Multidimensional Green's Functions: Applications

22

The previous chapter gathered together some general properties of the GFs and their companion, the Dirac delta function. This chapter considers the Green's functions for elliptic, parabolic, and hyperbolic equations that satisfy the BCs appropriate for each type of PDE.

22.1 Elliptic Equations

The most general linear PDE in m variables of the elliptic type was discussed in Sect. 21.1.2. We will not discuss this general case, because all elliptic PDOs encountered in mathematical physics are of a much simpler nature. In fact, the self-adjoint elliptic PDO of the form $\mathbf{L_x} = \nabla^2 + q(\mathbf{x})$ is sufficiently general for purposes of this discussion. Recall from Sect. 21.1.2 that the BCs associated with an elliptic PDE are of two types, Dirichlet and Neumann. Let us consider these separately.

22.1.1 The Dirichlet Boundary Value Problem

A Dirichlet BVP consists of an elliptic PDE together with a Dirichlet BC, such as

$$\begin{aligned} \mathbf{L_x}[u] &= \nabla^2 u + q(\mathbf{x})u = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in D, \\ u(\mathbf{x}_b) &= g(\mathbf{x}_b) \quad \text{for } \mathbf{x}_b \in \partial D, \end{aligned} \tag{22.1}$$

where $g(\mathbf{x}_b)$ is a given function defined on the closed hypersurface ∂D.

The Green's function for the Dirichlet BVP must satisfy the *homogeneous* BC, for the same reason as in the one-dimensional Green's function. Thus, the Dirichlet Green's function, denoted by $G_D(\mathbf{x}, \mathbf{y})$, must satisfy

$$\mathbf{L_x}\big[G_D(\mathbf{x}, \mathbf{y})\big] = \delta(\mathbf{x} - \mathbf{y}), \qquad G_D(\mathbf{x}_b, \mathbf{y}) = 0 \text{ for } \mathbf{x}_b \in S.$$

As discussed in Sect. 21.3.2, we can separate G_D into a singular part $G_D^{(s)}$ and a regular part H where $G_D^{(s)}$ satisfies the same DE as G_D and H satisfies the corresponding homogeneous DE and the BC $H(\mathbf{x}_b, \mathbf{y}) = -G_D^{(s)}(\mathbf{x}_b, \mathbf{y})$.

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Using Eq. (22.1) and the properties of $G_D(x, y)$ in Eq. (21.27), we obtain

$$u(\mathbf{x}) = \int_D d^m y G_D(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) + \int_{\partial D} g(\mathbf{y}_b) \frac{\partial G_D}{\partial n_y}(\mathbf{x}, \mathbf{y}_b)\, da, \tag{22.2}$$

where $\partial/\partial n_y$ indicates normal differentiation with respect to the second argument.

Historical Notes

Gustav Peter Lejeune Dirichlet 1805–1859

Gustav Peter Lejeune Dirichlet (1805–1859), the son of a postmaster, first attended public school, then a private school that emphasized Latin. He was precociously interested in mathematics; it is said that before the age of twelve he used his pocket money to buy mathematical books. In 1817 he entered the gymnasium in Bonn. He is reported to have been an unusually attentive and well-behaved pupil who was particularly interested in modern history as well as in mathematics.

After two years in Bonn, Dirichlet was sent to a Jesuit college in Cologne that his parents preferred. Among his teachers was the physicist Georg Simon Ohm, who gave him a thorough grounding in theoretical physics. Dirichlet completed his *Abitur* examination at the very early age of sixteen. His parents wanted him to study law, but mathematics was already his chosen field. At the time the level of pure mathematics in the German universities was at a low ebb: Except for the formidable Carl Gauss, in Göttingen, there were no outstanding mathematicians, while in Paris the firmament was studded by such luminaries as P.-S. Laplace, Adrien Legendre, Joseph Fourier, and Siméon Poisson.

Dirichlet arrived in Paris in May 1822. In the summer of 1823 he was fortunate in being appointed to a well-paid and pleasant position as tutor to the children of General Maximilien Fay, a national hero of the Napoleonic wars and then the liberal leader of the opposition in the Chamber of Deputies. Dirichlet was treated as a member of the family and met many of the most prominent figures in French intellectual life. Among the mathematicians, he was particularly attracted to Fourier, whose ideas had a strong influence upon his later works on trigonometric series and mathematical physics.

General Fay died in November 1825, and the next year Dirichlet decided to return to Germany, a plan strongly supported by Alexander von Humboldt, who was working for the strengthening of the natural sciences in Germany. Dirichlet was permitted to qualify for habilitation as Privatdozent at the University of Breslau; since he did not have the required doctorate, this was awarded honoris causa by the University of Cologne. His habilitation thesis dealt with polynomials whose prime divisors belong to special arithmetic series. A second paper from this period was inspired by Gauss's announcements on the biquadratic law of reciprocity.

Dirichlet was appointed extraordinary professor in Breslau, but the conditions for scientific work were not inspiring. In 1828 he moved to Berlin, again with the assistance of Humboldt, to become a teacher of mathematics at the military academy. Shortly afterward, at the age of twenty-three, he was appointed extraordinary (later ordinary) professor at the University of Berlin. In 1831 he became a member of the Berlin Academy of Sciences, and in the same year he married Rebecca Mendelssohn-Bartholdy, sister of Felix Mendelssohn, the composer.

Dirichlet spent twenty-seven years as a professor in Berlin and exerted a strong influence on the development of German mathematics through his lectures, through his many pupils, and through a series of scientific papers of the highest quality that he published during this period. He was an excellent teacher, always expressing himself with great clarity. His manner was modest; in his later years he was shy and at times reserved. He seldom spoke at meetings and was reluctant to make public appearances. In many ways he was a direct contrast to his lifelong friend, the mathematician Karl Gustav Jacobi.

One of Dirichlet's most important papers, published in 1850, deals with the boundary value problem, now known as *Dirichlet's boundary value problem*, in which one wishes to determine a potential function satisfying Laplace's equation and having prescribed values on a given surface, in Dirichlet's case a sphere.

In 1855, when Gauss died, the University of Göttingen was anxious to seek a successor of great distinction, and the choice fell upon Dirichlet. Dirichlet moved to Göttingen in the fall of 1855, bought a house with a garden, and seemed to enjoy the quieter life of a prominent university in a small city. He had a number of excellent pupils and relished the increased leisure for research. His work in this period was centered on general problems of mechanics. This new life, however, was not to last long. In the summer of 1858 Dirichlet traveled to a meeting in Montreux, Switzerland, to deliver a memorial speech in honor of Gauss. While there, he suffered a heart attack and was barely able to return to his family in Göttingen. During his illness his wife died of a stroke, and Dirichlet himself died the following spring.

Some special cases of (22.2) are worthy of mention.

1. The first is $u(\mathbf{x}_b) = 0$, the solution to an inhomogeneous DE satisfying the homogeneous BC. We obtain this by substituting zero for $g(\mathbf{x}_b)$ in (22.2) so that only the integration over D remains.
2. The second special case is when the DE is homogeneous, that is, when $f(\mathbf{x}) = 0$ but the BC is inhomogeneous. This yields an integration over the boundary ∂D alone.
3. Finally, the solution to the homogeneous DE with the homogeneous BC is simply $u = 0$, referred to as the zero solution. This is consistent with physical intuition: If the function is zero on the boundary and there is no source $f(\mathbf{x})$ to produce any "disturbance," we expect no nontrivial solution.

method of images and Dirichlet BVP

Example 22.1.1 (Method of Images and Dirichlet BVP) Let us find the Green's function for the three-dimensional Laplacian $\mathbf{L}_\mathbf{x} = \nabla^2$ satisfying the Dirichlet BC $G_D(\boldsymbol{\rho}, \mathbf{y}) = 0$ for $\boldsymbol{\rho}$, on the xy-plane. Here D is the upper half-space ($z \geq 0$) and ∂D is the xy-plane.

It is more convenient to use $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ instead of $\mathbf{x}$ and $\mathbf{y}$, respectively. Using (21.21) as $G_D^{(s)}$, we can write

$$
\begin{aligned}
G_D(\mathbf{r}, \mathbf{r}') &= -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}') \\
&= -\frac{1}{4\pi} \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\
&\quad + H(x, y, z; x', y', z').
\end{aligned}
$$

The requirement that G_D vanish on the xy-plane gives

$$
H(x, y, 0; x', y', z') = \frac{1}{4\pi} \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + z'^2}}.
$$

This fixes the dependence of H on all variables except z. On the other hand, $\nabla^2 H = 0$ in D implies that the form of H must be the same as that of $G_D^{(s)}$ because except at $\mathbf{r} = \mathbf{r}'$, the latter does satisfy Laplace's equation. Thus, because of the symmetry of $G_D^{(s)}$ in $\mathbf{r}$ and $\mathbf{r}'$ [$G_D(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}', \mathbf{r})$] and the evenness of the Laplacian in z (as well as x and y), we have two choices for the z-dependence: $(z - z')^2$ and $(z + z')^2$. The first gives $G_D = 0$, which is a trivial solution. Thus, we must choose

$$
H(x, y, z; x', y', z') = \frac{1}{4\pi} \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}.
$$

Note that with $\mathbf{r}'' \equiv (x', y', -z')$, this equation satisfies $\nabla^2 H = -\delta(\mathbf{r} - \mathbf{r}'')$, and it may appear that H does not satisfy the homogeneous DE, as it should. However, $\mathbf{r}''$ is outside D, and $\mathbf{r} \neq \mathbf{r}''$ as long as $\mathbf{r} \in D$. So H does satisfy the homogeneous DE in D. The Green's function for the given Dirichlet BC is therefore

$$G_D(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}''|}\right),$$

where $\mathbf{r}''$ is the *reflection* of $\mathbf{r}'$ in the xy-plane.

This result has a direct physical interpretation. If determining the solution of the Laplace equation is considered a problem in electrostatics, then $G_D^{(s)}(\mathbf{r}, \mathbf{r}')$ is simply the potential at $\mathbf{r}$ of a unit point charge located at $\mathbf{r}'$, and $G_D(\mathbf{r}, \mathbf{r}')$ is the potential of two point charges of opposite signs, one at $\mathbf{r}'$ and the other at the mirror image of $\mathbf{r}'$. The fact that the two charges are equidistant from the xy-plane ensures the vanishing of the potential in that plane. The introduction of image charges to ensure the vanishing of G_D at ∂D is common in electrostatics and is known as the **method of images**. This method reduces the Dirichlet problem for the Laplacian to finding appropriate point charges outside D that guarantee the vanishing of the potential on ∂D. For simple geometries, such as the one discussed in this example, determination of the magnitudes and locations of such image charges is easy, rendering the method extremely useful.

method of images

Having found the Green's function, we can pose the general Dirichlet BVP:

$$\nabla^2 u = -\rho(\mathbf{r}) \quad \text{and} \quad u(x, y, 0) = g(x, y), \quad \text{for } z > 0.$$

The solution is

$$\begin{aligned} u(\mathbf{r}) = {} & \frac{1}{4\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_0^{\infty} dz' \rho(\mathbf{r}')\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}''|}\right) \\ & + \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' g(x'y') \frac{\partial G_D}{\partial z}\bigg|_{z=0}, \end{aligned} \tag{22.3}$$

where $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', z')$, and $\mathbf{r}'' = (x', y', -z')$.

A typical application consists in introducing a number of charges in the vicinity of an infinite conducting sheet, which is held at a constant potential V_0. If there are N charges, $\{q_i\}_{i=1}^N$, located at $\{\mathbf{r}_i\}_{i=1}^N$, then $\rho(\mathbf{r}) = \sum_{i=1}^N q_i \delta(\mathbf{r} - \mathbf{r}_i)$, $g(x, y) = \text{const} = V_0$, and we get

$$u(\mathbf{r}) = \sum_{i=1}^N \frac{1}{4\pi}\left(\frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} - \frac{q_i}{|\mathbf{r} - \mathbf{r}_i'|}\right) + V_0 \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\partial G_D}{\partial z}\bigg|_{z=0}, \tag{22.4}$$

where $\mathbf{r}_i = (x_i, y_i, z_i)$ and $\mathbf{r}_i' = (x_i, y_i, -z_i)$. That the double integral in Eq. (22.4) is unity can be seen by direct integration or by noting that the sum vanishes when $z = 0$. On the other hand, $u(x, y, 0) = V_0$. Thus, the solution becomes

$$u(\mathbf{r})=\sum_{i=1}^{N}\frac{1}{4\pi}\left(\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}-\frac{q_i}{|\mathbf{r}-\mathbf{r}_i'|}\right)+V_0.$$

Example 22.1.2 (Dirichlet BVP for a Sphere) The method of images is also applicable when the boundary is a sphere. Inside a sphere of radius a with center at the origin, we wish to solve this Dirichlet BVP:

Dirichlet BVP for a sphere

$$\nabla^2 u=-\rho(r,\theta,\varphi)\quad \text{for } r<a,\quad \text{and}\quad u(a,\theta,\varphi)=g(\theta,\varphi).$$

The GF satisfies

$$\begin{aligned}\nabla^2 G_D\big(r,\theta,\varphi;r',\theta',\varphi'\big)&=\delta\big(\mathbf{r}-\mathbf{r}'\big)\quad \text{for } r<a,\\ G_D\big(a,\theta,\varphi;r',\theta',\varphi'\big)&=0.\end{aligned}\tag{22.5}$$

Thus, G_D can again be interpreted as the potential of point charges, of which one is in the sphere and the others are outside.

