Lie Groups and Lie Algebras

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The theory of differential equations had flourished to such a level by the 1860s that a systematic study of their solutions became possible. Sophus Lie, a Norwegian mathematician, undertook such a study using the same tool that was developed by Galois and others to study algebraic equations: group theory. The groups associated with the study of differential equations, now called Lie groups, unlike their algebraic counterparts, are uncountably infinite, and, as such, are both intricate and full of far-reaching structures. It was beyond the wildest dream of any 19th-century mathematician to imagine that a concept as abstract as Lie groups would someday find application in the study of the heart of matter. Yet, three of the four fundamental interactions are described by Lie groups, and the fourth one, gravity, is described in a language very akin to the other three.

29.1 Lie Groups and Their Algebras

Lie groups are infinite groups that have the extra property that their multiplication law is differentiable. We have seen that the natural setting for differentiation is the structure of a manifold. Thus, Lie groups must have manifold properties as well as group properties.

Definition 29.1.1 A **Lie group** G is a differentiable manifold endowed with a group structure such that the group operation $G \times G \to G$ and the map $G \to G$ given by $g \mapsto g^{-1}$ are differentiable. If the dimension of the underlying manifold is r, we say that G is an r-**parameter Lie group**.

Lie groups defined

Because of the dual nature of Lie groups, most of their mapping properties combine those of groups and manifolds. For instance, a **Lie group homomorphism** is a group homomorphism that is also $\mathcal{C}^\infty$, and a **Lie group isomorphism** is a group isomorphism that is also a diffeomorphism.

Example 29.1.2 ($GL(\mathcal{V})$ is a Lie group) As the paradigm of Lie groups, we consider $GL(\mathcal{V})$, the set of invertible operators on an n-dimensional real vector space $\mathcal{V}$, and show that it is indeed a Lie group. The set $\mathcal{L}(\mathcal{V})$ is a

$GL(\mathcal{V})$ is a Lie group

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vector space of dimension n^2 (Proposition 26.1.1), and therefore, by Example 28.1.7, a manifold of the same dimension. The map $\det : \mathcal{L}(\mathcal{V}) \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map because the determinant, when expressed in terms of a matrix, is a polynomial. In particular, it is continuous. Now note that

$$GL(\mathcal{V}) = \det^{-1}\big(\mathbb{R} - \{0\}\big)$$

and that $\mathbb{R} - \{0\}$ is open. It follows that $GL(\mathcal{V})$ is an open submanifold of $\mathcal{L}(\mathcal{V})$. Thus, $GL(\mathcal{V})$ is an n^2-dimensional manifold. Choosing a basis B for $\mathcal{V}$ and representing operators (points) $\mathbf{A}$ of $GL(\mathcal{V})$ as matrices (a_{ij}) in that basis provides a coordinate patch for $GL(\mathcal{V})$. We denote this coordinate patch by $\{x^{ij}\}$, where $x^{ij}(\mathbf{A}) = a_{ij}$.

To show that $GL(\mathcal{V})$ is a Lie group, we need to prove that if $\mathbf{A}, \mathbf{B} \in GL(\mathcal{V})$, then

$$\mathbf{AB} : GL(\mathcal{V}) \times GL(\mathcal{V}) \to GL(\mathcal{V}) \quad \text{and} \quad \mathbf{A}^{-1} : GL(\mathcal{V}) \to GL(\mathcal{V})$$

are $\mathcal{C}^\infty$ maps of manifolds. This is done by showing that the coordinate representations of these maps are $\mathcal{C}^\infty$. These representations are simply the matrix representations of operators. Since AB is a linear function of elements of the two matrices, it has derivatives of all orders. It follows that $\mathbf{AB}$ is $\mathcal{C}^\infty$. The case of A^{-1} is only slightly more complicated. We note that

$$\mathsf{A}^{-1} = \frac{P(a_{ij})}{\det \mathsf{A}}, \qquad P(a_{ij}) = \text{a polynomial in } a_{ij}.$$

Thus, since $\det \mathsf{A}$ is also a polynomial in a_{ij}, the kth derivative of A^{-1} is of the form $Q(a_{ij})/(\det \mathsf{A})^k$, where Q is another polynomial. The fact that $\det \mathsf{A} \neq 0$ establishes the $\mathcal{C}^\infty$ property of $\mathbf{A}^{-1}$.

One can similarly show that if $\mathcal{V}$ is a complex vector space, then $GL(\mathcal{V})$ is a manifold of dimension $2n^2$.

$SL(\mathcal{V})$ is a Lie group

Example 29.1.3 ($SL(\mathcal{V})$ is a Lie group) Recall that $SL(\mathcal{V})$ is the subgroup of $GL(\mathcal{V})$ whose elements have unit determinant. Since $\det : GL(\mathcal{V}) \to \mathbb{R}$ is $\mathcal{C}^\infty$, Theorem 28.3.8 and the example after it show that $SL(\mathcal{V}) = \det^{-1}(1)$ is a submanifold of $GL(\mathcal{V})$ of dimension $\dim GL(\mathcal{V}) - \dim \mathbb{R} = n^2 - 1$. Since it is already a subgroup, we conclude that $SL(\mathcal{V})$ is also a Lie group (Problem 29.5). Similarly, when $\mathcal{V}$ is a complex vector space, one can show that $\dim SL(\mathcal{V}) = 2n^2 - 2$.

Example 29.1.4 (Other examples of Lie groups) The reader may verify the following:

(a) Any finite-dimensional vector space is a Lie group under vector addition.
(b) The unit circle S^1, as a subset of nonzero multiplicative complex numbers is a Lie group under multiplication.
(c) The product $G \times H$ of two Lie groups is itself a Lie group with the product manifold structure and the direct product group structure.
(d) $GL(n, \mathbb{R})$, the set of invertible $n \times n$ matrices, is a Lie group under matrix multiplication.

(e) Let $G = GL(n, \mathbb{R}) \times \mathbb{R}^n$ be the product *manifold*. Define the group operation by $(\mathsf{A}, \mathbf{u})(\mathsf{B}, \mathbf{v}) \equiv (\mathsf{AB}, \mathsf{A}\mathbf{v} + \mathbf{u})$. The reader may verify that this operation indeed defines a group structure on G. In fact, G becomes a Lie group, called the **group of affine motions of** $\mathbb{R}^n$, for if we identify $(\mathsf{A}, \mathbf{u})$ with the affine motion[1] $\mathbf{x} \mapsto \mathsf{A}\mathbf{x} + \mathbf{u}$ of $\mathbb{R}^n$, then the group operation in G is composition of affine motions. We shall study in some detail the Poincaré group, a subgroup of the group of affine motions, in which the matrices are (pseudo) orthogonal.

group of affine motions of $\mathbb{R}^n$

In calculations, one translates all group operations to the corresponding operations of charts. This is particularly useful when the group multiplication can be defined only locally. One then speaks of an r-parameter **local Lie group**. To be precise, one considers a neighborhood U of the origin of $\mathbb{R}^r$ and defines an *associative* "multiplication" $m : U \times U \to \mathbb{R}^r$ and an inversion $i : U_0 \to U$ where U_0 is a subset of U. We therefore write the multiplication as

local Lie groups

$$m(\mathbf{a}, \mathbf{b}) = \mathbf{c}, \quad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^r,$$

where $\mathbf{a} = (a^1, a^2, \dots, a^r)$, etc. are coordinates of elements of G. The coordinates of the identity element of G are taken to be all zero. Thus, $m(\mathbf{a}, \mathbf{0}) = \mathbf{a}$ and $m(\mathbf{a}, i(\mathbf{a})) = \mathbf{0}$. In component forms,

$$c^k = m^k(\mathbf{a}, \mathbf{b}), \qquad m^k\big(\mathbf{a}, i(\mathbf{a})\big) = 0, \quad k = 1, 2, \dots, r. \tag{29.1}$$

The fact that G is a manifold implies that all functions in Eq. (29.1) are infinitely differentiable.

Example 29.1.5 As an example of a local 1-parameter Lie group, consider the multiplication rule $m : U \times U \to \mathbb{R}$ where $U = \{x \in \mathbb{R} \mid |x| < 1\}$ and

$$m(x, y) = \frac{2xy - x - y}{xy - 1}, \quad x, y \in U.$$

The reader can check that $m(x, (y, z)) = m((x, y), z)$, so that the multiplication is associative. Moreover, $m(0, x) = m(x, 0) = x$ for all $x \in U$, and $i(x) = x/(2x - 1)$, defined for $U_0 = \{x \in \mathbb{R} \mid |x| < \frac{1}{2}\}$.

29.1.1 Group Action

As mentioned in our discussion of finite groups, the action of a group on a set is more easily conceived than abstract groups. In the case of Lie groups, the natural action is not on an arbitrary set, but on a manifold.

Definition 29.1.6 Let M be a manifold. A **local group of transformations** acting on M is a (local) Lie group G, an (open) subset U with the property $\{e\} \times M \subset U \subset G \times M$, and a map $\Psi : U \to M$ satisfying the following conditions:

local group of transformations

[1] These consist of a linear transformation followed by a translation.

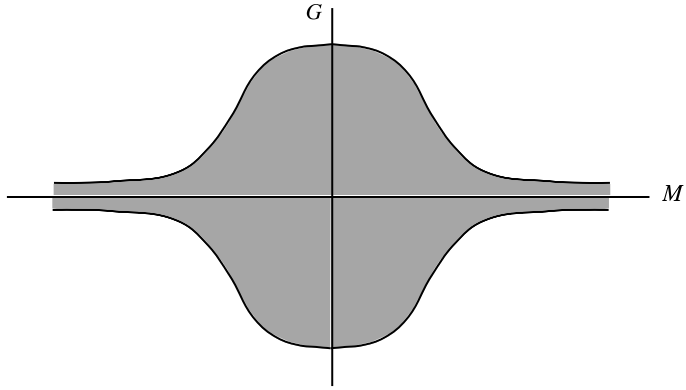

Fig. 29.1 For small regions of M, we may be able to include a large portion of G. However, if we want to include all of M, as we should, then only a small neighborhood of the identity may be available

1. If $(g, P) \in U$, $(h, \Psi(g, P)) \in U$, and $(hg, P) \in U$, then
$$\Psi\big(h, \Psi(g, P)\big) = \Psi(hg, P).$$
2. $\Psi(e, P) = P$ for all $P \in M$.
3. If $(g, P) \in U$, then $(g^{-1}, \Psi(g, P)) \in U$ and $\Psi(g^{-1}, \Psi(g, P)) = P$.

Normally, we shall denote $\Psi(g, P)$ by $g \cdot P$, or gP. Then the conditions of the definition above take the simple form

$$\begin{aligned} g \cdot (h \cdot P) &= (gh) \cdot P, \quad g, h \in G,\ P \in M, \\ e \cdot P &= P \quad \text{for all } P \in M, \\ g^{-1} \cdot (g \cdot P) &= P, \quad g \in G,\ P \in M, \end{aligned} \tag{29.2}$$

whenever $g \cdot P$ is defined. Note that the word "local" refers to G and not M, i.e., we may have to choose a very small neighborhood of the identity before all the elements of that neighborhood can act on *all* points of M (see Fig. 29.1).

orbit; stabilizer; transitive, effective, and free action

All the properties of a group action described in Chap. 23 can be applied here as well. So, one talks about the **orbit** of G as the collection of points in M obtained from one another by the action of G; the **stabilizer** G_x of a point $x \in M$ as the collection of all group elements leaving x fixed; **transitive** action of G on M when there is only one orbit; **free** action of G on M when $G_x = \{e\}$ for all $x \in M$; and **effective** action of G on M when $g \cdot x = x$ for all $x \in M$ implies that $g = e$. The only extra condition one has to be aware of is that the group action is not defined for all elements of G, and that a sufficiently small neighborhood of the identity needs to be chosen. Since "belonging to the same orbit" is an equivalence relation on M, the set of orbits of M is denoted by M/G.

M/G is the set of orbits in M

An important consequence of the free action of a group is the following

Theorem 29.1.7 *If G acts freely on M, then G is diffeomorphic to Gx for any $x \in M$.*

Proof We assume a left action. The proof for the right action is identical to this proof. Consider the map $\phi : Gx \to G$ given by $\phi(y) = g$ for $y = gx$. For this map to make sense, g must be determined uniquely from y. If $g_2x = y = g_1x$, then $x = g_2^{-1}g_1x$, and *because the action is free*, we conclude that $g_2^{-1}g_1 = e$ and $g_1 = g_2$, so that indeed g is determined uniquely by y. Now, we have to show that ϕ is a bijection:

Surjectivity: If $g \in G$, then clearly $gx \in Gx$ and $\phi(gx) = g$.
Injectivity: If $\phi(y_1) = \phi(y_2)$, with $y_1 = g_1x$ and $y_2 = g_2x$, then $g_1 = g_2$. □

In the old literature, the group action is described in terms of coordinates. Although for calculations this is desirable, it can be very clumsy for formal discussions, as we shall see later. Let $\mathbf{a} = (a^1, \dots, a^r)$ be a coordinate system on G and $\mathbf{x} = (x^1, \dots, x^n)$ a coordinate system on M. Then the group action $\Psi : G \times M \to M$ becomes a set of n functions described by

$$\mathbf{x}' = \Psi(\mathbf{a}, \mathbf{x}), \qquad \mathbf{x}'' = \Psi\big(\mathbf{b}, \mathbf{x}'\big) = \Psi\big(m(\mathbf{b}, \mathbf{a}), \mathbf{x}\big), \tag{29.3}$$

where m is the multiplication law of the Lie group written in terms of coordinates as given in Eq. (29.1). It is assumed that Ψ is infinitely differentiable.

Box 29.1.8 *Equation (29.3) can be used to unravel the multiplication law for the Lie group when the latter is given in terms of transformations.*

Example 29.1.9 (Examples of groups of transformation)

(a) The two-dimensional rotation group acts on the xy-plane as

$$\Phi(\theta, \mathbf{r}) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

If we write $\mathbf{r}' = \Phi(\theta_1, \mathbf{r})$ and $\mathbf{r}'' = \Phi(\theta_2, \mathbf{r}')$, then a simple calculation shows that

$$\mathbf{r}'' = \big(x\cos(\theta_1 + \theta_2) - y\sin(\theta_1 + \theta_2), x\sin(\theta_1 + \theta_2) + y\cos(\theta_1 + \theta_2)\big).$$

With $\mathbf{r}'' = (m(\theta_1, \theta_2); \mathbf{r})$, we recognize the "multiplication" law as $m(\theta_1, \theta_2) = \theta_1 + \theta_2$. The orbits are circles centered at the origin.

(b) Let $M = \mathbb{R}^n$, $\mathbf{a}$ a fixed vector in $\mathbb{R}^n$, and $G = \mathbb{R}$. Define $\Psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Psi(t, \mathbf{x}) = \mathbf{x} + t\mathbf{a}, \quad \mathbf{x} \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

This group action is globally defined. The orbits are straight lines parallel to $\mathbf{a}$. The group is the set of **translations** in the direction $\mathbf{a}$ in $\mathbb{R}^n$. translations
The reader may verify that the "multiplication" law is addition of t's.

