# Mathematical Preliminaries

# 1

This introductory chapter gathers together some of the most basic tools and notions that are used throughout the book. It also introduces some common vocabulary and notations used in modern mathematical physics literature. Readers familiar with such concepts as sets, maps, equivalence relations, and metric spaces may wish to skip this chapter.

## 1.1 Sets

Modern mathematics starts with the basic (and undefinable) concept of **set**. We think of a set as a structureless family, or collection, of objects. We speak, for example, of the set of students in a college, of men in a city, of women working for a corporation, of vectors in space, of points in a plane, or of events in the continuum of space-time. Each member $a$ of a set $A$ is called an **element** of that set. This relation is denoted by $a \in A$ (read "$a$ is an element of $A$" or "$a$ belongs to $A$" ), and its negation by $a \notin A$. Sometimes $a$ is called a **point** of the set $A$ to emphasize a geometric connotation.

concept of set elaborated

A set is usually designated by enumeration of its elements between braces. For example, $\{2, 4, 6, 8\}$ represents the set consisting of the first four even natural numbers; $\{0, \pm 1, \pm 2, \pm 3, \dots\}$ is the set of all integers; $\{1, x, x^2, x^3, \dots\}$ is the set of all nonnegative powers of $x$; and $\{1, i, -1, -i\}$ is the set of the four complex fourth roots of unity. In many cases, a set is defined by a (mathematical) statement that holds for all of its elements. Such a set is generally denoted by $\{x \mid P(x)\}$ and read "the set of all $x$'s such that $P(x)$ is true." The foregoing examples of sets can be written alternatively as follows:

$$\{n \mid n \text{ is even and } 1 < n < 9\}$$

$$\{\pm n \mid n \text{ is a natural number}\}$$

$$\bigl\{y \mid y = x^n \text{ and } n \text{ is a natural number}\bigr\}$$

$$\bigl\{z \mid z^4 = 1 \text{ and } z \text{ is a complex number}\bigr\}.$$

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_1,

In a frequently used shorthand notation, the last two sets can be abbreviated as $\{x^n \mid n \geq 0$ and $n$ is an integer$\}$ and $\{z \in \mathbb{C} \mid z^4 = 1\}$. Similarly, the unit circle can be denoted by $\{z \in \mathbb{C} \mid |z| = 1\}$, the closed interval $[a, b]$ as $\{x \mid a \leq x \leq b\}$, the open interval $(a, b)$ as $\{x \mid a < x < b\}$, and the set of all nonnegative powers of $x$ as $\{x^n\}_{n=0}^{\infty}$ or $\{x^n\}_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers (i.e., nonnegative integers). This last notation will be used frequently in this book. A set with a single element is called a **singleton**.

singleton

If $a \in A$ whenever $a \in B$, we say that $B$ is a **subset** of $A$ and write $B \subset A$ or $A \supset B$. If $B \subset A$ and $A \subset B$, then $A = B$. If $B \subset A$ and $A \neq B$, then $B$ is called a **proper** subset of $A$. The set defined by $\{a \mid a \neq a\}$ is called the **empty set** and is denoted by $\emptyset$. Clearly, $\emptyset$ contains no elements and is a subset of any arbitrary set. The collection of all subsets (including $\emptyset$) of a set $A$ is denoted by $2^A$. The reason for this notation is that the number of subsets of a set containing $n$ elements is $2^n$ when $n$ is finite (Problem 1.1).

(proper) subset

empty set

If $A$ and $B$ are sets, their **union**, denoted by $A \cup B$, is the set containing all elements that belong to $A$ or $B$ or both. The **intersection** of the sets $A$ and $B$, denoted by $A \cap B$, is the set containing all elements belonging to both $A$ and $B$. If $\{B_\alpha\}_{\alpha\in I}$ is a collection of sets,[1] we denote their union by $\bigcup_{\alpha\in I} B_\alpha$ and their intersection by $\bigcap_{\alpha\in I} B_\alpha$.

union, intersection, complement

The **complement** of a set $A$ is denoted by $\sim A$ and defined as

$$\sim A \equiv \{a \mid a \notin A\}.$$

The complement of $B$ in $A$ (or their difference) is

$$A \sim B \equiv \{a \mid a \in A \text{ and } a \notin B\}.$$

In any application of set theory there is an underlying **universal set** whose subsets are the objects of study. This universal set is usually clear from the context. For example, in the study of the properties of integers, the set of integers, denoted by $\mathbb{Z}$, is the universal set. The set of reals, $\mathbb{R}$, is the universal set in real analysis, and the set of complex numbers, $\mathbb{C}$, is the universal set in complex analysis. To emphasize the presence of a universal set $X$, one can write $X \sim A$ instead of $\sim A$.

universal set

From two given sets $A$ and $B$, it is possible to form the **Cartesian product** of $A$ and $B$, denoted by $A \times B$, which is the set of **ordered pairs** $(a, b)$, where $a \in A$ and $b \in B$. This is expressed in set-theoretic notation as

Cartesian product

ordered pairs

$$A \times B = \big\{(a, b) \mid a \in A \text{ and } b \in B\big\}.$$

We can generalize this to an arbitrary number of sets. If $A_1, A_2, \ldots, A_n$ are sets, then the Cartesian product of these sets is

$$A_1 \times A_2 \times \cdots \times A_n = \big\{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\big\},$$

[1] Here $I$ is an index set—or a counting set—with its typical element denoted by $\alpha$. In most cases, $I$ is the set of (nonnegative) integers, but, in principle, it can be any set, for example, the set of real numbers.

which is a set of ordered $n$-tuples. If $A_1 = A_2 = \cdots = A_n = A$, then we write $A^n$ instead of $A \times A \times \cdots \times A$, and

$$A^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A\}.$$

The most familiar example of a Cartesian product occurs when $A = \mathbb{R}$. Then $\mathbb{R}^2$ is the set of pairs $(x_1, x_2)$ with $x_1, x_2 \in \mathbb{R}$. This is simply the points in the plane. Similarly, $\mathbb{R}^3$ is the set of triplets $(x_1, x_2, x_3)$, or the points in space, and $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{R}\}$ is the set of real $n$-tuples.