We write $G_D=G_D^{(s)}+H$ and choose H in such a way that the second equation in (22.5) is satisfied. As in the case of the xy-plane, let[1] $H(\mathbf{r},\mathbf{r}'')=-\frac{k}{4\pi|\mathbf{r}-\mathbf{r}''|}$, where k is a constant to be determined. If $\mathbf{r}''$ is *outside* the sphere, $\nabla^2 H$ will vanish everywhere *inside* the sphere. The problem has been reduced to finding k and $\mathbf{r}''$ (the location of the image charge). We want to choose $\mathbf{r}''$ such that

$$\left.\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right|_{r=a}=\left.\frac{k}{|\mathbf{r}-\mathbf{r}''|}\right|_{r=a}\quad\Rightarrow\quad k\big(|\mathbf{r}-\mathbf{r}'|\big)_{r=a}=\big(|\mathbf{r}-\mathbf{r}''|\big)_{r=a}.$$

This shows that k must be positive. Squaring both sides and expanding the result yields

$$k^2\big(a^2+r'^2-2ar'\cos\gamma\big)=a^2+r''^2-2ar''\cos\gamma,$$

where γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, and we have assumed that $\mathbf{r}'$ and $\mathbf{r}''$ are in the same direction. If this equation is to hold for arbitrary γ, we must have $k^2r'=r''$ and $k^2(a^2+r'^2)=a^2+r''^2$. Combining these two equations yields $k^4r'^2-k^2(a^2+r'^2)+a^2=0$, whose positive solutions are $k=1$ and $k=a/r$. The first choice implies that $r''=r'$, which is impossible because r'' must be outside the sphere. We thus choose $k=a/r'$, which gives $\mathbf{r}''=(a^2/r'^2)\mathbf{r}'$. We then have

$$G_D\big(\mathbf{r},\mathbf{r}'\big)=-\frac{1}{4\pi}\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}-\frac{ar'}{|r'^2\mathbf{r}-a^2\mathbf{r}'|}\right].\tag{22.6}$$

[1]Actually, to be general, we must add an arbitrary function $f(\mathbf{r}'')$ to this. However, as the reader can easily verify, the following argument will show that $f(\mathbf{r}'')=0$. Besides, we are only interested in *a* solution, not the most general one. All simplifying assumptions that follow are made for the same reason.

Substituting this in Eq. (22.2), and noting that $\partial G/\partial n_y = (\partial G/\partial r')_{r'=a}$, yields

$$
\begin{aligned}
u(\mathbf{r}) = {} & \frac{1}{4\pi}\int_0^a r'^2 dr' \int_0^\pi \sin\theta'\, d\theta' \\
& \times \int_0^{2\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{ar'}{|r'^2\mathbf{r}-a^2\mathbf{r}'|}\right)\rho(\mathbf{r}')\, d\varphi' \\
& + \frac{a(a^2-r^2)}{4\pi}\int_0^{2\pi} d\varphi' \int_0^\pi \sin\theta'\, d\theta' \frac{g(\theta',\varphi')}{|\mathbf{r}-\mathbf{a}|^3},
\end{aligned}
\tag{22.7}
$$

where $\mathbf{a} = (a, \theta', \varphi')$ is a vector from the origin to a point on the sphere. For the Laplace equation $\rho(\mathbf{r}') = 0$, and only the double integral in Eq. (22.7) will contribute.

It can be shown that if $g(\theta', \varphi') = \text{const} = V_0$, then $u(\mathbf{r}) = V_0$. This is the familiar fact shown in electromagnetism: If the potential on a sphere is kept constant, the potential inside the sphere will be constant and equal to the potential at the surface.

Dirichlet BVP for a circle

Example 22.1.3 (Dirichlet BVP for a Circle) In this example we find the Dirichlet GF for a circle of radius a centered at the origin. The GF is logarithmic [see Eq. (21.22)]. Therefore, H is also logarithmic, and its most general form is

$$
H(\mathbf{r},\mathbf{r}'') = -\frac{1}{2\pi}\ln\left(|\mathbf{r}-\mathbf{r}''|\right) - \frac{1}{2\pi}\ln\left[f(\mathbf{r}'')\right] = -\frac{1}{2\pi}\ln\left(|\mathbf{r}-\mathbf{r}''|f(\mathbf{r}'')\right),
$$

so that

$$
\begin{aligned}
G_D(\mathbf{r},\mathbf{r}') &= \frac{1}{2\pi}\ln\left(|\mathbf{r}-\mathbf{r}'|\right) - \frac{1}{2\pi}\ln\left(|\mathbf{r}-\mathbf{r}''|f(\mathbf{r}'')\right) \\
&= \frac{1}{2\pi}\ln\left|\frac{\mathbf{r}-\mathbf{r}'}{(\mathbf{r}-\mathbf{r}'')f(\mathbf{r}'')}\right|.
\end{aligned}
$$

For G_D to vanish at all points on the circle, we must have

$$
\left|\frac{\mathbf{a}-\mathbf{r}'}{(\mathbf{a}-\mathbf{r}'')f(\mathbf{r}'')}\right| = 1 \quad \Rightarrow \quad |\mathbf{a}-\mathbf{r}'| = \left|(\mathbf{r}-\mathbf{r}'')f(\mathbf{r}'')\right|,
$$

where $\mathbf{a}$ is a vector from origin to a point on the circle. Assuming that $\mathbf{r}''$ and $\mathbf{r}'$ are in the same direction, squaring both sides of the last equation and expanding the result, we obtain

$$
\left(a^2 + r''^2 - 2ar''\cos\gamma\right)f^2(\mathbf{r}'') = a^2 + r'^2 - 2ar'\cos\gamma,
$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{r}'$ (or $\mathbf{r}''$). This equation must hold for arbitrary γ. Hence, we have $f^2(\mathbf{r}'')r'' = r'$ and $f^2(\mathbf{r}'')(a^2 + r''^2) = a^2 + r'^2$. These can be solved for $f(\mathbf{r}'')$ and $\mathbf{r}''$. The result is

$$
\mathbf{r}'' = \frac{a^2}{r'^2}\mathbf{r}', \qquad f(\mathbf{r}'') = \frac{a}{r''} = \frac{r'}{a}.
$$

Substituting these formulas in the expression for G_D, we obtain

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln(|\mathbf{r}-\mathbf{r}'|) - \frac{1}{2\pi}\ln\left(\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|\frac{r'}{a}\right).$$

To write the solution to the Dirichlet BVP, we also need $\partial G_D/\partial n = \partial G_D/\partial r'$. Using polar coordinates, we express G_D as

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\ln\left|\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{r^2r'^2/a^2+a^2-2rr'\cos(\theta-\theta')}\right|.$$

Differentiation with respect to r' yields

$$\left.\frac{\partial G_D}{\partial n}\right|_{r'=a} = \left.\frac{\partial G_D}{\partial r'}\right|_{r'=a} = \frac{1}{2\pi a}\frac{a^2-r^2}{r^2+a^2-2ra\cos(\theta-\theta')},$$

from which we can immediately write the solution to the two-dimensional Dirichlet BVP $\nabla^2 u = \rho$, $u(r=a) = g(\theta')$ as

$$u(\mathbf{r}) = \int_0^{2\pi} d\theta' \int_0^a r' G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dr' + \frac{a^2-r^2}{2\pi a}\int_0^{2\pi} d\theta' \frac{g(\theta')}{r^2+a^2-2ra\cos(\theta-\theta')}.$$

In particular, for Laplace's equation $\rho(\mathbf{r}') = 0$, and we get

Poisson integral formula

$$u(r,\theta) = \frac{a^2-r^2}{2\pi a}\int_0^{2\pi} d\theta' \frac{g(\theta')}{r^2+a^2-2ra\cos(\theta-\theta')}. \tag{22.8}$$

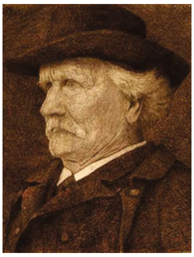
Equation (22.8) is called the **Poisson integral formula**.

22.1.2 The Neumann Boundary Value Problem

The Neumann BVP is not as simple as the Dirichlet BVP because it requires the normal derivative of the solution. But the normal derivative is related to the Laplacian through the divergence theorem. Thus, the BC and the DE are tied together, and unless we impose some solvability conditions, we may have no solution at all. These points are illustrated clearly if we consider the Laplacian operator.

Historical Notes

Carl Gottfried Neumann (1832–1925) was the son of Franz Ernst Neumann, a professor of physics and mineralogy at Königsberg; his mother, Luise Florentine Hagen, was a sister-in-law of the astronomer Bessel. Neumann received his primary and secondary education in Königsberg, attended the university, and formed particularly close friendships with the analyst F.J. Richelot and the geometer L.O. Hesse. After passing the examination for secondary-school teaching, he obtained his doctorate in 1855; in 1858 he qualified for lecturing in mathematics at Halle, where he became Privatdozent and, in 1863, assistant professor. In the latter year he was called to Basel, and in 1865 to Tübingen. From the



Carl Gottfried Neumann 1832–1925

autumn of 1868 until his retirement in 1911 he was at the University of Leipzig. In 1864 he married Hermine Mathilde Elise Kloss; she died in 1875.

Neumann, who led a quiet life, was a successful university teacher and a productive researcher. More than two generations of future gymnasium teachers received their basic mathematical education from him. As a researcher he was especially prominent in the field of potential theory. His investigations into *boundary value problems* resulted in pioneering achievements; in 1870 he began to develop the method of the arithmetical mean for their solution. He also coined the term "logarithmic potential." The second boundary value problem of potential theory still bears his name; a generalization of it was later provided by H. Poincaré.

Neumann was a member of the Berlin Academy and the Societies of Göttingen, Munich, and Leipzig. He performed a valuable service in founding and editing the important German mathematics periodical *Mathematische Annalen*.

Consider the Neumann BVP

$$\nabla^2 u = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in D, \quad \text{and} \quad \frac{\partial u}{\partial n} = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in \partial D.$$

Integrating the first equation over D and using the divergence theorem, we obtain

$$\int_D f(\mathbf{x})\, d^m x = \int_D \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u)\, d^m x = \int_{\partial D} \hat{\mathbf{e}}_n \cdot \boldsymbol{\nabla} u\, da = \int_{\partial D} \frac{\partial u}{\partial n}\, da.$$

It follows that we cannot arbitrarily assign values of $\partial u/\partial n$ on the boundary. In particular, if the BC is homogeneous, as in the case of Green's functions, the RHS is zero, and we must have $\int_D f(\mathbf{x})\, d^m x = 0$. This relation is a restriction on the DE, and is a solvability condition, as mentioned above. To satisfy this condition, it is necessary to subtract from the inhomogeneous term its average value over the region D. Thus, if V_D is the volume of the region D, then

$$\nabla^2 u = f(\mathbf{x}) - \bar{f} \quad \text{where} \quad \bar{f} = \frac{1}{V_D} \int_D f(\mathbf{x})\, d^m x$$

ensures that the Neumann BVP is solvable. In particular, the inhomogeneous term for the Green's function is not simply $\delta(\mathbf{x}-\mathbf{y})$ but $\delta(\mathbf{x}-\mathbf{y}) - \bar{\delta}$, where

$$\bar{\delta} = \frac{1}{V_D} \int_D \delta(\mathbf{x}-\mathbf{y})\, d^m x = \frac{1}{V_D} \quad \text{if } \mathbf{y} \in D.$$

Thus, the Green's function for the Neumann BVP, $G_N(\mathbf{x}, \mathbf{y})$, satisfies

$$\begin{aligned} \nabla^2 G_N(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}) - \frac{1}{V_D}, \\ \frac{\partial G_N}{\partial n}(\mathbf{x}, \mathbf{y}) &= 0 \quad \text{for } \mathbf{x} \in \partial D. \end{aligned}$$

Applying Green's identity, Eq. (21.27), we get

$$u(\mathbf{x}) = \int_D d^m y\, G_N(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) - \int_{\partial D} G_N(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial n}\, da + \bar{u}, \qquad (22.9)$$

where $\bar{u} = (1/V_D)\int_D u(\mathbf{x})\, d^m x$ is the average value of u in D. Equation (22.9) is valid only for the Laplacian operator, although a similar result can be obtained for a general self-adjoint SOLPDO with constant coefficients. We will not pursue that result, however, since it is of little practical use.

interior vs exterior BVP

Throughout the discussion so far we have assumed that D is bounded; that is, we have considered points inside D with BCs on the boundary ∂D specified. This is called an **interior BVP**. In many physical situations we are interested in points outside D. We are then dealing with an **exterior BVP**. In dealing with such a problem, we must specify the behavior of the Green's function at infinity. In most cases, the physics of the problem dictates such behavior. For instance, for the case of an exterior Dirichlet BVP, where

$$u(\mathbf{x}) = \int_D d^m y G_D(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) + \int_{\partial D} u(\mathbf{y}_b) \frac{\partial G_D}{\partial n_y}(\mathbf{x}, \mathbf{y}_b)\, da$$

and it is desired that $u(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, the vanishing of $G_D(\mathbf{x}, \mathbf{y})$ at infinity guarantees that the second integral vanishes, as long as ∂D is a finite hypersurface. To guarantee the disappearance of the first integral, we must demand that $G_D(\mathbf{x}, \mathbf{y})$ tend to zero faster than $f(\mathbf{y}) d^m y$ tends to infinity. For most cases of physical interest, the calculation of the exterior Green's functions is not conceptually different from that of the interior ones. However, the algebra may be more involved.

