ and v satisfies the BVP with data $\{g; \mu_1, \ldots, \mu_n\}$, then $\alpha u + \beta v$ satisfies the BVP with data $\{\alpha f + \beta g; \alpha\gamma_1 + \beta\mu_1, \ldots, \alpha\gamma_n + \beta\mu_n\}$. It follows that if u and v both satisfy the BVP with data $\{f; \gamma_1, \ldots, \gamma_n\}$, then $u - v$ satisfies the BVP with data $\{0; 0, 0, \ldots, 0\}$, which is called the **completely homogeneous problem**.

completely homogeneous problem

Unlike the IVP, the BVP with data $\{0; 0, 0, \ldots, 0\}$ may have a nontrivial solution. If the completely homogeneous problem has no nontrivial solution, then the BVP with data $\{f; \gamma_1, \ldots, \gamma_n\}$ has at most one solution (a solution exists for any set of data). On the other hand, if the completely homogeneous problem has nontrivial solutions, then the BVP with data $\{f; \gamma_1, \ldots, \gamma_n\}$ either has no solutions or has more than one solution (see [Stak 79, pp. 203–204]).

Recall that when a differential (unbounded) operator $\mathbf{L}_x$ acts in a Hilbert space, such as $\mathcal{L}_w^2(a, b)$, it acts only on its domain. In the context of the present discussion, this means that not all functions in $\mathcal{L}_w^2(a, b)$ satisfy the BCs necessary for defining $\mathbf{L}_x$. Thus, the functions for which the operator is defined (those that satisfy the BCs) form a subset of $\mathcal{L}_w^2(a, b)$, which we called the domain of $\mathbf{L}_x$ and denoted by $\mathcal{D}(\mathbf{L}_x)$. From a formal standpoint it is important to distinguish among maps that have different domains. For instance, the Hilbert-Schmidt integral operators, which are defined on a finite interval, are compact, while those defined on the entire real line are not.

Definition 20.2.4 Let $\mathbf{L}_x$ be the DO of Eq. (20.8). Suppose there exists a DO $\mathbf{L}_x^\dagger$, with the property that

adjoint of a differential operator

$$w\big\{v^*\big(\mathbf{L}_x[u]\big) - u\big(\mathbf{L}_x^\dagger[v]\big)^*\big\} = \frac{d}{dx} Q\big[u, v^*\big] \quad \text{for } u, v \in \mathcal{D}(\mathbf{L}_x) \cap \mathcal{D}\big(\mathbf{L}_x^\dagger\big),$$

conjunct

where $Q[u, v^*]$, called the **conjunct** of the functions u and v, depends on u, v, and their derivatives of order up to $n - 1$. The DO $\mathbf{L}_x^\dagger$ is then called the **formal adjoint** of $\mathbf{L}_x$. If $\mathbf{L}_x^\dagger = \mathbf{L}_x$ (without regard to the BCs imposed on their solutions), then $\mathbf{L}_x$ is said to be **formally self-adjoint**. If $\mathcal{D}(\mathbf{L}_x^\dagger) \supset \mathcal{D}(\mathbf{L}_x)$ and $\mathbf{L}_x^\dagger = \mathbf{L}_x$ on $\mathcal{D}(\mathbf{L}_x)$, then $\mathbf{L}_x$ is said to be **hermitian**. If $\mathcal{D}(\mathbf{L}_x^\dagger) = \mathcal{D}(\mathbf{L}_x)$ and $\mathbf{L}_x^\dagger = \mathbf{L}_x$, then $\mathbf{L}_x$ is said to be **self-adjoint**.

The relation given in the definition above involving the conjunct is a generalization of the Lagrange identity and can also be written in integral form:

generalized Green's identity

$$\int_a^b dx\, w\big\{v^*\big(\mathbf{L}_x[u]\big)\big\} - \int_a^b dx\, w\big\{u\big(\mathbf{L}_x^\dagger[v]\big)^*\big\} = Q\big[u, v^*\big]\Big|_a^b \qquad (20.11)$$

This form is sometimes called the **generalized Green's identity**.

Historical Notes

George Green (1793?–1841) was not appreciated in his lifetime. His date of birth is unknown (however, it is known that he was baptized on 14 July 1793), and no portrait of him survives. He left school, after only one year's attendance, to work in his father's bakery. When the father opened a windmill in Nottingham, the boy used an upper room as a study in which he taught himself physics and mathematics from library books. In 1828, when he was thirty-five years old, he published his most important work, *An Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism* at his own expense. In it Green apologized for any shortcomings in the paper due to his minimal formal education or the limited resources available to him, the latter being apparent in the few previous works he cited. The introduction explained the importance Green placed on the "potential" function. The body of the paper generalizes this idea to electricity and magnetism.

George Green
1793?–1841

In addition to the physics of electricity and magnetism, Green's first paper also contained the monumental mathematical contributions for which he is now famous: The relationship between surface and volume integrals we now call *Green's theorem*, and the *Green's function*, a ubiquitous solution to partial differential equations in almost every area of physics. With little appreciation for the future impact of this work, one of Green's contemporaries declared the publication "a complete failure". The "Essay", which received little notice because of poor circulation, was saved by Lord Kelvin, who tracked it down in a German journal.

When his father died in 1829, some of George's friends urged him to seek a college education. After four years of self-study, during which he closed the gaps in his elementary education, Green was admitted to Caius College of Cambridge University at the age of 40, from which he graduated four years later after a disappointing performance on his final examinations. Later, however, he was appointed Perce Fellow of Caius College. Two years after his appointment he died, and his famous 1828 paper was republished, this time reaching a much wider audience. This paper has been described as "the beginning of mathematical physics in England".

He published only ten mathematical works. In 1833 he wrote three further papers. Two on electricity were published by the Cambridge Philosophical Society. One on hydrodynamics was published by the Royal Society of Edinburgh (of which he was a Fellow) in 1836. He also had two papers on hydrodynamics (in particular wave motion in canals), two papers on reflection and refraction of light, and two papers on reflection and refraction of sound published in Cambridge.

In 1923 the Green windmill was partially restored by a local businessman as a gesture of tribute to Green. Einstein came to pay homage. Then a fire in 1947 destroyed the renovations. Thirty years later the idea of a memorial was once again mooted, and sufficient money was raised to purchase the mill and present it to the sympathetic Nottingham City Council. In 1980 the George Green Memorial Appeal was launched to secure £20,000 to get the sails turning again and the machinery working once more. Today, Green's restored mill stands as a mathematics museum in Nottingham.

20.2.1 Second-Order Linear DOs

Since second-order linear differential operators (SOLDOs) are sufficiently general for most physical applications, we will concentrate on them. Because homogeneous BCs are important in constructing Green's functions, let us first consider BCs of the form

$$\begin{aligned}\mathbf{R}_1[u] &\equiv \alpha_{11}u(a)+\alpha_{12}u'(a)+\beta_{11}u(b)+\beta_{12}u'(b)=0,\\ \mathbf{R}_2[u] &\equiv \alpha_{21}u(a)+\alpha_{22}u'(a)+\beta_{21}u(b)+\beta_{22}u'(b)=0,\end{aligned} \tag{20.12}$$

where it is assumed, as usual, that $(\alpha_{11},\alpha_{12},\beta_{11},\beta_{12})$ and $(\alpha_{21},\alpha_{22},\beta_{21},\beta_{22})$ are linearly independent.

If we define the inner product as an integral with weight w, Eq. (20.11) can be formally written as

$$\langle v|\mathbf{L}|u\rangle = \langle u|\mathbf{L}^\dagger|v\rangle^* + Q\big[u, v^*\big]\big|_a^b.$$

This would coincide with the usual definition of the adjoint if the surface term vanishes, that is, if

$$Q\big[u, v^*\big]\big|_{x=b} = Q\big[u, v^*\big]\big|_{x=a}. \tag{20.13}$$

For this to happen, we need to impose BCs on v. To find these BCs, let us rewrite Eq. (20.12) in a more compact form. Linear independence of the two row vectors of coefficients implies that the 2×4 matrix of coefficients has rank two. This means that the 2×4 matrix has an invertible 2×2 submatrix. By rearranging the terms in Eq. (20.12) if necessary, we can assume that the second of the two 2×2 submatrices is invertible. The homogeneous BCs can then be conveniently written as

$$\mathbf{R}[u] = \begin{pmatrix}\mathbf{R}_1[u]\\ \mathbf{R}_2[u]\end{pmatrix} = \begin{pmatrix}\mathsf{A} & \mathsf{B}\end{pmatrix}\begin{pmatrix}\mathsf{u}_a\\ \mathsf{u}_b\end{pmatrix} = \mathsf{A}\mathsf{u}_a + \mathsf{B}\mathsf{u}_b = 0, \tag{20.14}$$

where

$$\mathsf{A} \equiv \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}, \qquad \mathsf{B} \equiv \begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix}, \qquad \mathsf{u}_a \equiv \begin{pmatrix}u(a)\\ u'(a)\end{pmatrix},$$
$$\mathsf{u}_b \equiv \begin{pmatrix}u(b)\\ u'(b)\end{pmatrix},$$

and B is invertible.

