Representation of Groups

Group action is extremely important in quantum mechanics. Suppose the Hamiltonian of a quantum system is invariant under a symmetry transformation of its independent parameters such as position, momentum, and time. This invariance will show up as certain properties of the solutions of the Schrödinger equation.

Moreover, the very act of labeling quantum-mechanical states often involves groups and their actions. For example, labeling atomic states by eigenvalues of angular momentum assumes invariance of the Hamiltonian under the action of the rotation group (see Chap. 29) on the Hilbert space of the quantum-mechanical system under consideration.

24.1 Definitions and Examples

In the language of group theory, we have the following situation. Put all the parameters $x_1, \ldots, x_p$ of the Hamiltonian **H** together to form a space, say $\mathbb{R}^p$, and write $\mathbf{H} = \mathbf{H}(x_1, \ldots, x_p) \equiv \mathbf{H}(\mathbf{x})$. A **group of symmetry** of **H** is a group G whose action on $\mathbb{R}^p$ leaves **H** unchanged,[1] i.e., $\mathbf{H}(\mathbf{x} \cdot g) = \mathbf{H}(\mathbf{x})$. For example, a one-dimensional harmonic oscillator, with $\mathbf{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$, has, among other things, parity P (defined by $Px = -x$) as a symmetry. Thus, the group $G = \{e, P\}$ is a group of symmetry of **H**.

The Hamiltonian **H** of a quantum-mechanical system is an operator in a Hilbert space, such as $\mathcal{L}^2(\mathbb{R}^3)$, the space of square-integrable functions. The important question is: What is the proper way of transporting the action of G from $\mathbb{R}^p$ to $\mathcal{L}^2(\mathbb{R}^3)$? This is a relevant question because the solutions of the Schrödinger equation are, in general, functions of the parameters of the Hamiltonian, and as such will be affected by the symmetry operation on the Hamiltonian. The answer is provided in the following definition.[2]

---

[1] It will become clear shortly that the appropriate direction for the action is from the right.

[2] We have already encountered the notion of representation in the context of algebras. Groups are much more widely used in physics than algebras, and group representations have a wider application in physics than their algebraic counterparts. Since some readers may have skipped the section on the representation of algebras, we'll reintroduce the ideas here at the risk of being redundant.

Definition 24.1.1 Let G be a group and $\mathcal{H}$ a Hilbert space. A **representation of G on $\mathcal{H}$** is a homomorphism $T : G \rightarrow GL(\mathcal{H})$. The representation is **faithful** if the homomorphism is 1–1. We often denote $T(g)$ by $\mathbf{T}_g$. $\mathcal{H}$ is called the **carrier space of T**. The trivial homomorphism $T : G \rightarrow \{\mathbf{1}\}$ is also called the **identity representation**. The dimension of $\mathcal{H}$ is called the **dimension of the representation T**.

We do not want to distinguish between representations that differ only by isomorphic vector spaces, because otherwise we can generate an infinite set of representations that are trivially related to one another. A *vector space* isomorphism $f : \mathcal{H} \rightarrow \mathcal{H}'$ induces a *group* isomorphism $\phi : GL(\mathcal{H}) \rightarrow GL(\mathcal{H}')$ defined by

$$\phi(\mathbf{T}) = f \circ \mathbf{T} \circ f^{-1} \quad \text{for } \mathbf{T} \in GL(\mathcal{H}).$$

This motivates the following definition.

Definition 24.1.2 Two representations $T : G \rightarrow GL(\mathcal{H})$ and $T' : G \rightarrow GL(\mathcal{H}')$ are called **equivalent** if there exists an isomorphism $f : \mathcal{H} \rightarrow \mathcal{H}'$ such that $\mathbf{T}'_g = f \circ \mathbf{T}_g \circ f^{-1}$ for all $g \in G$.