(c) Let $G = \mathbb{R}^+$ be the multiplicative group of nonzero positive real numbers. Fix real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, not all zero. Define the action of G on $\mathbb{R}^n$ by

$$\Psi(\lambda, \mathbf{x}) \equiv \lambda \cdot \mathbf{x} = \big(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n\big),$$
$$\lambda \in \mathbb{R}^+, \ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

The orbits are obtained by choosing a point in $\mathbb{R}^n$ and applying G to it for different λ's. The result is a curve in $\mathbb{R}^n$. For example, if $n = 2$, $\alpha_1 = 1$, and $\alpha_2 = 2$, we get, as the orbit containing $\mathbf{x}_0$ the curve

$$\lambda \cdot \mathbf{x}_0 = \big(\lambda x_0, \lambda^2 y_0\big) \quad \Rightarrow \quad y = \frac{y_0}{x_0^2} x^2,$$

which is a parabola going through the origin and the point (x_0, y_0). Note that the orbit containing the origin has only one point. This group is called the group of **scale transformations**. The multiplication law is ordinary multiplication of (positive) real numbers.

scale transformations

(d) Let $G = \mathbb{R}^4$ act on $M = \mathbb{R}$ by

one-dimensional projective group

$$\Phi(\mathbf{a}, x) \equiv \frac{a_1 x + a_2}{a_3 x + a_4}, \quad \mathbf{a} = (a_1, a_2, a_3, a_4), \ a_1 a_4 - a_2 a_3 \neq 0.$$

The reader may verify that this is indeed the action of a group (catch where the condition $a_1 a_4 - a_2 a_3 \neq 0$ is used!), and if $x' = \Phi(\mathbf{b}, x)$ and $x'' = \Phi(\mathbf{a}, x')$, then

$$x'' = \frac{(a_1 b_1 + a_2 b_3)x + a_1 b_2 + a_2 b_4}{(a_3 b_1 + a_4 b_3)x + a_3 b_2 + a_4 b_4},$$

so that the multiplication rule is

$$m(\mathbf{a}, \mathbf{b}) = (a_1 b_1 + a_2 b_3, a_1 b_2 + a_2 b_4, a_3 b_1 + a_4 b_3, a_3 b_2 + a_4 b_4).$$

This group is called the **one-dimensional projective group**.

29.1.2 Lie Algebra of a Lie Group

The group property of a Lie group G provides a natural diffeomorphism on G that determines a substantial part of its structure.

left translation, left-invariant vector fields, left-invariant forms, and their "right" counterparts

Definition 29.1.10 Let G be a Lie group and $g \in G$. The **left translation by** g is a diffeomorphism $L_g : G \to G$ defined by

$$L_g(h) = gh \quad \forall h \in G.$$

A vector field $\boldsymbol{\xi}$ on G is called **left-invariant** if for each $g \in G$, $\boldsymbol{\xi}$ is L_g-related to itself; i.e.,[2]

$$L_{g*} \circ \boldsymbol{\xi} = \boldsymbol{\xi} \circ L_g, \quad \text{or} \quad L_{g*}\big(\boldsymbol{\xi}(h)\big) = \boldsymbol{\xi}(gh) \quad \forall g, h \in G.$$

[2]When there is no danger of confusion, we shall use $\boldsymbol{\xi}(h)$ for $\boldsymbol{\xi}|_h$.

The set of left-invariant vector fields on G is denoted by $\mathfrak{g}$. A 1-form whose pairing with a left-invariant vector field gives a constant function on G is called a **left-invariant** 1-form.

The **right translation** by g, $R_g : G \to G$, and **right-invariant** vector fields and 1-forms are defined similarly.

The reader may easily check that right and left translations commute:

$$R_g \circ L_h = L_h \circ R_g \quad \forall g, h \in G. \tag{29.4}$$

It is convenient to have a coordinate representation of L_{g*}. The coordinate representation of L_g is simply the multiplication law $L_g(h) = m(\mathbf{g}, \mathbf{h})$, where we have used the same symbol for coordinates as for group elements. Equation (28.11) can now be used to write the coordinate representation of L_{g*}:

$$L_{g*} \to \begin{pmatrix} \partial m^1/\partial h^1 & \partial m^1/\partial h^2 & \dots & \partial m^1/\partial h^r \\ \partial m^2/\partial h^1 & \partial m^2/\partial h^2 & \dots & \partial m^2/\partial h^r \\ \vdots & \vdots & & \vdots \\ \partial m^r/\partial h^1 & \partial m^r/\partial h^2 & \dots & \partial m^r/\partial h^r \end{pmatrix}, \tag{29.5}$$

where all the derivatives in the matrix are evaluated at $(\mathbf{g}, \mathbf{h})$.

We have already mentioned in Chap. 28 that for a general manifold M, $\mathcal{X}(M)$ is an infinite-dimensional Lie algebra under the Lie bracket "multiplication". In general, $\mathcal{X}(M)$ has no finite-dimensional **subalgebra**. However, Lie groups are an exception:

Proposition 29.1.11 *Let G be a Lie group and $\mathfrak{g}$ the set of its left-invariant vector fields. Then $\mathfrak{g}$ is a real vector space, and the map $\phi : \mathfrak{g} \to \mathcal{T}_e(G)$, defined by $\phi(\boldsymbol{\xi}) = \boldsymbol{\xi}(e)$, is a linear isomorphism. Therefore,* $\dim \mathfrak{g} = \dim \mathcal{T}_e(G) = \dim G$. *Furthermore, $\mathfrak{g}$ is closed under Lie brackets; i.e., $\mathfrak{g}$ is a Lie algebra.*

Proof It is clear that $\mathfrak{g}$ is a real vector space. If $\phi(\boldsymbol{\xi}) = \phi(\boldsymbol{\eta})$ for $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{g}$, then

$$\boldsymbol{\xi}(g) = L_{g*}\big(\boldsymbol{\xi}(e)\big) = L_{g*}\big(\boldsymbol{\eta}(e)\big) = \boldsymbol{\eta}(g) \quad \forall g \in G \Rightarrow \boldsymbol{\xi} = \boldsymbol{\eta}.$$

This shows that ϕ is injective. To show that ϕ is surjective, suppose that $\mathbf{v} \in \mathcal{T}_e(G)$ and define the vector field $\boldsymbol{\xi}$ on G by $\boldsymbol{\xi}(g) = L_{g*}(\mathbf{v})$ for all $g \in G$. Then $\phi(\boldsymbol{\xi}) = \mathbf{v}$ and $\boldsymbol{\xi} \in \mathfrak{g}$, because

$$L_{g*} \circ \boldsymbol{\xi}(h) = L_{g*} \circ L_{h*}(\mathbf{v}) = L_{gh*}(\mathbf{v}) = \boldsymbol{\xi}(gh) \equiv \boldsymbol{\xi}(L_g h) = \boldsymbol{\xi} \circ L_g(h).$$

This proves the first part of the proposition. The second part follows immediately from the definition of a left-invariant vector field and Theorem 28.4.4. □

The flow of $\boldsymbol{\xi}$ at $g \in G$ can be shown to be

$$F_t = g \exp(t\boldsymbol{\xi}) = R_{\exp(t\boldsymbol{\xi})} g. \tag{29.6}$$

Indeed, let $\mathbf{X}_\xi$ be the vector field associated with this flow. The action of this vector field on a function f is

$$\mathbf{X}_\xi|_g(f) = \frac{d}{dt}\big(f(g\exp(t\boldsymbol{\xi}))\big)\Big|_{t=0}.$$

Therefore,

$$\begin{aligned}(L_{h*}\mathbf{X}_\xi|_g)(f) &= \mathbf{X}_\xi|_g(f \circ L_h) = \frac{d}{dt}\big(f \circ L_h(g\exp(t\boldsymbol{\xi}))\big)\Big|_{t=0} \\ &= \frac{d}{dt}\big(f(hg\exp(t\boldsymbol{\xi}))\big)\Big|_{t=0} = \mathbf{X}_\xi(hg)(f) = \big[\mathbf{X}_\xi \circ L_h(g)\big](f).\end{aligned}$$

Since this is true for all f and g, and $\mathbf{X}_\xi|_e = \boldsymbol{\xi}(e)$, we conclude that $\mathbf{X}_\xi$ is the unique left-invariant vector field corresponding to $\boldsymbol{\xi}(e)$.

The Lie algebra of a Lie group

Definition 29.1.12 The **Lie algebra** of the Lie group G is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. Sometimes we think of $\boldsymbol{\xi}$ as a vector in $\mathcal{T}_e(G)$. In that case, we denote by $\mathbf{X}_\xi$ the left-invariant vector field whose value at the identity is $\boldsymbol{\xi}$.

The isomorphism of $\mathfrak{g}$ with $\mathcal{T}_e(G)$ induces a Lie bracket on $\mathcal{T}_e(G)$ and turns it into a Lie algebra. In many cases of physical interest, it is this interpretation of the Lie algebra of G that is most useful.

If two groups stand in some algebraic relation to one another, their Lie algebras will inherit such relations. More precisely, let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose $\phi : G \to H$ is a Lie group homomorphism. Then identifying $\mathfrak{g}$ with $\mathcal{T}_e(G)$ and $\mathfrak{h}$ with $\mathcal{T}_e(H)$, and using Theorem 28.4.4, we conclude that $\phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, i.e., it preserves the Lie brackets:

$$\phi_*[\boldsymbol{\xi}, \boldsymbol{\eta}] = [\phi_*\boldsymbol{\xi}, \phi_*\boldsymbol{\eta}] \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{g}. \tag{29.7}$$

Theorem 29.1.13 *If $\phi : G \to H$ is a Lie group homomorphism, then $\phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

In particular, if ϕ is a Lie group isomorphism, then ϕ_* is a Lie algebra isomorphism.

Example 29.1.14 Let $\mathcal{V}$ be a complex vector space with its general linear group $GL(\mathcal{V})$, a $2n^2$-dimensional Lie group. Recall that $GL(\mathcal{V})$ is an open submanifold of $\mathcal{L}(\mathcal{V})$. By Eq. (28.4), $\mathcal{T}_e(GL(\mathcal{V}))\mathcal{T}_e(\mathcal{L}(\mathcal{V}))$, where e is the unit operator. Now note that on the one hand, we can identify $\mathcal{T}_e(\mathcal{L}(\mathcal{V}))$ with $\mathcal{L}(\mathcal{V})$ [see the box after Eq. (28.4)]. On the other hand, $\mathcal{T}_e(GL(\mathcal{V}))$ can be identified with $\mathfrak{gl}(\mathcal{V})$, the Lie algebra of $GL(\mathcal{V})$. Therefore, $\mathfrak{gl}(\mathcal{V}) \cong \mathcal{L}(\mathcal{V})$. We use the notation $\mathsf{A}(t)$ for a curve in $GL(\mathcal{V})$ and $\dot{\mathsf{A}}$ for the vector tangent to the curve.

It is instructive to construct the coordinate representation of vector fields on $GL(\mathcal{V})$. Let $f : GL(\mathcal{V}) \to \mathbb{R}$ be a function and $\dot{\mathbf{A}}$ a vector field. Then, we have

$$\dot{\mathbf{A}}(f) = \frac{d}{dt}\big(f\big(\mathbf{A}(t)\big)\big) = \frac{da_{ij}}{dt}\frac{\partial f}{\partial x^{ij}},$$

or, since f is arbitrary,

$$\dot{\mathbf{A}} = \frac{da_{ij}}{dt}\frac{\partial}{\partial x^{ij}} \equiv \dot{a}_{ij}\frac{\partial}{\partial x^{ij}} \equiv \frac{d\mathbf{A}}{dt}(t),$$

where summation over repeated indices is understood and we introduced $d\mathbf{A}/dt$ *as an abbreviation* for $\dot{a}_{ij}(\partial/\partial x^{ij})$. However, the one-to-one correspondence between matrices and operators makes this more than just an abbreviation. Indeed, we can interpret $d\mathbf{A}/dt$ as the derivative of $\mathbf{A}$ and perform such differentiation whenever it is possible. The equation above states that

Box 29.1.15 *To obtain the matrix elements* (*coordinates*) *of the operator* $\dot{\mathbf{A}}$, *one differentiates the t-dependent elements of the* (*matrix representation of the*) *operator* $\mathbf{A}(t)$.

Of particular interest are the left invariant vector fields, or equivalently, the vectors belonging to $\mathcal{T}_e(GL(\mathcal{V}))$. This amounts to substituting $t = 0$ in the formulas above. Thus, if $\dot{\mathbf{A}} \in \mathcal{T}_e(GL(\mathcal{V}))$,

$$\dot{\mathbf{A}} = \dot{a}_{ij}(0)\frac{\partial}{\partial x^{ij}} \equiv \frac{d\mathbf{A}}{dt}(0). \tag{29.8}$$

For the product of two operators, we get

$$\begin{aligned}\widehat{\mathbf{AB}} &= \frac{d}{dt} f\big(\mathbf{A}(t)\mathbf{B}(t)\big)\bigg|_{t=0} = \frac{d}{dt}(a_{ik}b_{kj})\bigg|_{t=0}\frac{\partial}{\partial x^{ij}} \\ &= \big(\dot{a}_{ik}(0)\underbrace{b_{kj}(0)}_{\delta_{kj}} + \underbrace{a_{ik}(0)}_{\delta_{ik}}\dot{b}_{kj}(0)\big)\frac{\partial}{\partial x^{ij}} \\ &= \dot{a}_{ij}(0)\frac{\partial}{\partial x^{ij}} + \dot{b}_{ij}(0)\frac{\partial}{\partial x^{ij}} = \frac{d\mathbf{A}}{dt}(0) + \frac{d\mathbf{B}}{dt}(0).\end{aligned} \tag{29.9}$$

Many of the Lie groups used in physics are subgroups of $GL(\mathcal{V})$. A characterization of the Lie algebras of these subgroups is essential for understanding the subgroups themselves and applying them to physical situations. These subgroups are typically defined in terms of maps $\phi : GL(\mathcal{V}) \to M$ for which M is a manifold and ϕ_* is surjective. To construct the Lie algebra of subgroups of $GL(\mathcal{V})$, we need to concentrate on the map ϕ_{*e} as defined on $\mathcal{T}_e(GL(\mathcal{V}))$.

An important map is $\det : GL(\mathcal{V}) \to \mathbb{C}$ for a complex vector space $\mathcal{V}$. We are interested in evaluating the map $\det_* : \mathcal{T}_e(GL(\mathcal{V})) \to \mathcal{T}_1(\mathbb{C})$ in which we consider $\mathbb{C} \cong \mathbb{R}^2$ to be a manifold. For an operator $\dot{\mathbf{A}} \in \mathcal{T}_e(GL(\mathcal{V})) \cong \mathfrak{gl}(\mathcal{V})$

Differential of the determinant map is the trace: $\det_* = \mathrm{tr}$

and a complex-valued function, we have

$$\begin{aligned}\det_*(\dot{\mathbf{A}})f &\equiv \frac{d}{dt} f\big(\det \mathbf{A}(t)\big)\bigg|_{t=0} \equiv \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} \\ &= \frac{d}{dt}\operatorname{Re}\det \mathbf{A}(t)\bigg|_{t=0}\frac{\partial f}{\partial x} + \frac{d}{dt}\operatorname{Im}\det \mathbf{A}(t)\bigg|_{t=0}\frac{\partial f}{\partial y} \\ &= \operatorname{Re}\operatorname{tr}\dot{\mathbf{A}}\frac{\partial f}{\partial x} + \operatorname{Im}\operatorname{tr}\dot{\mathbf{A}}\frac{\partial f}{\partial y},\end{aligned}$$

where we used Eq. (5.34). Since f is arbitrary and $\{\partial/\partial x, \partial/\partial y\}$ can be identified with $\{1, i\}$, we have

$$\det_*(\dot{\mathbf{A}}) = \operatorname{tr}\dot{\mathbf{A}}. \tag{29.10}$$

Example 29.1.16 (Lie algebra of $SL(\mathcal{V})$) The special linear group $SL(\mathcal{V})$ is characterized by the fact that all its elements have unit determinant.