The most general form of the conjunct for a SOLDO is

$$\begin{aligned}Q\big[u, v^*\big](x) &= q_{11}(x)u(x)v^*(x) + q_{12}(x)u(x)v'^*(x)\\ &\quad + q_{21}(x)u'(x)v^*(x) + q_{22}(x)u'(x)v'^*(x),\end{aligned}$$

which can be written in matrix form as

$$Q\big[u, v^*\big](x) = \mathsf{u}_x^t\mathsf{Q}_x\mathsf{v}_x^* \quad \text{where} \quad \mathsf{Q}_x = \begin{pmatrix}q_{11}(x) & q_{12}(x)\\ q_{21}(x) & q_{22}(x)\end{pmatrix}, \tag{20.15}$$

and u_x and v_x^* have similar definitions as u_a and u_b above. The vanishing of the surface term becomes

$$\mathsf{u}_b^t \mathsf{Q}_b \mathsf{v}_b^* = \mathsf{u}_a^t \mathsf{Q}_a \mathsf{v}_a^*. \tag{20.16}$$

We need to translate this equation into a condition on v^* *alone*.[2] This is accomplished by solving for two of the four quantities $u(a)$, $u'(a)$, $u(b)$, and $u'(b)$ in terms of the other two, substituting the result in Eq. (20.16), and setting the coefficients of the other two equal to zero. Let us assume, as before, that the submatrix B is invertible, i.e., $u(b)$ and $u'(b)$ are expressible in terms of $u(a)$ and $u'(a)$. Then $\mathsf{u}_b = -\mathsf{B}^{-1}\mathsf{A}\mathsf{u}_a$, or $\mathsf{u}_b^t = -\mathsf{u}_a^t \mathsf{A}^t(\mathsf{B}^t)^{-1}$, and we obtain

$$-\mathsf{u}_a^t \mathsf{A}^t \left(\mathsf{B}^t\right)^{-1} \mathsf{Q}_b \mathsf{v}_b^* = \mathsf{u}_a^t \mathsf{Q}_a \mathsf{v}_a^* \quad \Rightarrow \quad \mathsf{u}_a^t \left[\mathsf{A}^t \left(\mathsf{B}^t\right)^{-1} \mathsf{Q}_b \mathsf{v}_b^* + \mathsf{Q}_a \mathsf{v}_a^*\right] = 0,$$

and the condition on v^* becomes

$$\mathsf{A}^t \left(\mathsf{B}^t\right)^{-1} \mathsf{Q}_b \mathsf{v}_b^* + \mathsf{Q}_a \mathsf{v}_a^* = 0. \tag{20.17}$$

We see that all factors of u have disappeared, as they should. The expanded version of the BCs on v^* are written as

$$\begin{aligned} \mathbf{B}_1\left[v^*\right] &\equiv \sigma_{11} v^*(a) + \sigma_{12} v'^*(a) + \eta_{11} v^*(b) + \eta_{12} v'^*(b) = 0, \\ \mathbf{B}_2\left[v^*\right] &\equiv \sigma_{21} v^*(a) + \sigma_{22} v'^*(a) + \eta_{21} v^*(b) + \eta_{22} v'^*(b) = 0. \end{aligned} \tag{20.18}$$

These homogeneous BCs are said to be **adjoint** to those of (20.12). Because of the difference between BCs and their adjoints, the domain of a differential operator need not be the same as that of its adjoint.

adjoint boundary conditions

Example 20.2.5 Let $\mathbf{L}_x = d^2/dx^2$ with the homogeneous BCs

$$\mathbf{R}_1[u] = \alpha u(a) - u'(a) = 0 \quad \text{and} \quad \mathbf{R}_2[u] = \beta u(b) - u'(b) = 0. \tag{20.19}$$

We want to calculate $Q[u, v^*]$ and the adjoint BCs for v. By repeated integration by parts [or by using Eq. (14.22)], we obtain $Q[u, v^*] = u'v^* - uv'^*$. For the surface term to vanish, we must have

$$u'(a)v^*(a) - u(a)v'^*(a) = u'(b)v^*(b) - u(b)v'^*(b).$$

Substituting from (20.19) in this equation, we get

$$u(a)\left[\alpha v^*(a) - v'^*(a)\right] = u(b)\left[\beta v^*(b) - v'^*(b)\right],$$

which holds for arbitrary u if and only if

$$\mathbf{B}_1\left[v^*\right] = \alpha v^*(a) - v'^*(a) = 0 \quad \text{and} \quad \mathbf{B}_2\left[v^*\right] = \beta v^*(b) - v'^*(b) = 0. \tag{20.20}$$

This is a special case, in which the adjoint boundary conditions are the same as the original BCs (substitute u for v^* to see this).

[2] The boundary conditions on v^* should not depend on the choice of u.

To see that the original BCs and their adjoints need not be the same, we consider

$$\mathbf{R}_1[u] = u'(a) - \alpha u(b) = 0 \quad \text{and} \quad \mathbf{R}_2[u] = \beta u(a) - u'(b) = 0, \tag{20.21}$$

from which we obtain $u(a)[\beta v^*(b) + v'^*(a)] = u(b)[\alpha v^*(a) + v'^*(b)]$. Thus,

mixed and unmixed BCs

$$\mathbf{B}_1\big[v^*\big] = \alpha v^*(a) + v'^*(b) = 0 \quad \text{and} \quad \mathbf{B}_2\big[v^*\big] = \beta v^*(b) + v'^*(a) = 0, \tag{20.22}$$

which is not the same as (20.21). Boundary conditions such as those in (20.19) and (20.20), in which each equation contains the function and its derivative evaluated at the same point, are called **unmixed BCs**. On the other hand, (20.21) and (20.22) are mixed BCs.

20.2.2 Self-adjoint SOLDOs

In Chap. 14, we showed that a SOLDO satisfies the generalized Green's identity with $w(x) = 1$. In fact, since u and v are real, Eq. (14.23) is identical to (20.11) if we set $w = 1$ and

$$Q[u, v] = p_2 v u' - (p_2 v)' u + p_1 u v. \tag{20.23}$$

Also, we have seen that any SOLDO can be made (formally) self-adjoint. Thus, let us consider the formally self-adjoint SOLDO

$$\mathbf{L}_x = \mathbf{L}_x^\dagger = \frac{d}{dx}\left(p \frac{d}{dx}\right) + q$$

where both $p(x)$ and $q(x)$ are real functions and the inner product is defined with weight $w = 1$. If we are interested in formally self-adjoint operators with respect to a general weight $w > 0$, we can construct them as follows. We first note that if $\mathbf{L}_x$ is formally self-adjoint with respect to a weight of unity, then $(1/w)\mathbf{L}_x$ is self-adjoint with respect to weight w. Next, we note that $\mathbf{L}_x$ is formally self-adjoint for all functions q, in particular, for wq. Now we define

$$\mathbf{L}_x^{(w)} = \frac{d}{dx}\left(p \frac{d}{dx}\right) + qw$$

and note that $\mathbf{L}_x^{(w)}$ is formally self-adjoint with respect to a weight of unity, and therefore

$$\mathbf{L}_x \equiv \frac{1}{w}\mathbf{L}_x^{(w)} = \frac{1}{w}\frac{d}{dx}\left(p \frac{d}{dx}\right) + q \tag{20.24}$$

is formally self-adjoint with respect to weight $w(x) > 0$.

For SOLDOs that are formally self-adjoint with respect to weight w, the conjunct given in (20.23) becomes

$$Q[u, v] = p(x)w(x)\big(vu' - uv'\big). \tag{20.25}$$

Thus, the surface term in the generalized Green's identity vanishes if and only if

$$p(b)w(b)\big[v(b)u'(b)-u(b)v'(b)\big] \\ = p(a)w(a)\big[v(a)u'(a)-u(a)v'(a)\big]. \tag{20.26}$$

common types of boundary conditions for a SOLDE

The DO becomes self-adjoint if u and v satisfy Eq. (20.26) as well as *the same BCs*. It can easily be shown that the following four types of BCs on $u(x)$ assure the validity of Eq. (20.26) and therefore define a self-adjoint operator $\mathbf{L}_x$ given by (20.24):

1. The **Dirichlet** BCs: $u(a)=u(b)=0$
2. The **Neumann** BCs: $u'(a)=u'(b)=0$
3. **General unmixed** BCs: $\alpha u(a)-u'(a)=\beta u(b)-u'(b)=0$
4. **Periodic** BCs: $u(a)=u(b)$ and $u'(a)=u'(b)$