Later we will develop general methods for finding the Green's functions for certain partial differential operators that satisfy appropriate BCs. At this point, let us simply mention what are called mixed BCs for elliptic PDEs. A general mixed BC is of the form

$$\alpha(\mathbf{x}) u(\mathbf{x}) + \beta(\mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{x}) = \gamma(\mathbf{x}). \tag{22.10}$$

Problem 22.6 examines the conditions that the GF must satisfy in such a case.

22.2 Parabolic Equations

Elliptic partial differential equations arise in static problems, where the solution is independent of time. Of the two major time-dependent equations, the wave equation and the heat (or diffusion) equation,[2] the latter is a parabolic PDE and the former a hyperbolic PDE. This section examines the heat equation, which is of the form $\nabla^2 u = a^2 \partial u/\partial t$. By changing t to t/a^2, we can write the equation as $\mathbf{L}_{\mathbf{x},t}[u] \equiv (\partial/\partial t - \nabla^2) u(\mathbf{x}, t) = 0$. We wish to calculate the Green's function associated with $\mathbf{L}_{\mathbf{x},t}$ and the homogeneous BCs. Because of the time variable, we must also specify the solution at $t = 0$.

[2] The heat equation turns into the Schrödinger equation if t is changed to $\sqrt{-1}\,t$; thus, the following discussion incorporates the Schrödinger equation as well.

Thus, we consider the BVP

$$\begin{aligned} \mathbf{L}_{\mathbf{x},t}[u] &\equiv \left(\frac{\partial}{\partial t} - \nabla^2\right) u(\mathbf{x},t) = 0 \quad \text{for } \mathbf{x} \in D, \\ u(\mathbf{x}_b, t) = 0, &\qquad u(\mathbf{x}, 0) = h(\mathbf{x}) \quad \text{for } \mathbf{x}_b \in \partial D,\ \mathbf{x} \in D. \end{aligned} \tag{22.11}$$

To find a solution to (22.11), we can use a method that turns out to be useful for evaluating Green's functions in general—the method of eigenfunctions. Let $\{u_n\}_{n=1}^{\infty}$ be the eigenfunctions of ∇^2 with eigenvalues $\{-\lambda_n\}_{n=1}^{\infty}$. Let the BC be $u_n(\mathbf{x}_b) = 0$ for $\mathbf{x}_b \in \partial D$. Then

$$\begin{aligned} \nabla^2 u_n(\mathbf{x}) + \lambda_n u_n(\mathbf{x}) = 0 &\quad \text{for } n = 1, 2, \ldots,\ \mathbf{x} \in D, \\ u_n(\mathbf{x}_b) = 0 &\quad \text{for } \mathbf{x}_b \in \partial D. \end{aligned} \tag{22.12}$$

Equation (22.12) constitutes a Sturm-Liouville problem in m dimensions, which we assume to have a solution with $\{u_n\}_{n=1}^{\infty}$ as a complete orthonormal set. We can therefore write

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} C_n(t) u_n(\mathbf{x}). \tag{22.13}$$

This is possible because at each specific value of t, $u(\mathbf{x}, t)$ is a function of $\mathbf{x}$ and therefore can be written as a linear combination of the *same set*, $\{u_n\}_{n=1}^{\infty}$. The coefficients $C_n(t)$ are given by

$$C_n(t) = \int_D u(\mathbf{x}, t) u_n(\mathbf{x})\, d^m x. \tag{22.14}$$

To calculate $C_n(t)$, we differentiate (22.14) with respect to time and use (22.11) to obtain

$$\dot{C}_n(t) \equiv \frac{dC_n}{dt} = \int_D \frac{\partial u}{\partial t}(\mathbf{x}, t) u_n(\mathbf{x})\, d^m x = \int_D \big[\nabla^2 u(\mathbf{x}, t)\big] u_n(\mathbf{x})\, d^m x.$$

Using Green's identity for the operator ∇^2 yields

$$\int_D \big[u_n \nabla^2 u - u \nabla^2 u_n\big]\, d^m x = \int_{\partial D} \left(u_n \frac{\partial u}{\partial n} - u \frac{\partial u_n}{\partial n}\right) da.$$

Since both u and u_n vanish on ∂D, the RHS is zero, and we get

$$\dot{C}_n(t) = \int_D u \nabla^2 u_n\, d^m x = -\lambda_n \int_D u(\mathbf{x}, t) u_n(\mathbf{x})\, d^m x = -\lambda_n C_n.$$

This has the solution $C_n(t) = C_n(0) e^{-\lambda_n t}$, where

$$C_n(0) = \int_D u(\mathbf{y}, 0) u_n(\mathbf{y})\, d^m y = \int_D h(\mathbf{y}) u_n(\mathbf{y})\, d^m y,$$

so that

$$C_n(t) = e^{-\lambda_n t} \int_D h(\mathbf{y}) u_n(\mathbf{y})\, d^m y.$$

Substituting this in (22.13) and switching the order of integration and summation, we get

$$u(\mathbf{x}, t) = \int_D \left[\sum_{n=1}^{\infty} e^{-\lambda_n t} u_n(\mathbf{x}) u_n(\mathbf{y}) \right] h(\mathbf{y})\, d^m y$$

and read off the GF as $\sum_{n=1}^{\infty} e^{-\lambda_n t} u_n(\mathbf{x}) u_n(\mathbf{y}) \theta(t)$, where we also introduced the theta function to ensure that the solution vanishes for $t < 0$. More generally, we have

$$G(\mathbf{x}, \mathbf{y}; t - \tau) = \sum_{n=1}^{\infty} e^{-\lambda_n (t-\tau)} u_n(\mathbf{x}) u_n(\mathbf{y}) \theta(t - \tau). \tag{22.15}$$

Note the property

$$\lim_{\tau \to t} G(\mathbf{x}, \mathbf{y}; t - \tau) = \sum_{n=1}^{\infty} u_n(\mathbf{x}) u_n(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}),$$

which is usually written as

$$G\big(\mathbf{x}, \mathbf{y}; 0^+\big) = \delta(\mathbf{x} - \mathbf{y}). \tag{22.16}$$

The reader may also check that

$$\mathbf{L}_{\mathbf{x},t} G(\mathbf{x}, \mathbf{y}; t - \tau) = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau). \tag{22.17}$$

This is precisely what we expect for the Green's function of an operator in the variables $\mathbf{x}$ and t. Another property of $G(\mathbf{x}, \mathbf{y}; t - \tau)$ is that it vanishes on ∂D, as it should.

Having found the Green's function and noted its properties, we are in a position to solve the inhomogeneous analogue of Eq. (22.11), in which the RHS of the first equation is $f(\mathbf{x}, t)$, and the zero on the RHS of the second equation is replaced by $g(\mathbf{x}_b, t)$. Experience with similar but simpler problems indicates that to make any progress toward a solution, we must come up with a form of Green's identity involving $\mathbf{L}_{\mathbf{x},t}$ and its adjoint. It is easy to show that

$$v \mathbf{L}_{\mathbf{x},t}[u] - u \mathbf{L}^{\dagger}_{\mathbf{x},t}[v] = \frac{\partial}{\partial t}(uv) - \boldsymbol{\nabla} \cdot (v \boldsymbol{\nabla} u - u \boldsymbol{\nabla} v), \tag{22.18}$$

where $\mathbf{L}^{\dagger}_{\mathbf{x},t} = -\partial/\partial t - \nabla^2$.

Now consider the $(m + 1)$-dimensional "cylinder" one of whose bases is at $t = \epsilon$, where ϵ is a small positive number. This base is barely above the m-dimensional hyperplane $\mathbb{R}^m$. The other base is at $t = \tau - \epsilon$ and is a duplicate of $D \subset \mathbb{R}^m$ (see Fig. 22.1). Let a^μ, where $\mu = 0, 1, \dots, m$, be the components of an $(m + 1)$-dimensional vector $\mathsf{a} = (a^0, a^1, \dots, a^m)$. Define an inner product by

$$\mathsf{a} \cdot \mathsf{b} \equiv \sum_{\mu=0}^{m} a^\mu b_\mu \equiv a^0 b^0 - a^1 b^1 - \cdots - a^m b^m \equiv a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$$

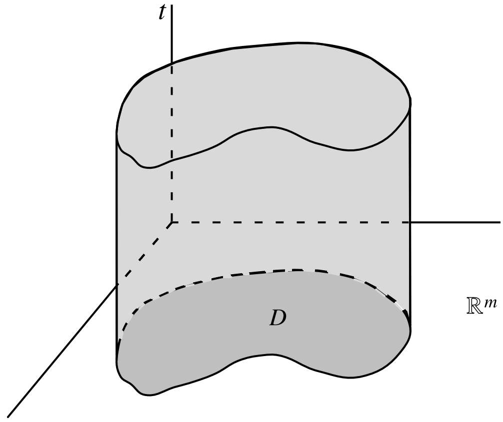


Fig. 22.1 The "cylinder" used in evaluating the GF for the diffusion and wave equations. Note that the bases are not planes, but hyperplanes (that is, spaces such as $\mathbb{R}^m$)

and the $(m+1)$-dimensional vector $\mathsf{Q} \equiv (Q^0, \mathbf{Q})$ by $Q^0 = uv$, $\mathbf{Q} = v\nabla u - u\nabla v$. Then (22.18) can be expressed as

$$v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v] = \sum_{\mu=0}^{m} \frac{\partial Q^{\mu}}{\partial x^{\mu}} \equiv \frac{\partial Q^0}{\partial x^0} - \frac{\partial Q^1}{\partial x^1} - \cdots - \frac{\partial Q^m}{\partial x^m}. \tag{22.19}$$

We recognize the RHS as a divergence in $(m+1)$-dimensional space. Denoting the volume of the $(m+1)$-dimensional cylinder by $\mathcal{D}$ and its boundary by $\partial\mathcal{D}$ and integrating (22.19) over $\mathcal{D}$, we obtain

$$\begin{aligned}\int_{\mathcal{D}} \left(v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v]\right) d^{m+1}x &= \int_{\mathcal{D}} \sum_{\mu=0}^{m} \frac{\partial Q^{\mu}}{\partial x^{\mu}} d^{m+1}x \\ &= \int_{\partial\mathcal{D}} \sum_{\mu=0}^{m} Q^{\mu} n_{\mu}\, dS,\end{aligned} \tag{22.20}$$

where dS is an element of "area" of $\partial\mathcal{D}$. Note that the divergence theorem was used in the last step. The LHS is an integration over t and $\mathbf{x}$, which can be written as

$$\int_{\mathcal{D}} \left(v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v]\right) d^{m+1}x = \int_{\epsilon}^{\tau-\epsilon} dt \int_{D} d^m x \left(v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v]\right).$$

The RHS of (22.20), on the other hand, can be split into three parts: a base at $t = \epsilon$, a base at $t = \tau - \epsilon$, and the lateral surface. The base at $t = \epsilon$ is simply the region D, whose outward-pointing normal is in the negative t direction. Thus, $n_0 = -1$, and $n_i = 0$ for $i = 1, 2, \ldots, m$. The base at $t = \tau - \epsilon$ is also the region D; however, its normal is in the positive t direction. Thus, $n_0 = 1$, and $n_i = 0$ for $i = 1, 2, \ldots, m$. The element of "area" for these two bases is simply $d^m x$. The unit normal to the lateral surface has no time component and is simply the unit normal to the boundary of D. The element of "area" for the lateral surface is $dt\, da$, where da is an element of "area" for ∂D. Putting everything together, we can write (22.20) as

$$\int_{\epsilon}^{\tau-\epsilon} dt \int_D d^m x \big(v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v]\big)$$
$$= \int_D \big(-Q^0\big)\big|_{t=\epsilon} d^m x + \int_D Q^0\big|_{t=\tau-\epsilon} d^m x - \int_{\partial D} da \int_{\epsilon}^{\tau-\epsilon} dt \mathbf{Q}\cdot\hat{\mathbf{e}}_n.$$

The minus sign for the last term is due to the definition of the inner product. Substituting for **Q** yields

$$\int_{\epsilon}^{\tau-\epsilon} dt \int_D d^m x \big(v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v]\big)$$
$$= -\int_D u(\mathbf{x},\epsilon)v(\mathbf{x},\epsilon)d^m x + \int_D u(\mathbf{x},\tau-\epsilon)v(\mathbf{x},\tau-\epsilon)d^m x$$
$$- \int_{\partial D} da \int_{\epsilon}^{\tau-\epsilon} dt \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right). \tag{22.21}$$

Let v be $g(\mathbf{x},\mathbf{y};t-\tau)$, the GF associated with the adjoint operator. Then Eq. (22.21) gives

$$\int_{\epsilon}^{\tau-\epsilon} dt \int_D d^m x \big[g(\mathbf{x},\mathbf{y};t-\tau)f(\mathbf{x},t) - u(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)\big]$$
$$= -\int_D u(\mathbf{x},\epsilon)g(\mathbf{x},\mathbf{y};\epsilon-\tau)d^m x + \int_D u(\mathbf{x},\tau-\epsilon)g(\mathbf{x},\mathbf{y};-\epsilon)d^m x$$
$$- \int_{\partial D} da \int_{\epsilon}^{\tau-\epsilon} dt \left[g(\mathbf{x}_b,\mathbf{y};t-\tau)\frac{\partial u}{\partial n} - u(\mathbf{x}_b,t)\frac{\partial g}{\partial n}\right]. \tag{22.22}$$

We now use the following facts:

1. $\delta(t-\tau)=0$ in the second integral on the LHS of Eq. (22.22), because t can never be equal to τ in the range of integration.
2. Using the symmetry property of the Green's function and the fact that $\mathbf{L}_{\mathbf{x},t}$ is real, we have $g(\mathbf{x},\mathbf{y};t-\tau) = G(\mathbf{y},\mathbf{x};\tau-t)$, where we have used the fact that t and τ are the time components of $\mathbf{x}$ and $\mathbf{y}$, respectively. In particular, by (22.16), $g(\mathbf{x},\mathbf{y};-\epsilon) = G(\mathbf{y},\mathbf{x};\epsilon) = \delta(\mathbf{x}-\mathbf{y})$.
3. The function $g(\mathbf{x},\mathbf{y};t-\tau)$ satisfies the same homogeneous BC as $G(\mathbf{x},\mathbf{y};t-\tau)$. Thus, $g(\mathbf{x}_b,\mathbf{y};t-\tau)=0$ for $\mathbf{x}_b\in\partial D$.