Box 29.1.17 *The Lie algebra $\mathfrak{sl}(\mathcal{V})$ of the special linear group is the set of all traceless operators.*

This is because if we use (29.10) and (28.12) and the fact that $SL(\mathcal{V}) = \det^{-1}(1)$, we can conclude that $\det_*(\dot{\mathbf{A}}) = \operatorname{tr}\dot{\mathbf{A}} = 0$ for all $\dot{\mathbf{A}} \in \mathfrak{sl}(\mathcal{V})$.

Example 29.1.18 (Lie algebras of unitary and related groups) Let us first show that the set of unitary operators on $\mathcal{V}$, denoted by $U(\mathcal{V})$, is a Lie subgroup of $GL(\mathcal{V})$, called the **unitary group** of $\mathcal{V}$. Consider the map $\psi : GL(\mathcal{V}) \to \mathbb{H}$, where $\mathbb{H}$ is the set of hermitian operators considered as a vector space (therefore, a manifold) *over the reals*, defined by $\psi(\mathbf{A}) = \mathbf{A}\mathbf{A}^\dagger$. Using Eq. (29.9), the reader may verify that ψ_* is surjective and

unitary group

$$\psi_*(\dot{\mathbf{A}}) = \dot{\mathbf{A}} + \dot{\mathbf{A}}^\dagger. \tag{29.11}$$

It follows from Theorem 28.3.8 that $U(\mathcal{V}) \equiv \psi^{-1}(\mathbf{1})$ is a subgroup of $GL(\mathcal{V})$. Using Eq. (28.12), we conclude that $\psi_*(\dot{\mathbf{A}}) = \dot{\mathbf{A}} + \dot{\mathbf{A}}^\dagger = 0$ for all $\dot{\mathbf{A}} \in \mathfrak{u}(\mathcal{V})$, i.e.,

Box 29.1.19 *The Lie algebra $\mathfrak{u}(\mathcal{V})$ of the unitary group is the set of all anti-hermitian operators.*

When the vector space is $\mathbb{C}^n$, we write $U(n)$ instead of $U(\mathbb{C}^n)$. By counting the number of independent *real* parameters of a matrix representing a hermitian operator, we can conclude that $\dim \mathbb{H} = n^2$. It follows from Theorem 28.3.8 that $\dim U(\mathcal{V}) = n^2$.

The intersection of $SL(\mathcal{V})$ and $U(\mathcal{V})$, denoted by $SU(\mathcal{V})$, is called the **special unitary group**. When the vector space is $\mathbb{C}^n$, we write $SU(n)$ instead of $SU(\mathbb{C}^n)$. The previous two results yields

special unitary group

Box 29.1.20 *The Lie algebra* $\mathfrak{su}(\mathcal{V})$ *consists of anti-hermitian traceless operators. If* $\dim \mathcal{V} = n$, *then* $\dim \mathfrak{su}(\mathcal{V}) = n^2 - 1$. *We write* $\mathfrak{su}(n)$ *for* $\mathfrak{su}(\mathcal{V})$ *if* $\mathcal{V} = \mathbb{C}^n$.

The reader is asked to check that $\dim \mathfrak{su}(\mathcal{V}) = n^2 - 1$.

orthogonal and special orthogonal groups

If we restrict ourselves to real vector spaces, then unitary and special unitary groups become the **orthogonal group** $O(\mathcal{V})$ and **special orthogonal group** $SO(\mathcal{V})$, respectively. Their algebras consist of antisymmetric and traceless antisymmetric operators, respectively. When $\mathcal{V} = \mathbb{R}^n$, we use the notation $O(n)$ and $SO(n)$.

Let $\mathbf{X}$ be a vector field on G. We know from our discussion of flows that $\mathbf{X}$ has a flow $F_t \equiv \exp(t\mathbf{X})$ at every point g of G with $-\epsilon < t < \epsilon$. Now, since $F_t(g) \neq g$ is in G, it follows from the group property of G that $(F_t)^n(g) = F_{nt}(g) \in G$ for all n. This shows that the flow of every vector field on a Lie group is defined for all $t \in \mathbb{R}$, i.e., all vector fields on a Lie group are complete. Now consider $\mathfrak{g}$ as a vector space *and manifold* and define a map $\exp : \mathfrak{g} \to G$ that is simply the flow evaluated at $t = 1$. It can be shown that the following result holds ([Warn 83, pp. 103–104]):

exponential map of a Lie algebra

Theorem 29.1.21 $\exp : \mathfrak{g} \to G$, *called the* ***exponential map***, *is a diffeomorphism of a neighborhood of the origin of* $\mathfrak{g}$ *with a neighborhood of the identity element of* G.

This theorem states that in a neighborhood of the identity element, a Lie group, as a manifold, "looks like" its tangent space there. In particular,

Box 29.1.22 *Two Lie groups that have identical Lie algebras are locally diffeomorphic.*

Example 29.1.23 (Why exp is called the exponential map) Let $\mathcal{V}$ be a finite-dimensional vector space and $\mathbf{A} \in \mathfrak{gl}(\mathcal{V})$. Define, as in Chap. 4,

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!} = \mathbf{1} + t\mathbf{A} + \cdots$$

and note that

$$\frac{d}{dt} e^{t\mathbf{A}} = \mathbf{A} e^{t\mathbf{A}} \quad \Rightarrow \quad \left.\frac{d}{dt} e^{t\mathbf{A}}\right|_{t=0} = \mathbf{A}.$$

Furthermore,

$$e^{t\mathbf{A}} e^{s\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!} \sum_{n=0}^{\infty} \frac{s^n \mathbf{A}^n}{n!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^k s^n}{k! n!} \mathbf{A}^{k+n}$$

$$= \sum_{m=0}^{\infty} \underbrace{\left(\sum_{n=0}^{m} \frac{t^{m-n} s^n}{(m-n)!n!} \right)}_{=(t+s)^m/m!} \mathbf{A}^m = e^{(t+s)\mathbf{A}}.$$

It follows that $e^{t\mathbf{A}}$ has all the properties expected of the flow of the vector field $\mathbf{A}$.

The exponential map has some important properties that we shall have occasion to use later. The first of these properties is the content of the following proposition, whose proof is left as an exercise for the reader.

Proposition 29.1.24 *Let $\phi : H \to G$ be a Lie group homomorphism. Then, for all $\boldsymbol{\eta} \in \mathfrak{h}$, we have $\phi(\exp_H \boldsymbol{\eta}) = \exp_G(\phi_* \boldsymbol{\eta})$.*

inner automorphism of a Lie group

For every $g \in G$, let $I_g \equiv R_g^{-1} \circ L_g$. The reader may readily verify that I_g, which takes $x \in G$ to $gxg^{-1} \in G$, is an isomorphism of G, i.e., $I_g(xy) = I_g(x)I_g(y)$ and I_g is bijective. It is called the **inner automorphism** associated with g.

adjoint map of a Lie algebra

Definition 29.1.25 The Lie algebra isomorphism $I_{g*} = R_{g*}^{-1} \circ L_{g*} : \mathfrak{g} \to \mathfrak{g}$ is denoted by Ad_g and is called the **adjoint map** associated with g.

Since $\mathfrak{g}$ is a vector space, the adjoint map can be used to construct a representation of G.

Definition 29.1.26 The **adjoint representation** of a Lie group G is $Ad : G \to GL(\mathfrak{g})$ given by $Ad(g) = Ad_g \equiv I_{g*}$.

Using Proposition 29.1.24, we have the following corollary.

Corollary 29.1.27 $\exp(Ad_g \boldsymbol{\xi}) = I_g \exp \boldsymbol{\xi} = g \exp \boldsymbol{\xi} g^{-1}$ *for all $\boldsymbol{\xi} \in \mathfrak{g}$ and $g \in G$.*

Let $\{\boldsymbol{\xi}_i\}$ be a basis for the (finite-dimensional) Lie algebra of the Lie group G. The Lie bracket of two basis vectors, being itself a left-invariant vector field, can be written as a linear combination of $\{\boldsymbol{\xi}_i\}$:

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \sum_{k=1}^{n} c_{ij}^k \boldsymbol{\xi}_k.$$

On a general manifold, c_{ij}^k will depend on the point at which the fields are being evaluated. However, on Lie groups, they are independent of the point, as the following manipulation shows:

$$\begin{aligned}\big[\boldsymbol{\xi}_i(g), \boldsymbol{\xi}_j(g)\big] &= \big[L_{g*}\boldsymbol{\xi}_i(e), L_{g*}\boldsymbol{\xi}_j(e)\big] = L_{g*}\big[\boldsymbol{\xi}_i(e), \boldsymbol{\xi}_j(e)\big] \\ &= L_{g*} \sum_{k=1}^{n} c_{ij}^k(e) \boldsymbol{\xi}_k(e) = \sum_{k=1}^{n} c_{ij}^k(e) L_{g*}\boldsymbol{\xi}_k(e)\end{aligned}$$

$$= \sum_{k=1}^{n} c_{ij}^{k}(e)\boldsymbol{\xi}_k(g).$$

Therefore, the value of c_{ij}^k at any point $g \in G$ is the same as its value at the identity, i.e., c_{ij}^k is a constant. This statement is called **Lie's second theorem**.

Lie's second theorem

Definition 29.1.28 Let $\{\boldsymbol{\xi}_i\}_{i=1}^n$ be a basis for the Lie algebra $\mathfrak{g}$ of the Lie group G. Then

structure constants of a Lie algebra

$$\bigl[\boldsymbol{\xi}_i(g), \boldsymbol{\xi}_j(g)\bigr] = \sum_{k=1}^{n} c_{ij}^{k}\boldsymbol{\xi}_k(g), \tag{29.12}$$

where c_{ij}^k, which are independent of g, are called the **structure constants** of G.

The structure constants satisfy certain relations that are immediate consequences of the commutation relations. The antisymmetry of the Lie bracket and the Jacobi identity lead directly to

Lie's third theorem

$$\begin{aligned} c_{\rho\sigma}^{\kappa} &= -c_{\sigma\rho}^{\kappa}, \\ c_{\rho\sigma}^{\kappa}c_{\kappa\mu}^{\nu} + c_{\sigma\mu}^{\kappa}c_{\kappa\rho}^{\nu} + c_{\mu\rho}^{\kappa}c_{\kappa\sigma}^{\nu} &= 0. \end{aligned} \tag{29.13}$$

The fact that $\{c_{\sigma\rho}^{\kappa}\}$ obey Eq. (29.13) is the content of **Lie's third theorem**.

29.1.3 Invariant Forms

If $\boldsymbol{\omega}|_e$ is a 1-form on $\mathcal{T}_e(G)$, then $\boldsymbol{\omega} \in \Lambda^1(G)$, given by $\boldsymbol{\omega}|_g \equiv L_{g^{-1}}^{*}\boldsymbol{\omega}|_e$, is a left-invariant 1-form:

$$\begin{aligned} \boldsymbol{\omega}|_g(\mathbf{X}|_g) &= L_{g^{-1}}^{*}\boldsymbol{\omega}|_e(\mathbf{X}|_g) = \boldsymbol{\omega}|_e(L_{g^{-1}*}\mathbf{X}|_g) \\ &= \boldsymbol{\omega}|_e(\mathbf{X}|_{g^{-1}g}) = \boldsymbol{\omega}|_e(\mathbf{X}|_e) \end{aligned} \tag{29.14}$$

independent of g. A differential form $\boldsymbol{\omega}$ on G is called left-invariant if $L_g^{*}(\boldsymbol{\omega}) = \boldsymbol{\omega}$ for every $g \in G$. If $\boldsymbol{\omega}$ is a left-invariant p-form and $\{\boldsymbol{\xi}_i\}_{i=1}^p$ a set of left-invariant vector fields, then, as in (29.14), the function $\boldsymbol{\omega}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_p)$ is constant on G. The exterior derivative of a left-invariant p-form satisfies

$$\begin{aligned} &d\boldsymbol{\omega}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{p+1}) \\ &\quad = \sum_{1 \le i < j \le p+1} (-1)^{i+j}\boldsymbol{\omega}\bigl([\boldsymbol{\xi}_i, \boldsymbol{\xi}_j], \boldsymbol{\xi}_1, \dots, \hat{\boldsymbol{\xi}}_i, \dots, \hat{\boldsymbol{\xi}}_j, \dots, \boldsymbol{\xi}_{p+1}\bigr) \end{aligned}$$

where we used Theorem 28.5.11 and the fact that vector fields give zero when acting on a constant function. For a left-invariant 1-form, this yields

$$d\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) = -\boldsymbol{\omega}\bigl([\boldsymbol{\xi}, \boldsymbol{\eta}]\bigr). \tag{29.15}$$

Definition 29.1.29 The **canonical** 1**-form** $\boldsymbol{\theta}$ on a Lie group G is the left-invariant $\mathfrak{g}$-valued 1-form uniquely determined by $\boldsymbol{\theta}(\boldsymbol{\xi}) = \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in \mathfrak{g}$.

Let $\{\boldsymbol{\xi}_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$. Then, $\boldsymbol{\theta} = \sum_{i=1}^n \boldsymbol{\theta}^i \boldsymbol{\xi}_i$, where $\boldsymbol{\theta}^i$ are real-valued 1-forms. Now the definition of $\boldsymbol{\theta}$ gives

$$\boldsymbol{\xi}_j = \boldsymbol{\theta}(\boldsymbol{\xi}_j) = \sum_{i=1}^n \boldsymbol{\theta}^i(\boldsymbol{\xi}_j)\boldsymbol{\xi}_i.$$

This implies that $\boldsymbol{\theta}^i(\boldsymbol{\xi}_j) = \delta^i_j$, i.e., that $\{\boldsymbol{\theta}^i\}_{i=1}^n$ is the dual basis of $\{\boldsymbol{\xi}_i\}_{i=1}^n$.