20.3 Green's Functions for SOLDOs

We are now in a position to find the Green's function for a SOLDO. First, note that a complete specification of $\mathbf{L}_x$ requires not only knowledge of $p_0(x)$, $p_1(x)$, and $p_2(x)$—its coefficient functions—but also knowledge of the BCs imposed on the solutions. The most general BCs for a SOLDO are of the type given in Eq. (20.10) with $n=2$. Thus, to specify $\mathbf{L}_x$ uniquely, we consider the system $(\mathbf{L};\mathbf{R}_1,\mathbf{R}_2)$ with data $(f;\gamma_1,\gamma_2)$. This system defines a unique BVP:

$$\begin{aligned} \mathbf{L}_x[u] &= p_2(x)\frac{d^2u}{dx^2}+p_1(x)\frac{du}{dx}+p_0(x)u=f(x), \\ \mathbf{R}_i[u] &= \gamma_i, \quad i=1,2. \end{aligned} \tag{20.27}$$

A necessary condition for $\mathbf{L}_x$ to be invertible is that the homogeneous DE $\mathbf{L}_x[u]=0$ have only the trivial solution $u=0$. For $u=0$ to be the *only* solution, it must be *a* solution. This means that it must meet all the conditions in Eq. (20.27). In particular, since $\mathbf{R}_i$ are linear functionals of u, we must have $\mathbf{R}_i[0]=0$. This can be stated as follows:

Lemma 20.3.1 *A necessary condition for a second-order linear DO to be invertible is for its associated BCs to be homogeneous.*[3]

Thus, to study Green's functions we must restrict ourselves to problems with homogeneous BCs. This at first may seem restrictive, since not all problems have homogeneous BCs. Can we solve the others by the Green's function method? The answer is yes, as will be shown later in this chapter.

The above discussion clearly indicates that the Green's function of $\mathbf{L}_x$, being its "inverse", is defined only if we consider the system $(\mathbf{L};\mathbf{R}_1,\mathbf{R}_2)$ with data $(f;0,0)$. If the Green's function exists, it must satisfy the DE of

[3]The lemma applies to all linear DOs, not just second order ones.

Theorem 20.2.3, in which $\mathbf{L}_x$ acts on $G(x,y)$. But part of the definition of $\mathbf{L}_x$ are the BCs imposed on the solutions. Thus, if the LHS of the DE is to make any sense, $G(x,y)$ must also satisfy those same BCs. We therefore make the following definition:

formal definition of Green's function

Definition 20.3.2 The **Green's function** of a DO $\mathbf{L}_x$ is a function $G(x,y)$ that satisfies both the DE

$$\mathbf{L}_x G(x,y) = \frac{\delta(x-y)}{w(x)}$$

and, as a function of x, the homogeneous BCs $\mathbf{R}_i[G] = 0$ for $i = 1, 2$ where the $\mathbf{R}_i$ are defined as in Eq. (20.12).

It is convenient to study the Green's function for the adjoint of $\mathbf{L}_x$ simultaneously. Denoting this by $g(x,y)$, we have

$$\mathbf{L}_x^\dagger g(x,y) = \frac{\delta(x-y)}{w(x)}, \qquad \mathbf{B}_i[g] = 0, \quad \text{for } i = 1, 2, \tag{20.28}$$

where $\mathbf{B}_i$ are the boundary functionals adjoint to $\mathbf{R}_i$ and given in Eq. (20.18). The function $g(x,y)$ is known as the **adjoint Green's function** associated with the DE of (20.27).

adjoint Green's function

We can now use (20.27) and (20.28) to find the solutions to

$$\begin{aligned} \mathbf{L}_x[u] &= f(x), \qquad \mathbf{R}_i[u] = 0 \quad \text{for } i = 1, 2, \\ \mathbf{L}_x^\dagger[v] &= h(x), \qquad \mathbf{B}_i\big[v^*\big] = 0 \quad \text{for } i = 1, 2. \end{aligned} \tag{20.29}$$

With $v(x) = g(x,y)$ in Eq. (20.11)—whose RHS is assumed to be zero—we get $\int_a^b wg^*(x,y)\mathbf{L}_x[u]\,dx = \int_a^b wu(x)(\mathbf{L}_x^\dagger[g])^*dx$. Using (20.28) on the RHS and (20.29) on the LHS, we obtain

$$u(y) = \int_a^b g^*(x,y)w(x)f(x)\,dx.$$

Similarly, with $u(x) = G(x,y)$, Eq. (20.11) gives

$$v^*(y) = \int_a^b G(x,y)w(x)h^*(x)\,dx,$$

or, since $w(x)$ is a (positive) real function,

$$v(y) = \int_a^b G^*(x,y)w(x)h(x)\,dx.$$

These equations for $u(y)$ and $v(y)$ are not what we expect [see, for instance, Eq. (20.1)]. However, if we take into account certain properties of Green's functions that we will discuss next, these equations become plausible.

20.3.1 Properties of Green's Functions

Let us rewrite the generalized Green's identity [Eq. (20.11)], with the RHS equal to zero, as

$$\int_a^b dt\, w(t)\big\{v^*(t)\big(\mathbf{L}_t[u]\big)\big\} = \int_a^b dt\, w(t)\big\{u(t)\big(\mathbf{L}_t^\dagger[v]\big)^*\big\}. \qquad (20.30)$$

This is sometimes called **Green's identity**. Substituting $G(t, y)$ for $u(t)$ and $g(t, x)$ for $v(t)$ gives

Green's identity

$$\int_a^b dt\, w(t) g^*(t, x) \frac{\delta(t-y)}{w(t)} = \int_a^b dt\, w(t) G(t, y) \frac{\delta(t-x)}{w(t)},$$

or $g^*(y, x) = G(x, y)$. A consequence of this identity is

Proposition 20.3.3 *$G(x, y)$ must satisfy the adjoint boundary conditions with respect to its second argument.*

If for the time being we assume that the Green's function associated with a system $(\mathbf{L}; \mathbf{R}_1, \mathbf{R}_2)$ is unique, then, since for a self-adjoint differential operator, $\mathbf{L}_x$ and $\mathbf{L}_x^\dagger$ are identical and u and v both satisfy the same BCs, we must have $G(x, y) = g(x, y)$ or, using $g^*(y, x) = G(x, y)$, we get $G(x, y) = G^*(y, x)$. In particular, if the coefficient functions of $\mathbf{L}_x$ are all real, $G(x, y)$ will be real, and we have $G(x, y) = G(y, x)$. We thus have

Proposition 20.3.4 *The Green's function is a symmetric function of its two arguments*: $G(x, y) = G(y, x)$.

The last property is related to the continuity of $G(x, y)$ and its derivative at $x = y$. For a SOLDO, we have

$$\mathbf{L}_x G(x, y) = p_2(x)\frac{\partial^2 G}{\partial x^2} + p_1(x)\frac{\partial G}{\partial x} + p_0(x) G = \frac{\delta(x-y)}{w(x)},$$

where p_0, p_1, and p_2 are assumed to be real and continuous in the interval $[a, b]$, and $w(x)$ and $p_2(x)$ are assumed to be positive for all $x \in [a, b]$. We multiply both sides of the DE by

$$h(x) = \frac{\mu(x)}{p_2(x)}, \quad \text{where} \quad \mu(x) \equiv \exp\left[\int_a^x \frac{p_1(t)}{p_2(t)}\, dt\right],$$

noting that $d\mu/dx = (p_1/p_2)\mu$. This transforms the DE into

$$\frac{\partial}{\partial x}\left[\mu(x)\frac{\partial}{\partial x} G(x, y)\right] + \frac{p_0(x)\mu(x)}{p_2(x)} G(x, y) = \frac{\mu(y)}{p_2(y) w(y)} \delta(x-y).$$

Integrating this equation gives

$$\mu(x)\frac{\partial}{\partial x} G(x, y) + \int_a^x \frac{p_0(t)\mu(t)}{p_2(t)} G(t, y)\, dt = \frac{\mu(y)}{p_2(y) w(y)} \theta(x-y) + \alpha(y) \qquad (20.31)$$

because the primitive of $\delta(x-y)$ is $\theta(x-y)$. Here $\alpha(y)$ is the "constant" of integration. First consider the case where $p_0 = 0$, for which the Green's function will be denoted by $G_0(x, y)$. Then Eq. (20.31) becomes

$$\mu(x)\frac{\partial}{\partial x}G_0(x, y) = \frac{\mu(y)}{p_2(y)w(y)}\theta(x-y) + \alpha_1(y),$$

which (since μ, p_2, and w are continuous on $[a, b]$, and $\theta(x-y)$ has a discontinuity only at $x = y$) indicates that $\partial G_0/\partial x$ is continuous everywhere on $[a, b]$ except at $x = y$. Now divide the last equation by μ and integrate the result to get

$$G_0(x, y) = \frac{\mu(y)}{p_2(y)w(y)}\int_a^x \frac{\theta(t-y)}{\mu(t)}\,dt + \alpha_1(y)\int_a^x \frac{dt}{\mu(t)} + \alpha_2(y).$$

Every term on the RHS is continuous except possibly the integral involving the θ-function. However, that integral can be written as

$$\int_a^x \frac{\theta(t-y)}{\mu(t)}\,dt = \theta(x-y)\int_y^x \frac{dt}{\mu(t)}. \tag{20.32}$$

The θ-function in front of the integral is needed to ensure that $a \le y \le x$ as demanded by the LHS of Eq. (20.32). The RHS of Eq. (20.32) is continuous at $x = y$ with limit being zero as $x \to y$.

