18 Integral Equations

The beginning of Chap. 17 showed that to solve a vector-operator equation one transforms it into an equation involving a sum over a discrete index [the matrix equation of Eq. (17.1)], or an equation involving an integral over a continuous index [Eq. (17.2)]. The latter is called an **integral equation**, which we shall investigate here using the machinery of Chap. 17.

18.1 Classification

Integral equations can be divided into two major groups. Those that have a variable limit of integration are called **Volterra equations**; those that have constant limits of integration are called **Fredholm equations**. If the unknown function appears only inside the integral, the integral equation is said to be **of the first kind**. Integral equations having the unknown function outside the integral as well as inside are said to be **of the second kind**. The four kinds of equations can be written as follows.

Volterra and Fredholm equations of first and second kind

$$\int_a^x K(x,t)u(t)\,dt = v(x), \qquad \text{Volterra equation of the 1st kind,}$$

$$\int_a^b K(x,t)u(t)\,dt = v(x), \qquad \text{Fredholm equation of the 1st kind,}$$

$$u(x) = v(x) + \int_a^x K(x,t)u(t)\,dt, \qquad \text{Volterra equation of the 2nd kind,}$$

$$u(x) = v(x) + \int_a^b K(x,t)u(t)\,dt, \qquad \text{Fredholm equation of the 2nd kind.}$$

In all these equations, $K(x,t)$ is called the **kernel** of the integral equation.

kernel of an integral equation

In the theory of integral equations of the second kind, one usually multiplies the integral by a nonzero complex number λ. Thus, the Fredholm equation of the second kind becomes

$$u(x) = v(x) + \lambda \int_a^b K(x,t)u(t)\,dt, \qquad (18.1)$$

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and for the Volterra equation of the second kind one obtains

$$u(x) = v(x) + \lambda \int_a^x K(x,t)u(t)\,dt. \tag{18.2}$$

characteristic value of an integral equation

A λ that satisfies (18.1) or (18.2) with $v(x) = 0$ is called a **characteristic value** of the integral equation. In the abstract operator language both equations are written as

$$|u\rangle = |v\rangle + \lambda \mathbf{K}|u\rangle \quad \Rightarrow \quad \big(\mathbf{K} - \lambda^{-1}\mathbf{1}\big)|u\rangle = -\lambda^{-1}|v\rangle. \tag{18.3}$$

Thus λ is a characteristic value if and only if λ^{-1} is an eigenvalue of $\mathbf{K}$. Recall that when the interval of integration (a, b) is finite, $K(x,t)$ is called a *Hilbert-Schmidt* kernel. Example 17.5.10 showed that $\mathbf{K}$ is a compact operator, and by Theorem 17.5.11, the eigenvalues of $\mathbf{K}$ either form a finite set or a sequence that converges to zero.

Theorem 18.1.1 *The characteristic values of a Fredholm equation of the second kind either form a finite set or a sequence of complex numbers increasing beyond limit in absolute value.*

Our main task in this chapter is to study methods of solving integral equations of the second kind. We treat the Volterra equation first because it is easier to solve. Let us introduce the notation

$$K[u](x) \equiv \int_a^x K(x,t)u(t)\,dt \quad \text{and} \quad K^n[u](x) = K\big[K^{n-1}[u]\big](x) \tag{18.4}$$

whereby $K[u]$ denotes a function whose value at x is given by the integral on the RHS of the first equation in (18.4). One can show with little difficulty that the associated operator $\mathbf{K}$ is compact. Let $M = \max\{|K(x,t)| \mid a \le t \le x \le b\}$ and note that

$$\big|\lambda K[u](x)\big| = \left|\lambda \int_a^x K(x,t)u(t)\,dt\right| \le |\lambda||M|\|u\|_\infty(x-a),$$

where $\|u\|_\infty \equiv \max\{|u(x)| \mid x \in (a,b)\}$.

Using mathematical induction, one can show that (see Problem 18.1)

$$\big|(\lambda K)^n[u](x)\big| \le |\lambda|^n|M|^n\|u\|_\infty \frac{(x-a)^n}{n!}. \tag{18.5}$$

Since $b \ge x$, we can replace x with b and still satisfy the inequality. Then the inequality of Eq. (18.5) will hold for all x, and we can write the equation as an operator norm inequality: $\|(\lambda\mathbf{K})^n\| \le |\lambda|^n|M|^n\|u\|_\infty(b-a)^n/n!$. Therefore,

$$\left\|\sum_{n=0}^{\infty}(\lambda\mathbf{K})^n\right\| \le \sum_{n=0}^{\infty}\big\|(\lambda\mathbf{K})^n\big\| \le \sum_{n=0}^{\infty}\frac{|\lambda|^n|M|^n(b-a)^n}{n!} = e^{M|\lambda|(b-a)},$$

and the series $\sum_{n=0}^{\infty}(\lambda\mathbf{K})^n$ converges for all λ. In fact, a direct calculation shows that the series converges to the inverse of $\mathbf{1}-\lambda\mathbf{K}$. Thus, the latter is invertible and the spectrum of $\mathbf{K}$ has no nonzero points. We have just shown the following.

Theorem 18.1.2 *The Volterra equation of the second kind has no nonzero characteristic value. In particular, the operator* $\mathbf{1}-\lambda\mathbf{K}$ *is invertible, the equation always has a unique solution given by the convergent infinite series*

Volterra equation of the second kind has a unique solution and no nonzero characteristic value

$$u(x)=\sum_{j=0}^{\infty}\lambda^j\int_a^x K^j(x,t)v(t)\,dt$$

where $K^j(x,t)$ *is defined inductively in Eq.* (18.4).

Historical Notes

Vito Volterra (1860–1940) was only 11 when he became interested in mathematics while reading Legendre's *Geometry*. At the age of 13 he began to study the three body problem and made some progress.

Vito Volterra 1860–1940

His family were extremely poor (his father had died when Vito was two years old) but after attending lectures at Florence he was able to proceed to Pisa in 1878. At Pisa he studied under Betti, graduating as a doctor of physics in 1882. His thesis on hydrodynamics included some results of Stokes, discovered later but independently by Volterra.

He became Professor of Mechanics at Pisa in 1883, and upon Betti's death, he occupied the chair of mathematical physics. After spending some time at Turin as the chair of mechanics, he was awarded the chair of mathematical physics at the University of Rome in 1900.

Volterra conceived the idea of a theory of functions that depend on a continuous set of values of another function in 1883. Hadamard was later to introduce the word "functional", which replaced Volterra's original terminology. In 1890 Volterra used his functional calculus to show that the theory of Hamilton and Jacobi for the integration of the differential equations of dynamics could be extended to other problems of mathematical physics.

His most famous work was done on **integral equations**. He began this study in 1884, and in 1896 he published several papers on what is now called the Volterra integral equation. He continued to study functional analysis applications to integral equations producing a large number of papers on composition and permutable functions.