Substituting all the above in (22.22), taking the limit $\epsilon\to 0$, and switching $\mathbf{x}$ and $\mathbf{y}$ and t and τ, we obtain

$$u(\mathbf{x},t) = \int_0^t d\tau \int_D d^m y G(\mathbf{x},\mathbf{y};t-\tau)f(\mathbf{y},\tau) + \int_D u(\mathbf{y},0)G(\mathbf{x},\mathbf{y};t)d^m y$$
$$- \int_0^t d\tau \int_{\partial D} u(\mathbf{y}_b,\tau)\frac{\partial G}{\partial n_y}(\mathbf{x},\mathbf{y}_b;t-\tau)\,da, \tag{22.23}$$

where $\partial/\partial n_y$ in the last integral means *normal* differentiation with respect to the second argument of the Green's function.

Equation (22.23) gives the complete solution to the BVP associated with a parabolic PDE. If $f(\mathbf{y}, \tau) = 0$ and u vanishes on the hypersurface ∂D, then Eq. (22.23) gives

$$u(\mathbf{x}, t) = \int_D u(\mathbf{y}, 0) G(\mathbf{x}, \mathbf{y}; t) d^m y, \tag{22.24}$$

GF as evolution operator or propagator

which is the solution to the BVP of Eq. (22.11), which led to the general Green's function of (22.15). Equation (22.24) lends itself nicely to a physical interpretation. The RHS can be thought of as an integral operator with kernel $G(\mathbf{x}, \mathbf{y}; t)$. This integral operator acts on $u(\mathbf{y}, 0)$ and gives $u(\mathbf{x}, t)$; that is, given the shape of the solution at $t = 0$, the integral operator produces the shape for all subsequent time. That is why $G(\mathbf{x}, \mathbf{y}; t)$ is called the **evolution operator**, or **propagator**.

22.3 Hyperbolic Equations

The hyperbolic equation we will discuss is the wave equation

$$\mathbf{L}_{\mathbf{x},t}[u] \equiv \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) u(\mathbf{x}, t) = 0, \tag{22.25}$$

where we have set the speed of the wave equal to unity.

We wish to calculate the Green's function for $\mathbf{L}_{\mathbf{x},t}$ subject to appropriate BCs. Let us proceed as we did for the parabolic equation and write

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}; t) &= \sum_{n=1}^{\infty} C_n(\mathbf{y}; t) u_n(\mathbf{x}) \\ C_n(\mathbf{y}; t) &= \int_D G(\mathbf{x}, \mathbf{y}; t) u_n(\mathbf{x})\, d^m x, \end{aligned} \tag{22.26}$$

where $u_n(\mathbf{x})$ are orthonormal eigenfunctions of ∇^2 with eigenvalues $-\lambda_n$, satisfying certain, as yet unspecified, BCs. As usual, we expect G to satisfy

$$\mathbf{L}_{\mathbf{x},t}[G] = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) G(\mathbf{x}, \mathbf{y}; t - \tau) = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau). \tag{22.27}$$

Substituting (22.26) in (22.27) with $\tau = 0$ and using $\nabla^2 u_n = -\lambda_n u_n$, gives

$$\sum_{n=1}^{\infty} \left\{ \frac{\partial^2}{\partial t^2} C_n(\mathbf{y}; t) + \lambda_n C_n(\mathbf{y}; t) \right\} u_n(\mathbf{x}) = \sum_{n=1}^{\infty} \left[u_n(\mathbf{y}) \delta(t) \right] u_n(\mathbf{x}),$$

where we used $\delta(\mathbf{x} - \mathbf{y}) = \sum_{n=1}^{\infty} u_n(\mathbf{x}) u_n(\mathbf{y})$ on the RHS. The orthonormality of u_n now gives $\ddot{C}_n(\mathbf{y}; t) + \lambda_n C_n(\mathbf{y}; t) = u_n(\mathbf{y}) \delta(t)$. It follows that $C_n(\mathbf{y}; t)$ is separable. In fact,

$$C_n(\mathbf{y}; t) = u_n(\mathbf{y}) T_n(t) \quad \text{where} \quad \left(\frac{d^2}{dt^2} + \lambda_n \right) T_n(t) = \delta(t).$$

This equation describes a one-dimensional Green's function and can be solved using the methods of Chap. 20. Assuming that $T_n(t) = 0$ for $t \le 0$, we obtain $T_n(t) = (\sin\omega_n t/\omega_n)\theta(t)$, where $\omega_n^2 = \lambda_n$. Substituting all the above results in (22.26), we obtain

$$G(\mathbf{x}, \mathbf{y}; t) = \sum_{n=1}^{\infty} u_n(\mathbf{x}) u_n(\mathbf{y}) \frac{\sin\omega_n t}{\omega_n}\theta(t),$$

or, more generally,

$$G(\mathbf{x}, \mathbf{y}; t-\tau) = \sum_{n=1}^{\infty} u_n(\mathbf{x}) u_n(\mathbf{y}) \frac{\sin\omega_n (t-\tau)}{\omega_n}\theta(t-\tau). \tag{22.28}$$

We note that

$$G\big(\mathbf{x}, \mathbf{y}; 0^+\big) = 0 \quad \text{and} \quad \frac{\partial G}{\partial t}(\mathbf{x}, \mathbf{y}; t)\bigg|_{t\to 0^+} = \delta(\mathbf{x}-\mathbf{y}), \tag{22.29}$$

as can easily be verified.

With the Green's function for the operator $\mathbf{L}_{\mathbf{x},t}$ of Eq. (22.25) at our disposal, we can attack the BVP given by

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{x}, t) = f(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in D, \\ &u(\mathbf{x}_b, t) = h(\mathbf{x}_b, t), \qquad u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad \text{for } \mathbf{x}_b \in \partial D,\ \mathbf{x} \in D, \\ &\frac{\partial u}{\partial t}(\mathbf{x}, t)\bigg|_{t=0} = \psi(\mathbf{x}) \quad \text{for } \mathbf{x} \in D. \end{aligned} \tag{22.30}$$

As in the case of the parabolic equation, we first derive an appropriate expression of Green's identity. This can be done by noting that

$$v\mathbf{L}_{\mathbf{x},t}[u] - u\mathbf{L}^{\dagger}_{\mathbf{x},t}[v] = \frac{\partial}{\partial t}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) - \boldsymbol{\nabla}\cdot(u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u).$$

Thus, $\mathbf{L}_{\mathbf{x},t}$ is formally self-adjoint. Furthermore, we can identify

$$Q^0 = u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \quad \text{and} \quad \mathbf{Q} = u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u.$$

Following the procedure used for the parabolic case step by step, we can easily derive a Green's identity and show that

$$\begin{aligned} u(\mathbf{x}, t) &= \int_0^t d\tau \int_D d^m y\, G(\mathbf{x}, \mathbf{y}; t-\tau) f(\mathbf{y}, \tau) \\ &\quad + \int_D \left[\psi(\mathbf{y}) G(\mathbf{x}, \mathbf{y}; t) - \phi(\mathbf{y}) \frac{\partial G}{\partial t}(\mathbf{x}, \mathbf{y}; t)\right] d^m y \\ &\quad - \int_0^t d\tau \int_{\partial D} h(\mathbf{y}_b, \tau) \frac{\partial G}{\partial n_y}(\mathbf{x}, \mathbf{y}_b; t-\tau)\, da. \end{aligned} \tag{22.31}$$

The details are left as Problem 22.11.

For the homogeneous PDE with the homogeneous BC $h = 0 = \psi$, we get

$$u(\mathbf{x}, t) = -\int_D \phi(\mathbf{y}) \frac{\partial G}{\partial t}(\mathbf{x}, \mathbf{y}; t) d^m y.$$

Note the difference between this equation and Eq. (22.24). Here the propagator is the time derivative of the Green's function. There is another difference between hyperbolic and parabolic equations. When the solution to a parabolic equation vanishes on the boundary and is initially zero, and the PDE is homogeneous $[f(\mathbf{x}, t) = 0]$, the solution must be zero. This is clear from Eq. (22.23). On the other hand, Eq. (22.31) indicates that under the same circumstance, there may be a nonzero solution for a hyperbolic equation if ψ is nonzero. In such a case we obtain

$$u(\mathbf{x}, t) = \int_D \psi(\mathbf{y}) G(\mathbf{x}, \mathbf{y}; t) d^m y.$$

This difference in the two types of equations is due to the fact that hyperbolic equations have second-order time derivatives. Thus, the initial shape of a solution is not enough to uniquely specify it. The initial velocity profile is also essential. We saw examples of this in Chap. 19.

The discussion of Green's functions has so far been formal. The main purpose of the remaining sections is to bridge the gap between formalism and concrete applications. Several powerful techniques are used in obtaining Green's functions, but we will focus only on two: the Fourier transform technique, and the eigenfunction expansion technique.

22.4 The Fourier Transform Technique

Recall that any Green's function can be written as a sum of a singular part and a regular part: $G = G_s + H$. Since we have already discussed homogeneous equations in detail in Chap. 19, we will not evaluate H in this section but will concentrate on the singular parts of various Green's functions.

The BCs play no role in evaluating G_s. Therefore, the Fourier transform technique (FTT), which involves integration over all space, can be utilized. The FTT has a drawback—it does not work if the coefficient functions are not constants. For most physical applications treated in this book, however, this will not be a shortcoming.

Let us consider the most general SOLPDO with constant coefficients,

$$\mathbf{L_x} = a_0 + \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} + \sum_{j,k=1}^{m} b_{jk} \frac{\partial^2}{\partial x_j \partial x_k}, \tag{22.32}$$

where a_0, a_j, and b_{jk} are constants. The corresponding Green's function has a singular part that satisfies the usual PDE with the delta function on the RHS. The FTT starts with assuming a Fourier integral representation in the variable $\mathbf{x}$ for the singular part and for the delta function:

$$G_s(\mathbf{x}, \mathbf{y}) = \frac{1}{(2\pi)^{m/2}} \int d^m k \tilde{G}_s(\mathbf{k}, \mathbf{y}) e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^m}\int d^m k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Substituting these equations in the PDE for the GF, we get

$$\tilde{G}_s(\mathbf{k},\mathbf{y}) = \frac{1}{(2\pi)^{m/2}}\left(\frac{e^{-i\mathbf{k}\cdot\mathbf{y}}}{a_0 + i\sum_{j=1}^m a_j k_j - \sum_{j,l=1}^m b_{jk}k_j k_l}\right)$$

and

$$G_s(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^m}\int d^m k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{a_0 + i\sum_{j=1}^m a_j k_j - \sum_{j,l=1}^m b_{jk}k_j k_l}. \qquad (22.33)$$

If we can evaluate the integral in (22.33), we can find G.

The following examples apply Eq. (22.33) to specific problems. Note that (22.33) indicates that G_s depends only on $\mathbf{x}-\mathbf{y}$. This point was mentioned in Chap. 20, where it was noted that such dependence occurs when the BCs play no part in an evaluation of the singular part of the Green's function of a DE with constant coefficients; and this is exactly the situation here.