### 1.1.1 Equivalence Relations

There are many instances in which the elements of a set are naturally grouped together. For example, all vector potentials that differ by the gradient of a scalar function can be grouped together because they all give the same magnetic field. Similarly, all quantum state functions (of unit "length") that differ by a multiplicative complex number of unit length can be grouped together because they all represent the same physical state. The abstraction of these ideas is summarized in the following definition.

relation and equivalence relation

**Definition 1.1.1** Let $A$ be a set. A **relation** on $A$ is a comparison test between members of ordered pairs of elements of $A$. If the pair $(a, b) \in A \times A$ passes this test, we write $a \rhd b$ and read "$a$ is related to $b$". An **equivalence relation** on $A$ is a relation that has the following properties:

$$a \rhd a \quad \forall a \in A, \qquad \text{(reflexivity)}$$

$$a \rhd b \Rightarrow b \rhd a \quad a, b \in A, \qquad \text{(symmetry)}$$

$$a \rhd b, \text{ and } b \rhd c \Rightarrow a \rhd c \quad a, b, c \in A, \qquad \text{(transivity).}$$

When $a \rhd b$, we say that "$a$ is equivalent to $b$". The set $[\![a]\!] = \{b \in A \mid b \rhd a\}$ of all elements that are equivalent to $a$ is called the **equivalence class** of $a$.

equivalence class

The reader may verify the following property of equivalence relations.

**Proposition 1.1.2** *If $\rhd$ is an equivalence relation on $A$ and $a, b \in A$, then either $[\![a]\!] \cap [\![b]\!] = \emptyset$ or $[\![a]\!] = [\![b]\!]$.*

Therefore, $a' \in [\![a]\!]$ implies that $[\![a']\!] = [\![a]\!]$. In other words, any element of an equivalence class can be chosen to be a **representative** of that class. Because of the symmetry of equivalence relations, sometimes we denote them by $\bowtie$.

representative of an equivalence class

**Example 1.1.3** Let $A$ be the set of human beings. Let $a \rhd b$ be interpreted as "$a$ is older than $b$." Then clearly, $\rhd$ is a relation but not an equivalence relation. On the other hand, if we interpret $a \rhd b$ as "$a$ and $b$ live in the same city," then $\rhd$ is an equivalence relation, as the reader may check. The equivalence class of $a$ is the population of that city.

Let $V$ be the set of vector potentials. Write $\mathbf{A} \rhd \mathbf{A}'$ if $\mathbf{A} - \mathbf{A}' = \nabla f$ for some function $f$. The reader may verify that $\rhd$ is an equivalence relation, and that $[\![\mathbf{A}]\!]$ is the set of all vector potentials giving rise to the same magnetic field.

Let the underlying set be $\mathbb{Z} \times (\mathbb{Z} \sim \{0\})$. Say "$(a, b)$ is related to $(c, d)$" if $ad = bc$. Then this relation is an equivalence relation. Furthermore, $[\![(a, b)]\!]$ can be identified as the ratio $a/b$.

partition of a set

**Definition 1.1.4** Let $A$ be a set and $\{B_\alpha\}$ a collection of subsets of $A$. We say that $\{B_\alpha\}$ is a **partition** of $A$, or $\{B_\alpha\}$ **partitions** $A$, if the $B_\alpha$'s are disjoint, i.e., have no element in common, and $\bigcup_\alpha B_\alpha = A$.

quotient set

Now consider the collection $\{[\![a]\!] \mid a \in A\}$ of all equivalence classes of $A$. These classes are disjoint, and evidently their union covers all of $A$. Therefore, *the collection of equivalence classes of $A$ is a partition of $A$*. This collection is denoted by $A/\bowtie$ and is called the **quotient set** or **factor set** of $A$ under the equivalence relation $\bowtie$.

projective space

**Example 1.1.5** Let the underlying set be $\mathbb{R}^3$. Define an equivalence relation on $\mathbb{R}^3$ by saying that $P_1 \in \mathbb{R}^3$ and $P_2 \in \mathbb{R}^3$ are equivalent if they lie on the same line passing through the origin. Then $\mathbb{R}^3/\bowtie$ is the set of all lines in space passing through the origin. If we choose the unit vector with positive third coordinate along a given line as the representative of that line, then $\mathbb{R}^3/\bowtie$, called the **projective space** associated with $\mathbb{R}^3$, is almost (but not quite) the same as the upper unit hemisphere. The difference is that any two points on the edge of the hemisphere which lie on the same diameter ought to be identified as the same to turn it into the projective space.