During the First World War Volterra joined the Air Force. He made many journeys to France and England to promote scientific collaboration. After the war he returned to the University of Rome, and his interests moved to mathematical biology. He studied the Verhulst equation and the logistic curve. He also wrote on predator–prey equations.

In 1922 Fascism seized Italy, and Volterra fought against it in the Italian Parliament. However, by 1930 the Parliament was abolished, and when Volterra refused to take an oath of allegiance to the Fascist government in 1931, he was forced to leave the University of Rome. From the following year he lived mostly abroad, mainly in Paris, but also in Spain and other countries.

Differential equations can be transformed into integral equations. For instance, consider the SOLDE

$$\frac{d^2u}{dx^2}+p_1(x)\frac{du}{dx}+p_0(x)u=r(x),\qquad u(a)=c_1,\qquad u'(a)=c_2. \tag{18.6}$$

By integrating the DE once, we obtain

$$\frac{du}{dx}=-\int_a^x p_1(t)u'(t)\,dt-\int_a^x p_0(t)u(t)\,dt+\int_a^x r(t)\,dt+c_2.$$

Integrating the first integral by parts gives

$$\begin{aligned} u'(x) = -p_1(x)u(x) + \underbrace{\int_a^x \big[p_1'(t) - p_0(t)\big]u(t)\,dt}_{\equiv f(x)} \\ + \underbrace{\int_a^x r(t)\,dt}_{\equiv g(x)} + p_1(a)c_1 + c_2. \end{aligned}$$

Integrating once more yields

$$\begin{aligned} u(x) &= -\int_a^x p_1(t)u(t)\,dt + \int_a^x f(s)\,ds + \int_a^x g(s)\,ds \\ &\quad + (x-a)\big[p_1(a)c_1 + c_2\big] \\ &= -\int_a^x p_1(t)u(t)\,dt + \int_a^x ds \int_a^s \big[p_1'(t) - p_0(t)\big]u(t)\,dt \\ &\quad + \int_a^x ds \int_a^s r(t)\,dt + (x-a)\big[p_1(a)c_1 + c_2\big] + c_1 \\ &= \int_a^x \big\{(x-t)\big[p_1'(t) - p_0(t)\big] - p_1(t)\big\}u(t)\,dt \\ &\quad + \int_a^x (x-t)r(t)\,dt + (x-a)\big[p_1(a)c_1 + c_2\big] + c_1, \end{aligned} \tag{18.7}$$

where we have used the formula

$$\int_a^x ds \int_a^s f(t)\,dt = \int_a^x (x-t)f(t)\,dt,$$

which the reader may verify by interchanging the order of integration on the LHS.

Proposition 18.1.3 *A SOLDE of the form* (18.6) *is equivalent to a Volterra equation of the second kind with kernel*

$$K(x,t) \equiv (x-t)\big[p_1'(t) - p_0(t)\big] - p_1(t)$$

and

$$v(x) \equiv \int_a^x (x-t)r(t)\,dt + (x-a)\big[p_1(a)c_1 + c_2\big] + c_1.$$

Neumann series solution

We now outline a systematic approach to obtaining the infinite series of Theorem 18.1.2, which also works for the Fredholm equation of the second kind. In the latter case, the series is guaranteed to converge only if $|\lambda|\|\mathbf{K}\| < 1$. This approach has the advantage that in each successive step, we obtain a better approximation to the solution. Writing the equation as

$$|u\rangle = |v\rangle + \lambda\mathbf{K}|u\rangle, \tag{18.8}$$

we can interpret it as follows. The difference between $|u\rangle$ and $|v\rangle$ is $\lambda\mathbf{K}|u\rangle$. If $\lambda\mathbf{K}$ were absent, the two vectors $|u\rangle$ and $|v\rangle$ would be equal. The effect of $\lambda\mathbf{K}$ is to change $|u\rangle$ in such a way that when the result is added to $|v\rangle$, it gives $|u\rangle$. As our initial approximation, therefore, we take $|u\rangle$ to be equal to $|v\rangle$ and write $|u_0\rangle = |v\rangle$, where the index reminds us of the order (in this case zeroth, because $\lambda\mathbf{K} = 0$) of the approximation. To find a better approximation, we always substitute the latest approximation for $|u\rangle$ in the RHS of Eq. (18.8). At this stage, we have $|u_1\rangle = |v\rangle + \lambda\mathbf{K}|u_0\rangle = |v\rangle + \lambda\mathbf{K}|v\rangle$. Still a better approximation is achieved if we substitute this expression in (18.8):

$$|u_2\rangle = |v\rangle + \lambda\mathbf{K}|u_1\rangle = |v\rangle + \lambda\mathbf{K}\big(|v\rangle + \lambda\mathbf{K}|v\rangle\big) = |v\rangle + \lambda\mathbf{K}|v\rangle + \lambda^2\mathbf{K}^2|v\rangle.$$

The procedure is now clear. Once $|u_n\rangle$, the nth approximation, is obtained, we can get $|u_{n+1}\rangle$ by substituting in the RHS of (18.8).

Before continuing, let us write the above equations in integral form. In what follows, we shall concentrate on the Fredholm equation. To obtain the result for the Volterra equation, one simply replaces b, the upper limit of integration, with x. The first approximation can be obtained by substituting $v(t)$ for $u(t)$ on the RHS of Eq. (18.1). This yields

$$u_1(x) = v(x) + \lambda \int_a^b K(x,t)v(t)\,dt.$$

Substituting this back in Eq. (18.1) gives

$$\begin{aligned}
u_2(x) &= v(x) + \lambda \int_a^b ds\,K(x,s)u_1(s) \\
&= v(x) + \lambda \int_a^b ds\,K(x,s)v(s) \\
&\quad + \lambda^2 \int_a^b dt\left[\int_a^b K(x,s)K(s,t)\,ds\right]v(t) \\
&= v(x) + \lambda \int_a^b dt\,K(x,t)v(t) + \lambda^2 \int_a^b dt\,K^2(x,t)v(t),
\end{aligned}$$

where $K^2(x,t) \equiv \int_a^b K(x,s)K(s,t)\,ds$. Similar expressions can be derived for $u_3(x)$, $u_4(x)$, and so forth. The integrals expressing various "powers" of K can be obtained using Dirac notation and vectors with continuous indices, as discussed in Sect. 7.3. Thus, for instance,

$$\begin{aligned}
K^3(x,t) &\equiv \langle x|\mathbf{K}\underbrace{\left(\int_a^b |s_1\rangle\langle s_1|\,ds_1\right)}_{=\mathbf{1}}\mathbf{K}\underbrace{\left(\int_a^b |s_2\rangle\langle s_2|\,ds_2\right)}_{=\mathbf{1}}\mathbf{K}|t\rangle \\
&= \int_a^b ds_1 \int_a^b ds_2 \langle x|\mathbf{K}|s_1\rangle\langle s_1|\mathbf{K}|s_2\rangle\langle s_2|\mathbf{K}|t\rangle \\
&= \int_a^b ds_1 \int_a^b ds_2\,K(x,s_1)K(s_1,s_2)K(s_2,t).
\end{aligned}$$

We can always use this technique to convert an equation in kets into an equation in functions and integrals. Therefore, we can concentrate on the abstract operator equation and its various approximations.