Next, we write $G(x, y) = G_0(x, y) + H(x, y)$, and apply $\mathbf{L}_x$ to both sides. This gives

$$\begin{aligned}\frac{\delta(x-y)}{w(x)} &= \left(p_2\frac{d^2}{dx^2} + p_1\frac{d}{dx}\right)G_0 + p_0G_0 + \mathbf{L}_x H(x, y)\\ &= \frac{\delta(x-y)}{w(x)} + p_0G_0 + \mathbf{L}_x H(x, y),\end{aligned}$$

or $p_2H'' + p_1H' + p_0H = -p_0G_0$. The continuity of G_0, p_0, p_1, and p_2 on $[a, b]$ implies the continuity of H, because a discontinuity in H would entail a delta function discontinuity in dH/dx, which is impossible because there are no delta functions in the equation for H. Since both G_0 and H are continuous, G must also be continuous on $[a, b]$.

We can now calculate the jump in $\partial G/\partial x$ at $x = y$. We denote the jump as $\Delta G'(y)$ and define it as follows:

$$\Delta G'(y) \equiv \lim_{\epsilon\to 0}\left[\frac{\partial G}{\partial x}(x, y)\bigg|_{x=y+\epsilon} - \frac{\partial G}{\partial x}(x, y)\bigg|_{x=y-\epsilon}\right].$$

Dividing (20.31) by $\mu(x)$ and taking the above limit for all terms, we obtain

$$\begin{aligned}\Delta G'(y) + \lim_{\epsilon\to 0}\bigg[&\frac{1}{\mu(y+\epsilon)}\int_a^{y+\epsilon}\frac{p_0(t)\mu(t)}{p_2(t)}G(t, y)\,dt\\ &- \frac{1}{\mu(y-\epsilon)}\int_a^{y-\epsilon}\frac{p_0(t)\mu(t)}{p_2(t)}G(t, y)\,dt\bigg]\end{aligned}$$

$$= \frac{\mu(y)}{p_2(y)w(y)} \lim_{\epsilon\to 0}\left[\frac{\overbrace{\theta(+\epsilon)}^{=1}}{\mu(y+\epsilon)} - \frac{\overbrace{\theta(-\epsilon)}^{=0}}{\mu(y-\epsilon)}\right].$$

The second term on the LHS is zero because all functions are continuous at y. The limit on the RHS is simply $1/\mu(y)$. We therefore obtain

$$\Delta G'(y) = \frac{1}{p_2(y)w(y)}. \tag{20.33}$$

20.3.2 Construction and Uniqueness of Green's Functions

We are now in a position to calculate the Green's function for a general SOLDO and show that it is unique.

Theorem 20.3.5 *Consider the system* $(\mathbf{L}; \mathbf{R}_1, \mathbf{R}_2)$ *with data* $(f; 0, 0)$, *in which* $\mathbf{L}_x$ *is a SOLDO. If the homogeneous DE* $\mathbf{L}_x[u] = 0$ *has no nontrivial solution, then the GF associated with the given system exists and is unique. The solution of the system is* $u(x) = \int_a^b dy w(y)G(x, y)f(y)$ *and is also unique.*

existence and uniqueness of GF for a second order linear differential operator

Proof The GF satisfies the DE $\mathbf{L}_x G(x, y) = 0$ for all $x \in [a, b]$ except $x = y$. We thus divide $[a, b]$ into two intervals, $I_1 = [a, y)$ and $I_2 = (y, b]$, and note that a general solution to the above homogeneous DE can be written as a linear combination of a basis of solutions, u_1 and u_2. Thus, we can write the solution of the DE as

$$G_l(x, y) = c_1 u_1(x) + c_2 u_2(x) \quad \text{for } x \in I_1$$
$$G_r(x, y) = d_1 u_1(x) + d_2 u_2(x) \quad \text{for } x \in I_2$$

and define the GF as

$$G(x, y) = \begin{cases} G_l(x, y) & \text{if } x \in I_1, \\ G_r(x, y) & \text{if } x \in I_2, \end{cases} \tag{20.34}$$

where c_1, c_2, d_1, and d_2 are, in general, functions of y. To determine $G(x, y)$ we must determine four unknowns. We also have four relations: the continuity of G, the jump in $\partial G/\partial x$ at $x = y$, and the two BCs $\mathbf{R}_1[G] = \mathbf{R}_2[G] = 0$. The continuity of G gives

$$c_1(y)u_1(y) + c_2(y)u_2(y) = d_1(y)u_1(y) + d_2(y)u_2(y).$$

The jump of $\partial G/\partial x$ at $x = y$ yields

$$c_1(y)u_1'(y) + c_2(y)u_2'(y) - d_1(y)u_1'(y) - d_2(y)u_2'(y) = -\frac{1}{p_2(y)w(y)}.$$

Introducing $b_1 = c_1 - d_1$ and $b_2 = c_2 - d_2$ changes the two preceding equations to

$$b_1 u_1 + b_2 u_2 = 0,$$
$$b_1 u_1' + b_2 u_2' = -\frac{1}{p_2 w}.$$

These equations have a unique solution iff

$$\det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \neq 0.$$

But the determinant is simply the Wronskian of the two independent solutions and therefore cannot be zero. Thus, $b_1(y)$ and $b_2(y)$ are determined in terms of u_1, u_1', u_2, u_2', p_2, and w.

We now define

$$h(x, y) \equiv \begin{cases} b_1(y)u_1(x) + b_2(y)u_2(x) & \text{if } x \in I_1, \\ 0 & \text{if } x \in I_2. \end{cases}$$

so that $G(x, y) = h(x, y) + d_1(y)u_1(x) + d_2(y)u_2(x)$. We have reduced the number of unknowns to two, d_1 and d_2. Imposing the BCs gives two more relations:

$$\mathbf{R}_1[G] = \mathbf{R}_1[h] + d_1\mathbf{R}_1[u_1] + d_2\mathbf{R}_1[u_2] = 0,$$
$$\mathbf{R}_2[G] = \mathbf{R}_2[h] + d_1\mathbf{R}_2[u_1] + d_2\mathbf{R}_2[u_2] = 0.$$

Can we solve these equations and determine d_1 and d_2 uniquely? We can, if

$$\det\begin{pmatrix} \mathbf{R}_1[u_1] & \mathbf{R}_1[u_2] \\ \mathbf{R}_2[u_1] & \mathbf{R}_2[u_2] \end{pmatrix} \neq 0.$$

It can be shown that this determinant is nonzero (see Problem 20.5).

Having found the unique $\{b_i, d_i\}_{i=1}^2$, we can calculate c_i uniquely, substitute all of them in Eq. (20.34), and obtain the unique $G(x, y)$. That $u(x)$ is also unique can be shown similarly. □

Example 20.3.6 Let us calculate the GF for $\mathbf{L}_x = d^2/dx^2$ with BCs $u(a) = u(b) = 0$. We note that $\mathbf{L}_x[u] = 0$ with the given BCs has no nontrivial solution (verify this). Thus, the GF exists. The DE for $G(x, y)$ is $G'' = 0$ for $x \neq y$, whose solutions are

$$G(x, y) = \begin{cases} c_1 x + c_2 & \text{if } a \le x < y, \\ d_1 x + d_2 & \text{if } y < x \le b. \end{cases} \tag{20.35}$$

Continuity at $x = y$ gives $c_1 y + c_2 = d_1 y + d_2$ or $b_1 y + b_2 = 0$ with $b_i = c_i - d_i$. The discontinuity of dG/dx at $x = y$ gives

$$d_1 - c_1 = \frac{1}{p_2 w} = 1 \quad \Rightarrow \quad b_1 = -1$$

assuming that $w = 1$. From the equations above we also get $b_2 = y$. $G(x, y)$ must also satisfy the given BCs. Thus, $G(a, y) = 0 = G(b, y)$. Since $a \le y$ and $b \ge y$, we obtain $c_1 a + c_2 = 0$ and $d_1 b + d_2 = 0$, or, after substituting $c_i = b_i + d_i$,

$$ad_1 + d_2 = a - y, \qquad bd_1 + d_2 = 0.$$

The solution to these equations is $d_1 = (y - a)/(b - a)$ and $d_2 = -b(y - a)/(b - a)$. With b_1, b_2, d_1, and d_2 as given above, we find

$$c_1 = b_1 + d_1 = -\frac{b - y}{b - a} \quad \text{and} \quad c_2 = b_2 + d_2 = a\frac{b - y}{b - a}.$$

Writing Eq. (20.35) as

$$G(x, y) = (c_1 x + c_2)\theta(y - x) + (d_1 x + d_2)\theta(x - y)$$

and using the identity $\theta(y - x) = 1 - \theta(x - y)$, we get

$$G(x, y) = c_1 x + c_2 - (b_1 x + b_2)\theta(x - y).$$

Using the values found for the b's and c's, we obtain

$$G(x, y) = (a - x)\left(\frac{b - y}{b - a}\right) + (x - y)\theta(x - y),$$

which is the same as the GF obtained in Example 20.1.4.