22.4.1 GF for the m-Dimensional Laplacian

We calculated the GF for the m-dimensional Laplacian in Sect. 21.2.2 using a different method. With $a_0 = 0 = a_j$, $b_{jl} = \delta_{jl}$, and $\mathbf{r} = \mathbf{x}-\mathbf{y}$, Eq. (22.33) reduces to

$$G_s(\mathbf{r}) = \frac{1}{(2\pi)^m}\int d^m k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{-k^2}, \qquad (22.34)$$

where $k^2 = k_1^2 + \cdots + k_m^2 = \mathbf{k}\cdot\mathbf{k}$. To integrate (22.34), we choose spherical coordinates in the m-dimensional k-space. Furthermore, to simplify calculations we let the k_m-axis lie along $\mathbf{r}$ so that $\mathbf{r} = (0, 0, \ldots, |\mathbf{r}|)$ and $\mathbf{k}\cdot\mathbf{r} = k|\mathbf{r}|\cos\theta_1$ [see Eq. (21.12)]. Substituting this in (22.34) and writing $d^m k$ in spherical coordinates yields

$$G_s(\mathbf{r}) = \frac{-1}{(2\pi)^m}\int \frac{e^{ik|\mathbf{r}|\cos\theta_1}}{k^2} \times k^{m-1}(\sin\theta_1)^{m-2}\cdots\sin\theta_{m-2}\,dk\,d\theta_1\cdots d\theta_{m-1}. \qquad (22.35)$$

From Eq. (21.15) we note that $d\Omega_m = (\sin\theta_1)^{m-2}d\theta_1 d\Omega_{m-1}$. Thus, after integrating over the angles $\theta_2, \ldots, \theta_{m-1}$, Eq. (22.35) becomes

$$G_s(\mathbf{r}) = -\frac{1}{(2\pi)^m}\Omega_{m-1}\int_0^\infty k^{m-3}dk\int_0^\pi (\sin\theta_1)^{m-2}e^{ik|\mathbf{r}|\cos\theta_1}d\theta_1.$$

The inner integral can be looked up in an integral table (see [Grad 65, p. 482]):

$$\int_0^\pi (\sin\theta_1)^{m-2}e^{ik|\mathbf{r}|\cos\theta_1}d\theta_1 = \sqrt{\pi}\left(\frac{2}{kr}\right)^{m/2-1}\Gamma\left(\frac{m-1}{2}\right)J_{m/2-1}(kr).$$

Substituting this and (21.16) in the preceding equation and using the result (see [Grad 65, p. 684])

$$\int_0^\infty x^\mu J_\nu(ax)\,dx = 2^\mu a^{-\mu-1}\frac{\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(\frac{\mu-\nu+1}{2})},$$

we obtain

$$G_s(\mathbf{r}) = -\frac{\Gamma(m/2-1)}{4\pi^{m/2}}\left(\frac{1}{r^{m-2}}\right) \quad \text{for } m>2,$$

which agrees with (21.20) since $\Gamma(m/2) = (m/2-1)\Gamma(m/2-1)$.

22.4.2 GF for the m-Dimensional Helmholtz Operator

For the Helmholtz operator $\nabla^2 - \mu^2$, Eq. (22.33) reduces to

$$G_s(\mathbf{r}) = -\frac{1}{(2\pi)^m}\int d^m k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mu^2+k^2}.$$

Following the same procedure as in the previous subsection, we find

$$\begin{aligned} G_s(\mathbf{r}) &= -\frac{\Omega_{m-1}}{(2\pi)^m}\int_0^\infty \frac{k^{m-1}dk}{\mu^2+k^2}\int_0^\pi (\sin\theta_1)^{m-2}e^{ikr\cos\theta_1}d\theta_1 \\ &= -\frac{\Omega_{m-1}}{(2\pi)^m}\sqrt{\pi}\left(\frac{2}{r}\right)^{m/2-1}\Gamma\left(\frac{m-1}{2}\right)\int_0^\infty \frac{k^{m/2}}{\mu^2+k^2}J_{m/2-1}(kr)\,dk. \end{aligned}$$

Here we can use the integral formula (see [Grad 65, pp. 686 and 952])

$$\int_0^\infty \frac{J_\nu(bx)x^{\nu+1}}{(x^2+a^2)^{\eta+1}}\,dx = \frac{a^{\nu-\eta}b^\eta}{2^\eta\Gamma(\eta+1)}K_{\nu-\eta}(ab),$$

where

$$K_\nu(z) = \frac{i\pi}{2}e^{i\nu\pi/2}H_\nu^{(1)}(iz),$$

to obtain

$$G_s(\mathbf{r}) = -\frac{\Omega_{m-1}}{(2\pi)^m}\sqrt{\pi}\left(\frac{2}{r}\right)^{m/2-1}\Gamma\left(\frac{m-1}{2}\right)\mu^{m/2-1}\frac{\pi}{2}e^{im\pi/4}H_{m/2-1}^{(1)}(i\mu r),$$

which simplifies to

$$G_s(\mathbf{r}) = -\frac{\pi/2}{(2\pi)^{m/2}}\left(\frac{\mu}{r}\right)^{m/2-1}e^{im\pi/4}H_{m/2-1}^{(1)}(i\mu r). \tag{22.36}$$

It can be shown (see Problem 22.8) that for $m=3$ this reduces to $G_s(\mathbf{r}) = -\frac{e^{-\mu r}}{4\pi r}$, which is the Yukawa potential due to a unit charge.

We can easily obtain the GF for $\nabla^2 + \mu^2$ by substituting $\pm i\mu$ for μ in Eq. (22.36). The result is

$$G_s(\mathbf{r}) = i^{m+1} \frac{\pi/2}{(2\pi)^{m/2}} \left(\frac{\mu}{r}\right)^{m/2-1} H^{(1)}_{m/2-1}(\pm\mu r). \tag{22.37}$$

For $m = 3$ this yields $G_s(\mathbf{r}) = -e^{\pm i\mu r}/(4\pi r)$. The two signs in the exponent correspond to the so-called incoming and outgoing "waves".

Example 22.4.1 For a non-local potential, the time-independent Schrödinger equation is

Non-local potentials depend not only on the observation point, but also on some other "non-local" variables.

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi + \int_{\mathbb{R}^3} V(\mathbf{r}, \mathbf{r}')\Psi(\mathbf{r}')\, d^3r' = E\Psi(\mathbf{r}).$$

Then, the integral equation associated with this differential equation is (see Sect. 21.4)

$$\Psi(\mathbf{r}) = Ae^{i\mathbf{k}\cdot\mathbf{r}} - \frac{\mu}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\int_{\mathbb{R}^3} d^3r'' V(\mathbf{r}', \mathbf{r}'')\Psi(\mathbf{r}''). \tag{22.38}$$

For a separable potential, for which $V(\mathbf{r}', \mathbf{r}'') = -g^2U(\mathbf{r}')U(\mathbf{r}'')$, we can solve (22.38) exactly. We substitute for $V(\mathbf{r}', \mathbf{r}'')$ in (22.38) to obtain

$$\Psi(\mathbf{r}) = Ae^{i\mathbf{k}\cdot\mathbf{r}} + \frac{\mu g^2}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}')\int_{\mathbb{R}^3} d^3r'' U(\mathbf{r}'')\Psi(\mathbf{r}''). \tag{22.39}$$

Defining the quantities

$$Q(\mathbf{r}) \equiv \frac{\mu g^2}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}'), \qquad C \equiv \int_{\mathbb{R}^3} d^3r'' U(\mathbf{r}'')\Psi(\mathbf{r}'') \tag{22.40}$$

and substituting them in (22.39) yields $\Psi(\mathbf{r}) = Ae^{i\mathbf{k}\cdot\mathbf{r}} + CQ(\mathbf{r})$. Multiplying both sides of this equation by $U(\mathbf{r})$ and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} C &= A\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{r}}U(\mathbf{r})d^3r + C\int_{\mathbb{R}^3} U(\mathbf{r})Q(\mathbf{r})\, d^3r \\ &= (2\pi)^{3/2}A\tilde{U}(-\mathbf{k}) + C\int_{\mathbb{R}^3} U(\mathbf{r})Q(\mathbf{r})\, d^3r, \end{aligned}$$

from which we obtain

$$C = \frac{(2\pi)^{3/2}A\tilde{U}(-\mathbf{k})}{1 - \int_{\mathbb{R}^3} U(\mathbf{r})Q(\mathbf{r})\, d^3r},$$

leading to the solution

$$\Psi(\mathbf{r}) = Ae^{i\mathbf{k}\cdot\mathbf{r}} + \frac{(2\pi)^{3/2}A\tilde{U}(-\mathbf{k})}{1 - \int_{\mathbb{R}^3} U(\mathbf{r}')Q(\mathbf{r}')\, d^3r'}Q(\mathbf{r}). \tag{22.41}$$

In principle, $\tilde{U}(-\mathbf{k})$ [the Fourier transform of $U(\mathbf{r})$] and $Q(\mathbf{r})$ can be calculated once the functional form of $U(\mathbf{r})$ is known. Equations (22.40) and (22.41) give the solution to the Schrödinger equation in closed form.

22.4.3 GF for the m-Dimensional Diffusion Operator

When dealing with parabolic and hyperbolic equations, we will find it convenient to consider the "different" variable (usually t) as the zeroth coordinate. In the Fourier transform we then use $\omega = -k_0$ and write

$$\begin{aligned} G_s(\mathbf{r},t) &= \frac{1}{(2\pi)^{(m+1)/2}} \int_{-\infty}^{\infty} d\omega \int d^m k \tilde{G}_s(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \\ \delta(\mathbf{r})\delta(t) &= \frac{1}{(2\pi)^{m+1}} \int_{-\infty}^{\infty} d\omega \int d^m k e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \end{aligned} \tag{22.42}$$

where $\mathbf{r}$ is the m-dimensional position vector.

We substitute (22.42) in $(\partial/\partial t - \nabla^2)G_s(\mathbf{r},t) = \delta(\mathbf{r})\delta(t)$ to obtain

$$G_s(\mathbf{r},t) = \frac{1}{(2\pi)^{m+1}} \int d^m k e^{i\mathbf{k}\cdot\mathbf{r}} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + ik^2}, \tag{22.43}$$

where as usual, $k^2 = \sum_{i=1}^m k_i^2$. The ω integration can be done using the calculus of residues. The integrand has a simple pole at $\omega = -ik^2$, that is, in the lower half of the complex ω-plane (LHP). To integrate, we must know the sign of t. If $t > 0$, the exponential factor dictates that the contour be closed in the LHP, where there is a pole and, therefore, a contribution to the residues. On the other hand, if $t < 0$, the contour must be closed in the UHP. The integral is then zero because there are no poles in the UHP. We must therefore introduce a step function $\theta(t)$ in the Green's function. Evaluating the residue, the ω integration yields $-2\pi i e^{-k^2 t}$. (The minus sign arises because of clockwise contour integration in the LHP.) Substituting this in Eq. (22.43), using spherical coordinates in which the last k-axis is along $\mathbf{r}$, and integrating over all angles except θ_1, we obtain

$$\begin{aligned} G_s(\mathbf{r},t) &= \theta(t) \frac{\Omega_{m-1}}{(2\pi)^m} \int_0^{\infty} k^{m-1} dk e^{-k^2 t} \int_0^{\pi} (\sin\theta_1)^{m-2} e^{ikr\cos\theta_1} d\theta_1 \\ &= \theta(t) \frac{\Omega_{m-1}}{(2\pi)^m} \sqrt{\pi} \left(\frac{2}{r}\right)^{m/2-1} \Gamma\left(\frac{m-1}{2}\right) \\ &\quad \times \int_0^{\infty} k^{m/2} e^{-k^2 t} J_{m/2-1}(kr)\, dk. \end{aligned}$$

For the θ_1 integration, we used the result quoted in Sect. 22.4.1.

Using the integral formula (see [Grad 65, pp. 716 and 1058])

$$\begin{aligned} &\int_0^{\infty} x^{\mu} e^{-\alpha x^2} J_{\nu}(\beta x)\, dx \\ &\quad = \frac{\beta^{\nu} \Gamma(\frac{\mu+\nu+1}{2})}{2^{\nu+1} \alpha^{(\mu+\nu+1)/2} \Gamma(\nu+1)} \Phi\left(\frac{\mu+\nu+1}{2}, \nu+1; -\frac{\beta^2}{4\alpha}\right), \end{aligned}$$

where Φ is the confluent hypergeometric function, we obtain

$$G_s(\mathbf{r},t) = \theta(t) \frac{2\pi^{(m-1)/2}}{(2\pi)^m} \sqrt{\pi} \left(\frac{2}{r}\right)^{m/2-1} \frac{r^{m/2-1}}{2^{m/2} t^{m/2}} \Phi\left(\frac{m}{2}, \frac{m}{2}; -\frac{r^2}{4t}\right). \tag{22.44}$$

The power-series expansion for the confluent hypergeometric function Φ shows that $\Phi(\alpha,\alpha;z)=e^z$. Substituting this result in (22.44) and simplifying, we finally obtain

$$G_s(\mathbf{r},t)=\frac{e^{-r^2/4t}}{(4\pi t)^{m/2}}\theta(t). \tag{22.45}$$

22.4.4 GF for the m-Dimensional Wave Equation

The difference between this example and the preceding one is that here the time derivative is of second order. Thus, instead of Eq. (22.43), we start with

$$G_s(\mathbf{r},t)=-\frac{1}{(2\pi)^{m+1}}\int d^mk e^{i\mathbf{k}\cdot\mathbf{r}}\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega t}}{\omega^2-k^2}. \tag{22.46}$$

The ω integration can be done using the method of residues. Since the singularities of the integrand, $\omega=\pm k$, are on the real axis, it seems reasonable to use the principal value as the value of the integral. This, in turn, depends on the sign of t. If $t>0$ ($t<0$), we have to close the contour in the LHP (UHP): to avoid the explosion of the exponential. If one also insists on not including the poles inside the contour,[3] then one can show that

$$P\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega t}}{\omega^2-k^2}=-\pi\frac{\sin kt}{k}\epsilon(t),$$

where

$$\epsilon(t)\equiv\theta(t)-\theta(-t)=\begin{cases}1 & \text{if } t>0,\\ -1 & \text{if } t<0.\end{cases}$$