Continuing to the nth-order approximation, we easily obtain

$$|u_n\rangle = |v\rangle + \lambda \mathbf{K}|v\rangle + \cdots + \lambda^n \mathbf{K}^n|v\rangle = \sum_{j=0}^{n} (\lambda \mathbf{K})^j |v\rangle, \tag{18.9}$$

whose integral form is

$$u_n(x) = \sum_{j=0}^{n} \lambda^j \int_a^b K^j(x,t) v(t)\, dt. \tag{18.10}$$

Here $K^j(x,t)$ is defined inductively by

$$K^0(x,t) = \langle x|\mathbf{K}^0|t\rangle = \langle x|\mathbf{1}|t\rangle = \langle x|t\rangle = \delta(x-t),$$

$$K^j(x,t) = \langle x|\mathbf{K}\mathbf{K}^{j-1}|t\rangle = \langle x|\mathbf{K}\left(\int_a^b |s\rangle\langle s|\, ds\right)\mathbf{K}^{j-1}|t\rangle$$

$$= \int_a^b K(x,s) K^{j-1}(s,t)\, ds.$$

The limit of $u_n(x)$ as $n \to \infty$ gives

$$u(x) = \sum_{j=0}^{\infty} \lambda^j \int_a^b K^j(x,t) v(t)\, dt. \tag{18.11}$$

Neumann series

The convergence of this series, called the **Neumann series**, is always guaranteed for the Volterra equation. For the Fredholm equation, we need to impose the extra condition $|\lambda| \|\mathbf{K}\| < 1$.

Example 18.1.4 As an example, let us find the solution of $u(x) = 1 + \lambda \int_0^x u(t)\, dt$, a Volterra equation of the second kind. Here, $v(x) = 1$ and $K(x,t) = 1$, and it is straightforward to calculate approximations to $u(x)$:

$$u_0(x) = v(x) = 1, \qquad u_1(x) = 1 + \lambda \int_0^x K(x,t) u_0(t)\, dt = 1 + \lambda x,$$

$$u_2(x) = 1 + \lambda \int_0^x K(x,t) u_1(t)\, dt = 1 + \lambda \int_0^x (1 + \lambda t)\, dt$$

$$= 1 + \lambda x + \frac{\lambda^2 x^2}{2}.$$

It is clear that the nth term will look like

$$u_n(x) = 1 + \lambda x + \frac{\lambda^2 x^2}{2} + \cdots + \frac{\lambda^n x^n}{n!} = \sum_{j=0}^{n} \frac{\lambda^j x^j}{j!}.$$

As $n \to \infty$, we obtain $u(x) = e^{\lambda x}$. By direct substitution, it is readily checked that this is indeed a solution of the original integral equation.

18.2 Fredholm Integral Equations

We can use our knowledge of compact operators gained in the previous chapter to study Fredholm equations of the second kind. With $\lambda \neq 0$ a complex number, we consider the characteristic equation

$$(\mathbf{1} - \lambda \mathbf{K})|u\rangle = |v\rangle, \quad \text{or} \quad u(x) - \lambda K[u](x) = v(x), \tag{18.12}$$

where all functions are square-integrable on $[a, b]$, and $K(x, t)$, the Hilbert-Schmidt kernel, is square-integrable on the rectangle $[a, b] \times [a, b]$.

Using Proposition 17.2.9, we immediately see that Eq. (18.12) has a unique solution if $|\lambda| \|\mathbf{K}\| < 1$, and the solution is of the form

$$|u\rangle = (\mathbf{1} - \lambda \mathbf{K})^{-1}|v\rangle = \sum_{n=0}^{\infty} \lambda^n \mathbf{K}^n |v\rangle, \tag{18.13}$$

or $u(x) = \sum_{n=0}^{\infty} \lambda^n K^n[v](x)$, where $K^n[v](x)$ is defined as in Eq. (18.4) except that now b replaces x as the upper limit of integration.

Example 18.2.1 Consider the integral equation

$$u(x) - \int_0^1 K(x, t) u(t)\, dt = x, \quad \text{where} \quad K(x, t) = \begin{cases} x & \text{if } 0 \leq x < t, \\ t & \text{if } t < x \leq 1. \end{cases}$$

Here $\lambda = 1$; therefore, a Neumann series solution exists if $\|\mathbf{K}\| < 1$. It is convenient to write K in terms of the theta function:[1]

$$K(x, t) = x\theta(t - x) + t\theta(x - t). \tag{18.14}$$

This gives $|K(x, t)|^2 = x^2\theta(t - x) + t^2\theta(x - t)$ because $\theta^2(x - t) = \theta(x - t)$ and $\theta(x - t)\theta(t - x) = 0$. Thus, we have

$$\begin{aligned} \|\mathbf{K}\|^2 &= \int_0^1 dx \int_0^1 dt \bigl|K(x, t)\bigr|^2 \\ &= \int_0^1 dx \int_0^1 x^2\theta(t - x)\, dt + \int_0^1 dx \int_0^1 t^2\theta(x - t)\, dt \\ &= \int_0^1 dt \int_0^t x^2\, dx + \int_0^1 dx \int_0^x t^2\, dt \\ &= \int_0^1 dt \left(\frac{t^3}{3}\right) + \int_0^1 dx \left(\frac{x^3}{3}\right) = \frac{1}{6}. \end{aligned}$$

Since this is less than 1, the Neumann series converges, and we have[2]

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[1] Recall that the theta function is defined to be 1 if its argument is positive, and 0 if it is negative.