Example 20.3.7 Let us find the GF for $\mathbf{L}_x = d^2/dx^2 + 1$ with the BCs $u(0) = u(\pi/2) = 0$. The general solution of $\mathbf{L}_x[u] = 0$ is

$$u(x) = A \sin x + B \cos x.$$

If the BCs are imposed, we get $u = 0$. Thus, $G(x, y)$ exists. The general form of $G(x, y)$ is

$$G(x, y) = \begin{cases} c_1 \sin x + c_2 \cos x & \text{if } 0 \le x < y, \\ d_1 \sin x + d_2 \cos x & \text{if } y < x \le \pi/2. \end{cases} \tag{20.36}$$

Continuity of G at $x = y$ gives $b_1 \sin y + b_2 \cos y = 0$ with $b_i = c_i - d_i$. The discontinuity of the derivative of G at $x = y$ gives $b_1 \cos y - b_2 \sin y = -1$, where we have set $w(x) = 1$. Solving these equations yields $b_1 = -\cos y$ and $b_2 = \sin y$. The BCs give

$$G(0, y) = 0 \quad \Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad d_2 = -b_2 = -\sin y,$$
$$G(\pi/2, y) = 0 \quad \Rightarrow \quad d_1 = 0 \quad \Rightarrow \quad c_1 = -b_1 = -\cos y.$$

Substituting in Eq. (20.36) gives

$$G(x, y) = \begin{cases} -\cos y \sin x & \text{if } x < y, \\ -\sin y \cos x & \text{if } y < x, \end{cases}$$

or, using the theta function,

$$\begin{aligned}G(x,y) &= -\theta(y-x)\cos y\sin x - \theta(y-x)\sin y\cos x \\ &= -\big[1-\theta(x-y)\big]\cos y\sin x - \theta(x-y)\sin y\cos x \\ &= -\cos y\sin x + \theta(x-y)\sin(x-y).\end{aligned}$$

It is instructive to verify directly that $G(x,y)$ satisfies $\mathbf{L}_x[G] = \delta(x-y)$:

$$\begin{aligned}\mathbf{L}_x[G] &= -\cos y \underbrace{\left(\frac{d^2}{dx^2}+1\right)\sin x}_{=0} + \left(\frac{d^2}{dx^2}+1\right)\big[\theta(x-y)\sin(x-y)\big] \\ &= \frac{d^2}{dx^2}\big[\theta(x-y)\sin(x-y)\big] + \theta(x-y)\sin(x-y) \\ &= \frac{d}{dx}\big[\underbrace{\delta(x-y)\sin(x-y)}_{=0} + \theta(x-y)\cos(x-y)\big] \\ &\quad + \theta(x-y)\sin(x-y).\end{aligned}$$

The first term vanishes because the sine vanishes at the only point where the delta function is nonzero. Thus, we have

$$\begin{aligned}\mathbf{L}_x[G] &= \big[\delta(x-y)\cos(x-y) - \theta(x-y)\sin(x-y)\big] \\ &\quad + \theta(x-y)\sin(x-y) \\ &= \delta(x-y)\end{aligned}$$

because the delta function demands that $x = y$, for which $\cos(x-y) = 1$.

The existence and uniqueness of the Green's function $G(x,y)$ in conjunction with its properties and its adjoint, imply the existence and uniqueness of the adjoint Green's function $g(x,y)$. Using this fact, we can show that the condition for the absence of a nontrivial solution for $\mathbf{L}_x[u] = 0$ is also a necessary condition for the existence of $G(x,y)$. That is, if $G(x,y)$ exists, then $\mathbf{L}_x[u] = 0$ implies that $u = 0$. Suppose $G(x,y)$ exists; then $g(x,y)$ also exists. In Green's identity let $v = g(x,y)$. This gives an identity:

$$\begin{aligned}\int_a^b w(x)g^*(x,y)\big(\mathbf{L}_x[u]\big)\,dx &= \int_a^b w(x)u(x)\big(\mathbf{L}_x^\dagger[g]\big)^*\,dx \\ &= \int_a^b w(x)u(x)\frac{\delta(x-y)}{w(x)}\,dx = u(y).\end{aligned}$$

In particular, if $\mathbf{L}_x[u] = 0$, then $u(y) = 0$ for all y. We have proved the following result.

Proposition 20.3.8 *The DE $\mathbf{L}_x[u] = 0$ implies that $u \equiv 0$ if and only if the GF corresponding to $\mathbf{L}_x$ and the homogeneous BCs exist.*

It is sometimes stated that the Green's function of a SOLDO with constant coefficients depends on the difference $x - y$. This statement is motivated by the observation that if $u(x)$ is a solution of

$$\mathbf{L}_x[u] = a_2\frac{d^2u}{dx^2} + a_1\frac{du}{dx} + a_0u = f(x),$$

then $u(x-y)$ is the solution of $a_2u'' + a_1u' + a_0u = f(x-y)$ if a_0, a_1, and a_2 are constant. Thus, if $G(x)$ is a solution of $\mathbf{L}_x[G] = \delta(x)$ [again assuming that $w(x) = 1$], then it seems that the solution of $\mathbf{L}_x[G] = \delta(x-y)$ is simply $G(x-y)$. This is clearly wrong, as Examples 20.3.6 and 20.3.7 showed. The reason is, of course, the BCs. The fact that $G(x-y)$ satisfies the right DE does not guarantee that it also satisfies the right BCs. The following example, however, shows that the conjecture is true for a homogeneous *initial value problem*.

Example 20.3.9 The most general form for the GF is

$$G(x,y) = \begin{cases} c_1u_1(x) + c_2u_2(x) & \text{if } a \le x < y, \\ d_1u_1(x) + d_2u_2(x) & \text{if } y < x \le b. \end{cases}$$

The IVP condition $G(a,y) = 0 = G'(a,y)$ implies

$$c_1u_1(a) + c_2u_2(a) = 0 \quad \text{and} \quad c_1u_1'(a) + c_2u_2'(a) = 0.$$

Linear independence of u_1 and u_2 implies

$$\det\begin{pmatrix} u_1(a) & u_2(a) \\ u_1'(a) & u_2'(a) \end{pmatrix} = W(a; u_1, u_2) \ne 0.$$

Hence, $c_1 = c_2 = 0$ is the only solution. This gives

$$G(x,y) = \begin{cases} 0 & \text{if } a \le x < y, \\ d_1u_1(x) + d_2u_2(x) & \text{if } y < x \le b. \end{cases} \tag{20.37}$$

Continuity of G at $x = y$ yields $d_1u_1(y) + d_2u_2(y) = 0$, while the discontinuity jump condition in the derivative gives $d_1u_1'(y) + d_2u_2'(y) = 1$. Solving these two equations, we get

$$d_1 = \frac{u_2(y)}{u_1'(y)u_2(y) - u_2'(y)u_1(y)}, \qquad d_2 = -\frac{u_1(y)}{u_1'(y)u_2(y) - u_2'(y)u_1(y)}.$$

Substituting this in (20.37) gives

$$G(x,y) = \left[\frac{u_2(y)u_1(x) - u_1(y)u_2(x)}{u_1'(y)u_2(y) - u_2'(y)u_1(y)}\right]\theta(x-y). \tag{20.38}$$

Equation (20.38) holds for any SOLDO with the given BCs. We now use the fact that the SOLDO has *constant coefficients*. In that case, we know the exact form of u_1 and u_2. There are two cases to consider:

1. If the characteristic polynomial of $\mathbf{L}_x$ has two distinct roots λ_1 and λ_2, then $u_1(x) = e^{\lambda_1 x}$ and $u_2(x) = e^{\lambda_2 x}$. Writing $\lambda_1 = a + b$ and $\lambda_2 = a - b$ and substituting the exponential functions and their derivatives in Eq. (20.38) yields

$$G(x, y) = \left[\frac{e^{(a-b)y}e^{(a+b)x} - e^{(a+b)y}e^{(a-b)x}}{2be^{2ay}}\right]\theta(x - y)$$
$$= \frac{1}{2b}\left[e^{(a+b)(x-y)} - e^{(a-b)(x-y)}\right]\theta(x - y),$$

which is a function of $x - y$ alone.
2. If $\lambda_1 = \lambda_2 = \lambda$, then $u_1(x) = e^{\lambda x}$, $u_2(x) = xe^{\lambda x}$, and substitution of these functions in Eq. (20.38) gives

$$G(x, y) = (x - y)e^{\lambda(x-y)}\theta(x - y).$$

20.3.3 Inhomogeneous BCs

So far we have concentrated on problems with homogeneous BCs, $\mathbf{R}_i[u] = 0$, for $i = 1, 2$. What if the BCs are inhomogeneous? It turns out that the Green's function method, even though it was derived for homogeneous BCs, solves this kind of problem as well! The secret of this success is the generalized Green's identity. Suppose we are interested in solving the DE