On the set $\mathbb{Z}$ of integers, define a relation by writing $m \rhd n$ for $m, n \in \mathbb{Z}$ if $m - n$ is divisible by $k$, where $k$ is a fixed integer. Then $\rhd$ is not only a relation, but an equivalence relation. In this case, we have

$$\mathbb{Z}/\rhd = \big\{[\![0]\!], [\![1]\!], \ldots, [\![k-1]\!]\big\},$$

as the reader is urged to verify.

For the equivalence relation defined on $\mathbb{Z} \times (\mathbb{Z} \sim \{0\})$ of Example 1.1.3, the set $(\mathbb{Z} \times (\mathbb{Z} \sim \{0\}))/\bowtie$ can be identified with $\mathbb{Q}$, the set of rational numbers.

## 1.2 Maps

map, domain, codomain, image

To communicate between sets, one introduces the concept of a map. A **map** $f$ from a set $X$ to a set $Y$, denoted by $f : X \to Y$ or $X \xrightarrow{f} Y$, is a correspondence between elements of $X$ and those of $Y$ in which all the elements of $X$ participate, and each element of $X$ corresponds to only one element of $Y$ (see Fig. 1.1). If $y \in Y$ is the element that corresponds to $x \in X$ via the map $f$, we write

$$y = f(x) \quad \text{or} \quad x \mapsto f(x) \quad \text{or} \quad x \xmapsto{f} y$$

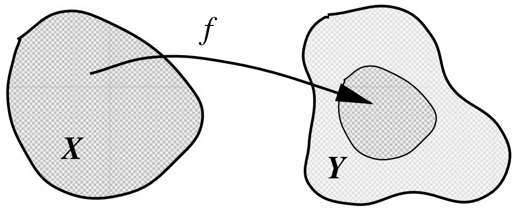


**Fig. 1.1** The map $f$ maps all of the set $X$ onto a subset of $Y$. The *shaded area* in $Y$ is $f(X)$, the range of $f$

and call $f(x)$ the **image** of $x$ under $f$. Thus, by the definition of map, $x \in X$ can have only one image. The set $X$ is called the **domain**, and $Y$ the **codomain** or the **target space**. Two maps $f: X \to Y$ and $g: X \to Y$ are said to be equal if $f(x) = g(x)$ for all $x \in X$.

function

**Definition 1.2.1** A map whose codomain is the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$ is commonly called a **function**.

A special map that applies to all sets $A$ is $\mathrm{id}_A : A \to A$, called the **identity map** of $A$, and defined by

identity map

$$\mathrm{id}_A(a) = a \quad \forall a \in A.$$

The **graph** $\Gamma_f$ of a map $f: A \to B$ is a subset of $A \times B$ defined by

graph of a map

$$\Gamma_f = \big\{\big(a, f(a)\big) \mid a \in A\big\} \subset A \times B.$$

This general definition reduces to the ordinary graphs encountered in algebra and calculus where $A = B = \mathbb{R}$ and $A \times B$ is the $xy$-plane.

If $A$ is a subset of $X$, we call $f(A) = \{f(x) \mid x \in A\}$ the *image* of $A$. Similarly, if $B \subset f(X)$, we call $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ the inverse image, or **preimage**, of $B$. In words, $f^{-1}(B)$ consists of all elements in $X$ whose images are in $B \subset Y$. If $B$ consists of a single element $b$, then $f^{-1}(b) = \{x \in X \mid f(x) = b\}$ consists of all elements of $X$ that are mapped to $b$. Note that it is possible for many points of $X$ to have the same image in $Y$. The subset $f(X)$ of the codomain of a map $f$ is called the **range** of $f$ (see Fig. 1.1).

preimage

If $f: X \to Y$ and $g: Y \to W$, then the mapping $h: X \to W$ given by $h(x) = g(f(x))$ is called the **composition** of $f$ and $g$, and is denoted by $h = g \circ f$ (see Fig. 1.2).[2] It is easy to verify that

composition of two maps

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

If $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, we call $f$ **injective**, or *one-to-one* (denoted 1-1). For an injective map only one element of $X$ corresponds to an element of $Y$. If $f(X) = Y$, the mapping is said to be **surjective**, or

injection, surjection, and bijection, or 1-1 correspondence

[2]Note the importance of the order in which the composition is written. The reverse order may not even exist.

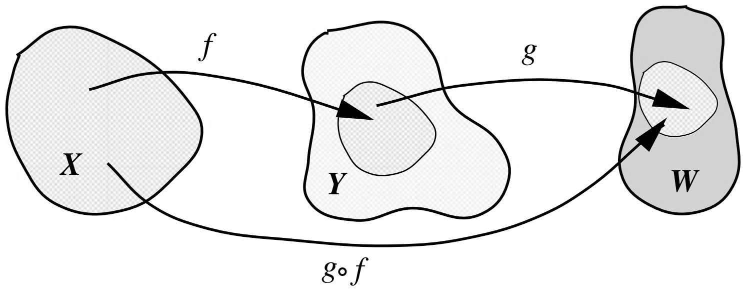


**Fig. 1.2** The composition of two maps is another map

*onto*. A map that is both injective and surjective is said to be **bijective**, or to be a **one-to-one correspondence**. Two sets that are in one-to-one correspondence, have, by definition, the same number of elements. If $f : X \to Y$ is a bijection from $X$ onto $Y$, then for each $y \in Y$ there is one and only one element $x$ in $X$ for which $f(x) = y$. Thus, there is a mapping $f^{-1} : Y \to X$ given by $f^{-1}(y) = x$, where $x$ is the unique element such that $f(x) = y$. This mapping is called the **inverse** of $f$. The inverse of $f$ is also identified as the map that satisfies $f \circ f^{-1} = \text{id}_Y$ and $f^{-1} \circ f = \text{id}_X$. For example, one can easily verify that $\ln^{-1} = \exp$ and $\exp^{-1} = \ln$, because $\ln(e^x) = x$ and $e^{\ln x} = x$.