We now want to express the exterior derivative of $\boldsymbol{\theta}^i$ in the basis $\{\boldsymbol{\theta}^i\}_{i=1}^n$. Since it is a 2-form, $d\boldsymbol{\theta}^i = \alpha^i_{lm}\boldsymbol{\theta}^l \wedge \boldsymbol{\theta}^m$, for some $\alpha^i_{lm} \in \mathbb{R}$ to be determined. Since it is invariant, it must satisfy Eq. (29.15). So, we must have

$$\alpha^i_{lm}\boldsymbol{\theta}^l \wedge \boldsymbol{\theta}^m(\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) = -\boldsymbol{\theta}^i[\boldsymbol{\xi}_j, \boldsymbol{\xi}_k].$$

For the right-hand side, using Eq. (29.12), we have

$$\boldsymbol{\theta}^i[\boldsymbol{\xi}_j, \boldsymbol{\xi}_k] = c^l_{jk}\boldsymbol{\theta}^i(\boldsymbol{\xi}_l) = c^l_{jk}\delta^i_l = c^i_{jk}.$$

For the left-hand side, we obtain

$$\begin{aligned}\alpha^i_{lm}\boldsymbol{\theta}^l \wedge \boldsymbol{\theta}^m(\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) &= \alpha^i_{lm}\left[\boldsymbol{\theta}^l(\boldsymbol{\xi}_j)\boldsymbol{\theta}^m(\boldsymbol{\xi}_k) - \boldsymbol{\theta}^l(\boldsymbol{\xi}_k)\boldsymbol{\theta}^m(\boldsymbol{\xi}_l)\right] \\ &= \alpha^i_{lm}\left[\delta^l_j\delta^m_k - \delta^l_k\delta^m_j\right] = 2\alpha^i_{jk}\end{aligned}$$

because α^i_{jk} is antisymmetric in its lower indices. The last two equations imply that $\alpha^i_{jk} = \frac{1}{2}c^i_{jk}$. We thus arrive at the **Maurer-Cartan equation**:

Maurer-Cartan equation

$$d\boldsymbol{\theta}^i = -\frac{1}{2}c^i_{jk}\boldsymbol{\theta}^j \wedge \boldsymbol{\theta}^k. \tag{29.16}$$

Multiplying both sides by $\boldsymbol{\xi}_i$ and summing over i, the left-hand side becomes $d\boldsymbol{\theta}$. The right-hand side gives

$$-\frac{1}{2}c^i_{jk}\boldsymbol{\theta}^j \wedge \boldsymbol{\theta}^k\boldsymbol{\xi}_i = -\frac{1}{2}\boldsymbol{\theta}^j \wedge \boldsymbol{\theta}^k[\boldsymbol{\xi}_j, \boldsymbol{\xi}_k] \equiv -\frac{1}{2}[\boldsymbol{\theta}, \boldsymbol{\theta}]$$

where the last expression is defined by the middle expression. Thus the Maurer-Cartan equation can also be written as

$$d\boldsymbol{\theta} = -\frac{1}{2}[\boldsymbol{\theta}, \boldsymbol{\theta}]. \tag{29.17}$$

29.1.4 Infinitesimal Action

The action $\Phi : G \times M \to M$ of a Lie group on a manifold M induces a homomorphism of its algebra with $\mathfrak{X}(M)$. If $\boldsymbol{\xi} \in \mathfrak{g}$, then $\exp(t\boldsymbol{\xi}) \in G$ can act on M at a point P to produce a curve $\gamma(t) = \exp(t\boldsymbol{\xi}) \cdot P$ going through P. The tangent to this curve at P is defined to be the image of this homomorphism.

Definition 29.1.30 Let $\Phi : G \times M \to M$ be an action. If $\xi \in \mathfrak{g}$, then $\Phi(\exp t\xi, P)$ is a flow on M. The corresponding vector field on M given by

$$\xi_M|_P \equiv \xi_M(P) \equiv \frac{d}{dt}\Phi(\exp t\xi, P)\bigg|_{t=0}$$

is called the **infinitesimal generator** of the action induced by ξ.

infinitesimal generators of an action

In particular,

Infinitesimal generators of representations of G form a representation of $\mathfrak{g}$.

Box 29.1.31 *If M happens to be a vector space, and the action a representation as given in Box* 24.1.3, *then the infinitesimal generators constitute a representation of the Lie algebra of the group.*

Example 29.1.32 One can think of left translation on a Lie group G as an action of G on itself. Let $\Phi : G \times G \to G$ be given by $\Phi(g, h) = L_g(h)$. Then Definition 29.1.30 gives

$$\begin{aligned}\xi_G(g) &= \frac{d}{dt}\Phi(\exp t\xi, g)\bigg|_{t=0} = \frac{d}{dt}\exp t\xi g\bigg|_{t=0} = \frac{d}{dt}R_g(\exp t\xi)\bigg|_{t=0}\\ &= R_{g*}\xi\end{aligned}$$

by the first equation in (28.24). It follows that ξ_G is *right-invariant*. Indeed,

$$\begin{aligned}\xi_G \circ R_h(g) = \xi_G(gh) &= (R_{gh})_*\xi = (R_h \circ R_g)_*\xi = R_{h*} \circ R_{g*}\xi\\ &= R_{h*} \circ \xi_G(g).\end{aligned}$$

Since this holds for all $g \in G$, it follows that $\xi_G \circ R_h = R_{h*} \circ \xi_G$, demonstrating that ξ_G is right-invariant.

The adjoint map of Definition 29.1.25 induces a natural action on the Lie algebra $\mathfrak{g}$ with some important properties that we now explore. Define the **adjoint action** $\Phi : G \times \mathfrak{g} \to \mathfrak{g}$ of G on $\mathfrak{g} \cong \mathcal{T}_e(G)$ by $\Phi(g, \xi) = Ad_g(\xi)$.

adjoint action

Theorem 29.1.33 *The infinitesimal generator $\xi_{\mathfrak{g}}$ of the adjoint action is $\mathfrak{ad}_\xi$, where $\mathfrak{ad}_\xi(\eta) \equiv [\xi, \eta]$.*

Proof In fact,

$$\begin{aligned}\xi_{\mathfrak{g}}(\eta) &= \frac{d}{dt}\Phi(\exp t\xi, \eta)\bigg|_{t=0} = \frac{d}{dt}Ad_{\exp t\xi}(\eta)\bigg|_{t=0}\\ &= \frac{d}{dt}R^{-1}_{\exp t\xi *} \circ L_{\exp t\xi *}(\eta)\bigg|_{t=0} = \frac{d}{dt}F^{-1}_{t*}\eta(\exp t\xi)\bigg|_{t=0}\\ &= L_\xi(\eta) = [\xi, \eta] \equiv \mathfrak{ad}_\xi(\eta),\end{aligned} \tag{29.18}$$

where we used Eq. (29.6) as well as the definition of Lie derivative, Eq. (28.31). □

If $\varPhi : G \times M \to M$ is an action, then $\varPhi_g : M \to M$, defined by $\varPhi_g(P) = \varPhi(g, P)$, is a diffeomorphism of M. Consequently, $\varPhi_{g*} : \mathcal{T}_P(M) \to \mathcal{T}_{g \cdot P}(M)$ is an isomorphism for every $P \in M$ whose inverse is $\varPhi_{g*}^{-1} = \varPhi_{g^{-1}*}$.

Proposition 29.1.34 *Let $\varPhi : G \times M \to M$ be an action. Then for every $g \in G$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{g}$, we have*

$$(Ad_g \boldsymbol{\xi})_M = \varPhi_{g*}^{-1} \boldsymbol{\xi}_M \quad \textit{and} \quad [\boldsymbol{\xi}_M, \boldsymbol{\eta}_M] = -[\boldsymbol{\xi}, \boldsymbol{\eta}]_M.$$

Proof Let P be any point in M. Then,

$$\begin{aligned}
(Ad_g \boldsymbol{\xi})_M(P) &= \frac{d}{dt} \varPhi(\exp t Ad_g \boldsymbol{\xi}, P)\Big|_{t=0} && \text{(by Definition 29.1.30)} \\
&= \frac{d}{dt} \varPhi\big(g(\exp t \boldsymbol{\xi}) g^{-1}, P\big)\Big|_{t=0} && \text{(by Corollary 29.1.27)} \\
&= \frac{d}{dt} \varPhi\big(g(\exp t \boldsymbol{\xi}), \varPhi_{g^{-1}}(P)\big)\Big|_{t=0} && \text{(by definition of action)} \\
&= \frac{d}{dt} \varPhi_g \circ \varPhi\big(\exp t \boldsymbol{\xi}, \varPhi_{g^{-1}}(P)\big)\Big|_{t=0} && \text{(by definition of } \varPhi_g) \\
&= \varPhi_{g*}|_{\varPhi_{g^{-1}}(P)} \frac{d}{dt} \varPhi\big(\exp t \boldsymbol{\xi}, \varPhi_{g^{-1}}(P)\big)\Big|_{t=0} && \text{[by (28.24)]} \\
&= \varPhi_{g*}|_{\varPhi g^{-1}(P)} \boldsymbol{\xi}_M\big(\varPhi_{g^{-1}}(P)\big) = (\varPhi_{g*} \boldsymbol{\xi}_M)(P). && \text{[by (28.28)]}
\end{aligned}$$

The second part of the proposition follows by replacing g with $\exp t\boldsymbol{\eta}$, so that

$$(Ad_{\exp t \boldsymbol{\eta}} \boldsymbol{\xi})_M = \varPhi_{\exp t \boldsymbol{\eta} *} \boldsymbol{\xi}_M = \varPhi_{\exp t(-\boldsymbol{\eta})*}^{-1} \boldsymbol{\xi}_M.$$

Differentiate both sides with respect to t and note that the LHS gives $[\boldsymbol{\eta}, \boldsymbol{\xi}]_M$. The derivative of the RHS is the Lie derivative of $\boldsymbol{\xi}_M$ with respect to $-\boldsymbol{\eta}_M$, which is $-[\boldsymbol{\eta}_M, \boldsymbol{\xi}_M]$. □

Proposition 29.1.34 calculated the infinitesimal action of I_g for a fixed $g \in G$. This can be considered as a kind of partial derivative. Proposition 28.3.7 shows us how to find the total derivative for a general action $\varPhi$. For $\boldsymbol{\xi} \in \mathcal{T}_g(G)$ and $\mathbf{X} \in \mathcal{T}_P(M)$, Proposition 28.3.7 yields

$$\varPhi_*(\boldsymbol{\xi}, \mathbf{X}) = \varPhi_{g*}(\mathbf{X}) + \varPhi_{P*}(\boldsymbol{\xi}) \equiv g \cdot \mathbf{X} + \boldsymbol{\xi} \cdot P \tag{29.19}$$

where the last identity defines the symbols on its left. Note that if $\boldsymbol{\xi}_0 = \boldsymbol{\xi}(e)$, then $\boldsymbol{\xi} = L_{g*} \boldsymbol{\xi}_0$ in the equation above. We will have some occasions to use Eq. (29.19) later.

As mentioned earlier, a Lie group action is usually described in terms of the parameters of the group, which are simply coordinate functions on the group G, as well as coordinate functions on the manifold M. The infinitesimal generators, being vector fields on M, will then be expressed as a linear combination of coordinate frames.

comparison of coordinate manipulations with geometric (coordinate-free) analysis

In the older literature, no mention of the manifold structure is made. A Lie group is *defined* in terms of multiplication functions and other functions that represent the action of the group on the manifold. Thus, an r-parameter Lie group G is a collection of two sets of functions, $m^\rho : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$, $\rho = 1, 2, \dots, r$, representing the group multiplication, and $\phi^i : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, n$, representing the action of G on the n-dimensional manifold M. We sketch the procedure below, leaving most of the calculations as exercises for the reader. As we develop the theory, the reader is urged to compare this "coordinate-dependent" procedure with the "geometric" procedure—which does not use coordinates—described so far.

The action of the group is described by the coordinate transformations[3]

$$\begin{aligned} x'_i &= \phi_i(a_1, \dots, a_r; x_1, \dots, x_n), \quad i = 1, \dots, n, \\ x_i &= \phi_i(0, \dots, 0; x_1, \dots, x_n), \end{aligned} \tag{29.20}$$

as well as the group multiplication properties

$$\begin{aligned} c_\rho &= m_\rho(a_1, \dots, a_r; b_1, \dots, b_r), \quad \rho = 1, \dots, r, \\ a_\rho &= m_\rho(0, \dots, 0; a_1, \dots, a_r) = m_\rho(a_1, \dots, a_r; 0, \dots, 0), \\ m_\rho\big(\mathbf{a}; m(\mathbf{b}; \mathbf{c})\big) &= m_\rho\big(m(\mathbf{a}; \mathbf{b}); \mathbf{c}\big). \end{aligned} \tag{29.21}$$

Equation (29.20) is to be interpreted as a rule that takes the second set of arguments and transforms them via the first set into the LHS. Now suppose that we translate from x'_i to a neighboring point $x'_i + dx'_i$ via a set of group parameters $\{\delta a_\rho\}_{\rho=1}^r$. We can also get to $x'_i + dx'_i$ from x_i via a new set of parameters,[4] which have to be slightly different from $\{a_\rho\}_{\rho=1}^r$, say $\{a_\rho + da_\rho\}_{\rho=1}^r$. We then have

$$\begin{aligned} x'_i + dx'_i &= \phi_i\big(\delta a_1, \dots, \delta a_r; x'_1, \dots, x'_n\big), \\ x'_i + dx'_i &= \phi_i(a_1 + da_1, \dots, a_r + da_r; x_1, \dots, x_n), \\ a_\rho + da_\rho &= m_\rho(\delta a_1, \dots, \delta a_r; a_1, \dots, a_r), \end{aligned} \tag{29.22}$$

and, with summation over repeated indices understood,

$$\begin{aligned} dx'_i &= \left.\frac{\partial \phi_i(\mathbf{a}; \mathbf{x}')}{\partial a_\kappa}\right|_{\mathbf{a}=0} \delta a_\kappa \equiv u_{i\kappa}\big(\mathbf{x}'\big)\delta a_\kappa, \\ da_\lambda &= \left.\frac{\partial m_\lambda(\mathbf{b}; \mathbf{a})}{\partial b_\kappa}\right|_{\mathbf{b}=0} \delta a_\kappa \equiv \theta_{\lambda\kappa}(\mathbf{a})\delta a_\kappa. \end{aligned} \tag{29.23}$$

[3] We use subscripts for coordinate functions here for typographical convenience.

[4] Here we are assuming that the action of the group is **transitive**, i.e., that every point of the manifold can be connected to any other point via a transformation.

Inverting the second equation and substituting the resulting δa's in the first equation yields

$$dx'_i = u_{i\kappa}\big(\mathbf{x}'\big)\theta^{-1}_{\kappa\lambda}(\mathbf{a})da_\lambda, \quad \text{or} \quad dx_i = u_{i\kappa}(\mathbf{x})\theta^{-1}_{\kappa\lambda}(\mathbf{a})da_\lambda,$$

where in the last equation, we changed the free coordinate variable on both sides. It then follows that

$$\frac{\partial x_i}{\partial a_\lambda} = \sum_{\kappa=1}^{r} u_{i\kappa}(\mathbf{x})\theta^{-1}_{\kappa\lambda}(\mathbf{a}). \tag{29.24}$$

Lie's first theorem

Equation (29.24) and establishing that $u_{i\kappa}$ is $\mathcal{C}^\infty$ is the content of **Lie's first theorem**.

The change of an arbitrary function $f(\mathbf{x})$ due to an infinitesimal transformation is

$$df = \frac{\partial f}{\partial x_i}dx_i = \frac{\partial f}{\partial x_i}u_{i\kappa}(\mathbf{x})\delta a_\kappa = \delta a_\kappa\left(u_{i\kappa}(\mathbf{x})\frac{\partial}{\partial x_i}\right)f.$$

This suggests calling

$$\mathbf{X}_\kappa = \sum_{i=1}^{n} u_{i\kappa}(\mathbf{x})\frac{\partial}{\partial x_i} \tag{29.25}$$

infinitesimal generators as vector fields on M

the **infinitesimal generators** of the Lie group. The commutator of two of these generators is

$$[\mathbf{X}_\rho, \mathbf{X}_\sigma] = \left[u_{i\rho}\frac{\partial u_{j\sigma}}{\partial x_i} - u_{i\sigma}\frac{\partial u_{j\rho}}{\partial x_i}\right]\frac{\partial}{\partial x_j}. \tag{29.26}$$

This commutator does not appear to be similar to the one in Definition 29.1.28, which is necessary if the generators are to form a Lie algebra. However, through a long and tortuous manipulation, outlined in Problem 29.9, one can show that

$$[\mathbf{X}_\rho, \mathbf{X}_\sigma] = c^\kappa_{\rho\sigma}\mathbf{X}_\kappa \tag{29.27}$$

where $c^\kappa_{\rho\sigma}$ are *constants*.