GF solves inhomogeneous BCs as well

$$\mathbf{L}_x[u] = f(x) \quad \text{with} \quad \mathbf{R}_i[u] = \gamma_i \quad \text{for } i = 1, 2,$$

and we have the GF for $\mathbf{L}_x$ (with homogeneous BCs, of course). We can substitute $v = g(x, y) = G^*(y, x)$ in the generalized Green's identity and use the DE to obtain

$$\int_a^b w(x)G(y, x)f(x)\,dx - \int_a^b w(x)u(x)\left(\mathbf{L}_x^\dagger[g]\right)^* dx$$
$$= Q\left[u, g^*(x, y)\right]\Big|_{x=a}^{x=b},$$

or, using $\mathbf{L}_x^\dagger[g(x, y)] = \delta(x - y)/w(y)$,

$$u(y) = \int_a^b w(x)G(y, x)f(x)\,dx - Q\left[u, g^*(x, y)\right]\Big|_{x=a}^{x=b}.$$

To evaluate the surface term, let us write the BCs in matrix form [see Eq. (20.17)]:

$$\mathsf{A}\mathsf{u}_a + \mathsf{B}\mathsf{u}_b = \gamma \quad \Rightarrow \quad \mathsf{u}_b = \mathsf{B}^{-1}\gamma - \mathsf{B}^{-1}\mathsf{A}\mathsf{u}_a,$$
$$\mathsf{A}\mathsf{G}_a + \mathsf{B}\mathsf{G}_b = 0 \quad \Rightarrow \quad \mathsf{A}^t\left(\mathsf{B}^t\right)^{-1}\mathsf{Q}_b\mathsf{g}_b^* + \mathsf{Q}_a\mathsf{g}_a^* = 0,$$

where γ is a column vector composed of γ_1 and γ_2, and we have assumed that $G(x, y)$ and $g^*(x, y)$ satisfy, respectively, the homogeneous BCs (with $\gamma = 0$) and their adjoints. We have also assumed that the 2×4 matrix of coefficients has rank 2, and without loss of generality, let B be the invertible 2×2 submatrix. Then, assuming the general form of the surface term as in Eq. (20.15), we obtain

$$\begin{aligned} Q\big[u, g^*(x, y)\big]\big|_{x=a}^{x=b} &= \mathsf{u}_b^t \mathsf{Q}_b \mathsf{g}_b^* - \mathsf{u}_a^t \mathsf{Q}_a \mathsf{g}_a^* \\ &= \big(\mathsf{B}^{-1}\gamma - \mathsf{B}^{-1}\mathsf{A}\mathsf{u}_a\big)^t \mathsf{Q}_b \mathsf{g}_b^* - \mathsf{u}_a^t \mathsf{Q}_a \mathsf{g}_a^* \\ &= \gamma^t \big(\mathsf{B}^t\big)^{-1} \mathsf{Q}_b \mathsf{g}_b^* - \mathsf{u}_a^t \underbrace{\big[\mathsf{A}^t \big(\mathsf{B}^t\big)^{-1} \mathsf{Q}_b \mathsf{g}_b^* + \mathsf{Q}_a \mathsf{g}_a^*\big]}_{\substack{= 0 \text{ because } g^* \text{ satisfies} \\ \text{homogeneous adjoint BC}}} \\ &= \gamma^t \big(\mathsf{B}^t\big)^{-1} \mathsf{Q}_b \mathsf{g}_b^*, \end{aligned} \tag{20.39}$$

where

$$g_b^* \equiv \begin{pmatrix} g^*(b, y) \\ \frac{\partial}{\partial x} g^*(x, y)|_{x=b} \end{pmatrix} = \begin{pmatrix} G(y, b) \\ \frac{\partial}{\partial x} G(y, x)|_{x=b} \end{pmatrix}.$$

It follows that $Q[u, g^*(x, y)]|_{x=a}^{x=b}$ is given entirely in terms of G, its derivative, the coefficient functions of the DE (hidden in the matrix Q), the homogeneous BCs (hidden in B), and the constants γ_1 and γ_2. The fact that g^* and $\partial g^*/\partial x$ appear to be evaluated at $x = b$ is due to the simplifying (but harmless) assumption that B is invertible, i.e., that $u(b)$ and $u'(b)$ can be written in terms of $u(a)$ and $u'(a)$. Of course, this may not be possible; then we have to find another pair of the four quantities in terms of the other two, in which case the matrices and the vectors will change but the argument, as well as the conclusion, will remain valid. We can now write

$$u(y) = \int_a^b w(x) G(y, x) f(x)\, dx - \gamma^t \mathsf{M} \mathsf{g}^*, \tag{20.40}$$

where a general matrix M has been introduced, and the subscript b has been removed to encompass cases where submatrices other than B are invertible. Equation (20.40) shows that u can be determined completely once we know $G(x, y)$, even though the BCs are inhomogeneous. In practice, there is no need to calculate M. We can use the expression for $Q[u, g^*]$ obtained from the Lagrange identity of Chap. 14 and evaluate it at b and a. This, in general, involves evaluating u and G and their derivatives at a and b. We know how to handle the evaluation of G because we can actually construct it (if it exists). We next find two of the four quantities corresponding to u in terms of the other two and insert the result in the expression for $Q[u, g^*]$. Equation (20.39) then guarantees that the coefficients of the other two terms will be zero. Thus, we can simply drop all the terms in $Q[u, g^*]$ containing a factor of the other two terms.

Specifically, we use the conjunct for a formally self-adjoint SOLDO [see Eq. (20.26)] and $g^*(x, y) = G(y, x)$ to obtain

$$\begin{aligned} u(y) = &\int_a^b w(x)G(y, x)f(x)\,dx \\ &- \left\{p(x)w(x)\left[G(y, x)\frac{du}{dx} - u(x)\frac{\partial G}{\partial x}(y, x)\right]\right\}_{x=a}^{x=b}. \end{aligned}$$

Interchanging x and y gives

$$\begin{aligned} u(x) = &\int_a^b w(y)G(x, y)f(y)\,dy \\ &+ \left\{p(y)w(y)\left[u(y)\frac{\partial G}{\partial y}(x, y) - G(x, y)\frac{du}{dy}\right]\right\}_{y=a}^{y=b}. \end{aligned} \tag{20.41}$$

This equation is valid only for a self-adjoint SOLDO. That is, using it requires casting the SOLDO into a self-adjoint form (a process that is always possible, in light of Theorem 14.5.4).

By setting $f(x) = 0$, we can also obtain the solution to a homogeneous DE $\mathbf{L}_x[u] = 0$ that satisfies the inhomogeneous BCs.

Example 20.3.10 Let us find the solution of the simple DE $d^2u/dx^2 = f(x)$ subject to the simple inhomogeneous BCs $u(a) = \gamma_1$ and $u(b) = \gamma_2$. The GF for this problem has been calculated in Examples 20.1.5 and 20.3.6. Let us begin by calculating the surface term in Eq. (20.41). We have $p(y) = 1$, and we set $w(y) = 1$, then

$$\begin{aligned} \text{surface term} &= u(b)\frac{\partial G}{\partial y}\bigg|_{y=b} - G(x, b)u'(b) - u(a)\frac{\partial G}{\partial y}\bigg|_{y=a} \\ &\quad + G(x, a)u'(a) \\ &= \gamma_2\frac{\partial G}{\partial y}\bigg|_{y=b} - \gamma_1\frac{\partial G}{\partial y}\bigg|_{y=a} + G(x, a)u'(a) - G(x, b)u'(b). \end{aligned}$$

That the unwanted (and unspecified) terms are zero can be seen by observing that $G(x, a) = g^*(a, x) = (g(a, x))^*$, and that $g(x, y)$ satisfies the BCs adjoint to the homogeneous BCs (obtained when $\gamma_i = 0$). In this particular and simple case, the BCs happen to be self-adjoint (Dirichlet BCs). Thus, $u(a) = u(b) = 0$ implies that $g(a, x) = g(b, x) = 0$ for all $x \in [a, b]$. (In a more general case the coefficient of $u'(a)$ would be more complicated, but still zero.) Thus, we finally have

$$\text{surface term} = \gamma_2\frac{\partial G}{\partial y}\bigg|_{y=b} - \gamma_1\frac{\partial G}{\partial y}\bigg|_{y=a}.$$