inverse of a map

Given a map $f : X \to Y$, we can define a relation $\bowtie$ on $X$ by saying $x_1 \bowtie x_2$ if $f(x_1) = f(x_2)$. The reader may check that this is in fact an *equivalence* relation. The equivalence classes are subsets of $X$ all of whose elements map to the same point in $Y$. In fact, $[\![x]\!] = f^{-1}(f(x))$. Corresponding to $f$, there is a map $\tilde{f} : X/\bowtie \to Y$, called **quotient map** or **factor map**, given by $\tilde{f}([\![x]\!]) = f(x)$. This map is injective because if $\tilde{f}([\![x_1]\!]) = \tilde{f}([\![x_2]\!])$, then $f(x_1) = f(x_2)$, so $x_1$ and $x_2$ belong to the same equivalence class; therefore, $[\![x_1]\!] = [\![x_2]\!]$. It follows that

quotient or factor map

**Proposition 1.2.2** *The map* $\tilde{f} : X/\bowtie \to f(X)$ *is bijective.*

If $f$ and $g$ are both bijections with inverses $f^{-1}$ and $g^{-1}$, respectively, then $g \circ f$ also has an inverse, and verifying that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is straightforward.

**Example 1.2.3** As an example of the preimage of a set, consider the sine and cosine functions: $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$. Then it should be clear that

$$\sin^{-1} 0 = \{n\pi\}_{n=-\infty}^{\infty}, \qquad \cos^{-1} 0 = \left\{\frac{\pi}{2} + n\pi\right\}_{n=-\infty}^{\infty}.$$

Similarly, $\sin^{-1}[0, \frac{1}{2}]$, the preimage of the closed interval $[0, \frac{1}{2}] \subset \mathbb{R}$, consists of all the intervals on the $x$-axis marked by heavy line segments in Fig. 1.3, i.e., all the points whose sine lies between 0 and $\frac{1}{2}$.

**Example 1.2.4** Let $X$ be any set on which an equivalence relation $\bowtie$ is defined. Then there is a natural map $\pi$, called **projection** $\pi : X \to X/\bowtie$

projection

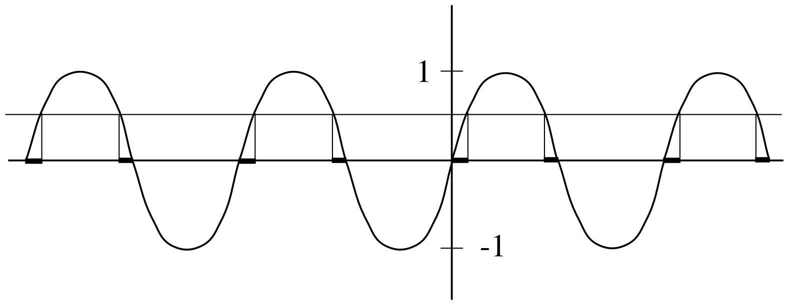


**Fig. 1.3** The union of all the intervals on the $x$-axis marked by heavy line segments is $\sin^{-1}[0, \frac{1}{2}]$

given by $\pi(x) = [\![x]\!]$. This map is obviously surjective, but not injective, as $\pi(y) = \pi(x)$ if $y \bowtie x$. It becomes injective only if the equivalence relation becomes the identity map: $\bowtie = \mathrm{id}_X$. Then the map becomes bijective, and we write $X \cong X/\mathrm{id}_X$.

**Example 1.2.5** As further examples of maps, we consider functions $f : \mathbb{R} \to \mathbb{R}$ studied in calculus. The two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to (-1, +1)$ given, respectively, by $f(x) = x^3$ and $g(x) = \tanh x$ are bijective. The latter function, by the way, shows that there are as many points in the whole real line as there are in the interval $(-1, +1)$. If we denote the set of positive real numbers by $\mathbb{R}^+$, then the function $f : \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = x^2$ is surjective but not injective (both $x$ and $-x$ map to $x^2$). The function $g : \mathbb{R}^+ \to \mathbb{R}$ given by the same rule, $g(x) = x^2$, is injective but not surjective. On the other hand, $h : \mathbb{R}^+ \to \mathbb{R}^+$ again given by $h(x) = x^2$ is bijective, but $u : \mathbb{R} \to \mathbb{R}$ given by the same rule is neither injective nor surjective.

injectivity and surjectivity depend on the domain and codomain

Let $\mathcal{M}^{n\times n}$ denote the set of $n \times n$ real matrices. Define a function $\det : \mathcal{M}^{n\times n} \to \mathbb{R}$ by $\det(\mathsf{A}) = \det \mathsf{A}$. This function is clearly surjective (why?) but not injective. The set of all matrices whose determinant is 1 is $\det^{-1}(1)$. Such matrices occur frequently in physical applications.