One can also obtain this same result by the much simpler method of applying Proposition 29.1.34 to both sides of Eq. (29.12):

$$\big[(\boldsymbol{\xi}_i)_M, (\boldsymbol{\xi}_j)_M\big] = -[\boldsymbol{\xi}, \boldsymbol{\eta}]_M = -\left(\sum_{k=1}^{n} c^k_{ij}\boldsymbol{\xi}_k\right)_M = -\sum_{k=1}^{n} c^k_{ij}(\boldsymbol{\xi}_k)_M.$$

This equation is equivalent to (29.27) if we identify the $\mathbf{X}_\rho$'s with the $(\boldsymbol{\xi}_i)_M$'s and ignore the irrelevant minus sign.

The reader has hopefully been able to appreciate the power and elegance of the geometric approach to Lie groups and Lie algebras. The above illustration (Problem 29.9) brings out the tedium and the error-prone procedure of obtaining group-theoretic results through coordinate manipulations, a procedure used in the old literature including the work of Sophus Lie himself. Although such calculations are inevitable in practice, where most Lie groups are given in terms of parameters, they are not suitable for obtaining formal results.

Example 29.1.35 The two-dimensional rotation group $SO(2)$ is a 1-parameter Lie group defined by

$$
\begin{aligned}
x_1' &\equiv \phi_1(x_1, x_2; \theta) = x_1 \cos\theta - x_2 \sin\theta, \\
x_2' &\equiv \phi_2(x_1, x_2; \theta) = x_1 \sin\theta + x_2 \cos\theta.
\end{aligned}
$$

Using Eq. (29.25), we find the (only) generator of this group:

$$
\mathbf{X} = u_i \frac{\partial}{\partial x_i} \quad \text{where} \quad u_i = \left.\frac{\partial \phi_i}{\partial \theta}\right|_{\theta=0}.
$$

Explicitly, we have

$$
\begin{aligned}
u_1 &= \left.\frac{\partial \phi_1}{\partial \theta}\right|_{\theta=0} = (-x_1 \sin\theta - x_2 \cos\theta)|_{\theta=0} = -x_2, \\
u_2 &= \left.\frac{\partial \phi_2}{\partial \theta}\right|_{\theta=0} = (x_1 \cos\theta - x_2 \sin\theta)|_{\theta=0} = x_1,
\end{aligned}
$$

and

$$
\mathbf{X} = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}.
$$

The reader recognizes this, within a factor of i, as the z-component of the angular momentum operator in quantum mechanics. In fact, with $\mathbf{p}_n = -i\partial/\partial x_n$, we have

$$
\mathbf{X} = i(x_1 \mathbf{p}_2 - x_2 \mathbf{p}_1) = i\mathbf{L}_3 \quad \text{or} \quad \mathbf{L}_3 = -i\mathbf{X}
$$

where $\mathbf{L}_3$ is the third component of angular momentum operator $\mathbf{r} \times \mathbf{p}$. Therefore,

Box 29.1.36 *Angular momentum operators are the infinitesimal generators of rotation.*

Inclusion of the other two rotations about the x-axis and the y-axis completes the set of infinitesimal generators of the rotation group in three dimensions. Let us obtain the commutation relation between these components. First we note that the x-and y-components can also be calculated as

$$
\mathbf{X}_x = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \qquad \mathbf{X}_y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z},
$$

and all the three results can be summarized as $\mathbf{X}_j = \epsilon_{jmn} x_m \partial/\partial x_n$, with summation over the repeated indices understood. In terms of momentum, this becomes $\mathbf{X}_j = i\epsilon_{jmn} x_m \mathbf{p}_n$ and $\mathbf{L}_j = -i\mathbf{X}_j = \epsilon_{jmn} x_m \mathbf{p}_n$. The commutation relation among the components of angular momentum can now be calculated

$$\begin{aligned}[\mathbf{L}_j, \mathbf{L}_k] &= \epsilon_{jmn}\epsilon_{krs}[x_m \mathbf{p}_n, x_r \mathbf{p}_s] \\ &= \epsilon_{jmn}\epsilon_{krs}\big(x_m[\mathbf{p}_n, x_r]\mathbf{p}_s + x_r[x_m, \mathbf{p}_s]\mathbf{p}_n\big) \\ &= \epsilon_{jmn}\epsilon_{krs}(-i\delta_{rn}x_m\mathbf{p}_s + i\delta_{sm}x_r\mathbf{p}_n) \\ &= -i\epsilon_{jmn}\epsilon_{kns}x_m\mathbf{p}_s + i\epsilon_{jmn}\epsilon_{krm}x_r\mathbf{p}_n.\end{aligned}$$

Using

$$\epsilon_{jmn}\epsilon_{ksn} = \delta_{jk}\delta_{ms} - \delta_{js}\delta_{km} \quad \text{and} \quad x_j\mathbf{p}_k - x_k\mathbf{p}_j = \epsilon_{jkm}\mathbf{L}_m,$$

we obtain

$$[\mathbf{L}_j, \mathbf{L}_k] = i\epsilon_{jkm}\mathbf{L}_m$$

which is the desired result.

We obtained the generators of rotation by obtaining each component separately from equations connecting two x's to two xs involving only one angle. We could have used three equations connecting the three x's directly to the three xs. Such an equation writes each x' in terms of the three xs and trigonometric functions of three angles, the Euler angles. The matrix connecting the two sets of coordinates is given in Example 5.2.7. Problem 29.10, which the reader is asked to solve as a very illuminating exercise, calculates the three components of angular momentum directly.

The action of a Lie group on M can be reconstructed from its infinitesimal action. The flow of $\mathbf{X}_\kappa$ is the solution of the DE

$$\frac{dx_i'}{dt} = u_{i\kappa}\big(\mathbf{x}'\big), \qquad x_i'(0) = x_i. \tag{29.28}$$

Once the solution is obtained, one can replace t with a_κ for each κ. In some applications, $u_{i\kappa}(\mathbf{x})$ will be given implicitly in terms of certain parameters of integration of some DEs [unrelated to (29.28)]. The solution of these DEs are typically generators of coordinate transformations that can be written linearly in terms of the parameters. To be more precise, suppose that after solving some DEs, we obtain

$$X_i = \sum_{\kappa=1}^{r} c_{i\kappa} f_\kappa^{(i)}(x_1, \ldots, x_n), \tag{29.29}$$

where $\{c_{i\kappa}\}$ are the parameters of integration, and X_i are components of the vector field that generates the coordinate transformation. This means that for small parameters, one can write

$$x_i' = x_i + \sum_{\kappa=1}^{r} c_{i\kappa} f_\kappa^{(i)}(x_1, \ldots, x_n)$$

and read off $u_{i\kappa}(\mathbf{x}) = f_\kappa^{(i)}(\mathbf{x})$. In that case, we have

$$\frac{dx'_i}{dt} = f_\kappa^{(i)}\bigl(x'_1, \dots, x'_n\bigr), \quad x'_i(0) = x_i. \tag{29.30}$$

We shall have occasion to use this formula later.

29.1.5 Integration on Lie Groups

As any other manifold, one can define integration on Lie groups; i.e., one can construct nonvanishing n-forms and use Eq. (28.47) to define integrals on a Lie group G. Because of the left-invariant property of objects on G, it would be helpful if the integration process were also left-invariant. For this to happen, the n-form would have to be left-invariant. It turns out that this can be accomplished more or less uniquely:

Proposition 29.1.37 *Let G be a Lie group of dimension n. Then there exists a left-invariant nonvanishing n-form $\boldsymbol{\mu}$ that is unique up to a nonzero multiplicative constant. If G is compact, then $\boldsymbol{\mu}$ is also right-invariant and the multiplicative constant can be chosen to be* 1.

Proof Let $\boldsymbol{\mu}_e$ be any nonzero n-form on $\mathcal{T}_e(G)$. The desired n-form is the left translation of this form, i.e., $L^*_{g^{-1}}\boldsymbol{\mu}_e$. Indeed, let $\{\mathbf{X}_i\}_{i=1}^n$ be left invariant. Then

$$\begin{aligned}\boldsymbol{\mu}_g(\mathbf{X}_1|_g, \dots, \mathbf{X}_n|_g) &= L^*_{g^{-1}}\boldsymbol{\mu}_e(\mathbf{X}_1|_g, \dots, \mathbf{X}_n|_g)\\ &= \boldsymbol{\mu}_e(L_{g^{-1}*}\mathbf{X}_1|_g, \dots, L_{g^{-1}*}\mathbf{X}_n|_g)\\ &= \boldsymbol{\mu}_e(\mathbf{X}_1|_{g^{-1}g}, \dots, \mathbf{X}_n|_{g^{-1}g}) = \boldsymbol{\mu}_e(\mathbf{X}_1|_e, \dots, \mathbf{X}_n|_e).\end{aligned}$$

This shows that $\boldsymbol{\mu}$ is left-invariant. Now note that any other n-form $\boldsymbol{\mu}'_e$ on $\mathcal{T}_e(G)$ is a constant multiple of $\boldsymbol{\mu}_e$. Therefore, the corresponding n-form $\boldsymbol{\mu}'_g$ will be a constant multiple of $\boldsymbol{\mu}_g$.

Let $x \in G$ and consider $\boldsymbol{\mu}' \equiv R^*_x\boldsymbol{\mu}$. We have

$$L^*_g\boldsymbol{\mu}' = L^*_g \circ R^*_x\boldsymbol{\mu} = R^*_x \circ L^*_g\boldsymbol{\mu} = R^*_x\boldsymbol{\mu} = \boldsymbol{\mu}',$$

where we used the fact that L_g and R_x commute and that $\boldsymbol{\mu}$ is left invariant. The equation above shows that $\boldsymbol{\mu}'$ is also left-invariant. Therefore, $\boldsymbol{\mu}' = c\boldsymbol{\mu}$. If G is compact, we can integrate both sides and note that $\int_G \boldsymbol{\mu} = \int_G \boldsymbol{\mu}'$ because $\boldsymbol{\mu}'$ is related to $\boldsymbol{\mu}$ by a change of variable. Therefore, $c = 1$ and $R^*_x\boldsymbol{\mu} = \boldsymbol{\mu}$. □

The left-invariant volume element (nonvanishing n-form) guaranteed by the proposition above is called **Haar measure**. Since all calculations are done using some coordinate system, we give an explicit expression of the Haar measure in terms of coordinates (parameters) of a general Lie group. Let $\mathbf{y} = (y^1, \dots, y^r)$ be the coordinates of the translation of $\mathbf{x} = (x^1, \dots, x^r)$

Haar measure

by $g \in G$. Then we can write $\mathbf{y} = m(\mathbf{g}, \mathbf{x})$, so that $dy^j = (\partial y^j/\partial x^i)dx^i = (\partial m^j/\partial x^i)dx^i$. Therefore,

$$dy^1 \wedge \cdots \wedge dy^r = \det\left(\frac{\partial m^j(\mathbf{g}, \mathbf{x})}{\partial x^i}\right)dx^1 \wedge \cdots \wedge dx^r.$$

In particular, if $\mathbf{x} = 0$, the coordinates of the identity, then $\mathbf{y}$ will be the coordinates of g. So, the volume element at g, denoted by $d^r y$, will be given by

$$d^r y = \det\left(\frac{\partial m^j(\mathbf{g}, \mathbf{x})}{\partial x^i}\right)\bigg|_{\mathbf{x}=0} d^r x.$$

Note that this is consistent with the geometric definition of the invariant measure given in Proposition 29.1.37 because $L_{g^{-1}*} = L_{g*}^{-1}$ and the matrix of L_g^* is the inverse of the matrix of L_{g*}. The volume element at g, which is invariant on G—and therefore has the same value as at the identity—and which we denote by $d\boldsymbol{\mu}(g)$, will be given by

$$d\boldsymbol{\mu}(g) = d\boldsymbol{\mu}(e) \equiv d^r x = \det{}^{-1}\left(\frac{\partial m^j(\mathbf{g}, \mathbf{x})}{\partial x^i}\right)\bigg|_{\mathbf{x}=0} d^r g, \tag{29.31}$$

density functions associated with Haar measure

where we have replaced y with the more suggestive g. The volume element $d^r g$ is the ordinary Euclidean volume element of $\mathbb{R}^r$ evaluated at the parameters corresponding to g. The quantity multiplying $d^r g$ is called the **density function**. Note that since we are interested in the derivatives of m^j at small values of $\mathbf{x}$, we can take the components of $\mathbf{x}$ to be small, and retain them only up to the first order. This will sometimes simplify the calculation of the invariant Haar measure.

Example 29.1.38 From the multiplication rule for the one-dimensional projective group given in Example 29.1.9, we easily find

$$\det\left(\frac{\partial m_i}{\partial b_j}\right)\bigg|_{\mathbf{b}=0} = \det\begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & a_3 & 0 & a_4 \end{pmatrix} = (a_1 a_4 - a_2 a_3)^2.$$

Thus the density function is $(a_1 a_4 - a_2 a_3)^{-2}$, and the invariant Haar measure is

$$d\boldsymbol{\mu}(\mathbf{a}) = (a_1 a_4 - a_2 a_3)^{-2} d^4 a.$$

29.2 An Outline of Lie Algebra Theory

The notion of a Lie algebra has appeared on a number of occasions both in our study of vector fields on manifolds and, more recently, in the study of Lie groups in the vicinity of their identity elements. Lie algebras play an important role in the representation theory of Lie groups as well. It is therefore worth our effort to spend some time getting acquainted with the formal structure and properties of these algebras. We shall restrict our discussion to finite-dimensional Lie algebras.

Definition 29.2.1 A finite-dimensional vector space $\mathcal{V}$ over $\mathbb{R}$ (or $\mathbb{C}$) is called a **Lie algebra over** $\mathbb{R}$ (or $\mathbb{C}$) if there is a binary operation, called **Lie multiplication**, $[\cdot,\cdot]:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ on $\mathcal{V}$, satisfying

Lie algebra defined

1. $[\mathbf{X},\mathbf{Y}]=-[\mathbf{Y},\mathbf{X}]$ for all $\mathbf{X},\mathbf{Y}\in\mathcal{V}$ (antisymmetry).
2. $[\alpha\mathbf{X}+\beta\mathbf{Y},\mathbf{Z}]=\alpha[\mathbf{X},\mathbf{Z}]+\beta[\mathbf{Y},\mathbf{Z}]$ for $\alpha,\beta\in\mathbb{R}$ (or $\mathbb{C}$) (linearity).
3. $[\mathbf{X},[\mathbf{Y},\mathbf{Z}]]+[\mathbf{Z},[\mathbf{X},\mathbf{Y}]]+[\mathbf{Y},[\mathbf{Z},\mathbf{X}]]=0$ (Jacobi identity).

The concepts of a homomorphism, its kernel, its range, etc. are the same as before.

To distinguish Lie algebras from vector spaces, we shall denote the former by lowercase German letters as we have done for the Lie algebras of Lie groups.

Example 29.2.2 Recall from Chap. 3 that an algebra is a vector space with a product. If this product is associative, then one can construct a Lie algebra out of the associative algebra by defining $[\mathbf{a},\mathbf{b}]\equiv\mathbf{ab}-\mathbf{ba}$. In particular, the matrix algebra under commutation of matrices becomes a Lie algebra, which we denote by $\mathfrak{gl}(n,\mathbb{R})$ [or $\mathfrak{gl}(n,\mathbb{C})$].