[2] Note that in this case (Fredholm equation), we can calculate the jth term in isolation. In the Volterra case, it was more natural to calculate the solution *up to* a given order.

$$u(x) = \sum_{j=0}^{\infty} \lambda^j \int_a^b K^j(x,t)v(t)\,dt = \sum_{j=0}^{\infty} \int_0^1 K^j(x,t)t\,dt \equiv \sum_{j=0}^{\infty} f_j(x).$$

The first few terms are evaluated as follows:

$$\begin{aligned}
f_0(x) &= \int_0^1 K^0(x,t)t\,dt = \int_0^1 \delta(x,t)t\,dt = x \\
f_1(x) &= \int_0^1 K(x,t)t\,dt = \int_0^1 \big[x\theta(t-x) + t\theta(x-t)\big]t\,dt \\
&= x\int_x^1 t\,dt + \int_0^x t^2\,dt = \frac{x}{2} - \frac{x^3}{6}.
\end{aligned}$$

The next term is trickier than the first two because of the product of the theta functions. We first substitute Eq. (18.14) in the integral for the second-order term, and simplify

$$\begin{aligned}
f_2(x) &= \int_0^1 K^2(x,t)t\,dt = \int_0^1 t\,dt \int_0^1 K(x,s)K(s,t)\,ds \\
&= \int_0^1 t\,dt \int_0^1 \big[x\theta(s-x) + s\theta(x-s)\big]\big[s\theta(t-s) + t\theta(s-t)\big]\,ds \\
&= x\int_0^1 t\,dt \int_0^1 s\theta(s-x)\theta(t-s)\,ds \\
&\quad + x\int_0^1 t^2\,dt \int_0^1 \theta(s-x)\theta(s-t)\,ds \\
&\quad + \int_0^1 t\,dt \int_0^1 s^2\theta(x-s)\theta(t-s)\,ds \\
&\quad + \int_0^1 t^2\,dt \int_0^1 s\theta(x-s)\theta(s-t)\,ds.
\end{aligned}$$

It is convenient to switch the order of integration at this point. This is because of the presence of $\theta(x-s)$ and $\theta(s-x)$, which do not involve t and are best integrated last. Thus, we have

$$\begin{aligned}
f_2(x) &= x\int_0^1 s\theta(s-x)\,ds \int_s^1 t\,dt + x\int_0^1 \theta(s-x)\,ds \int_0^s t^2\,dt \\
&\quad + \int_0^1 s^2\theta(x-s)\,ds \int_s^1 t\,dt + \int_0^1 s\theta(x-s)\,ds \int_0^s t^2\,dt \\
&= x\int_x^1 s\,ds\left(\frac{1}{2} - \frac{s^2}{2}\right) + x\int_x^1 ds\frac{s^3}{3} + \int_0^x s^2\,ds\left(\frac{1}{2} - \frac{s^2}{2}\right) \\
&\quad + \int_0^x s\,ds\frac{s^3}{3} \\
&= \frac{5}{24}x - \frac{1}{12}x^3 + \frac{1}{120}x^5.
\end{aligned}$$

As a test of his/her knowledge of θ-function manipulation, the reader is urged to perform the integration in reverse order. Adding all the terms, we obtain an approximation for $u(x)$ that is valid for $0 \le x \le 1$:

$$u(x) \approx f_0(x) + f_1(x) + f_2(x) = \frac{41}{24}x - \frac{1}{4}x^3 + \frac{1}{120}x^5.$$

We have seen that the Volterra equation of the second kind has a unique solution which can be written as an infinite series (see Theorem 18.1.2). The case of the Fredholm equation of the second kind is more complicated because of the existence of eigenvalues. The general solution of Eq. (18.12) is discussed in the following:

Theorem 18.2.2 (Fredholm Alternative) *Let* $\mathbf{K}$ *be a Hilbert-Schmidt operator and* λ *a complex number. Then either*

Fredholm alternative

1. λ *is a regular value of Eq.* (18.12)*—or* λ^{-1} *is a regular point of the operator* $\mathbf{K}$*—in which case the equation has the unique solution* $|u\rangle = (\mathbf{1} - \lambda\mathbf{K})^{-1}|v\rangle$, *or*
2. λ *is a characteristic value of Eq.* (18.12) (λ^{-1} *is an eigenvalue of the operator* $\mathbf{K}$), *in which case the equation has a solution if and only if* $|v\rangle$ *is in the orthogonal complement of the* (*finite-dimensional*) *null space of* $\mathbf{1} - \lambda^*\mathbf{K}^\dagger$.

Proof The first part is trivial if we recall that by definition, regular points of $\mathbf{K}$ are those complex numbers μ which make the operator $\mathbf{K} - \mu\mathbf{1}$ invertible.

For part (2), we first show that the null space of $\mathbf{1} - \lambda^*\mathbf{K}^\dagger$ is finite-dimensional. We note that $\mathbf{1} - \lambda\mathbf{K}$ is invertible if and only if its adjoint $\mathbf{1} - \lambda^*\mathbf{K}^\dagger$ is invertible, and $\lambda \in \rho(\mathbf{K})$ iff $\lambda^* \in \rho(\mathbf{K}^\dagger)$. Since the spectrum of an operator is composed of all points that are not regular, we conclude that λ is in the spectrum of $\mathbf{K}$ if and only if λ^* is in the spectrum of $\mathbf{K}^\dagger$. For compact operators, all nonzero points of the spectrum are eigenvalues. Therefore, the nonzero points of the spectrum of $\mathbf{K}^\dagger$, a compact operator by Theorem 17.5.7, are all eigenvalues of $\mathbf{K}^\dagger$, and the null space of $\mathbf{1} - \lambda^*\mathbf{K}^\dagger$ is finite-dimensional (Theorem 17.5.11). Next, we note that the equation itself requires that $|v\rangle$ be in the range of the operator $\mathbf{1} - \lambda\mathbf{K}$, which, by Theorem 17.6.5, is the orthogonal complement of the null space of $\mathbf{1} - \lambda^*\mathbf{K}^\dagger$. □

Historical Notes

Erik Ivar Fredholm (1866–1927) was born in Stockholm, the son of a well-to-do merchant family. He received the best education possible and soon showed great promise in mathematics, leaning especially toward the applied mathematics of practical mechanics in a year of study at Stockholm's Polytechnic Institute. Fredholm finished his education at the University of Uppsala, obtaining his doctorate in 1898. He also studied at the University of Stockholm during this same period and eventually received an appointment to the faculty there. Fredholm remained there the rest of his professional life.

His first contribution to mathematics was contained in his doctoral thesis, in which he studied a first-order partial differential equation in three variables, a problem that arises in the deformation of anisotropic media. Several years later he completed this work by finding the fundamental solution to a general elliptic partial differential equation with constant coefficients.

Erik Ivar Fredholm
1866–1927

Fredholm is perhaps best known for his studies of the integral equation that bears his name. Such equations occur frequently in physics. Fredholm's genius led him to note the similarity between his equation and a relatively familiar matrix-vector equation, resulting in his identification of a quantity that plays the same role in his equation as the determinant plays in the matrix-vector equation. He thus obtained a method for determining the existence of a solution and later used an analogous expression to derive a solution to his equation akin to the Cramer's rule solution to the matrix-vector equation. He further showed that the solution could be expressed as a power series in a complex variable. This latter result was considered important enough that Poincaré assumed it without proof (in fact he was unable to prove it) in a study of related partial differential equations.
Fredholm then considered the homogeneous form of his equation. He showed that under certain conditions, the vector space of solutions is finite-dimensional. David Hilbert later extended Fredholm's work to a complete eigenvalue theory of the Fredholm equation, which ultimately led to the discovery of Hilbert spaces.