Another example of interest is $f : \mathbb{C} \to \mathbb{R}$ given by $f(z) = |z|$. This function is also neither injective nor surjective. Here $f^{-1}(1)$ is the **unit circle**, the circle of radius 1 in the complex plane. It is clear that $f(\mathbb{C}) = \{0\} \cup \mathbb{R}^+$. Furthermore, $f$ induces an equivalence relation on $\mathbb{C}$: $z_1 \bowtie z_2$ if $z_1$ and $z_2$ belong to the same circle. Then $\mathbb{C}/\bowtie$ is the set of circles centered at the origin of the complex plane and $\tilde{f} : \mathbb{C}/\bowtie \to \{0\} \cup \mathbb{R}^+$ is bijective, associating each circle to its radius.

unit circle

The domain of a map can be a Cartesian product of a set, as in $f : X \times X \to Y$. Two specific cases are worthy of mention. The first is when $Y = \mathbb{R}$. An example of this case is the dot product on vectors. Thus, if $X$ is the set of vectors in space, we can define $f(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$. The second case is when $Y = X$. Then $f$ is called a **binary operation** on $X$, whereby an element in $X$ is associated with two elements in $X$. For instance, let $X = \mathbb{Z}$, the set of all integers; then the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(m, n) = mn$ is

binary operation

the binary operation of multiplication of integers. Similarly, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $g(x, y) = x + y$ is the binary operation of addition of real numbers.

## 1.3 Metric Spaces

Although sets are at the root of modern mathematics, by themselves they are only of formal and abstract interest. To make sets useful, it is necessary to introduce some structures on them. There are two general procedures for the implementation of such structures. These are the abstractions of the two major branches of mathematics—algebra and analysis.

We can turn a set into an algebraic structure by introducing a binary operation on it. For example, a vector space consists, among other things, of the binary operation of vector addition. A group is, among other things, a set together with the binary operation of "multiplication". There are many other examples of algebraic systems, and they constitute the rich subject of algebra.

When analysis, the other branch of mathematics, is abstracted using the concept of sets, it leads to topology, in which the concept of continuity plays a central role. This is also a rich subject with far-reaching implications and applications. We shall not go into any details of these two areas of mathematics. Although some algebraic systems will be discussed and the ideas of limit and continuity will be used in the sequel, this will be done in an intuitive fashion, by introducing and employing the concepts when they are needed. On the other hand, some general concepts will be introduced when they require minimum prerequisites. One of these is a metric space:

metric space defined

**Definition 1.3.1** A **metric space** is a set $X$ together with a real-valued function $d : X \times X \to \mathbb{R}$ such that

(a) $d(x, y) \geq 0 \; \forall x, y$, and $d(x, y) = 0$ iff $x = y$.
(b) $d(x, y) = d(y, x)$ (symmetry).
(c) $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality).

It is worthwhile to point out that $X$ is a completely arbitrary set and needs no other structure. In this respect, Definition 1.3.1 is very broad and encompasses many different situations, as the following examples will show. Before examining the examples, note that the function $d$ defined above is the abstraction of the notion of distance: (a) says that the distance between any two points is always nonnegative and is zero only if the two points coincide; (b) says that the distance between two points does not change if the two points are interchanged; (c) states the known fact that the sum of the lengths of two sides of a triangle is always greater than or equal to the length of the third side.

Euclidean and Minkowskian metric spaces

The fact that the distance between two points of a set is positive and real is a property of a **Euclidean** metric space. In relativity, on the other hand, one has to deal with the possibility of a *Minkowskian* metric space for which distance (squared) is negative.

**Example 1.3.2** Here are some examples of metric spaces:

1. Let $X = \mathbb{Q}$, the set of rational numbers, and define $d(x, y) = |x - y|$.
2. Let $X = \mathbb{R}$, and again define $d(x, y) = |x - y|$.
3. Let $X$ consist of the points on the surface of a sphere. We can define two distance functions on $X$. Let $d_1(P, Q)$ be the length of the chord joining $P$ and $Q$ on the sphere. We can also define another metric, $d_2(P, Q)$, as the length of the arc of the great circle passing through points $P$ and $Q$ on the surface of the sphere. It is not hard to convince oneself that $d_1$ and $d_2$ satisfy all the properties of a metric function. Note that for $d_2$, if two of the three points are the poles of the sphere, then the triangle inequality becomes an equality.
4. Let $\mathcal{C}^0[a, b]$ denote the set of continuous real-valued functions on the closed interval $[a, b]$. We can define $d(f, g) = \int_a^b |f(x) - g(x)|\, dx$ for $f, g \in \mathcal{C}^0(a, b)$.
5. Let $\mathcal{C}_B(a, b)$ denote the set of *bounded* continuous real-valued functions on the closed interval $[a, b]$. We then define

$$d(f, g) = \max_{x\in[a,b]} \left\{ \left| f(x) - g(x) \right| \right\} \quad \text{for } f, g \in \mathcal{C}_B(a, b).$$

   This notation says: Take the absolute value of the difference in $f$ and $g$ at all $x$ in the interval $[a, b]$ and then pick the maximum of all these values.