Definition 29.2.3 Let $\mathfrak{v}$ be a Lie algebra. A subspace $\mathfrak{u}$ of $\mathfrak{v}$ is called a **subalgebra** if $[\mathbf{X},\mathbf{Y}]\in\mathfrak{u}$ whenever $\mathbf{X},\mathbf{Y}\in\mathfrak{u}$. The subspace $\mathfrak{u}$ is called an **ideal** if $[\mathbf{X},\mathbf{Y}]\in\mathfrak{u}$ whenever either $\mathbf{X}\in\mathfrak{u}$ or $\mathbf{Y}\in\mathfrak{u}$. The **center** $\mathfrak{z}$ of $\mathfrak{v}$ is the collection of all $\mathbf{X}\in\mathfrak{v}$ whose Lie multiplication with all vectors of $\mathfrak{v}$ vanishes. A Lie algebra is **abelian**, or **commutative**, if $\mathfrak{z}=\mathfrak{v}$.

subalgebra, ideal, and center of a Lie algebra

If we choose a basis in the Lie algebra $\mathfrak{v}$, and express the Lie multiplication of basis vectors as a linear combination of basis vectors, we end up with *basis-dependent* structure constants that satisfy Eq. (29.13). The structure constants completely determine the Lie algebra: Given these constants, one can choose a vector space $\mathcal{V}$ of correct dimension, a basis in that space, and impose the Lie multiplication law among the basis vectors suggested by the structure constants. Once the Lie multiplication law for basis vectors is established, the law for arbitrary vectors follows from linearity of Lie multiplication. This procedure induces a binary operation on $\mathcal{V}$ and turns it into a Lie algebra $\mathfrak{v}$. Any other algebra so constructed will be isomorphic to $\mathfrak{v}$.

Knowing the structure constants, one can reconstruct the Lie algebra!

Example 29.2.4 We can classify all two-dimensional Lie algebras by analyzing their structure constants. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be any two linearly independent vectors of the two-dimensional Lie algebra $\mathfrak{v}$. Write the only nonzero Lie bracket as

$$[\mathbf{X}_1,\mathbf{X}_2]=c_1\mathbf{X}_1+c_2\mathbf{X}_2.$$

There are two cases to consider: Either $c_1 = 0 = c_2$ or at least one of the constants is nonzero. The first case corresponds to a 2-dimensional abelian Lie algebra:

$$[\mathbf{X}_i, \mathbf{X}_j] = 0 \quad \text{for } i, j = 1, 2.$$

For the second case, suppose $c_1 \neq 0$ and define the vectors

$$\mathbf{X} \equiv c_1\mathbf{X}_1 + c_2\mathbf{X}_2, \qquad \mathbf{Y} \equiv \mathbf{X}_2/c_1.$$

Then the nonzero Lie bracket becomes $[\mathbf{X}, \mathbf{Y}] = \mathbf{X}$.

The result of Example 29.2.4 is summarized as follows:

Box 29.2.5 *There are only two 2-dimensional Lie algebras given by either one of the following nonzero Lie bracket relations*:

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \quad \textit{or} \quad [\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1.$$

Example 29.2.6 The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a Lie algebra under the commutation relation given by

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l.$$

Thus, $c^l_{jk} = 2i\epsilon_{jkl}$. Pauli spin matrices are a basis for $\mathfrak{su}(2)$.

Example 29.2.7 The Lie group $GL(n, \mathbb{R})$ has $\mathfrak{gl}(n, \mathbb{R})$, the set of all real $n \times n$ matrices, as its Lie algebra. The standard basis of this Lie algebra, also called the **Weyl basis**, consists of matrices e_{ij} that have zeros everywhere except at the ijth position. We therefore have

Weyl basis for $\mathfrak{gl}(n, \mathbb{R})$

$$(\mathsf{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}. \tag{29.32}$$

We can readily find the Lie multiplication (commutation relations) for these matrices. We simply need to look at the elements of the matrix of the commutator:

$$\begin{aligned}
\big([\mathsf{e}_{ij}, \mathsf{e}_{kl}]\big)_{mn} &= (\mathsf{e}_{ij}\mathsf{e}_{kl})_{mn} - (\mathsf{e}_{kl}\mathsf{e}_{ij})_{mn} \\
&= (\mathsf{e}_{ij})_{mr}(\mathsf{e}_{kl})_{rn} - (\mathsf{e}_{kl})_{mr}(\mathsf{e}_{ij})_{rn} \\
&= \delta_{im}\delta_{jr}\delta_{kr}\delta_{ln} - \delta_{km}\delta_{lr}\delta_{ir}\delta_{jn} = \delta_{im}\delta_{jk}\delta_{ln} - \delta_{km}\delta_{li}\delta_{jn} \\
&= (\mathsf{e}_{il})_{mn}\delta_{jk} - (\mathsf{e}_{kj})_{mn}\delta_{li},
\end{aligned}$$

or

$$[\mathsf{e}_{ij}, \mathsf{e}_{kl}] = \delta_{jk}\mathsf{e}_{il} - \delta_{il}\mathsf{e}_{kj}. \tag{29.33}$$

The structure constants, which are naturally double-indexed, can be read off from Eq. (29.33):

$$c_{ij,kl}^{mn} = \delta_{jk}\delta_i^m\delta_l^n - \delta_{il}\delta_k^m\delta_j^n, \tag{29.34}$$

where we have used a superscript for some of the Kronecker deltas to conform to the position of the corresponding index on the LHS.

Example 29.2.8 An important datum is the dimension of the Lie group (or its associated Lie algebra, since they are the same). This datum is not apparent in most cases of interest in which the group is defined in terms of some geometric property. For example, the symplectic group is defined as all linear transformations **A** that leave a certain antisymmetric bilinear form invariant (Example 23.2.2). In terms of matrices, we have

$$\mathsf{x}'^t\mathsf{J}\mathsf{x}' = \mathsf{x}^t\mathsf{J}\mathsf{x} \quad \Rightarrow \quad \mathsf{x}^t\mathsf{A}^t\mathsf{J}\mathsf{A}\mathsf{x} = \mathsf{x}^t\mathsf{J}\mathsf{x} \quad \forall \mathsf{x} \in \mathbb{R}^{2n}, \quad \mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows that the symplectic group consists of all matrices A such that

$$\mathsf{A}^t\mathsf{J}\mathsf{A} = \mathsf{J}. \tag{29.35}$$

If we write A in block form,

$$\mathsf{A} = \begin{pmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{pmatrix},$$

where A_{ij} are $n \times n$ matrices, then, Eq. (29.35) becomes

$$\begin{pmatrix} \mathsf{A}_{11}^t & \mathsf{A}_{21}^t \\ \mathsf{A}_{12}^t & \mathsf{A}_{22}^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

or

$$\mathsf{A}_{11}^t\mathsf{A}_{21} = \mathsf{A}_{21}^t\mathsf{A}_{11}, \qquad \mathsf{A}_{22}^t\mathsf{A}_{12} = \mathsf{A}_{12}^t\mathsf{A}_{22}, \qquad \mathsf{A}_{11}^t\mathsf{A}_{22} - \mathsf{A}_{21}^t\mathsf{A}_{12} = \mathbf{1}. \tag{29.36}$$

For the symplectic algebra $\mathfrak{sp}(2n, \mathbb{R})$, we are interested in the matrix A when it is close to the identity. This means that

$$\mathsf{A}_{11} = \mathbf{1} + \epsilon\mathsf{X}_{11}, \qquad \mathsf{A}_{22} = \mathbf{1} + \epsilon\mathsf{X}_{22}, \qquad \mathsf{A}_{12} = \epsilon\mathsf{X}_{12}, \qquad \mathsf{A}_{21} = \epsilon\mathsf{X}_{21}.$$

Substituting these in Eq. (29.36) and keeping terms linear in ϵ, we obtain the following relations among X_{ij}:

$$\mathsf{X}_{22}^t = -\mathsf{X}_{11}, \qquad \mathsf{X}_{12}^t = \mathsf{X}_{12}, \qquad \mathsf{X}_{21}^t = \mathsf{X}_{21}. \tag{29.37}$$

It follows that we need n^2 parameters to describe the $n \times n$ matrices X_{11} and X_{22} simultaneously. For the symmetric matrices X_{12} and X_{21}, we need $n(n+1)/2$ independent parameters each. Therefore, the total number of independent parameters needed for (or the dimension of) the symplectic algebra $\mathfrak{sp}(2n, \mathbb{R})$ is

$$n^2 + 2\frac{n(n+1)}{2} = n(2n+1).$$

Although our attempt is to give a formal discussion of the Lie algebras and their structure in this section, we shall do this with an eye to the eventual utility of this discussion in a better understanding of the Lie algebras of Lie groups. To make the connection between the present formalism and the Lie algebras arising from Lie groups, we shall make heavy use of matrix groups, i.e., $GL(n, \mathbb{R})$ [or $GL(n, \mathbb{C})$] and its subgroups. Equation (29.8) gives a method of finding the matrices of the algebra if those of the group are known:

Box 29.2.9 *Differentiate the matrix with respect to a parameter at the identity (where all parameters are set equal to zero) to find the matrix "in the direction" of that parameter.*

29.2.1 The Lie Algebras $\mathfrak{o}(p, n-p)$ and $\mathfrak{p}(p, n-p)$

Many of the Lie groups encountered in physical applications are special cases of the (pseudo) orthogonal group $O(p, n-p)$ and its associated Poincaré group $P(p, n-p)$. It is therefore worthwhile to study their Lie algebras in some detail. Introduce the diagonal matrix

$$\boldsymbol{\eta} = \text{diag}(\underbrace{-1, -1, \dots, -1}_{p \text{ times}}, \underbrace{1, 1, \dots, 1}_{n-p \text{ times}})$$

and note that the (pseudo) orthogonal group $O(p, n-p)$ consists of $n \times n$ matrices that leave the bilinear form $\mathsf{x} \cdot \mathsf{x} \equiv \mathsf{x}^t \boldsymbol{\eta} \mathsf{x}$ invariant for $\mathsf{x} \in \mathbb{R}^n$. This means that the matrices A will have to satisfy

$$\mathsf{A}^t \boldsymbol{\eta} \mathsf{A} = \boldsymbol{\eta} \quad \Rightarrow \quad (\det \mathsf{A})^2 = 1. \tag{29.38}$$

$\boldsymbol{\eta}$-orthogonal matrices

Such matrices are called **$\boldsymbol{\eta}$-orthogonal**. The fact that $O(p, n-p)$ is a group and that $\boldsymbol{\eta}^{-1} = \boldsymbol{\eta}$ can be used to show that

$$\mathsf{A}\boldsymbol{\eta}\mathsf{A}^t = \boldsymbol{\eta}. \tag{29.39}$$

Example 29.2.10 (The Lorentz group) The group of the special theory of relativity is the full Lorentz group $O(3, 1)$. This is the group of transformations that leave the invariant length[5]

$$\eta^{ij} x_i x_j = -x_1^2 - x_2^2 - x_3^2 + x_4^2 \equiv x_0^2 - x_1^2 - x_2^2 - x_3^2$$

[5] It is common to label the time coordinate with index 0 rather than 4. We shall use this convention.

of a 4-vector $(x_1, x_2, x_3, x_0 = ct)$ invariant. The $(0,0)$-components of Eqs. (29.38) and (29.39) yield

$$\begin{aligned} a_{00}^2 - a_{10}^2 - a_{20}^2 - a_{30}^2 &= 1, \\ a_{00}^2 - a_{01}^2 - a_{02}^2 - a_{03}^2 &= 1. \end{aligned} \tag{29.40}$$

Either one of these equations implies that $a_{00} \geq 1$ or $a_{00} \leq -1$. Lorentz transformations for which $a_{00} \geq 1$ are called **orthochronous**. Since $\det \mathbf{1} = +1$ and $\mathbf{1}_{00} = +1$, the identity belongs to the subset consisting of transformations with $\det \mathsf{A} = +1$ and $a_{00} \geq 1$. Such transformations form a subgroup of $O(3,1)$ called the **proper orthochronous** Lorentz transformations, and have the property that they can be reached continuously from the identity.

orthochronous and proper orthochronous Lorentz transformations

Depending on whether $\mathsf{x}\cdot\mathsf{x} > 0$, $\mathsf{x}\cdot\mathsf{x} < 0$, or $\mathsf{x}\cdot\mathsf{x} = 0$, the vector x is called **timelike**, **spacelike**, or **null**, respectively. In the special theory of relativity $\mathbb{R}^4$ becomes the set of **events**. At every event x the set $\mathbb{R}^4$ is divided into 5 regions:

Timelike, spacelike, and null vectors; $\mathbb{R}^4$ as the set of events

1. All events $\mathsf{y} = (y_1, y_2, y_3, y_0)$ to which one can go from x by material objects, with speed less than c, lie to the future of x, i.e., $y_0 - x_0 > 0$, and are timelike:
$$(y_0 - x_0)^2 > (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2.$$
They form a 4-dimensional subset of $\mathbb{R}^4$ and are said to lie *inside* the **future light cone**.

future light cone

2. All events $\mathsf{y} = (y_1, y_2, y_3, y_0)$ to which one can go from x only by a light signal lie to the future of x, i.e., $y_0 - x_0 > 0$, and
$$(y_0 - x_0)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2 - (y_3 - x_3)^2 = 0.$$
They form a 3-dimensional subset of $\mathbb{R}^4$ and are said to lie *on* the future light cone.
3. All events $\mathsf{y} = (y_1, y_2, y_3, y_0)$ from which one can come to x by material objects, with speed less than c, lie in the past of x, i.e., $x_0 - y_0 > 0$, and are timelike:
$$(x_0 - y_0)^2 > (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2.$$
They form a 4-dimensional subset of $\mathbb{R}^4$ and are said to lie *inside* the **past light cone**.

past light cone

4. All events $\mathsf{y} = (y_1, y_2, y_3, y_0)$ from which one can come from x only by a light signal lie to the past of x, i.e., $x_0 - y_0 > 0$, and
$$(x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 = 0.$$
They form a 3-dimensional subset of $\mathbb{R}^4$ and are said to lie *on* the past light cone.
5. All events in the remaining part of $\mathbb{R}^4$ form a 4-dimensional subset, are spacelike, and cannot be connected to x by any means. They are said to belong to **elsewhere**.

elsewhere

From a physical standpoint, future and past are observer-independent. Therefore, if y lies in or on the future light cone of x with respect to one observer, it should also do so with respect to all observers. Since observers are connected by Lorentz transformations, we expect the latter to preserve this relation between x and y. Not all elements of $O(3,1)$ have this property. However, the proper orthochronous transformations do. The details are left as a problem for the reader (see Problem 29.15).

As a prototype of $\boldsymbol{\eta}$-orthogonal matrices, consider the matrix obtained from the unit matrix by removing the iith, ijth, jith, and jjth elements, and replacing them by an overall 2×2 matrix. The result, denoted by $\mathsf{A}^{(ij)}$, will look like

$$\mathsf{A}^{(ij)} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & a_{ii} & \dots & a_{ij} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & a_{ji} & \dots & a_{jj} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}.$$

This matrix will transform $(x_1, \dots, x_n) \in \mathbb{R}^n$ according to

$$\begin{aligned} x'_i &= a_{ii}x_i + a_{ij}x_j, \quad \text{(no summation!)} \\ x'_j &= a_{ji}x_i + a_{jj}x_j, \\ x'_k &= x_k \quad \text{for } k \neq i, j. \end{aligned}$$

In order for $\mathsf{A}^{(ij)}$ to leave the bilinear form $\mathsf{x}^t\boldsymbol{\eta}\mathsf{x}$ invariant, the 2×2 submatrix $\left(\begin{smallmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{smallmatrix}\right)$ must be either a rotation (corresponding to the case where $i, j \le p$ or $i, j > p$), or a Lorentz boost[6] (corresponding to the case where $i \le p$ and $j > p$). In the first case, we have

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and in the second case

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix},$$

where $\xi \equiv \tanh^{-1}(v/c)$ is the "rapidity".