Substituting this in (22.46) and integrating over all angles as done in the previous examples yields

$$G_s(\mathbf{r},t)=\frac{\epsilon(t)}{2(2\pi)^{m/2}r^{m/2-1}}\int_0^{\infty}k^{m/2-1}J_{m/2-1}(kr)\sin kt\,dk. \tag{22.47}$$

Physics determines the contour of integration

As Problem 22.25 shows, the Green's function given by Eq. (22.47) satisfies only the homogeneous wave equation with no delta function on the RHS. The reason for this is that the principal value of an integral chooses a specific contour that may not reflect the physical situation. In fact, the Green's function in (22.47) contains two pieces corresponding to the two different contours of integration, and it turns out that the physically interesting Green's functions are obtained, not from the principal value, but from giving small imaginary parts to the poles. Thus, replacing the ω integral with a contour integral for which the two poles are pushed in the LHP and using

[3] This will determine how to (semi)circle around the poles.

the method of residues, we obtain

$$I_{up} \equiv \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - k^2} = \int_{C_1} \frac{e^{-izt}}{z^2 - k^2} dz = \frac{2\pi}{k} \theta(t) \sin kt.$$

The integral is zero for $t < 0$ because for negative values of t, the contour must be closed in the UHP, where there are no poles inside C_1. Substituting this in (22.46) and working through as before, we obtain what is called the **retarded Green's function**:

retarded Green's function

$$G_s^{(\text{ret})}(\mathbf{r}, t) = \frac{\theta(t)}{(2\pi)^{m/2} r^{m/2-1}} \int_0^{\infty} k^{m/2-1} J_{m/2-1}(kr) \sin kt \, dk. \quad (22.48)$$

advanced Green's function

If the poles are pushed in the UHP we obtain the **advanced Green's function**:

$$G_s^{(\text{adv})}(\mathbf{r}, t) = -\frac{\theta(-t)}{(2\pi)^{m/2} r^{m/2-1}} \int_0^{\infty} k^{m/2-1} J_{m/2-1}(kr) \sin kt \, dk. \quad (22.49)$$

Unlike the elliptic and parabolic equations discussed earlier, the integral over k is not a function but a *distribution*, as will become clear below. To find the retarded and advanced Green's functions, we write the sine term in the integral in terms of exponentials and use the following (see [Grad 65, p. 712]):

$$\int_0^{\infty} x^{\nu} e^{-\alpha x} J_{\nu}(\beta x) \, dx = \frac{(2\beta)^{\nu} \Gamma(\nu + 1/2)}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu+1/2}} \quad \text{for } \operatorname{Re}(\alpha) > |\operatorname{Im}(\beta)|.$$

To ensure convergence at infinity, we add a small negative number to the exponential and define the integral

$$\begin{aligned} I_{\epsilon}^{\pm} &\equiv \int_0^{\infty} k^{\nu} e^{-(\mp it + \epsilon)k} J_{\nu}(kr) \, dk \\ &= \frac{(2r)^{\nu} \Gamma(\nu + 1/2)}{\sqrt{\pi}} \left[(\mp it + \epsilon)^2 + r^2 \right]^{-(\nu+1/2)}. \end{aligned}$$

For the GFs, we need to evaluate the (common) integral in (22.48) and (22.49). With $\nu = m/2 - 1$, we have

$$\begin{aligned} I^{(\nu)} &\equiv \int_0^{\infty} k^{\nu} J_{\nu}(kr) \sin kt \, dk = \frac{1}{2i} \lim_{\epsilon \to 0} \left(I_{\epsilon}^{+} - I_{\epsilon}^{-} \right) \\ &= \frac{(2r)^{\nu} \Gamma(\nu + 1/2)}{2i\sqrt{\pi}} \\ &\quad \times \lim_{\epsilon \to 0} \left\{ \frac{1}{[r^2 + (-it + \epsilon)^2]^{\nu+1/2}} - \frac{1}{[r^2 + (it + \epsilon)^2]^{\nu+1/2}} \right\}. \end{aligned}$$

At this point, it is convenient to discuss separately the two cases of m odd and m even. Let us derive the expression for odd m (the even case is left for Problem 22.26). Define the integer $n = (m-1)/2 = \nu + \frac{1}{2}$ and write $I^{(\nu)}$ as

$$I^{(n)} = \frac{(2r)^{n-1/2} \Gamma(n)}{2i\sqrt{\pi}} \lim_{\epsilon \to 0} \left\{ \frac{1}{[r^2 + (-it + \epsilon)^2]^n} - \frac{1}{[r^2 + (it + \epsilon)^2]^n} \right\}. \quad (22.50)$$

Define $u = r^2 + (-it + \epsilon)^2$. Then using the identity

$$\frac{1}{u^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{du^{n-1}}\left(\frac{1}{u}\right)$$

and the chain rule, $df/du = (1/2r)\partial f/\partial r$, we obtain $d/du = (1/2r)\partial/\partial r$ and

$$\frac{1}{[r^2 + (\pm it + \epsilon)^2]^n} = \frac{1}{(n-1)!}\left(-\frac{1}{2r}\frac{\partial}{\partial r}\right)^{n-1}\left[\frac{1}{r^2 + (\pm it + \epsilon)^2}\right].$$

Therefore, Eq. (22.50) can be written as

$$\begin{aligned} I^{(n)} &= \int_0^\infty k^{n-1/2} J_{n-1/2}(kr) \sin kt \, dk \\ &= \frac{(2r)^{n-1/2}\Gamma(n)}{2i\sqrt{\pi}} \frac{1}{(n-1)!} \\ &\quad \times \left(-\frac{1}{2r}\frac{\partial}{\partial r}\right)^{n-1}\left\{\lim_{\epsilon\to 0}\left[\frac{1}{[r^2 + (-it + \epsilon)^2]} - \frac{1}{[r^2 + (it + \epsilon)^2]}\right]\right\}. \end{aligned} \tag{22.51}$$

The limit in (22.51) is found in Problem 22.27. Using the result of that problem and $\Gamma(n) = (n-1)!$, we get

$$\begin{aligned} I^{(n)} &= \int_0^\infty k^{n-1/2} J_{n-1/2}(kr) \sin kt \, dk \\ &= -\frac{\sqrt{\pi}(2r)^{n-1/2}}{2}\left(-\frac{1}{2r}\frac{\partial}{\partial r}\right)^{n-1}\left\{\frac{1}{r}\big[\delta(t+r) - \delta(t-r)\big]\right\}. \end{aligned} \tag{22.52}$$

Employing this result in (22.48) and (22.49) yields

$$\begin{aligned} G_s^{(\text{ret})}(\mathbf{r}, t) &= \frac{1}{4\pi}\left(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\right)^{n-1}\left[\frac{\delta(t-r)}{r}\right] \quad \text{for } n = \frac{m-1}{2}, \\ G_s^{(\text{adv})}(\mathbf{r}, t) &= \frac{1}{4\pi}\left(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\right)^{n-1}\left[\frac{\delta(t+r)}{r}\right] \quad \text{for } n = \frac{m-1}{2}. \end{aligned} \tag{22.53}$$

The theta functions are not needed in (22.53) because the arguments of the delta functions already meet the restrictions imposed by the theta functions.

The two functions in (22.53) have an interesting physical interpretation. Green's functions are propagators (of signals of some sort), and $G_s^{(\text{ret})}(\mathbf{r}, t)$ is capable of propagating signals only for positive times. On the other hand, $G_s^{(\text{adv})}(\mathbf{r}, t)$ can propagate only in the negative time direction. Thus, if initially ($t = 0$) a signal is produced (by appropriate BCs), both $G_s^{(\text{ret})}(\mathbf{r}, t)$ and $G_s^{(\text{adv})}(\mathbf{r}, t)$ work to propagate it in their respective time directions. It may seem that $G_s^{(\text{adv})}(\mathbf{r}, t)$ is useless because every signal propagates forward in time. This is true, however, only for classical events. In relativistic quantum field theory antiparticles are interpreted mathematically as moving in the negative time direction! Thus, we cannot simply ignore $G_s^{(\text{adv})}(\mathbf{r}, t)$.

Feynman propagator

In fact, the correct propagator to choose in this theory is a linear combination of $G_s^{(\text{adv})}(\mathbf{r}, t)$ and $G_s^{(\text{ret})}(\mathbf{r}, t)$, called the **Feynman propagator** (see [Wein 95, pp. 274–280]).

The preceding example shows a subtle difference between Green's functions for second-order differential operators in one dimension and in higher dimensions. We saw in Chap. 20 that the former are continuous functions in the interval on which they are defined. Here, we see that higher dimensional Green's functions are not only discontinuous, but that they are not even *functions* in the ordinary sense; they contain a delta function. Thus, in general, Green's functions in higher dimensions ought to be treated as distributions (generalized functions).

22.5 The Eigenfunction Expansion Technique

Suppose that the differential operator $\mathbf{L_x}$, defined in a domain D with boundary ∂D, has discrete eigenvalues $\{\lambda_n\}_{n=1}^\infty$ with corresponding orthonormal eigenfunctions $\{u_m(\mathbf{x})\}_{m=1}^\infty$. These two sets may not be in one-to-one correspondence. Assume that the $u_m(\mathbf{x})$'s satisfy the same BCs as the Green's function to be defined below.

Now consider the operator $\mathbf{L_x} - \lambda\mathbf{1}$, where λ is different from all λ_n's. Then, as in the one-dimensional case, this operator is invertible, and we can define its Green's function by $(\mathbf{L_x} - \lambda)G_\lambda(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ where the weight function is set equal to one. The completeness of $\{u_n(\mathbf{x})\}_{m=1}^\infty$ implies that

$$\delta(\mathbf{x} - \mathbf{y}) = \sum_{n=1}^\infty u_n(\mathbf{x})u_n^*(\mathbf{y}) \quad \text{and} \quad G_\lambda(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^\infty a_n(\mathbf{y})u_n(\mathbf{x}).$$

Substituting these two expansions in the differential equation for GF yields

$$\sum_{n=1}^\infty (\lambda_n - \lambda)a_n(\mathbf{y})u_n(\mathbf{x}) = \sum_{n=1}^\infty u_n(\mathbf{x})u_n^*(\mathbf{y}).$$

The orthonormality of the u_n's gives $a_n(\mathbf{y}) = u_n^*(\mathbf{y})/(\lambda_n - \lambda)$. Therefore,

$$G_\lambda(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^\infty \frac{u_n(\mathbf{x})u_n^*(\mathbf{y})}{\lambda_n - \lambda}. \tag{22.54}$$

In particular, if zero is not an eigenvalue of $\mathbf{L_x}$, its Green's function can be written as

$$G(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^\infty \frac{u_n(\mathbf{x})u_n^*(\mathbf{y})}{\lambda_n}. \tag{22.55}$$

This is an expansion of the Green's function in terms of the eigenfunctions of $\mathbf{L_x}$.

It is instructive to consider a formal interpretation of Eq. (22.55). Recall that the spectral decomposition theorem permits us to write $f(\mathbf{A}) = \sum_i f(\lambda_i)\mathbf{P}_i$ for an operator $\mathbf{A}$ with (distinct) eigenvalues λ_i and projection operators $\mathbf{P}_i$. Allowing repetition of eigenvalues in the sum, we may write $f(\mathbf{A}) = \sum_n f(\lambda_n)|u_n\rangle\langle u_n|$, where n counts *all* the eigenfunctions corresponding to eigenvalues. Now, let $f(\mathbf{A}) = \mathbf{A}^{-1}$. Then

$$\mathbf{G} = \mathbf{A}^{-1} = \sum_n \lambda_n^{-1}|u_n\rangle\langle u_n| = \sum_n \frac{|u_n\rangle\langle u_n|}{\lambda_n},$$

or, in "matrix element" form,

$$G(\mathbf{x}, \mathbf{y}) = \langle\mathbf{x}|\mathbf{G}|\mathbf{y}\rangle = \sum_n \frac{\langle\mathbf{x}|u_n\rangle\langle u_n|\mathbf{y}\rangle}{\lambda_n} = \sum_n \frac{u_n(\mathbf{x})u_n^*(\mathbf{y})}{\lambda_n}.$$

This last expression coincides with the RHS of Eq. (22.55).

Equations (22.54) and (22.55) demand that the $u_n(\mathbf{x})$ form a complete discrete orthonormal set. We encountered many examples of such eigenfunctions in discussing Sturm-Liouville systems in Chap. 19. All the S-L systems there were, of course, one-dimensional. Here we are generalizing the S-L system to m dimensions. This is not a limitation, however, because—for the PDEs of interest—the separation of variables reduces an m-dimensional PDE to m one-dimensional ODEs. If the BCs are appropriate, the m ODEs will all be S-L systems. A review of Chap. 19 will reveal that homogeneous BCs always lead to S-L systems. In fact, Theorem 19.4.1 guarantees this claim. Since Green's functions must also satisfy homogeneous BCs, expansions such as those of (22.54) and (22.55) become possible.

Example 22.5.1 As a concrete example, let us obtain an eigenfunction expansion of the GF of the two-dimensional Laplacian, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, inside the rectangular region $0 \le x \le a$, $0 \le y \le b$ with Dirichlet BCs. Since the GF vanishes at the boundary, the eigenvalue problem becomes $\nabla^2 u = \lambda u$ with $u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0$. The method of separation of variables gives the orthonormal eigenfunctions[4]

$$u_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \quad \text{for } m, n = 1, 2, \ldots,$$

whose corresponding eigenvalues are $\lambda_{mn} = -[(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2]$.