The metric function creates a natural setting in which to test the "closeness" of points in a metric space. One occasion on which the idea of closeness becomes essential is in the study of a sequence. A **sequence** is a mapping $s : \mathbb{N} \to X$ from the set of natural numbers $\mathbb{N}$ into the metric space $X$. Such a mapping associates with a positive integer $n$ a point $s(n)$ of the metric space $X$. It is customary to write $s_n$ (or $x_n$ to match the symbol $X$) instead of $s(n)$ and to enumerate the values of the function by writing $\{x_n\}_{n=1}^\infty$.

sequence defined

Knowledge of the behavior of a sequence for large values of $n$ is of fundamental importance. In particular, it is important to know whether a sequence approaches a finite value as $n$ increases.

convergence defined

**Definition 1.3.3** Suppose that for some $x$ and for any positive real number $\epsilon$, there exists a natural number $N$ such that $d(x_n, x) < \epsilon$ whenever $n > N$. Then we say that the sequence $\{x_n\}_{n=1}^\infty$ **converges** to $x$ and write $\lim_{n\to\infty} d(x_n, x) = 0$ or $d(x_n, x) \to 0$ or simply $x_n \to x$.

It may not be possible to test directly for the convergence of a given sequence because this requires a knowledge of the limit point $x$. However, it is possible to do the next best thing—to see whether the points of the sequence get closer and closer as $n$ gets larger and larger.

**Definition 1.3.4** A **Cauchy sequence** is a sequence for which

Cauchy sequence

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

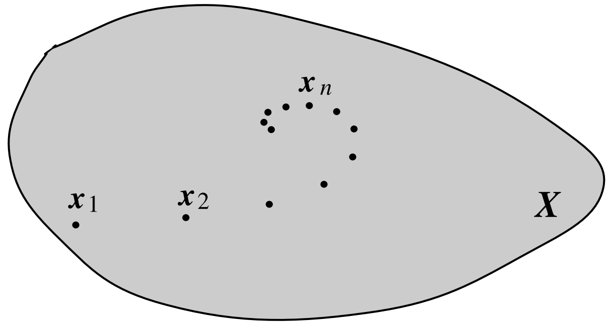


**Fig. 1.4** The distance between the elements of a Cauchy sequence gets smaller and smaller

Figure 1.4 shows a Cauchy sequence.

We can test directly whether or not a sequence is Cauchy. However, the fact that a sequence is Cauchy does not guarantee that it converges. For example, let the metric space be the set of rational numbers $\mathbb{Q}$ with the metric function $d(x, y) = |x - y|$, and consider the sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n = \sum_{k=1}^{n}(-1)^{k+1}/k$. It is clear that $x_n$ is a rational number for any $n$. Problem 1.7 shows how to prove that $|x_m - x_n| \to 0$. Thus, the sequence is Cauchy. However, it is probably known to the reader that $\lim_{n\to\infty} x_n = \ln 2$, which is not a rational number.

complete metric space

**Definition 1.3.5** A metric space in which every Cauchy sequence converges is called a **complete metric space**.

Complete metric spaces play a crucial role in modern analysis. The preceding example shows that $\mathbb{Q}$ is not a complete metric space. However, if the limit points of all Cauchy sequences are added to $\mathbb{Q}$, the resulting space becomes complete. This complete space is, of course, the real number system $\mathbb{R}$. It turns out that any incomplete metric space can be "enlarged" to a complete metric space.

## 1.4 Cardinality

cardinality

The process of counting is a one-to-one comparison of one set with another. If two sets are in one-to-one correspondence, they are said to have the same **cardinality**. Two sets with the same cardinality essentially have the same "number" of elements. The set $F_n = \{1, 2, \dots, n\}$ is finite and has cardinality $n$. Any set from which there is a bijection onto $F_n$ is said to be finite with $n$ elements.

**Historical Notes**

Although some steps had been taken before him in the direction of a definitive theory of sets, the creator of the theory of sets is considered to be **Georg Cantor** (1845–1918), who was born in Russia of Danish-Jewish parentage but moved to Germany with his parents. His father urged him to study engineering, and Cantor entered the University of Berlin in 1863 with that intention. There he came under the influence of Weierstrass and turned to pure mathematics. He became Privatdozent at Halle in 1869 and professor in 1879. When he was twenty-nine he published his first revolutionary paper on the theory of infinite sets in the *Journal für Mathematik*. Although some of its propositions were deemed faulty by the older mathematicians, its overall originality and brilliance attracted attention. He continued to publish papers on the theory of sets and on transfinite numbers until 1897.

One of Cantor's main concerns was to differentiate among infinite sets by "size" and, likeBolzano before him, he decided that one-to-one correspondence should be the basic principle. In his correspondence with Dedekind in 1873, Cantor posed the question of whether the set of real numbers can be put into one-to-one correspondence with the integers, and some weeks later he answered in the negative. He gave two proofs. The first is more complicated than the second, which is the one most often used today. In 1874 Cantor occupied himself with the equivalence of the points of a line and the points of $\mathbb{R}^n$ and sought to prove that a one-to-one correspondence between these two sets was impossible. Three years later he proved that there is such a correspondence. He wrote to Dedekind, "I see it but I do not believe it." He later showed that given any set, it is always possible to create a new set, the set of subsets of the given set, whose cardinal number is larger than that of the given set. For the natural numbers $\mathbb{N}$, whose cardinality is denoted by $\aleph_0$, the cardinal number of the set of subsets is denoted by $2^{\aleph_0}$. Cantor proved that $2^{\aleph_0} = c$, where $c$ is the cardinal number of the continuum; i.e., the set of real numbers.