Now, using the expression for $G(x,y)$ obtained in Examples 20.1.5 and 20.3.6, we get

$$\frac{\partial G}{\partial y} = -\frac{a-x}{b-a} - \theta(x-y) - \underbrace{(x-y)\delta(x-y)}_{=0} = \frac{x-a}{b-a} - \theta(x-y).$$

Thus,

$$\left.\frac{\partial G}{\partial y}\right|_{y=b} = \frac{x-a}{b-a}, \qquad \left.\frac{\partial G}{\partial y}\right|_{y=a} = \frac{x-a}{b-a} - 1 = \frac{x-b}{b-a}.$$

Substituting in Eq. (20.41), we get

$$u(x) = \int_a^b G(x,y)f(y)\,dy + \frac{\gamma_2-\gamma_1}{b-a}x + \frac{b\gamma_1 - a\gamma_2}{b-a}.$$

(Compare this with the result obtained in Example 20.1.5.)

relation of GF to sources

Green's functions have a very simple and enlightening physical interpretation. An inhomogeneous DE such as $\mathbf{L}_x[u] = f(x)$ can be interpreted as a black box ($\mathbf{L}_x$) that determines a physical quantity (u) when there is a source (f) of that physical quantity. For instance, electrostatic potential is a physical quantity whose source is charge; a magnetic field has an electric current as its source; displacements and velocities have forces as their sources; and so forth. Applying this interpretation and assuming that $w(x) = 1$, we have $G(x,y)$ as the physical quantity, evaluated at x when its source $\delta(x-y)$ is located at y. To be more precise, let us say that the strength of the source is S_1 and it is located at y_1; then the source becomes $S_1\delta(x-y_1)$. The physical quantity, the Green's function, is then $S_1G(x,y_1)$, because of the linearity of $\mathbf{L}_x$: If $G(x,y)$ is a solution of $\mathbf{L}_x[u] = \delta(x-y)$, then $S_1G(x,y_1)$ is a solution of $\mathbf{L}_x[u] = S_1\delta(x-y)$. If there are many sources located at $\{y_i\}_{i=1}^N$ with corresponding strengths $\{S_i\}_{i=1}^N$, then the overall source f as a function of x becomes $f(x) = \sum_{i=1}^N S_i\delta(x-y_i)$, and the corresponding physical quantity $u(x)$ becomes $u(x) = \sum_{i=1}^N S_iG(x,y_i)$.

Since the source S_i is located at y_i, it is more natural to define a function $S(x)$ and write $S_i = S(y_i)$. When the number of point sources goes to infinity and y_i becomes a smooth continuous variable, the sums become integrals, and we have

$$f(x) = \int_a^b S(y)\delta(x-y)\,dy, \qquad u(x) = \int_a^b S(y)G(x,y)\,dy.$$

The first integral shows that $S(x) = f(x)$. Thus, the second integral becomes $u(x) = \int_a^b f(y)G(x,y)\,dy$ which is precisely what we obtained formally.

20.4 Eigenfunction Expansion

Green's functions are inverses of differential operators. Inverses of operators in a Hilbert space are best studied in terms of resolvents. This is because if an operator $\mathbf{A}$ has an inverse, zero is in its resolvent set, and

$$\mathbf{R}_0(\mathbf{A}) = \mathbf{R}_\lambda(\mathbf{A})\big|_{\lambda=0} = (\mathbf{A} - \lambda\mathbf{1})^{-1}\big|_{\lambda=0} = \mathbf{A}^{-1}.$$

Thus, it is instructive to discuss Green's functions in the context of the resolvent of a differential operator. We will consider only the case where the eigenvalues are discrete, for example, when $\mathbf{L}_x$ is a Sturm-Liouville operator.

Formally, we have $(\mathbf{L} - \lambda\mathbf{1})\mathbf{R}_\lambda(\mathbf{L}) = \mathbf{1}$, which leads to the DE

$$(\mathbf{L}_x - \lambda)R_\lambda(x, y) = \frac{\delta(x - y)}{w(x)},$$

where $R_\lambda(x, y) = \langle x|\mathbf{R}_\lambda(\mathbf{L})|y\rangle$. The DE simply says that $R_\lambda(x, y)$ is the Green's function for the operator $\mathbf{L}_x - \lambda$. So we can rewrite the equation as

$$(\mathbf{L}_x - \lambda)G_\lambda(x, y) = \frac{\delta(x - y)}{w(x)},$$

where $\mathbf{L}_x - \lambda$ is a DO having some homogeneous BCs. The GF $G_\lambda(x, y)$ exists if and only if $(\mathbf{L}_x - \lambda)[u] = 0$ has no nontrivial solution, which is true only if λ is not an eigenvalue of $\mathbf{L}_x$. We choose the BCs in such a way that $\mathbf{L}_x$ becomes self-adjoint.

Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of the system $\mathbf{L}_x[u] = \lambda u$, $\{\mathbf{R}_i[u] = 0\}_{i=1}^2$, and let the $u_n^{(k)}(x)$ be the corresponding eigenfunctions. The index k distinguishes among the linearly independent vectors corresponding to the same eigenvalue λ_n. Assuming that $\mathbf{L}$ has compact resolvent (e.g., a Sturm-Liouville operator), these eigenfunctions form a complete set for the subspace of the Hilbert space that consists of those functions that satisfy the same BCs as the $u_n^{(k)}(x)$. In particular, $G_\lambda(x, y)$ can be expanded in terms of $u_n^{(k)}(x)$. The expansion coefficients are, of course, functions of y. Thus, we can write

$$G_\lambda(x, y) = \sum_k \sum_{n=1}^\infty a_n^{(k)}(y)u_n^{(k)}(x)$$

where $a_n^{(k)}(y) = \int_a^b w(x)u_n^{*(k)}(x)G_\lambda(x, y)\,dx$. Using Green's identity, Eq. (20.30), and the fact that λ_n is real, we have

$$\begin{aligned}
\lambda_n a_n^{(k)}(y) &= \int_a^b w(x)\big[\lambda_n u_n^{(k)}(x)\big]^* G_\lambda(x, y)\,dx \\
&= \int_a^b w(x)G_\lambda(x, y)\big\{\mathbf{L}_x\big[u_n^{(k)}(x)\big]\big\}^*\,dx \\
&= \int_a^b w(x)\big[u_n^{(k)}(x)\big]^*\mathbf{L}_x\big[G_\lambda(x, y)\big]\,dx
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b w(x)u_n^{*(k)}(x)\left[\frac{\delta(x-y)}{w(x)} + \lambda G_\lambda(x,y)\right]dx \\
&= u_n^{*(k)}(y) + \lambda \int_a^b w(x)u_n^{*(k)}(x)G_\lambda(x,y)\,dx \\
&= u_n^{*(k)}(y) + \lambda a_n^{(k)}(y).
\end{aligned}$$

Thus, $a_n^{(k)}(y) = u_n^{*(k)}(y)/(\lambda_n - \lambda)$, and the expansion for the Green's function is

$$G_\lambda(x,y) = \sum_k \sum_{n=1}^{\infty} \frac{u_n^{*(k)}(y)u_n^{(k)}(x)}{\lambda_n - \lambda}. \tag{20.42}$$

This expansion is valid as long as $\lambda_n \neq \lambda$ for any $n = 0, 1, 2, \ldots$. But this is precisely the condition that ensures the existence of an inverse for $\mathbf{L} - \lambda\mathbf{1}$.

An interesting result is obtained from Eq. (20.42) if λ is considered a complex variable. In that case, $G_\lambda(x,y)$ has (infinitely many) simple poles at $\{\lambda_n\}_{n=1}^{\infty}$. The residue at the pole λ_n is $-\sum_k u_n^{*(k)}(y)u_n^{(k)}(x)$. If C_m is a contour having the poles $\{\lambda_n\}_{n=1}^{m}$ in its interior, then, by the residue theorem, we have

$$\frac{1}{2\pi i}\oint_{C_m} G_\lambda(x,y)\,d\lambda = -\sum_k \sum_{n=1}^{m} u_n^{*(k)}(y)u_n^{(k)}(x).$$

In particular, if we let $m \to \infty$, we obtain

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{C_\infty} G_\lambda(x,y)\,d\lambda &= -\sum_k \sum_{n=1}^{\infty} u_n^{*(k)}(y)u_n^{(k)}(x) \\
&= -\frac{\delta(x-y)}{w(x)},
\end{aligned} \tag{20.43}$$

where C_∞ is any contour that encircles all the eigenvalues, and in the last step we used the completeness of the eigenfunctions. Equation (20.43) is the infinite-dimensional analogue of Eq. (17.10) with $f(\mathbf{A}) = \mathbf{1}$ when the latter equation is sandwiched between $\langle x|$ and $|y\rangle$.