Inserting the eigenfunctions and the eigenvalues in Eq. (22.55), we obtain

$$G(\mathbf{r}, \mathbf{r}') = G(x, y; x', y') = \sum_{m,n=1}^{\infty} \frac{u_{mn}(x, y)u_{mn}(x', y')}{\lambda_{mn}}$$

$$= -\frac{4}{ab} \sum_{m,n=1}^{\infty} \frac{\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(\frac{n\pi}{a}x')\sin(\frac{m\pi}{b}y')}{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2},$$

where we changed $\mathbf{x}$ to $\mathbf{r}$ and $\mathbf{y}$ to $\mathbf{r}'$. Note that the eigenvalues are never zero; thus, $G(\mathbf{r}, \mathbf{r}')$ is well-defined.

[4]The inner product is defined as a double integral over the rectangle.

In the preceding example, zero was not an eigenvalue of $\mathbf{L_x}$. This condition must hold when a Green's function is expanded in terms of eigenfunctions. In physical applications, certain conditions (which have nothing to do with the BCs) exclude the zero eigenvalue automatically when they are applied to the Green's function. For instance, the condition that the Green's function remain finite at the origin is severe enough to exclude the zero eigenvalue.

Example 22.5.2 Let us consider the two-dimensional Dirichlet BVP $\nabla^2 u = f$, with $u = 0$ on a circle of radius a. If we consider only the BCs and ask whether zero is an eigenvalue of ∇^2, the answer will be yes, as the following argument shows.

The most general solution to the zero-eigenvalue equation, $\nabla^2 u = 0$, in polar coordinates can be obtained by the method of separation of variables:

$$u(\rho,\varphi) = A + B\ln\rho + \sum_{n=1}^{\infty}\left(b_n\rho^n + b'_n\rho^{-n}\right)\cos n\varphi + \sum_{n=1}^{\infty}\left(c_n\rho^n + c'_n\rho^{-n}\right)\sin n\varphi. \tag{22.56}$$

Invoking the BC gives

$$0 = u(a,\varphi) = A + B\ln a + \sum_{n=1}^{\infty}\left(b_n a^n + b'_n a^{-n}\right)\cos n\varphi + \sum_{n=1}^{\infty}\left(c_n a^n + c'_n a^{-n}\right)\sin n\varphi,$$

which holds for arbitrary φ if and only if

$$A = -B\ln a, \qquad b'_n = -b_n a^{2n}, \qquad c'_n = -c_n a^{2n}.$$

Substituting in (22.56) gives

$$u(\rho,\varphi) = B\ln\left(\frac{\rho}{a}\right) + \sum_{n=1}^{\infty}\left(\rho^n - \frac{a^{2n}}{\rho^n}\right)(b_n\cos n\varphi + c_n\sin n\varphi). \tag{22.57}$$

Thus, if we demand nothing beyond the BCs, ∇^2 will have a nontrivial eigen-solution corresponding to the zero eigenvalue, given by Eq. (22.57).

Physical reality, however, demands that $u(\rho,\varphi)$ be well-behaved at the origin. This condition sets B, b'_n, and c'_n of Eq. (22.56) equal to zero. The BCs then make the remaining coefficients in (22.56) vanish. Thus, the demand that $u(\rho,\varphi)$ be well-behaved at $\rho = 0$ turns the situation completely around and ensures the nonexistence of a zero eigenvalue for the Laplacian, which in turn guarantees the existence of a GF.

In many cases the operator $\mathbf{L_x}$ as a whole is not amenable to a full Sturm-Liouville treatment, and as such will not yield orthonormal eigenvectors in terms of which the GF can be expanded. However, it may happen that

$\mathbf{L_x}$ can be broken up into two pieces one of which is an S-L operator. In such a case, the GF can be found as follows: Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are two *commuting* operators with $\mathbf{L}_2$ an S-L operator whose eigenvalues and eigenfunctions are known. Since $\mathbf{L}_2$ commutes with $\mathbf{L}_1$, it can be regarded as a constant as far as operations with (and on) $\mathbf{L}_1$ are concerned. In particular, $(\mathbf{L}_1 + \mathbf{L}_2)\mathbf{G} = \mathbf{1}$ can be regarded as an operator equation in $\mathbf{L}_1$ *alone* with $\mathbf{L}_2$ treated as a constant. Let $\mathbf{x}_1$ denote the subset of the variables on which $\mathbf{L}_1$ acts, and let $\mathbf{x}_2$ denote the remainder of the coordinates. Then we can write $\delta(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}_1-\mathbf{y}_1)\delta(\mathbf{x}_2-\mathbf{y}_2)$. Now let $G_1(\mathbf{x}_1, \mathbf{y}_1; k)$ denote the Green's function for $\mathbf{L}_1 + k$, where k is a constant. Then it is easily verified that

$$G(\mathbf{x}, \mathbf{y}) = G_1(\mathbf{x}_1, \mathbf{y}_1; \mathbf{L}_2)\delta(\mathbf{x}_2 - \mathbf{y}_2). \tag{22.58}$$

In fact,

$$(\mathbf{L}_1 + \mathbf{L}_2)G(\mathbf{x}, \mathbf{y}) = \underbrace{\big[(\mathbf{L}_1 + \mathbf{L}_2)G_1(\mathbf{x}_1, \mathbf{y}_1; \mathbf{L}_2)\big]}_{=\delta(\mathbf{x}_1-\mathbf{y}_1)\text{ by definition of } G_1}\delta(\mathbf{x}_2 - \mathbf{y}_2).$$

Once G_1 is found as a function of $\mathbf{L}_2$, it can operate on $\delta(\mathbf{x}_2-\mathbf{y}_2)$ to yield the desired Green's function. The following example illustrates the technique.

Example 22.5.3 Let us evaluate the Dirichlet GF for the two-dimensional Helmholtz operator $\nabla^2 - k^2$ in the infinite strip $0 \le x \le a$, $-\infty < y < \infty$. Let $\mathbf{L}_1 = \partial^2/\partial y^2 - k^2$ and $\mathbf{L}_2 = \partial^2/\partial x^2$. Then,

$$G(\mathbf{r}, \mathbf{r}') \equiv G(x, x', y, y') = G_1(y, y'; \mathbf{L}_2)\delta(x - x'),$$

where $(d^2/dy^2 - \mu^2)G_1 = \delta(y - y')$, $\mu^2 \equiv k^2 - \mathbf{L}_2$, and $G_1(y = -\infty) = G_1(y = \infty) = 0$. The GF G_1 can be readily found (see Problem 20.12):

$$G_1(y, y'; \mathbf{L}_2) = -\frac{e^{-\mu|y-y'|}}{2\mu} = -\frac{e^{-\sqrt{k^2-\mathbf{L}_2}|y-y'|}}{2\sqrt{k^2-\mathbf{L}_2}}.$$

The full GF is then

$$G(\mathbf{r}, \mathbf{r}') = \left(-\frac{e^{-\sqrt{k^2-\mathbf{L}_2}|y-y'|}}{2\sqrt{k^2-\mathbf{L}_2}}\right)\delta(x - x'). \tag{22.59}$$

The operator $\mathbf{L}_2$ constitutes an S-L system with eigenvalues $\lambda_n = -(n\pi x/a)^2$ and eigenfunctions $u_n(x) = \sqrt{2/a}\sin(n\pi x/a)$ where $n = 1, 2, \ldots$. Therefore, the delta function $\delta(x - x')$ can be expanded in terms of these eigenfunctions:

$$\delta(x - x') = \frac{2}{a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right).$$

As $\mu = \sqrt{k^2 - \mathbf{L}_2}$ acts on the delta function, $\mathbf{L}_2$ operates on the first factor in the above expansion and gives λ_n. Thus, $\mathbf{L}_2$ in Eq. (22.59) can be replaced by $-(n\pi x/a)^2$, and we have

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{a}\sum_{n=1}^{\infty}\left(-\frac{e^{-\sqrt{k^2+(n\pi x/a)^2}|y-y'|}}{\sqrt{k^2+(n\pi x/a)^2}}\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right).$$

Sometimes it is convenient to break an operator into more than two parts. In fact, in some cases it may be advantageous to define a set of commuting self-adjoint (differential) operators $\{\mathbf{M}_j\}$ such that the full operator $\mathbf{L}$ can be written as $\mathbf{L} = \sum_j \mathbf{L}_j \mathbf{M}_j$ where the differential operators $\{\mathbf{L}_j\}$ act on variables on which the $\mathbf{M}_j$ have no action. Since the $\mathbf{M}_j$'s commute among themselves, one can find simultaneous eigenfunctions for all of them. Then one expands part of the delta function in terms of these eigenfunctions in the hope that the ensuing problem becomes more manageable. The best way to appreciate this approach is via an example.

Example 22.5.4 Let us consider the Laplacian in spherical coordinates,

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{\partial^2 u}{\partial\varphi^2}\right].$$

If we introduce

$$\begin{aligned} \mathbf{M}_1 u &= u, \qquad \mathbf{L}_1 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \\ \mathbf{M}_2 u &= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{\partial^2 u}{\partial\varphi^2}\right], \qquad \mathbf{L}_2 u = \frac{1}{r^2}u, \end{aligned} \tag{22.60}$$

the Laplacian becomes $\nabla^2 = \mathbf{L}_1\mathbf{M}_1 + \mathbf{L}_2\mathbf{M}_2$. The mutual eigenfunctions of $\mathbf{M}_1$ and $\mathbf{M}_2$ are simply those of $\mathbf{M}_2$, which is (the negative of) the angular momentum operator discussed in Chap. 13, whose eigenfunctions are the spherical harmonics. We thus have $\mathbf{M}_2 Y_{lm}(\theta, \varphi) = -l(l+1)Y_{lm}(\theta, \varphi)$.

Let us expand the Green's function in terms of the spherical harmonics:

$$G(\mathbf{r}, \mathbf{r}') = \sum_{l,m} g_{lm}(r; r', \theta', \varphi')Y_{lm}(\theta, \varphi).$$

We also write the delta function as

$$\begin{aligned} \delta(\mathbf{r} - \mathbf{r}') &= \frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r'^2\sin\theta'} \\ &= \frac{\delta(r-r')}{r'^2}\sum_{l,m} Y_{lm}(\theta, \varphi)Y^*_{lm}(\theta', \varphi'), \end{aligned}$$

where we have used the completeness of the spherical harmonics. Substituting all of the above in $\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$, we obtain

$$\begin{aligned}\nabla^2 G(\mathbf{r},\mathbf{r}') &= (\mathbf{L}_1\mathbf{M}_1 + \mathbf{L}_2\mathbf{M}_2)\sum_{l,m} g_{lm}(r;r',\theta',\varphi')Y_{lm}(\theta,\varphi) \\ &= \sum_{l,m}\big\{\big[\mathbf{L}_1 - l(l+1)\mathbf{L}_2\big]g_{lm}(r;r',\theta',\varphi')\big\}Y_{lm}(\theta,\varphi) \\ &= \frac{\delta(r-r')}{r'^2}\sum_{l,m} Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi).\end{aligned}$$

The orthogonality of the $Y_{lm}(\theta,\varphi)$ yields

$$\big[\mathbf{L}_1 - l(l+1)\mathbf{L}_2\big]g_{lm}(r;r',\theta',\varphi') = \frac{\delta(r-r')}{r'^2}Y^*_{lm}(\theta',\varphi').$$

This shows that the angular part of g_{lm} is simply $Y^*_{lm}(\theta',\varphi')$. Separating this from the dependence on $\mathbf{r}$ and $\mathbf{r}'$ and substituting for $\mathbf{L}_1$ and $\mathbf{L}_2$, we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_{lm}}{dr}\right) - \frac{l(l+1)}{r^2}g_{lm} = \frac{\delta(r-r')}{r^2}, \tag{22.61}$$

where this last g_{lm} is a function of r and r' only. The techniques of Chap. 20 can be employed to solve Eq. (22.61) (see Problem 22.29).

The separation of the full operator into two "smaller" operators can also be used for cases in which both operators have eigenvalues and eigenvectors. The result of such an approach will, of course, be equivalent to the eigenfunction-expansion approach. However, there will be an arbitrariness in the operator approach: Which operator are we to choose as our $\mathbf{L}_1$? While in Example 22.5.3 the choice was clear (the operator that had no eigenfunctions), here either operator can be chosen as $\mathbf{L}_1$. The ensuing GFs will be equivalent, and the series representing them will be convergent, of course. However, the *rate* of convergence may be different for the two. It turns out, for example, that if we are interested in $G(x,y;x',y')$ for the two-dimensional Laplacian at points (x,y) whose y-coordinates are far from y', then the appropriate expansion is obtained by letting $\mathbf{L}_1 = \partial^2/\partial y^2$, that is, an expansion in terms of x eigenfunctions. On the other hand, if the Green's function is to be calculated for a point (x,y) whose x-coordinate is far away from the singular point (x',y'), then the appropriate expansion is obtained by letting $\mathbf{L}_1 = \partial^2/\partial x^2$.