Georg Cantor 1845–1918

Cantor's work, which resolved age-old problems and reversed much previous thought, could hardly be expected to receive immediate acceptance. His ideas on transfinite ordinal and cardinal numbers aroused the hostility of the powerful Leopold Kronecker, who attacked Cantor's theory savagely over more than a decade, repeatedly preventing Cantor from obtaining a more prominent appointment in Berlin. Though Kronecker died in 1891, his attacks left mathematicians suspicious of Cantor's work. Poincaré referred to set theory as an interesting "pathological case." He also predicted that "Later generations will regard [Cantor's] *Mengenlehre* as a disease from which one has recovered." At one time Cantor suffered a nervous breakdown, but resumed work in 1887.

Many prominent mathematicians, however, were impressed by the uses to which the new theory had already been put in analysis, measure theory, and topology. Hilbert spread Cantor's ideas in Germany, and in 1926 said, "No one shall expel us from the paradise which Cantor created for us." He praised Cantor's transfinite arithmetic as "the most astonishing product of mathematical thought, one of the most beautiful realizations of human activity in the domain of the purely intelligible." Bertrand Russell described Cantor's work as "probably the greatest of which the age can boast." The subsequent utility of Cantor's work in formalizing mathematics—a movement largely led by Hilbert—seems at odds with Cantor's Platonic view that the greater importance of his work was in its implications for metaphysics and theology. That his work could be so seamlessly diverted from the goals intended by its creator is strong testimony to its objectivity and craftsmanship.

Now consider the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. If there exists a bijection between a set $A$ and $\mathbb{N}$, then $A$ is said to be **countably infinite**. Some examples of countably infinite sets are the set of all integers, the set of even natural numbers, the set of odd natural numbers, the set of all prime numbers, and the set of energy levels of the bound states of a hydrogen atom.

countably infinite

It may seem surprising that a subset (such as the set of all even numbers) can be put into one-to-one correspondence with the full set (the set of all natural numbers); however, this is a property shared by all *infinite* sets. In fact, sometimes infinite sets are *defined* as those sets that are in one-to-one correspondence with at least one of their proper subsets. It is also astonishing to discover that there are as many rational numbers as there are natural numbers. After all, there are infinitely many rational numbers just in the interval (0, 1)—or between any two distinct real numbers![3]

[3]The proof involves writing $m/n$ as the $mn$th entry in an $\infty \times \infty$ matrix and starting the "count" with the $(1, 1)$ entry, going to the right to $(1, 2)$, then diagonally to $(2, 1)$,

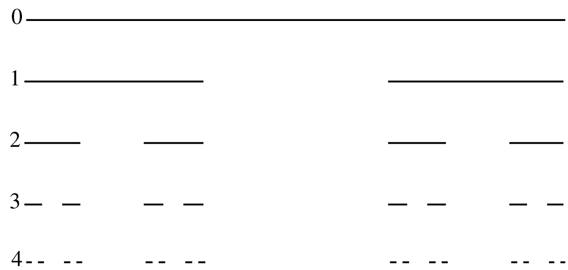


**Fig. 1.5** The Cantor set after one, two, three, and four "dissections"

uncountable sets

Sets that are neither finite nor countably infinite are said to be **uncountable**. In some sense they are "more infinite" than any countable set. Examples of uncountable sets are the points in the interval $(-1, +1)$, the real numbers, the points in a plane, and the points in space. It can be shown that these sets have the same cardinality: There are as many points in three-dimensional space—the whole universe—as there are in the interval $(-1, +1)$ or in any other finite interval.

Cantor set constructed

Cardinality is a very intricate mathematical notion with many surprising results. Consider the interval $[0, 1]$. Remove the open interval $(\frac{1}{3}, \frac{2}{3})$ from its middle (leaving the points $\frac{1}{3}$ and $\frac{2}{3}$ behind). From the remaining portion, $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, remove the two middle thirds; the remaining portion will then be

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

(see Fig. 1.5). Do this indefinitely. What is the cardinality of the remaining set, which is called the **Cantor set**? Intuitively we expect hardly anything to be left. We might persuade ourselves into accepting the fact that the number of points remaining is at most infinite but countable. The surprising fact is that the cardinality is that of the continuum! Thus, after removal of infinitely many middle thirds, the set that remains has as many points as the original set!

## 1.5 Mathematical Induction

Many a time it is desirable to make a mathematical statement that is true for all natural numbers. For example, we may want to establish a formula involving an integer parameter that will hold for all positive integers. One encounters this situation when, after experimenting with the first few positive integers, one recognizes a pattern and discovers a formula, and wants to make sure that the formula holds for all natural numbers. For this purpose, one uses **mathematical induction**. The essence of mathematical induction is stated as follows:

induction principle

then down to $(3, 1)$, then diagonally up, and so on. Obviously the set is countable and it exhausts all rational numbers. In fact, the process double counts some of the entries.