The matrices of the algebra are obtained by differentiation at $\theta = 0$ (or $\xi = 0$). Denoting these matrices by M_{ij}, we readily find that for the case of rotations, M_{ij} has -1 at the ijth position, $+1$ at the jith position, and 0

[6] The elementary Lorentz transformations involving only one space dimension.

everywhere else. For the case of boosts, M_{ij} has -1 at the ijth and the jith position, and 0 everywhere else. Both cases can be described by the single relation

$$(\mathsf{M}_{ij})^m{}_l = \eta_{il}\delta^m_j - \eta_{jl}\delta^m_i, \qquad \mathsf{M}_{ij} = -\mathsf{M}_{ji}.$$

It is convenient to have all indices in the lower position. So, we multiply both sides by η_{mk} to obtain

$$(\mathsf{M}_{ij})_{kl} = \eta_{il}\eta_{jk} - \eta_{jl}\eta_{ik}, \qquad \mathsf{M}_{ij} = -\mathsf{M}_{ji}. \tag{29.41}$$

We can use Eq. (29.41) to find the Lie multiplication (in this case, matrix commutation relations) for the algebra $\mathfrak{o}(p, n-p)$:

Lie brackets for the algebra $\mathfrak{o}(p, n-p)$

$$[\mathsf{M}_{ij}, \mathsf{M}_{kl}] = \eta_{ik}\mathsf{M}_{jl} - \eta_{il}\mathsf{M}_{jk} + \eta_{jl}\mathsf{M}_{ik} - \eta_{jk}\mathsf{M}_{il}. \tag{29.42}$$

The Lie group $O(p, n-p)$ includes rotations and Lorentz transformation. Another group with considerable significance in physics is the **Poincaré group** $P(p, n-p)$, which includes translations[7] in space and time as well. An element of $P(p, n-p)$ transforms $\mathsf{x} \in \mathbb{R}^n$ to $\mathsf{x}' = \mathsf{A}\mathsf{x} + \mathsf{u}$, where u is a column vector representing the translation part of the group. It is convenient to introduce matrices to represent these group operations. This is possible if we represent an element of $\mathbb{R}^n$ as an $(n+1)$-column whose last element is an insignificant 1. Then, the reader may check that a Poincaré transformation can be written as

Poincaré group as $(n+1) \times (n+1)$ matrices

$$\begin{pmatrix} \mathsf{x}' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{x} \\ 1 \end{pmatrix}, \tag{29.43}$$

where A is the $n \times n$ matrix of $O(p, n-p)$, and u is an n-dimensional column vector.

The Lie algebra of the Poincaré group is obtained by differentiating the $(n+1) \times (n+1)$ matrix of Eq. (29.43). The differentiation of the matrix A will give $\mathfrak{o}(p, n-p)$ of Eq. (29.42). The translation part will lead to matrices P_i with matrix elements given by

$$(\mathsf{P}_i)^k{}_l = \delta^k_i \delta^{n+1}_l \quad \Rightarrow \quad (\mathsf{P}_i)_{kl} = \eta_{ik}\delta^{n+1}_l. \tag{29.44}$$

These matrices satisfy the following Lie multiplication rules:

$$[\mathsf{P}_i, \mathsf{P}_j] = 0, \qquad [\mathsf{M}_{ij}, \mathsf{P}_k] = \eta_{ik}\mathsf{P}_j - \eta_{jk}\mathsf{P}_i.$$

It then follows that the full **Poincaré algebra** $\mathfrak{p}(p, n-p)$ is described by the following Lie brackets:

Lie brackets for the Poincaré algebra $\mathfrak{p}(p, n-p)$

$$\begin{aligned} [\mathsf{M}_{ij}, \mathsf{M}_{kl}] &= \eta_{ik}\mathsf{M}_{jl} - \eta_{il}\mathsf{M}_{jk} + \eta_{jl}\mathsf{M}_{ik} - \eta_{jk}\mathsf{M}_{il}, \\ [\mathsf{M}_{ij}, \mathsf{P}_k] &= \eta_{ik}\mathsf{P}_j - \eta_{jk}\mathsf{P}_i, \\ [\mathsf{P}_i, \mathsf{P}_j] &= 0. \end{aligned} \tag{29.45}$$

[7]One can think of the Poincaré group as a subgroup of the group of affine motions in which the matrices belong to $O(p, n-p)$ rather than $GL(n, \mathbb{R})$.

29.2.2 Operations on Lie Algebras

Definition 29.2.11 Let $\mathfrak{v}$ be a Lie algebra. A linear operator $\mathbf{D}: \mathfrak{v} \to \mathfrak{v}$ satisfying

$$\mathbf{D}[\mathbf{X}, \mathbf{Y}] = [\mathbf{D}\mathbf{X}, \mathbf{Y}] + [\mathbf{X}, \mathbf{D}\mathbf{Y}]$$

is called a **derivation** of $\mathfrak{v}$.

derivation algebra of a Lie algebra

Although the product of two derivations is not a derivation, their commutator is. Therefore, the set of derivations of a Lie algebra $\mathfrak{v}$ themselves form a Lie algebra $\mathfrak{D}_{\mathfrak{v}}$ under commutations, which is called the **derivation algebra**.

Recall that the infinitesimal generators of the adjoint action of a Lie group on its algebra were given by $\mathfrak{ad}_{\mathfrak{g}}$ [Eq. (29.18)]. We can apply this to a general Lie algebra $\mathfrak{v}$ by fixing a vector $\mathbf{X} \in \mathfrak{v}$ and defining the map $\mathrm{ad}_{\mathbf{X}}: \mathfrak{v} \to \mathfrak{v}$ given by $\mathrm{ad}_{\mathbf{X}}(\mathbf{Y}) = [\mathbf{X}, \mathbf{Y}]$. The reader may verify that $\mathrm{ad}_{\mathbf{X}}$ is a derivation of $\mathfrak{v}$ and that $\mathrm{ad}_{[\mathbf{X},\mathbf{Y}]} = [\mathrm{ad}_{\mathbf{X}}, \mathrm{ad}_{\mathbf{Y}}]$. Therefore, the set $\mathfrak{ad}_{\mathfrak{v}} \equiv \{\mathrm{ad}_{\mathbf{X}} \mid \mathbf{X} \in \mathfrak{v}\}$ is a Lie algebra, a subalgebra of the derivation algebra $\mathfrak{D}_{\mathfrak{v}}$ of $\mathfrak{v}$, and is called the **adjoint algebra** of $\mathfrak{v}$. There is a natural homomorphism $\psi: \mathfrak{v} \to \mathfrak{ad}_{\mathfrak{v}}$ given by $\psi(\mathbf{X}) = \mathrm{ad}_{\mathbf{X}}$ whose kernel is the center of $\mathfrak{v}$. Furthermore, $\mathfrak{ad}_{\mathfrak{v}}$ is an *ideal* of $\mathfrak{D}_{\mathfrak{v}}$.

adjoint algebra of a Lie algebra

Illustration of homomorphism of $\mathfrak{su}(2)$ and its adjoint using Pauli spin matrices

Example 29.2.12 We construct the matrix representation of the operators in the adjoint algebra of $\mathfrak{su}(2)$ with Pauli spin matrices as a basis. From

$$\mathrm{ad}_{\sigma_1}(\sigma_1) = [\sigma_1, \sigma_1] = 0$$

we conclude that the first column of the matrix of ad_{σ_1} is zero. From

$$\mathrm{ad}_{\sigma_1}(\sigma_2) = [\sigma_1, \sigma_2] = 2i\sigma_3$$

we conclude that the second column of the matrix of ad_{σ_1} has zeros for the first two elements and $2i$ for the last. Similarly, from

$$\mathrm{ad}_{\sigma_1}(\sigma_3) = [\sigma_1, \sigma_3] = -2i\sigma_2$$

we conclude that the third column of the matrix of ad_{σ_1} has zeros for the first and third elements and $-2i$ for the second. Thus, the matrix representation of ad_{σ_1} is

$$\mathrm{ad}_{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix}.$$

Likewise, we can obtain the other two matrix representations; they are

$$\mathrm{ad}_{\sigma_2} = \begin{pmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\sigma_3} = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The reader may readily verify that $[\mathrm{ad}_{\sigma_j}, \mathrm{ad}_{\sigma_k}] = 2i\epsilon_{jkl}\mathrm{ad}_{\sigma_l}$.

If ψ is an automorphism of $\mathfrak{v}$, i.e., an isomorphism of $\mathfrak{v}$ onto itself, then

$$\mathrm{ad}_{\psi(\mathbf{X})} = \psi \circ \mathrm{ad}_{\mathbf{X}} \circ \psi^{-1} \quad \forall \mathbf{X} \in \mathfrak{v}. \tag{29.46}$$

Since $\mathrm{ad}_{\mathbf{X}}$ is an operator on the vector space $\mathfrak{v}$, one can define the trace of $\mathrm{ad}_{\mathbf{X}}$. However, the notion of trace attains a far greater significance when it is combined with the notion of composition of operators. For $\mathbf{X}, \mathbf{Y} \in \mathfrak{v}$, define

$$(\mathbf{X} \mid \mathbf{Y}) \equiv \mathrm{tr}(\mathrm{ad}_{\mathbf{X}} \circ \mathrm{ad}_{\mathbf{Y}}). \tag{29.47}$$

Then one can show that $(\cdot|\cdot)$ is bilinear and symmetric. It becomes an inner product if the "vectors" of the Lie algebra are hermitian operators on some vector space $\mathcal{V}$, or if the underlying vector space is over $\mathbb{R}$ (see Proposition 5.6.6 in Chap. 5). Furthermore, $(\cdot|\cdot)$ satisfies

$$\big([\mathbf{X}, \mathbf{Y}] \mid \mathbf{Z}\big) + \big([\mathbf{X}, \mathbf{Z}] \mid \mathbf{Y}\big) = 0. \tag{29.48}$$

Definition 29.2.13 The symmetric bilinear form $(\cdot|\cdot) : \mathfrak{v} \times \mathfrak{v} \to \mathbb{C}$ defined by (29.47) is called the **Killing form** of $\mathfrak{v}$.

Killing form of a Lie algebra

It is an immediate consequence of this definition and Eq. (29.46) that the Killing form of $\mathfrak{v}$ is invariant under all automorphisms of $\mathfrak{v}$.

As noted above, the Killing form is an *inner product* if the Lie algebra consists of hermitian operators. This will certainly happen if the Lie algebra is that of a group whose elements are unitary operators on some vector space $\mathcal{V}$. We shall see shortly that such unitary operators are not only possible, but have extremely useful properties in the representation of compact Lie groups. A unitary representation of a Lie group induces a representation of its Lie algebra whose "vectors" are hermitian operators. Then the Killing form becomes an inner product. The natural existence of such Killing forms for the representation of a *compact* Lie group motivates the following:

Definition 29.2.14 A Lie algebra $\mathfrak{v}$ is **compact** if it has an *inner product* $(\cdot|\cdot)$ satisfying

compact Lie algebra

$$\big([\mathbf{X}, \mathbf{Y}] \mid \mathbf{Z}\big) + \big([\mathbf{X}, \mathbf{Z}] \mid \mathbf{Y}\big) = 0.$$

Choose a basis $\{\mathbf{X}_i\}$ for the Lie algebra $\mathfrak{v}$ and note that $(\mathrm{ad}_{\mathbf{X}_i})^k_j = c^k_{ij}$. Therefore,

$$(\mathbf{X}_i \mid \mathbf{X}_j) \equiv \mathrm{tr}(\mathrm{ad}_{\mathbf{X}_i} \circ \mathrm{ad}_{\mathbf{X}_j}) = (\mathrm{ad}_{\mathbf{X}_i})^k_l (\mathrm{ad}_{\mathbf{X}_j})^l{}_k = c^k_{il} c^l_{jk} \equiv g_{ij}, \tag{29.49}$$

where g_{ij} are components of the so-called **Cartan metric tensor** in the basis $\{\mathbf{X}_i\}$. If $\mathbf{A}, \mathbf{B} \in \mathfrak{v}$ have components $\{a^i\}$ and $\{b^i\}$ in the basis $\{\mathbf{X}_i\}$, then it follows from Eq. (29.49) that

Cartan metric tensor of a Lie algebra

$$(\mathbf{A} \mid \mathbf{B}) = a^i b^j g_{ij}, \tag{29.50}$$

as expected of a symmetric bilinear form. We can use the Cartan metric to lower the upper index of the structure constants: $c_{ijk} \equiv c^l_{ij} g_{lk}$. By virtue of

Eq. (29.49), the new constants may be written in the form

$$c_{ijk} = c_{ij}^{l} c_{ls}^{r} c_{kr}^{s} = \left(-c_{js}^{l} c_{li}^{r} - c_{si}^{l} c_{lj}^{r}\right) c_{kr}^{s} \quad \text{by (29.13)}$$
$$= c_{js}^{l} c_{il}^{r} c_{kr}^{s} + c_{si}^{l} c_{lj}^{r} c_{rk}^{s}.$$

The reader may now verify that the RHS is completely antisymmetric in i, j, and k. If the Lie algebra is compact, then one can choose an orthonormal basis in which $g_{lk} = \delta_{lk}$ (because the inner product is, by definition, positive definite) and obtain $c_{ij}^{k} = c_{ijk}$. We therefore have the following result.

Proposition 29.2.15 *Let $\mathfrak{v}$ be a compact Lie algebra. Then there exists a basis of $\mathfrak{v}$ in which the structure constants are represented by a third-order completely antisymmetric covariant tensor.*

Historical Notes

Wilhelm Karl Joseph Killing 1847–1923

Most mathematicians seem to have little or no interest in history, so that often the name attached to a key result is that of the follow-up person who exploited an idea or theorem rather than its originator (the Jordan form is due to Weierstrass, Wedderburn theory to Cartan and Molien). No one has suffered from this ahistoricism more than Killing. For example, the so-called "Cartan sub-algebra" and "Cartan matrix" were defined and exploited by Killing. He exhibited the characteristic equation of an arbitrary element of the Weyl group when Weyl was 3 years old and listed the orders of the Coxeter transformation 19 years before Coxeter was born!

Wilhelm Karl Joseph Killing (1847–1923) began university study in Münster in 1865 but quickly moved to Berlin and came under the influence of Kummer and Weierstrass. From 1868 to 1882 much of Killing's energy was devoted to teaching at the gymnasium level in Berlin and Brilon (south of Münster). At one stage, when Weierstrass was urging him to write up his research on space structures, he was spending as much as 36 hours per week in the classroom or tutoring. (Now many mathematicians consider 6 hours a week an intolerable burden!) On the recommendation of Weierstrass, Killing was appointed professor of mathematics at the Lyzeum Hosianum in Braunsberg, in East Prussia (now Braniewo in the region of Olsztyn, in Poland). This was a college founded in 1565 by Bishop Stanislaus Hosius, whose treatise on the Christian faith ran to 39 editions! The main object of the college was the training of Roman Catholic clergy, so Killing had to teach a wide range of topics, including the reconciliation of faith and science. Although he was isolated mathematically during his ten years in Braunsberg, this was the most creative period in his mathematical life. Killing produced his brilliant work despite worries about the health of his wife and seven children, demanding administrative duties as rector of the college and as a member and chairman of the City Council, and his active role in the Church of St. Catherine.