22.6 Problems

22.1 Find the GF for the Dirichlet BVP in two dimensions if D is the UHP and ∂D is the x-axis.

22.2 Add $f(\mathbf{r}'')$ to $H(\mathbf{r},\mathbf{r}'')$ in Example 22.1.2 and retrace the argument given there to show that $f(\mathbf{r}'') = 0$.

22.3 Use the method of images to find the GF for the Laplacian in the exterior region of a "sphere" of radius a in two and three dimensions.

22.4 Derive Eq. (22.7) from Eq. (22.6).

22.5 Using Eq. (22.7) with $\rho = 0$, show that if $g(\theta', \varphi') = V_0$, the potential at any point inside the sphere is V_0.

22.6 Find the BC that the GF must satisfy in order for the solution u to be representable in terms of the GF when the BC on u is mixed, as in Eq. (22.10). Assume a self-adjoint SOLPDO of the elliptic type, and consider the two cases $\alpha(\mathbf{x}) \neq 0$ and $\beta(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \partial D$. Hint: In each case, divide the mixed BC equation by the nonzero coefficient, substitute the result in the Green's identity, and set the coefficient of the u term in the ∂D integral equal to zero.

22.7 Show that the diffusion operator satisfies

$$\mathbf{L}_{\mathbf{x},t} G(\mathbf{x}, \mathbf{y}; t - \tau) = \delta(\mathbf{x} - \mathbf{y})\delta(t - \tau).$$

Hint: Use

$$\frac{\partial \theta}{\partial t}(t - \tau) = \delta(t - \tau).$$

22.8 Show that for $m = 3$ the expression for $G_s(\mathbf{r})$ given by Eq. (22.36) reduces to $G_s(\mathbf{r}) = -e^{-\mu r}/(4\pi r)$.

22.9 The time-independent Schrödinger equation can be rewritten as

$$\left(\nabla^2 + k^2\right)\Psi - \frac{2\mu}{\hbar^2} V(\mathbf{r})\Psi = 0,$$

where $k^2 = 2\mu E/\hbar^2$ and μ is the mass of the particle.

(a) Use techniques of Sect. 21.4 to write an integral equation for Ψ.

(b) Show that the Neumann series solution of the integral equation converges only if

$$\int_{\mathbb{R}^3} \left|V(\mathbf{r})\right|^2 d^3 r < \frac{2\pi \hbar^4 \operatorname{Im} k}{\mu^2}.$$

(c) Assume that the potential is of Yukawa type: $V(\mathbf{r}) = g^2 e^{-\kappa r}/r$. Find a condition between the (bound state) energy and the potential strength g that ensures convergence of the Neumann series.

22.10 Derive Eq. (22.29).

22.11 Derive Eq. (22.31) using the procedure outlined for parabolic equations.

22.12 Consider GF for the Helmholtz operator $\nabla^2 + \mu^2$ in two dimensions.

(a) Show that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{i}{4} H_0^{(1)}\left(\mu|\mathbf{r} - \mathbf{r}'|\right) + H(\mathbf{r}, \mathbf{r}'),$$

where $H(\mathbf{r}, \mathbf{r}')$ satisfies the homogeneous Helmholtz equation.

(b) Separate the variables and use the fact that H is regular at $\mathbf{r} = \mathbf{r}'$ to show that H can be written as

$$H(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} J_n(\mu r)\big[a_n(\mathbf{r}') \cos n\theta + b_n(\mathbf{r}') \sin n\theta\big].$$

(c) Now assume a circular boundary of radius a and the BC $G(\mathbf{a}, \mathbf{r}') = 0$, in which $\mathbf{a}$ is a vector from the origin to the circular boundary. Using this BC, show that

$$a_0(\mathbf{r}') = \frac{i}{8\pi J_0(\mu a)} \int_0^{2\pi} H_0^{(1)}\big(\mu\sqrt{a^2 + r'^2 - 2ar' \cos(\theta - \theta')}\big)\, d\theta,$$

$$a_n(\mathbf{r}') = \frac{i}{4\pi J_n(\mu a)} \times \int_0^{2\pi} H_0^{(1)}\big(\mu\sqrt{a^2 + r'^2 - 2ar' \cos(\theta - \theta')}\big) \cos n\theta\, d\theta,$$

$$b_n(\mathbf{r}') = \frac{i}{4\pi J_n(\mu a)} \times \int_0^{2\pi} H_0^{(1)}\big(\mu\sqrt{a^2 + r'^2 - 2ar' \cos(\theta - \theta')}\big) \sin n\theta\, d\theta.$$

These equations completely determine $H(\mathbf{r}, \mathbf{r}')$ and therefore $G(\mathbf{r}, \mathbf{r}')$.

22.13 Use the Fourier transform technique to find the singular part of the GF for the diffusion equation in one and three dimensions. Compare your results with that obtained in Sect. 22.4.3.

22.14 Show directly that both $G_s^{(\text{ret})}$ and $G_s^{(\text{adv})}$ satisfy $\nabla^2 G = \delta(\mathbf{r})\delta(t)$ in three dimensions.

22.15 Consider a rectangular box with sides a, b, and c located in the first octant with one corner at the origin. Let D denote the inside of this box.

(a) Show that zero cannot be an eigenvalue of the Laplacian operator with the Dirichlet BCs on ∂D.

(b) Find the GF for this Dirichlet BVP.

22.16 Find the GF for the two-dimensional Helmholtz equation $(\nabla^2 + k^2)u = 0$ on the rectangle $0 \le x \le a$, $0 \le y \le b$.

22.17 For the operator $ad^2/dx^2 + b$, where $a > 0$ and $b < 0$, find the singular part of the one-dimensional GF.

22.18 Calculate the GF of the two-dimensional Laplacian operator appropriate for Neumann BCs on the rectangle $0 \le x \le a$, $0 \le y \le b$.

22.19 For the Helmholtz operator $\nabla^2 - k^2$ in the half-space $z \ge 0$, find the three-dimensional Dirichlet GF.

22.20 For the Helmholtz operator $\nabla^2 - k^2$ in the half-space $z \le 0$, find the three-dimensional Neumann GF.

22.21 Using the integral form of the Schrödinger equation in three dimensions, show that an attractive delta-function potential $V(\mathbf{r}) = -V_0\delta(\mathbf{r} - \mathbf{a})$ does not have a bound state ($E < 0$). Contrast this with the result of Example 21.4.1.

22.22 By taking the Fourier transform of both sides of the integral form of the Schrödinger equation, show that for bound-state problems ($E < 0$), the equation in "momentum space" can be written as

$$\tilde{\psi}(\mathbf{p}) = -\frac{2\mu}{(2\pi)^{3/2}\hbar^2}\left(\frac{1}{\kappa^2 + p^2}\right)\int \tilde{V}(\mathbf{p} - \mathbf{q})\tilde{\psi}(\mathbf{q})\, d^3q,$$

where $\kappa^2 = -2\mu E/\hbar^2$.

22.23 Write the bound-state Schrödinger integral equation for a non-local potential, noting that $G(\mathbf{r}, \mathbf{r}') = e^{-\kappa|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r} - \mathbf{r}'|$, where $\kappa^2 = -2\mu E/\hbar^2$ and μ is the mass of the bound particle. The homogeneous solution is zero, as is always the case with bound states.

(a) Assuming that the potential is of the form $V(\mathbf{r}, \mathbf{r}') = -g^2 U(\mathbf{r}) U(\mathbf{r}')$, show that a solution to the Schrödinger equation exists iff

$$\frac{\mu g^2}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} U(\mathbf{r}) U\big(\mathbf{r}'\big) = 1. \tag{22.62}$$

(b) Taking $U(\mathbf{r}) = e^{-\alpha r}/r$, show that the condition in (22.62) becomes

$$\frac{4\pi\mu g^2}{\alpha\hbar^2}\left[\frac{1}{(\alpha + \kappa)^2}\right] = 1.$$

(c) Since $\kappa > 0$, prove that the equation in (b) has a unique solution only if $g^2 > \hbar^2\alpha^2/(4\pi\mu)$, in which case the bound-state energy is

$$E = -\frac{\hbar^2}{2\mu}\left[\left(\frac{4\pi\mu g^2}{\alpha\hbar^2}\right)^{1/2} - \alpha\right]^2.$$

22.24 Repeat calculations in Sects. 22.4.1 and 22.4.2 for $m = 2$.

22.25 In this problem, the dimension m is three.

(a) Derive the following identities:

$$\nabla^2\left[\frac{f(r)}{r}\right] = \frac{\nabla^2 f}{r} - \frac{2}{r^2}\frac{\partial f}{\partial r} + \nabla^2\left(\frac{1}{r}\right),$$

$$\frac{d\epsilon(t)}{dt} = 2\delta(t), \qquad \nabla^2\delta(t \pm r) = \delta''(t \pm r) \pm \frac{2}{r}\delta'(t \pm r),$$

where $\epsilon(t) = \theta(t) - \theta(-t)$.

(b) Use the results of (a) to show that the GF [Eq. (22.47)] derived from the principal value of the ω integration for the wave equation in three dimensions satisfies only the homogeneous PDE. Hint: Use $\nabla^2(1/r) = 4\pi\delta(\mathbf{r})$.

22.26 Calculate the retarded GF for the wave operator in two dimensions and show that it is equal to

$$G_s^{(\mathrm{ret})}(\mathbf{r}, t) = \frac{\theta(t)}{2\pi\sqrt{t^2 - r^2}}.$$

Now use this result to obtain the GF for any even number of dimensions:

$$G_s^{(\mathrm{ret})}(\mathbf{r}, t) = \frac{\theta(t)}{2\pi}\left(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\right)^{n-1}\left[\frac{1}{\sqrt{t^2 - r^2}}\right] \quad \text{for } n = m/2.$$

22.27 (a) Find the singular part of the retarded GF and the advanced GF for the wave equation in three dimensions using Eqs. (22.48) and (22.49). Hint: $J_{1/2}(kr) = \sqrt{2/\pi kr}\sin kr$.

(b) Use (a) and Eq. (22.51) to show that

$$\lim_{\epsilon\to 0}\left\{\frac{1}{[r^2 + (-it + \epsilon)^2]} - \frac{1}{[r^2 + (it + \epsilon)^2]}\right\} = -\frac{i\pi}{r}\big[\delta(t + r) - \delta(t - r)\big].$$

22.28 Show that the eigenfunction expansion of the GF for the Dirichlet BVP for the Laplacian operator in two dimensions for which the region of interest is the interior of a circle of radius a is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{2}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\epsilon_n J_n(\frac{\rho}{a}x_{nm})J_n(\frac{\rho'}{a}x_{nm})\cos n(\varphi - \varphi')}{J_{n+1}^2(x_{nm})x_{nm}^2},$$

where $\epsilon_0 = \frac{1}{2}$ and $\epsilon_n = 1$ for $n \geq 1$, and use has been made of Problem 15.39.

22.29 Go back to Example 22.5.4, and

(a) complete the calculations therein;
(b) find the GF for the Laplacian with Dirichlet BCs on two concentric spheres of radii a and b, with $a < b$.
(c) Consider the case where $a \to 0$ and $b \to \infty$ and compare the result with the singular part of the GF for the Laplacian.

22.30 Solve the Dirichlet BVP for the operator $\nabla^2 - k^2$ in the region $0 \leq x \leq a$, $0 \leq y \leq b$, $-\infty < z < \infty$. Hint: Separate the operator into $\mathbf{L}_1$ and $\mathbf{L}_2$.

22.31 Solve the problem of Example 22.5.1 using the separation of operator technique and show that the two results are equivalent.

22.32 Use the operator separation technique to calculate the Dirichlet GF for the two-dimensional operator $\nabla^2 - k^2$ on the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. Also obtain an eigenfunction expansion for this GF.

22.33 Use the operator separation technique to find the three-dimensional Dirichlet GF for the Laplacian in a circular cylinder of radius a and height h.

22.34 Calculate the singular part of the GF for the three-dimensional free Schrödinger operator

$$i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2\mu}\nabla^2.$$

22.35 Use the operator separation technique to show that

(a) the GF for the Helmholtz operator $\nabla^2 + k^2$ in three dimensions is

$$G(\mathbf{r}, \mathbf{r}') = -ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l(kr_>)Y_{lm}(\theta, \varphi)Y_{lm}^*(\theta', \varphi'),$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r' and j_l and h_l are the spherical Bessel and Hankel functions, respectively. No explicit BCs are assumed except that there is regularity at $r = 0$ and that $G(\mathbf{r}, \mathbf{r}') \to 0$ for $|\mathbf{r}| \to \infty$.

(b) Obtain the identity

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l(kr_>)Y_{lm}(\theta, \varphi)Y_{lm}^*(\theta', \varphi').$$

(c) Derive the plane wave expansion [see Eq. (19.46)]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kr)Y_{lm}^*(\theta', \varphi')Y_{lm}(\theta, \varphi),$$

where θ' and φ' are assumed to be the angular coordinates of $\mathbf{k}$. Hint: Let $|\mathbf{r}'| \to \infty$, and use

$$|\mathbf{r}-\mathbf{r}'| = \left(r'^2 + r^2 - 2\mathbf{r}\cdot\mathbf{r}'\right)^{1/2} \to r' - \frac{\mathbf{r}'\cdot\mathbf{r}}{r'}$$

and the asymptotic formula $h_l^{(1)}(z) \to (1/z)e^{i[z+(l+1)(\pi/2)]}$, valid for large z.