18.2.1 Hermitian Kernel

Of special interest are integral equations in which the kernel is hermitian, which occurs when the operator is hermitian. Such a kernel has the property that[3] $\langle x|\mathbf{K}|t\rangle^* = \langle t|\mathbf{K}|x\rangle$ or $[K(x,t)]^* = K(t,x)$. For such kernels we can use the spectral theorem for compact hermitian operators to find a series solution for the integral equation. First we recall that

$$\mathbf{K} = \sum_{j=1}^{N} \lambda_j^{-1}\mathbf{P}_j = \sum_{j=1}^{N} \lambda_j^{-1} \sum_{k=1}^{m_j} \big|e_k^{(j)}\big\rangle\big\langle e_k^{(j)}\big|,$$

where we have used λ_j^{-1} to denote the eigenvalue of the operator[4] and expanded the projection operator in terms of orthonormal basis vectors of the corresponding finite-dimensional eigenspace. Recall that N can be infinity. Instead of the double sum, we can sum once over all the basis vectors and write $\mathbf{K} = \sum_{n=1}^{\infty} \lambda_n^{-1}|u_n\rangle\langle u_n|$. Here n counts all the orthonormal eigenvectors of the Hilbert space, and λ_n^{-1} is the eigenvalue corresponding to the eigenvector $|u_n\rangle$. Therefore, λ_n^{-1} may be repeated in the sum. The action of $\mathbf{K}$ on a vector $|u\rangle$ is given by

$$\mathbf{K}|u\rangle = \sum_{n=1}^{\infty} \lambda_n^{-1}\langle u_n|u\rangle|u_n\rangle. \tag{18.15}$$

If the Hilbert space is $\mathcal{L}^2[a,b]$, we may be interested in the functional form of this equation. We obtain such a form by multiplying both sides by $\langle x|$:

$$K[u](x) \equiv \langle x|\mathbf{K}|u\rangle = \sum_{n=1}^{\infty} \lambda_n^{-1}\langle u_n|u\rangle\langle x|u_n\rangle = \sum_{n=1}^{\infty} \lambda_n^{-1}\langle u_n|u\rangle u_n(x).$$

Hilbert-Schmidt theorem

That this series converges *uniformly* in the interval $[a,b]$ is known as the **Hilbert-Schmidt theorem**.

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[3]Since we are dealing mainly with real functions, hermiticity of $\mathbf{K}$ implies the symmetry of K, i.e., $K(x,t) = K(t,x)$.

[4]λ_j is the characteristic value of the integral equation, or the inverse of the eigenvalue of the corresponding operator.

Example 18.2.3 Let us solve $u(x) = x + \lambda \int_a^b K(x,t)u(t)\,dt$, where $K(x,t) \equiv xt$ is a symmetric (hermitian) kernel, by the Neumann series method. We note that

$$\|\mathbf{K}\|^2 = \int_a^b \int_a^b \left|K(x,t)\right|^2 dx\,dt = \int_a^b \int_a^b x^2 t^2 dx\,dt$$
$$= \int_a^b x^2 dx \int_a^b t^2 dt = \left(\int_a^b x^2 dx\right)^2 = \frac{1}{9}\left(b^3 - a^3\right)^2,$$

or

$$\|\mathbf{K}\| = \int_a^b x^2 dx = \frac{1}{3}\left(b^3 - a^3\right),$$

and the Neumann series converges if $|\lambda|(b^3 - a^3) < 3$. Assuming that this condition holds, we have

$$u(x) = x + \sum_{j=1}^{\infty} \lambda^j \int_a^b K^j(x,t)t\,dt.$$

The special form of the kernel allows us to calculate $K^j(x,t)$ directly:

$$K^j(x,t) = \int_a^b \int_a^b \dots \int_a^b K(x,s_1)K(s_1,s_2)\cdots K(s_{j-1},t)\,ds_1 ds_2 \cdots ds_{j-1}$$
$$= \int_a^b \int_a^b \dots \int_a^b x s_1^2 s_2^2 \cdots s_{j-1}^2 t\,ds_1 ds_2 \cdots ds_{j-1}$$
$$= xt\left(\int_a^b s^2\,ds\right)^{j-1} = xt\|\mathbf{K}\|^{j-1}.$$

It follows that $\int_a^b K^j(x,t)t\,dt = x\|\mathbf{K}\|^{j-1}\frac{1}{3}(b^3 - a^3) = x\|\mathbf{K}\|^j$. Substituting this in the expression for $u(x)$ yields

$$u(x) = x + \sum_{j=1}^{\infty} \lambda^j x\|\mathbf{K}\|^j = x + x\lambda\|\mathbf{K}\| \sum_{j=1}^{\infty} \lambda^{j-1}\|\mathbf{K}\|^{j-1}$$
$$= x\left(1 + \lambda\|\mathbf{K}\|\frac{1}{1 - \lambda\|\mathbf{K}\|}\right) = \frac{x}{1 - \lambda\|\mathbf{K}\|} = \frac{3x}{3 - \lambda(b^3 - a^3)}.$$

Because of the simplicity of the kernel, we can solve the integral equation exactly. First we write

$$u(x) = x + \lambda \int_a^b xtu(t)\,dt = x + \lambda x \int_a^b tu(t)\,dt \equiv x(1 + \lambda A), \quad (18.16)$$

where $A = \int_a^b tu(t)\,dt$. Multiplying both sides by x and integrating, we obtain

$$A = \int_a^b xu(x)\,dx = (1 + \lambda A)\int_a^b x^2 dx = (1 + \lambda A)\|\mathbf{K}\|$$
$$\Rightarrow \quad A = \frac{\|\mathbf{K}\|}{1 - \lambda\|\mathbf{K}\|}.$$

Substituting A in Eq. (18.16) gives

$$u(x) = x\left(1 + \lambda \frac{\|\mathbf{K}\|}{1 - \lambda\|\mathbf{K}\|}\right) = \frac{x}{1 - \lambda\|\mathbf{K}\|}.$$

This solution is the same as the first one we obtained. However, no series was involved here, and therefore no assumption is necessary concerning $|\lambda|\|\mathbf{K}\|$.