What we now call Lie algebras were invented by the Norwegian mathematician Sophus Lie about 1870 *and independently* by Killing about 1880. Lie was seeking to develop an approach to the solution of differential equations analogous to the Galois theory of algebraic equations. Killing's consuming passion was non-Euclidean geometries and their generalizations, so he was led to the problem of classifying infinitesimal motions of a rigid body in any type of space (or *Raumformen*, as he called them).

In 1892 he was called back to his native Westphalia as professor of mathematics at the University of Münster, where he was quickly submerged in teaching, administration, and charitable activities. He was Rector Magnificus for some period and president of the St. Vincent de Paul charitable society for ten years. Killing's work was neglected partly because he was a modest man with high standards; he vastly underrated his own achievement. His interest was geometry, and for this he needed all real Lie algebras. To obtain merely the simple Lie algebras over the complex numbers did not appear to him to be very significant. Another reason was due to Lie, who was quite negative about Killing's work. At the top of page 770 of a three-volume joint work of Lie and Engel we find the

following less than generous comment about Killing: "With the exception of the preceding unproved theorem … all the theorems that are correct are due to Lie and all the false ones are due to Killing!"
Killing was conservative in his political views and vigorously opposed the attempt to reform the examination requirements for graduate students at the University of Münster by deleting the compulsory study of philosophy. Engel comments "Killing could not see that for most candidates the test in philosophy was completely worthless." He had a profound patriotic love of his country, so that his last years (1918–1923) were deeply pained by the collapse of social cohesion in Germany after the War of 1914–18.
(Taken from A.J. Coleman, "The Greatest Mathematical Paper of All Times," *Mathematical Intelligencer* **11**(3) (1989) 29–38.)

Example 29.2.16 We can calculate explicitly the Killing form of the Lie algebras $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{sl}(n,\mathbb{R})$. Choose the Weyl basis introduced in Example 29.2.7 and expand $\mathsf{A}, \mathsf{B} \in \mathfrak{gl}(n,\mathbb{R})$ in terms of the Weyl basis vectors: $\mathsf{A} = a^{ij}\mathsf{e}_{ij}$, $\mathsf{B} = b^{ij}\mathsf{e}_{ij}$. The Cartan metric tensor becomes

$$g_{ij,kl} = c_{ij,mn}^{rs} c_{kl,rs}^{mn} = \left(\delta_{jm}\delta_i^r\delta_n^s - \delta_{in}\delta_m^r\delta_j^s\right)\left(\delta_{lr}\delta_k^m\delta_s^n - \delta_{ks}\delta_r^m\delta_l^n\right),$$

where we have used Eq. (29.34). It follows from these relations, Eq. (29.50), and a simple index manipulation that

$$(\mathsf{A}\mid\mathsf{B}) \equiv a^{ij}b^{kl}g_{ij,kl} = 2n\,\mathrm{tr}(\mathsf{AB}) - 2\,\mathrm{tr}\,\mathsf{A}\,\mathrm{tr}\,\mathsf{B} \tag{29.51}$$

for $\mathsf{A}, \mathsf{B} \in \mathfrak{gl}(n,\mathbb{R})$, and

$$(\mathsf{A}\mid\mathsf{B}) = 2n\,\mathrm{tr}(\mathsf{AB}) \tag{29.52}$$

for $\mathsf{A}, \mathsf{B} \in \mathfrak{sl}(n,\mathbb{R})$, because all matrices in $\mathfrak{sl}(n,\mathbb{R})$ are traceless.

A Lie algebra $\mathfrak{v}$, as a vector space, may be written as a direct sum of its subspaces. We express this as

$$\mathfrak{v} = \mathfrak{u}_1 \oplus_V \mathfrak{u}_2 \oplus_V \cdots \oplus_V \mathfrak{u}_r = \sum_{k=1}^{r} \oplus_V \mathfrak{u}_k.$$

If in addition $\{\mathfrak{u}_k\}$ are Lie subalgebras every one of which commutes with the rest, we write

$$\mathfrak{v} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \cdots \oplus \mathfrak{u}_r \equiv \bigoplus_{k=1}^{r} \mathfrak{u}_k \tag{29.53}$$

and say that $\mathfrak{v}$ has been **decomposed** into a direct sum of Lie algebras. In this case, each $\mathfrak{u}_k$ is not only a subalgebra, but also an ideal of $\mathfrak{v}$ (see Proposition 3.2.11).

The study of the structure of Lie algebras boils down to the study of the "simplest" kind of Lie algebras in terms of which other Lie algebras can be decomposed. Intuitively, one would want to call a Lie algebra "simple" if it has no proper subalgebras. However, in terms of decomposition, such subalgebras are required to be ideals. So the natural definition of a simple Lie algebra would be the following (see Definition 3.2.12):

semisimple Lie algebras

Definition 29.2.17 A Lie algebra that has no proper ideal is called a **simple** Lie algebra. A Lie algebra is **semisimple** if it has no (nonzero) commutative ideal.

For example, the pseudo-orthogonal algebra $\mathfrak{o}(p, n-p)$ is semisimple, but the Poincaré algebra $\mathfrak{p}(p, n-p)$ is not because the translation generators $\mathbf{P}_j$ form a commutative ideal.

A useful criterion for semisimplicity is given by the following theorem due to Cartan, which we state without proof (for a proof, see [Baru 86, pp. 15–16]):

Theorem 29.2.18 (Cartan) *A Lie algebra $\mathfrak{v}$ is semisimple iff* $\det(g_{ij}) \neq 0$.

The importance of semisimple Lie algebras is embodied in the following theorem: [Baru 86, pp. 19–20].

Theorem 29.2.19 (Cartan) *A semisimple complex or real Lie algebra can be decomposed into a direct sum of pairwise orthogonal simple subalgebras. This decomposition is unique up to ordering.*

This is the analogue of Theorem 3.5.25.

The orthogonality is with respect to the Killing form. Theorem 29.2.19 reduces the study of semisimple Lie algebras to that of simple Lie algebras. What about a general Lie algebra $\mathfrak{v}$? If $\mathfrak{v}$ is compact, then it turns out that it can be written as $\mathfrak{v} = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is the center of $\mathfrak{v}$ and $\mathfrak{s}$ is semisimple. If $\mathfrak{v}$ is not compact, then the decomposition will not be in terms of a direct sum, but in terms of what is called a *semidirect* sum one of whose factors is semisimple. For details, the reader is referred to the fairly accessible treatment of Barut and Raczka, Chap. 2. From now on we shall restrict our discussion to semisimple Lie algebras. These algebras are completely known, because simple algebras have been completely classified. We shall not pursue the classification of Lie algebras. However, we simply state a definition that is used in such a classification, because we shall have an occasion to use it in the representation theory of Lie algebras.

Cartan subalgebra and the rank of a Lie algebra

Definition 29.2.20 Let $\mathfrak{v}$ be a Lie algebra. A subalgebra $\mathfrak{h}$ of $\mathfrak{v}$ is called a **Cartan subalgebra** if $\mathfrak{h}$ is the largest commutative subalgebra of $\mathfrak{v}$, and for all $\mathbf{X} \in \mathfrak{h}$, if $\mathrm{ad}_{\mathbf{X}}$ leaves a subspace of $\mathfrak{v}$ invariant, then it leaves the complement of $\mathfrak{v}$ invariant as well. The dimension of $\mathfrak{h}$ is called the **rank** of $\mathfrak{v}$.

29.3 Problems

29.1 Show that the set $G = GL(n, \mathbb{R}) \times \mathbb{R}^n$ equipped with the "product"

$$(\mathsf{A}, \mathbf{u})(\mathsf{B}, \mathbf{v}) \equiv (\mathsf{AB}, \mathsf{A}\mathbf{v} + \mathbf{u})$$

affine group

forms a group. This is called the **affine group**.

29.2 Show that $m : U \times U \to \mathbb{R}$ defined in Example 29.1.5 is a local Lie group.

29.3 Find the multiplication law for the groups in (b) and (c) of Example 29.1.9.

29.4 Show that the one-dimensional projective group of Example 29.1.9 satisfies all the group properties. In particular, find the identity and the inverse of an element in the group.

29.5 Let G be a Lie group. Let S be a subgroup of G that is also a submanifold of G. Show that S is a Lie group.

29.6 Show that the differential map of $\psi : GL(\mathcal{V}) \to \mathcal{H}(\mathcal{V})$, defined by $\psi(\mathbf{A}) = \mathbf{A}\mathbf{A}^\dagger$, where $\mathcal{H}(\mathcal{V})$ is the set of hermitian operators on $\mathcal{V}$, is surjective. Derive Eq. (29.11).

29.7 Verify that $I_g \equiv R_g^{-1} \circ L_g$ is an isomorphism.

29.8 Prove Proposition 29.1.24.

29.9 Start with Eq. (29.24) and use the fact that second derivative is independent of the order of differentiation to obtain

$$u_{i\kappa}\left[\frac{\partial\theta^{-1}_{\kappa\mu}}{\partial a_\lambda} - \frac{\partial\theta^{-1}_{\kappa\lambda}}{\partial a_\mu}\right] + \theta^{-1}_{\kappa\mu}\frac{\partial u_{i\kappa}}{\partial a_\lambda} - \theta^{-1}_{\kappa\lambda}\frac{\partial u_{i\kappa}}{\partial a_\mu} = 0.$$

Now use the chain rule $\partial u_{i\kappa}/\partial a_\lambda = (\partial u_{i\kappa}/\partial x_j)(\partial x_j/\partial a_\lambda)$ and Eq. (29.24) to get

$$u_{i\kappa}\left[\frac{\partial\theta^{-1}_{\kappa\mu}}{\partial a_\lambda} - \frac{\partial\theta^{-1}_{\kappa\lambda}}{\partial a_\mu}\right] + \left[u_{j\nu}\frac{\partial u_{i\kappa}}{\partial x_j} - u_{j\kappa}\frac{\partial u_{i\nu}}{\partial x_j}\right]\theta^{-1}_{\kappa\mu}\theta^{-1}_{\nu\lambda} = 0,$$

or

$$u_{j\sigma}\frac{\partial u_{i\tau}}{\partial x_j} - u_{j\tau}\frac{\partial u_{i\sigma}}{\partial x_j} = c^\kappa_{\sigma\tau}(\mathbf{a})u_{i\kappa}(\mathbf{x}), \tag{29.54}$$

where

$$c^\kappa_{\sigma\tau}(\mathbf{a}) = \left[\frac{\partial\theta^{-1}_{\kappa\mu}}{\partial a_\lambda} - \frac{\partial\theta^{-1}_{\kappa\lambda}}{\partial a_\mu}\right]\theta_{\mu\sigma}\theta_{\lambda\tau}.$$

Substituting Eq. (29.54) in Eq. (29.26) leads to (29.27). Now differentiate both sides of Eq. (29.54) with respect to a_ρ to get

$$\frac{\partial c^\kappa_{\sigma\tau}}{\partial a_\rho}u_{i\kappa} = 0.$$

With the assumption that the $u_{i\kappa}$ are linearly independent, conclude that the structure "constants" are indeed constants.

29.10 Using

$$x_1' \equiv \phi_1(x_1, x_2, x_3; \theta, \varphi, \psi),$$
$$x_2' \equiv \phi_2(x_1, x_2, x_3; \theta, \varphi, \psi),$$
$$x_3' \equiv \phi_3(x_1, x_2, x_3; \theta, \varphi, \psi),$$

obtained from the multiplication of the column vector consisting of x_1, x_2, and x_3 by the Euler matrix of Example 5.2.7 and employing Eq. (29.25), find the three components of the angular momentum.

29.11 Find the invariant Haar measure of the general linear group in two dimensions.

29.12 Show that the invariant Haar measure for a compact group satisfies $d\boldsymbol{\mu}_g = d\boldsymbol{\mu}_{g^{-1}}$. Hint: Define a measure $\boldsymbol{\nu}$ by $d\boldsymbol{\nu}_g \equiv d\boldsymbol{\mu}_{g^{-1}}$ and show that $\boldsymbol{\nu}$ is left-invariant. Now use the uniqueness of the left-invariant Haar measure for compact groups.

29.13 Show that $O(p, n-p)$ is a group. Use this and the fact that $\boldsymbol{\eta}^{-1} = \boldsymbol{\eta}$ to show that $\mathsf{A}\boldsymbol{\eta}\mathsf{A}^t = \boldsymbol{\eta}$.

29.14 Show that the orthogonal group $O(p, n-p)$ has dimension $n(n-1)/2$. Hint: Look at its algebra $\mathfrak{o}(p, n-p)$.

29.15 Let $\mathsf{x} = (x_1, x_2, x_3, x_0)$ be a timelike (null, isotropic) 4-vector with $x_0 > 0$. Let A be a proper orthochronous transformation. Show that $\mathsf{x}' = \mathsf{A}\mathsf{x}$ is also timelike (null). Hint: Consider the zeroth component of x' as an inner product of (x_1, x_2, x_3, x_0) and another vector and use Schwarz inequality.

29.16 Starting with the definition of each matrix, derive Eq. (29.45).

29.17 Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be derivations of a Lie algebra $\mathfrak{v}$. Show that $\mathbf{D}_1\mathbf{D}_2 \equiv \mathbf{D}_1 \circ \mathbf{D}_2$ is not a derivation, but $[\mathbf{D}_1, \mathbf{D}_2]$ is.

29.18 Let $\mathfrak{v}$ be a Lie algebra. Verify that $\mathrm{ad}_{\mathbf{X}}$ is a derivation of $\mathfrak{v}$ for any $\mathbf{X} \in \mathfrak{v}$, and that $\mathrm{ad}_{[\mathbf{X},\mathbf{Y}]} = [\mathrm{ad}_{\mathbf{X}}, \mathrm{ad}_{\mathbf{Y}}]$.

29.19 Show that $\psi : \mathfrak{v} \to \mathfrak{ad}_{\mathfrak{v}}$ given by $\psi(\mathbf{X}) = \mathrm{ad}_{\mathbf{X}}$ is (a) a homomorphism, (b) $\ker\psi$ is the center of $\mathfrak{v}$, and (c) $\mathfrak{ad}_{\mathfrak{v}}$ is an ideal of $\mathfrak{D}_{\mathfrak{v}}$.

29.20 Show that if ψ is an automorphism of $\mathfrak{v}$, then

$$\mathrm{ad}_{\psi(\mathbf{X})} = \psi \circ \mathrm{ad}_{\mathbf{X}} \circ \psi^{-1} \quad \forall \mathbf{X} \in \mathfrak{v}.$$

Hint: Apply both sides to an arbitrary element of $\mathfrak{v}$.

29.21 Show that for *any* Lie algebra,

$$c_{ijk} = c^l_{js} c^r_{il} c^s_{kr} + c^l_{si} c^r_{lj} c^s_{rk}$$

is completely antisymmetric in all its indices.

29.22 Show that the Killing form of $\mathfrak{v}$ is invariant under all automorphisms of $\mathfrak{v}$.

29.23 Show that the translation generators $\mathbf{P}_j$ of the Poincaré algebra $\mathfrak{p}(p, n-p)$ form a commutative ideal.

29.24 Find the Cartan metrics for $\mathfrak{o}(3, 1)$ and $\mathfrak{p}(3, 1)$, and show directly that the first is semisimple but the second is not.