Example 20.4.1 Consider the DO $\mathbf{L}_x = d^2/dx^2$ with BCs $u(0) = u(a) = 0$. This is an S-L operator with eigenvalues and normalized eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 \quad \text{and} \quad u_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \quad \text{for } n = 1, 2, \ldots.$$

Equation (20.42) becomes

$$G_\lambda(x,y) = -\frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/a)\sin(n\pi y/a)}{\lambda - (n\pi/a)^2},$$

which leads to

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{C_\infty} G_\lambda(x,y)\,d\lambda \\
&\quad = \frac{1}{2\pi i}\oint_{C_\infty}\left[-\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin(n\pi x/a)\sin(n\pi y/a)}{\lambda-(n\pi/a)^2}\right]d\lambda \\
&\quad = -\frac{1}{2\pi i}\left(\frac{2}{a}\right)\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)\oint_{C_\infty}\frac{d\lambda}{\lambda-(n\pi/a)^2} \\
&\quad = -\left(\frac{2}{a}\right)\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)\operatorname{Res}\left[\frac{1}{\lambda-(n\pi/a)^2}\right]_{\lambda=\lambda_n} \\
&\quad = -\left(\frac{2}{a}\right)\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right).
\end{aligned}$$

The RHS is recognized as $-\delta(x-y)$.

eigenfunction expansion of GF

If zero is not an eigenvalue of $\mathbf{L}_x$, Eq. (20.42) yields

$$G(x,y) \equiv G_0(x,y) = \sum_k\sum_{n=1}^{\infty}\frac{u_n^{*(k)}(y)u_n^{(k)}(x)}{\lambda_n}, \tag{20.44}$$

which is an expression for the Green's function of $\mathbf{L}_x$ in terms of its eigenvalues and eigenfunctions.

20.5 Problems

20.1 Using the GF method, solve the DE $\mathbf{L}_x u(x) = du/dx = f(x)$ subject to the BC $u(0) = a$. Hint: Consider the function $v(x) = u(x) - a$.

20.2 Solve the problem of Example 20.1.4 subject to the BCs $u(a) = u'(a) = 0$. Show that the corresponding GF also satisfies these BCs.

20.3 Show that the IVP with data $\{0; 0, 0, \dots, 0\}$ has only $u \equiv 0$ as a solution. Hint: Assume otherwise, add u to the solution of the inhomogeneous equation, and invoke uniqueness.

exact NOLDE

20.4 In this problem, we generalize the concepts of exactness and integrating factor to a NOLDE. The DO $\mathbf{L}_x^{(n)} \equiv \sum_{k=0}^{n} p_k(x)d^k/dx^k$ is said to be **exact** if there exists a DO $\mathbf{M}_x^{(n-1)} \equiv \sum_{k=0}^{n-1} a_k(x)d^k/dx^k$ such that

$$\mathbf{L}_x^{(n)}[u] = \frac{d}{dx}\left(\mathbf{M}_x^{(n-1)}[u]\right) \quad \forall u \in \mathcal{C}^n[a,b].$$

(a) Show that $\mathbf{L}_x^{(n)}$ is exact iff $\sum_{m=0}^{n}(-1)^m d^m p_m/dx^m = 0$.

(b) Show that there exists an integrating factor for $\mathbf{L}_x^{(n)}$—that is, a function $\mu(x)$ such that $\mu(x)\mathbf{L}_x^{(n)}$ is exact—if and only if $\mu(x)$ satisfies the DE

$$\mathbf{N}_x^{(n)}[\mu] \equiv \sum_{m=0}^{n} (-1)^m \frac{d^m}{dx^m}(\mu p_m) = 0.$$

The DO $\mathbf{N}_x^{(n)}$ is the formal adjoint of $\mathbf{L}_x^{(n)}$.

20.5 Let $\mathbf{L}_x$ be a SOLDO. Assuming that $\mathbf{L}_x[u] = 0$ has no nontrivial solution, show that the matrix

$$\mathsf{R} \equiv \begin{pmatrix} \mathbf{R}_1[u_1] & \mathbf{R}_1[u_2] \\ \mathbf{R}_2[u_1] & \mathbf{R}_2[u_2] \end{pmatrix},$$

where u_1 and u_2 are independent solutions of $\mathbf{L}_x[u] = 0$ and $\mathbf{R}_i$ are the boundary functionals, has a nonzero determinant. Hint: Assume otherwise and show that the system of homogeneous linear equations $\alpha\mathbf{R}_1[u_1] + \beta\mathbf{R}_1[u_2] = 0$ and $\alpha\mathbf{R}_2[u_1] + \beta\mathbf{R}_2[u_2] = 0$ has a nontrivial solution for (α, β). Reach a contradiction by considering $u = \alpha u_1 + \beta u_2$ as a solution of $\mathbf{L}_x[u] = 0$.

20.6 Determine the formal adjoint of each of the operators in (a) through (d) below (i) as a differential operator, and (ii) as an operator, that is, including the BCs. Which operators are formally self-adjoint? Which operators are self-adjoint?

(a) $\mathbf{L}_x = d^2/dx^2 + 1$ in [0, 1] with BCs $u(0) = u(1) = 0$.
(b) $\mathbf{L}_x = d^2/dx^2$ in [0, 1] with BCs $u(0) = u'(0) = 0$.
(c) $\mathbf{L}_x = d/dx$ in $[0, \infty]$ with BCs $u(0) = 0$.
(d) $\mathbf{L}_x = d^3/dx^3 - \sin x\, d/dx + 3$ in $[0, \pi]$ with BCs $u(0) = u'(0) = 0$, $u''(0) - 4u(\pi) = 0$.

20.7 Show that the Dirichlet, Neumann, general unmixed, and periodic BCs make the following formally self-adjoint SOLDO self-adjoint:

$$\mathbf{L}_x = \frac{1}{w}\frac{d}{dx}\left(p\frac{d}{dx}\right) + q.$$

20.8 Using a procedure similar to that described in the text for SOLDOs, show that for the FOLDO $\mathbf{L}_x = p_1 d/dx + p_0$

(a) the indefinite GF is

$$G(x, y) \equiv \frac{\mu(y)}{p_1(y)w(y)}\left[\frac{\theta(x-y)}{\mu(x)}\right] + C(y),$$

$$\text{where} \quad \mu(x) = \exp\left[\int^x \frac{p_0(t)}{p_1(t)}\,dt\right],$$

(b) and the GF itself is discontinuous at $x = y$ with

$$\lim_{\epsilon\to 0}\left[G(y+\epsilon, y) - G(y-\epsilon, y)\right] = \frac{1}{p_1(y)w(y)}.$$

(c) For the homogeneous BC

$$\mathbf{R}[u] \equiv \alpha_1 u(a) + \alpha_2 u'(a) + \beta_1 u(b) + \beta_2 u'(b) = 0$$

construct $G(x, y)$ and show that

$$G(x, y) = \frac{1}{p_1(y)w(y)v(y)} v(x)\theta(x - y) + C(y)v(x),$$

where $v(x)$ is any solution to the homogeneous DE $\mathbf{L}_x[u] = 0$ and

$$C(y) = \frac{\beta_1 v(b) + \beta_2 v'(b)}{\mathbf{R}[v] p_1(y)w(y)v(y)}, \quad \text{with } \mathbf{R}[v] \neq 0$$

(d) Show directly that $\mathbf{L}_x[G] = \delta(x - y)/w(x)$.

20.9 Let $\mathbf{L}_x$ be a NOLDO with constant coefficients. Show that if $u(x)$ satisfies $\mathbf{L}_x[u] = f(x)$, then $u(x - y)$ satisfies $\mathbf{L}_x[u] = f(x - y)$. (Note that no BCs are specified.)

20.10 Find the GF for $\mathbf{L}_x = d^2/dx^2 + 1$ with BCs $u(0) = u'(0) = 0$. Show that it can be written as a function of $x - y$ only.

20.11 Find the GF for $\mathbf{L}_x = d^2/dx^2 + k^2$ with BCs $u(0) = u(a) = 0$.

20.12 Find the GF for $\mathbf{L}_x = d^2/dx^2 - k^2$ with BCs $u(\infty) = u(-\infty) = 0$.

20.13 Find the GF for $\mathbf{L}_x = (d/dx)(xd/dx)$ given the condition that $G(x, y)$ is finite at $x = 0$ and vanishes at $x = 1$.

20.14 Evaluate the GF and the solutions for each of the following DEs in the interval [0, 1].

(a) $u'' - k^2 u = f; \qquad u(0) - u'(0) = a, \quad u(1) = b.$

(b) $u'' = f; \qquad u(0) = u'(0) = 0.$

(c) $u'' + 6u' + 9u = 0; \qquad u(0) = 0, \quad u'(0) = 1.$

(d) $u'' + \omega^2 u = f(x), \quad \text{for } x > 0; \qquad u(0) = a, \quad u'(0) = b.$

(e) $u^{(4)} = f; \qquad u(0) = 0, \quad u'(0) = 2u'(1), \quad u(1) = a, \quad u''(0) = 0.$

20.15 Use eigenfunction expansion of the GF to solve the BVP $u'' = x$, $u(0) = 0$, $u(1) - 2u'(1) = 0$.