If one can calculate the eigenvectors $|u_n\rangle$ and the eigenvalues λ_n^{-1}, then one can obtain a solution for the integral equation in terms of these eigenfunctions as follows: Substitute (18.15) in the Fredholm equation [Eq. (18.3)] to get

$$|u\rangle = |v\rangle + \lambda \sum_{n=1}^{\infty} \lambda_n^{-1} \langle u_n|u\rangle |u_n\rangle. \tag{18.17}$$

Multiply both sides by $\langle u_m|$:

$$\begin{aligned} \langle u_m|u\rangle &= \langle u_m|v\rangle + \lambda \sum_{n=1}^{\infty} \lambda_n^{-1} \langle u_n|u\rangle \underbrace{\langle u_m|u_n\rangle}_{=\delta_{mn}} \\ &= \langle u_m|v\rangle + \lambda\lambda_m^{-1}\langle u_m|u\rangle, \end{aligned} \tag{18.18}$$

or, if λ is not one of the eigenvalues,

$$\left(1 - \frac{\lambda}{\lambda_m}\right)\langle u_m|u\rangle = \langle u_m|v\rangle \quad \Rightarrow \quad \langle u_m|u\rangle = \frac{\lambda_m\langle u_m|v\rangle}{\lambda_m - \lambda}.$$

Substituting this in Eq. (18.17) gives

$$|u\rangle = |v\rangle + \lambda \sum_{n=1}^{\infty} \frac{\langle u_n|v\rangle}{\lambda_n - \lambda} |u_n\rangle, \tag{18.19}$$

and in the functional form,

$$u(x) = v(x) + \lambda \sum_{n=1}^{\infty} \frac{\langle u_n|v\rangle}{\lambda_n - \lambda} u_n(x), \quad \lambda \neq \lambda_n \ \forall n. \tag{18.20}$$

In case $\lambda = \lambda_m$ for some m, the Fredholm alternative (Theorem 18.2.2) says that we will have a solution only if $|v\rangle$ is in the orthogonal complement of the null space of[5] $\mathbf{1} - \lambda_m\mathbf{K}$. Moreover, Eq. (18.18) shows that $\langle u_m|u\rangle$, the expansion coefficients of the basis vectors of the eigenspace $\mathcal{M}_m$, cannot be specified. However, Eq. (18.18) does determine the rest of the coefficients as before. In this case, the solution can be written as

$$|u\rangle = |v\rangle + \sum_{k=1}^{r} c_k |u_m^{(k)}\rangle + \lambda \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \frac{\langle u_n|v\rangle}{\lambda_n - \lambda} |u_n\rangle, \tag{18.21}$$

[5] Remember that $\mathbf{K}$ is hermitian; therefore, λ_m is real.

where r is the (finite) dimension of $\mathcal{M}_m$, k labels the orthonormal basis $\{|u_m^{(k)}\rangle\}$ of $\mathcal{M}_m$, and $\{c_k\}_{k=1}^r$ are arbitrary constants. In functional form, this equation becomes

$$u(x) = v(x) + \sum_{k=1}^{r} c_k u_m^{(k)}(x) + \lambda \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{\langle u_n|v\rangle}{\lambda_n - \lambda} u_n(x). \tag{18.22}$$

Example 18.2.4 We now give an example of the application of Eq. (18.20). We want to solve $u(x) = 3\int_{-1}^{1} K(x,t)u(t)\,dt + x^2$ where

$$K(x,t) = \sum_{k=0}^{\infty} \frac{u_k(x)u_k(t)}{2^{k/2}}, \qquad u_k(x) = \sqrt{\frac{2k+1}{2}} P_k(x),$$

and $P_k(x)$ is a Legendre polynomial.

We first note that $\{u_k\}$ is an orthonormal set of functions, that $K(x,t)$ is real and symmetric (therefore, hermitian), and that

$$\begin{aligned}
&\int_{-1}^{1} dt \int_{-1}^{1} dx \big|K(x,t)\big|^2 \\
&\quad = \int_{-1}^{1} dt \int_{-1}^{1} dx \sum_{k,l=0}^{\infty} \frac{u_k(x)u_k(t)}{2^{k/2}} \frac{u_l(x)u_l(t)}{2^{l/2}} \\
&\quad = \sum_{k,l=0}^{\infty} \frac{1}{2^{k/2}} \frac{1}{2^{l/2}} \underbrace{\int_{-1}^{1} u_k(x)u_l(x)\,dx}_{=\delta_{kl}} \underbrace{\int_{-1}^{1} u_k(t)u_l(t)\,dt}_{=\delta_{kl}} \\
&\quad = \sum_{k=0}^{\infty} \frac{1}{2^k}\delta_{kk} = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 < \infty.
\end{aligned}$$

Thus, $K(x,t)$ is a Hilbert-Schmidt kernel.

Now note that

$$\begin{aligned}
\int_{-1}^{1} K(x,t)u_k(t)\,dt &= \int_{-1}^{1} \sum_{l=0}^{\infty} \frac{u_l(x)u_l(t)}{2^{l/2}} u_k(t)\,dt \\
&= \sum_{l=0}^{\infty} \frac{u_l(x)}{2^{l/2}} \underbrace{\int_{-1}^{1} u_l(t)u_k(t)\,dt}_{=\delta_{kl}} = \frac{1}{2^{k/2}} u_k(x).
\end{aligned}$$

This shows that u_k is an eigenfunction of $K(x,t)$ with eigenvalue $1/2^{k/2}$. Since $3 \neq 1/2^{k/2}$ for any integer k, we can use Eq. (18.20) to write

$$u(x) = x^2 + 3\sum_{k=0}^{\infty} \frac{\int_{-1}^{1} u_k(s)s^2 ds}{2^{k/2} - 3} u_k(x).$$

But $\int_{-1}^{1} u_k(s)s^2 ds = 0$ for $k \geq 3$. For $k \leq 2$, we use the first three Legendre polynomials to get

$$\int_{-1}^{1} u_0(s)s^2 ds = \frac{\sqrt{2}}{3}, \qquad \int_{-1}^{1} u_1(s)s^2 ds = 0,$$
$$\int_{-1}^{1} u_2(s)s^2 ds = \frac{2\sqrt{2}}{3\sqrt{5}}.$$

This gives $u(x) = \frac{1}{2} - 2x^2$. The reader is urged to substitute this solution in the original integral equation and verify that it works.

18.2.2 Degenerate Kernels

degenerate or separable kernel

The preceding example involves the simplest kind of degenerate, or separable, kernels. A kernel is called **degenerate**, or **separable**, if it can be written as a finite sum of products of functions of one variable:

$$K(x,t) = \sum_{j=1}^{n} \phi_j(x)\psi_j^*(t), \tag{18.23}$$

where ϕ_j and ψ_j are assumed to be square-integrable. Substituting (18.23) in the Fredholm integral equation of the second kind, we obtain

$$u(x) - \lambda \sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j^*(t)u(t)\,dt = v(x).$$

If we define $\mu_j = \int_a^b \psi_j^*(t)u(t)\,dt$, the preceding equation becomes

$$u(x) - \lambda \sum_{j=1}^{n} \mu_j \phi_j(x) = v(x). \tag{18.24}$$

Multiply this equation by $\psi_i^*(x)$ and integrate over x to get

$$\mu_i - \lambda \sum_{j=1}^{n} \mu_j A_{ij} = \nu_i \quad \text{for } i = 1, 2, \ldots, n, \tag{18.25}$$

where $A_{ij} = \int_a^b \psi_i^*(t)\phi_j(t)\,dt$ and $\nu_i = \int_a^b \psi_i^*(t)v(t)\,dt$. With μ_i, ν_i, and A_{ij} as components of column vectors u, v, and a matrix A, we can write the above linear system of equations as

$$\mathsf{u} - \lambda\mathsf{A}\mathsf{u} = \mathsf{v}, \quad \text{or} \quad (1 - \lambda\mathsf{A})\mathsf{u} = \mathsf{v}. \tag{18.26}$$

We can now determine the μ_i by solving the system of linear equations given by (18.25). Once the μ_i are determined, Eq. (18.24) gives $u(x)$. Thus, for a degenerate kernel the Fredholm problem reduces to a system of linear equations.