**Proposition 1.5.1** *Suppose that there is associated with each natural number (positive integer) $n$ a statement $S_n$. Then $S_n$ is true for every positive integer provided the following two conditions hold*:

1. *$S_1$ is true.*
2. *If $S_m$ is true for some given positive integer $m$, then $S_{m+1}$ is also true.*

**Example 1.5.2** We illustrate the use of mathematical induction by proving the **binomial theorem**:

binomial theorem

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} a^{m-k} b^k$$
$$= a^m + ma^{m-1}b + \frac{m(m-1)}{2!} a^{m-2} b^2 + \cdots + mab^{m-1} + b^m, \tag{1.1}$$

where we have used the notation

$$\binom{m}{k} \equiv \frac{m!}{k!(m-k)!}. \tag{1.2}$$

The mathematical statement $S_m$ is Eq. (1.1). We note that $S_1$ is trivially true: $(a+b)^1 = a^1 + b^1$. Now we assume that $S_m$ is true and show that $S_{m+1}$ is also true. This means starting with Eq. (1.1) and showing that

$$(a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m+1-k} b^k.$$

Then the induction principle ensures that the statement (equation) holds for all positive integers. Multiply both sides of Eq. (1.1) by $a+b$ to obtain

$$(a+b)^{m+1} = \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^{k+1}.$$

Now separate the $k=0$ term from the first sum and the $k=m$ term from the second sum:

$$(a+b)^{m+1} = a^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k + \underbrace{\sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+1}}_{\text{let } k=j-1 \text{ in this sum}} + b^{m+1}$$
$$= a^{m+1} + \sum_{k=1}^{m} \binom{m}{k} a^{m-k+1} b^k$$
$$+ \sum_{j=1}^{m} \binom{m}{j-1} a^{m-j+1} b^j + b^{m+1}.$$

The second sum in the last line involves $j$. Since this is a dummy index, we can substitute any symbol we please. The choice $k$ is especially useful because then we can unite the two summations. This gives

$$(a+b)^{m+1} = a^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} a^{m-k+1} b^k + b^{m+1}.$$

If we now use

$$\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1},$$

which the reader can easily verify, we finally obtain

$$\begin{aligned}(a+b)^{m+1} &= a^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} a^{m-k+1} b^k + b^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m-k+1} b^k.\end{aligned}$$

This complete the proof.

inductive definitions

Mathematical induction is also used in *defining* quantities involving integers. Such definitions are called **inductive definitions**. For example, inductive definition is used in defining powers: $a^1 = a$ and $a^m = a^{m-1}a$.

## 1.6 Problems

**1.1** Show that the number of subsets of a set containing $n$ elements is $2^n$.

**1.2** Let $A$, $B$, and $C$ be sets in a universal set $U$. Show that

(a) $A \subset B$ and $B \subset C$ implies $A \subset C$.
(b) $A \subset B$ iff $A \cap B = A$ iff $A \cup B = B$.
(c) $A \subset B$ and $B \subset C$ implies $(A \cup B) \subset C$.
(d) $A \cup B = (A \sim B) \cup (A \cap B) \cup (B \sim A)$.

Hint: To show the equality of two sets, show that each set is a subset of the other.

**1.3** For each $n \in \mathbb{N}$, let

$$I_n = \left\{ x \mid |x-1| < n \text{ and } |x+1| > \frac{1}{n} \right\}.$$

Find $\bigcup_n I_n$ and $\bigcap_n I_n$.

**1.4** Show that $a' \in [\![a]\!]$ implies that $[\![a']\!] = [\![a]\!]$.

**1.5** Can you define a binary operation of "multiplication" on the set of vectors in space? What about vectors in the plane? In each case, write the components of the product in terms of the components of the two vectors.

**1.6** Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ when $f$ and $g$ are both bijections.

**1.7** We show that the sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n = \sum_{k=1}^{n}(-1)^{k+1}/k$, is Cauchy. Without loss of generality, assume that $n > m$ and $n - m$ is even (the case of odd $n - m$ can be handled similarly).

(a) Show that

$$x_n - x_m = (-1)^m \sum_{j=1}^{n-m} \frac{(-1)^j}{j+m}.$$

(b) Separate the even and odd parts of the sum and show that

$$x_n - x_m = (-1)^m \left\{ -\sum_{k=1}^{(n-m)/2} \frac{1}{2k-1+m} + \sum_{k=1}^{(n-m)/2} \frac{1}{2k+m} \right\}.$$

(c) Add the two sums to obtain a single sum, showing that

$$x_n - x_m = -(-1)^m \left\{ \sum_{k=1}^{(n-m)/2} \frac{1}{(2k+m)(2k+m-1)} \right\},$$

and that

$$|x_n - x_m| \le \sum_{k=1}^{(n-m)/2} \frac{1}{(2k+m-1)^2}$$
$$= \frac{1}{(1+m)^2} + \sum_{k=2}^{(n-m)/2} \frac{1}{(2k+m-1)^2}.$$

(d) Convince yourself that $\int_1^s f(x)dx \ge \sum_{k=2}^{s} f(k)$ for any continuous function $f(x)$, and apply it to part (c) to get

$$|x_n - x_m| \le \frac{1}{(1+m)^2} + \int_1^{(n-m)/2} \frac{1}{(2x+m-1)^2} dx$$
$$= \frac{1}{n} + \frac{1}{(1+m)^2} - \frac{1}{2}\left( \frac{1}{n-1} - \frac{1}{m+1} \right).$$

Each term on the last line goes to zero independently as $m$ and $n$ go to infinity.

**1.8** Find a bijection $f : \mathbb{N} \to \mathbb{Z}$. Hint: Find an $f$ which maps even integers onto the positive integers and odd integers onto the negative integers.

**1.9** Take any two open intervals $(a, b)$ and $(c, d)$, and show that there are as many points in the first as there are in the second, regardless of the size of the intervals. Hint: Find a (linear) algebraic relation between points of the two intervals.

Leibniz rule **1.10** Use mathematical induction to derive the **Leibniz rule** for differentiating a product:

$$\frac{d^n}{dx^n}(f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}.$$

**1.11** Use mathematical induction to derive the following results:

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}, \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$