Example 18.2.5 As a concrete example of an integral equation with degenerate kernel, we solve $u(x) - \lambda \int_0^1 (1 + xt)u(t)\,dt = x$ for two different values of λ. The kernel, $K(x,t) = 1 + xt$, is separable with $\phi_1(x) = 1$, $\psi_1(t) = 1$, $\phi_2(x) = x$, and $\psi_2(t) = t$. This gives the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

For convenience, we define the matrix $\mathsf{B} \equiv \mathbf{1} - \lambda\mathsf{A}$.

(a) First assume that $\lambda = 1$. In that case B has a nonzero determinant. Thus, B^{-1} exists, and can be calculated to be

$$\mathsf{B}^{-1} = \begin{pmatrix} -\frac{8}{3} & -2 \\ -2 & 0 \end{pmatrix}.$$

With

$$\nu_1 = \int_0^1 \psi_1^*(t)v(t)\,dt = \int_0^1 t\,dt = \frac{1}{2} \quad \text{and}$$
$$\nu_2 = \int_0^1 \psi_2^*(t)v(t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$

we obtain

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mathsf{B}^{-1}\mathsf{v} = \begin{pmatrix} -\frac{8}{3} & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Equation (18.24) then gives $u(x) = \mu_1\phi_1(x) + \mu_2\phi_2(x) + x = -2$.

(b) Now, for the purpose of illustrating the other alternative of Theorem 18.2.2, let us take $\lambda = 8 + 2\sqrt{13}$. Then

$$\mathsf{B} = \mathbf{1} - \lambda\mathsf{A} = -\begin{pmatrix} 7 + 2\sqrt{13} & 4 + \sqrt{13} \\ 4 + \sqrt{13} & (5 + 2\sqrt{13})/3 \end{pmatrix},$$

and $\det \mathsf{B} = 0$. This shows that $8 + 2\sqrt{13}$ is a characteristic value of the equation. We thus have a solution only if $v(x) \equiv x$ is orthogonal to the null space of $\mathbf{1} - \lambda^*\mathsf{A}^\dagger = \mathsf{B}^\dagger$. To determine a basis for this null space, we have to find vectors $|z\rangle$ such that $\mathsf{B}^\dagger|z\rangle = 0$. Since λ is real, and B is real and symmetric, $\mathsf{B}^\dagger = \mathsf{B}$, and we must solve

$$\begin{pmatrix} 7 + 2\sqrt{13} & 4 + \sqrt{13} \\ 4 + \sqrt{13} & (5 + 2\sqrt{13})/3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

The solution to this equation is a multiple of $|z\rangle \equiv \left(\begin{smallmatrix} 3 \\ -2-\sqrt{13} \end{smallmatrix}\right)$. If the integral equation is to have a solution, the column vector v (whose corresponding ket we denote by $|v\rangle$) must be orthogonal to $|z\rangle$. But

$$\langle z|v\rangle = \begin{pmatrix} 3 & -2 - \sqrt{13} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \neq 0.$$

Therefore, the integral equation has no solution.

The reader may feel uneasy that the functions $\phi_j(x)$ and $\psi_j(t)$ appearing in a degenerate kernel are arbitrary to within a multiplicative function. After all, we can multiply $\phi_j(x)$ by a nonzero function, and divide $\psi_j(t)$ by the same function, and get the same kernel. Such a change clearly alters the matrices A and B and therefore seems likely to change the solution, $u(x)$. That this is not the case is demonstrated in Problem 18.2. In fact, it can be shown quite generally that the transformations described above do not change the solution.

As the alert reader may have noticed, we have been avoiding the problem of solving the eigenvalue (characteristic) problem for integral operators. Such a problem is nontrivial, and the analogue of the finite-dimensional case, where one works with determinants and characteristic polynomials, does not exist. An exception is a degenerate hermitian[6] kernel, i.e., a kernel of the form $K(x,t) = \sum_{i=1}^{n} h_i(x)h_i^*(t)$. Substituting this in the characteristic-value equation

$$u(x) = \lambda \int_a^b K(x,t)u(t)\,dt, \qquad \lambda \neq 0,$$

we obtain $u(x) = \lambda \sum_{i=1}^{n} h_i(x) \int_a^b h_i^*(t)u(t)\,dt$. Defining $\mu_i \equiv \int_a^b h_i^*(t) \times u(t)\,dt$ and substituting it back in the equation gives

$$u(x) = \lambda \sum_{i=1}^{n} h_i(x)\mu_i. \tag{18.27}$$

Multiplying this equation by $\lambda^{-1}h_k^*(x)$ and integrating over x yields

$$\lambda^{-1}\mu_k = \sum_{i=1}^{n} \left[\int_a^b h_k^*(x)h_i(x)\,dx\right]\mu_i \equiv \sum_{i=1}^{n} m_{ki}\mu_i.$$

This is an eigenvalue equation for the hermitian $n \times n$ matrix M with elements m_{ij}, which, by spectral theorem for hermitian operators, can be solved. In fact, the matrix need not be hermitian; as long as it is *normal*, the eigenvalue problem can be solved. Once the eigenvectors and the eigenvalues are found, we can substitute them in Eq. (18.27) and obtain $u(x)$. We expect to find a finite number of eigenfunctions and eigenvalues. Our analysis of compact operators included such a case. That analysis also showed that the entire (infinite-dimensional) Hilbert space could be written as the direct sum of eigenspaces that are finite-dimensional for nonzero eigenvalues. Therefore, we expect the eigenspace corresponding to the zero eigenvalue (or infinite characteristic value) to be infinite-dimensional. The following example illustrates these points.

Example 18.2.6 Let us find the nonzero characteristic values and corresponding eigenfunctions of the kernel $K(x,t) = 1 + \sin(x+t)$ for $-\pi \le x, t \le \pi$.

[6] Actually, the problem of a degenerate kernel that leads to a normal matrix, as described below, can also be solved.

We are seeking functions u and scalars λ satisfying $u(x) = \lambda K[u](x)$, or

$$u(x) = \lambda \int_{-\pi}^{\pi} \bigl[1 + \sin(x+t)\bigr] u(t)\, dt.$$

Expanding $\sin(x+t)$, we obtain

$$u(x) = \lambda \int_{-\pi}^{\pi} [1 + \sin x \cos t + \cos x \sin t] u(t)\, dt, \tag{18.28}$$

or

$$\lambda^{-1} u(x) = \mu_1 + \mu_2 \sin x + \mu_3 \cos x, \tag{18.29}$$

where $\mu_1 = \int_{-\pi}^{\pi} u(t)\, dt$, $\mu_2 = \int_{-\pi}^{\pi} u(t) \cos t\, dt$, and $\mu_3 = \int_{-\pi}^{\pi} u(t) \sin t\, dt$. Integrate both sides of Eq. (18.29) with respect to x from $-\pi$ to π to obtain $\lambda^{-1}\mu_1 = 2\pi\mu_1$. Similarly, multiplying by $\sin x$ and $\cos x$ and integrating yields

$$\lambda^{-1}\mu_2 = \pi\mu_3 \quad \text{and} \quad \lambda^{-1}\mu_3 = \pi\mu_2. \tag{18.30}$$

If $\mu_1 \neq 0$, we get $\lambda^{-1} = 2\pi$, which, when substituted in (18.30), yields $\mu_2 = \mu_3 = 0$. We thus have, as a first solution, $\lambda_1^{-1} = 2\pi$ and $|u_1\rangle = \alpha\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, where α is an arbitrary constant. Equation (18.29) now gives $\lambda_1^{-1} u_1(x) = \mu_1$, or $u_1(x) = c_1$, where c_1 is an arbitrary constant to be determined.

On the other hand, $\mu_1 = 0$ if $\lambda^{-1} \neq 2\pi$. Then Eq. (18.30) yields $\lambda^{-1} = \pm\pi$ and $\mu_2 = \pm\mu_3$. For $\lambda^{-1} \equiv \lambda_+^{-1} = \pi$, Eq. (18.29) gives

$$u(x) \equiv u_+(x) = c_+(\sin x + \cos x),$$

and for $\lambda^{-1} \equiv \lambda_-^{-1} = -\pi$, it yields $u(x) \equiv u_-(x) = c_-(\sin x - \cos x)$, where $c_\pm$ are arbitrary constants to be determined by normalization of eigenfunctions. The normalized eigenfunctions are

$$u_1 = \frac{1}{\sqrt{2\pi}}, \qquad u_\pm(x) = \frac{1}{\sqrt{2\pi}}(\sin x \pm \cos x).$$

Direct substitution in the original integral equation easily verifies that u_1, u_+, and u_- are eigenfunctions of the integral equation with the eigenvalues calculated above.

Let us now consider the zero eigenvalue (or infinite characteristic value). Divide both sides of Eq. (18.28) by λ and take the limit of $\lambda \to \infty$. Then the integral equation becomes

$$\int_{-\pi}^{\pi} [1 + \sin x \cos t + \cos x \sin t] u(t)\, dt = 0.$$

The solutions $u(t)$ to this equation would span the eigenspace corresponding to the zero eigenvalue, or infinite characteristic value. We pointed out above that this eigenspace is expected to be infinite-dimensional. This expectation is borne out once we note that all functions of the form $\sin nt$ or $\cos nt$ with $n \geq 2$ make the above integral zero; and there are infinitely many such functions.

18.3 Problems

18.1 Use mathematical induction to derive Eq. (18.5).

18.2 Repeat part (a) of Example 18.2.5 using

$$\phi_1(x) = \frac{1}{2}, \qquad \psi_1(t) = 2,$$
$$\phi_2(x) = x, \qquad \psi_2(t) = t$$

so that we still have $K(x,t) = \phi_1(x)\psi_1(t) + \phi_2(x)\psi_2(t)$.

18.3 Use the spectral theorem for compact hermitian operators to show that if the kernel of a Hilbert-Schmidt operator has a finite number of nonzero eigenvalues, then the kernel is separable. Hint: See the discussion at the beginning of Sect. 18.2.1.

18.4 Use the method of successive approximations to solve the Volterra equation $u(x) = \lambda \int_0^x u(t)dt$. Then derive a DE equivalent to the Volterra equation (make sure to include the initial condition), and solve it.

18.5 Regard the Fourier transform,

$$\mathbf{F}[f](x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y)\, dy$$

as an integral operator.

(a) Show that $\mathbf{F}^2[f](x) = f(-x)$.
(b) Deduce, therefore, that the only eigenvalues of this operator are $\lambda = \pm 1, \pm i$.
(c) Let $f(x)$ be any even function of x. Show that an appropriate choice of α can make $u = f + \alpha\mathbf{F}[f]$ an eigenfunction of $\mathbf{F}$. (This shows that the eigenvalues of $\mathbf{F}$ have infinite multiplicity.)

18.6 For what values of λ does the following integral equation have a solution?

$$u(x) = \lambda \int_0^{\pi} \sin(x+t)u(t)\, dt + x.$$

What is that solution? Redo the problem using a Neumann series expansion. Under what condition is the series convergent?

18.7 It is possible to multiply the functions $\phi_j(x)$ by $\gamma_j(x)$ and $\psi_j(t)$ by $1/\gamma_j(t)$ and still get the same degenerate kernel, $K(x,t) = \sum_{j=1}^{n} \phi_j(x)\psi_j(t)$. Show that such arbitrariness, although affecting the matrices A and B, does not change the solution of the Fredholm problem

$$u(x) - \lambda \int_a^b K(x,t)u(t)dt = f(x).$$

18.8 Show, by direct substitution, that the solution found in Example 18.2.4 does satisfy its integral equation.

18.9 Solve $u(x) = \frac{1}{2}\int_{-1}^{1}(x+t)u(t)\,dt + x$.

18.10 Solve $u(x) = \lambda\int_0^1 xtu(t)\,dt + x$ using the Neumann series method. For what values of λ is the series convergent? Now find the eigenvalues and eigenfunctions of the kernel and solve the problem using these eigenvalues and eigenfunctions.

18.11 Solve $u(x) = \lambda\int_0^\infty K(x,t)u(t)dt + x^\alpha$, where α is any real number except a negative integer, and $K(x,t) = e^{-(x+t)}$. For what values of λ does the integral equation have a solution?

18.12 Solve the integral equations

(a) $\displaystyle u(x) = e^x + \lambda\int_0^1 xtu(t)\,dt,$ (b) $\displaystyle u(x) = \lambda\int_0^\pi \sin(x-t)u(t)\,dt,$

(c) $\displaystyle u(x) = x^2 + \int_0^1 xtu(t)\,dt,$ (d) $\displaystyle u(x) = x + \int_0^x u(t)\,dt.$

18.13 Solve the integral equation $u(x) = x + \lambda\int_0^1 (x+t)tu(t)\,dt$, keeping terms up to λ^2.

18.14 Solve the integral equation $u(x) = e^{-|x|} + \lambda\int_{-\infty}^{\infty} e^{-|x-t|}u(t)\,dt$, assuming that f remains finite as $x \to \pm\infty$.

18.15 Solve the integral equation $u(x) = e^{-|x|} + \lambda\int_0^\infty u(t)\cos xt\,dt$, assuming that f remains finite as $x \to \pm\infty$.