> **Box 24.1.3** *Any representation $T : G \rightarrow GL(\mathcal{H})$ defines an action of the group G on the Hilbert space $\mathcal{H}$ by $\Phi(g, |a\rangle) \equiv \mathbf{T}_g |a\rangle$.*

As we saw in Chaps. 4 and 5, the transformation of an operator $\mathbf{A}$ under $\mathbf{T}_g$ would have to be defined by $\mathbf{T}_g \mathbf{A} (\mathbf{T}_g)^{-1}$. For a Hamiltonian with a group of symmetry G, this leads to the identity

$$\mathbf{T}_g \big[\mathbf{H}(\mathbf{x}) \big] (\mathbf{T}_g)^{-1} = \mathbf{H}(\mathbf{x} \cdot g).$$

Similarly, the action of the group on a vector (function) in $\mathcal{L}^2(\mathbb{R}^3)$ is defined by

$$(\mathbf{T}_g \psi)(\mathbf{x}) \equiv \psi(\mathbf{x} \cdot g), \tag{24.1}$$

where the parentheses around $\mathbf{T}_g \psi$ designate it as a *new* function. One can show that if G acts on the independent variables of a function *on the right* as in Eq. (24.1), then the vector space of such functions is the carrier space of a representation of G. In fact,

$$(\mathbf{T}_{g_1 g_2} \psi)(\mathbf{x}) \equiv \psi\big(\mathbf{x} \cdot (g_1 g_2)\big) = \psi\big((\mathbf{x} \cdot g_1) \cdot g_2\big) = (\mathbf{T}_{g_2} \psi)(\mathbf{x} \cdot g_1) \equiv \varphi(\mathbf{x} \cdot g_1),$$

where we have defined the new function φ by the last equality. Now note that

$$\varphi(\mathbf{x} \cdot g_1) = (\mathbf{T}_{g_1} \varphi)(\mathbf{x}) = \big(\mathbf{T}_{g_1}(\mathbf{T}_{g_2} \psi)\big)(\mathbf{x}) = (\mathbf{T}_{g_1} \mathbf{T}_{g_2} \psi)(\mathbf{x}).$$

It follows from the last two equations that

$$\mathbf{T}_{g_1 g_2} \psi = \mathbf{T}_{g_1} \mathbf{T}_{g_2} \psi.$$

Since this holds for arbitrary ψ, we must have $\mathbf{T}_{g_1 g_2} = \mathbf{T}_{g_1} \mathbf{T}_{g_2}$, i.e., that T is a representation. When the action of a group is "naturally" from the left, such as the action of a matrix on a column vector, we replace $\mathbf{x} \cdot g$ with $g^{-1} \cdot \mathbf{x}$. The reader can check that $T : G \to GL(\mathcal{H})$, given by $\mathbf{T}_g \psi(\mathbf{x}) = \psi(g^{-1} \cdot \mathbf{x})$, is indeed a representation.

Example 24.1.4 Let the Hamiltonian of the time-independent Schrödinger equation $\mathbf{H}|\psi\rangle = E|\psi\rangle$ be invariant under the action of a group G. This means that

$$\mathbf{T}_g \mathbf{H} \mathbf{T}_g^{-1} = \mathbf{H} \quad \Rightarrow \quad [\mathbf{H}, \mathbf{T}_g] = \mathbf{0},$$

i.e., that $\mathbf{H}$ and $\mathbf{T}_g$ are simultaneously diagonalizable (Theorem 6.4.18). It follows that we can choose the energy eigenstates to be eigenstates of $\mathbf{T}_g$ as well, and we can label the states not only by the energy "quantum numbers"—eigenvalues of $\mathbf{H}$—but also by the eigenvalues of $\mathbf{T}_g$. For example, if the Hamiltonian is invariant under the action of parity P, then we can choose the states to be *even*, corresponding to parity eigenvalue of $+1$, or *odd*, corresponding to parity eigenvalue of -1. Similarly, if G is the rotation group, then the states can be labeled by the eigenvalues of the rotation operators, which are, as we shall see, equivalent to the angular momentum operators discussed in Chap. 13.

> Energy eigenstates can be labeled by eigenvalues of the symmetry operators as well.

In crystallography and solid-state physics, the Hamiltonian of an (infinite) lattice is invariant under translation by an integer multiple of each socalled *primitive lattice translation*, the three noncoplanar vectors that define a primitive cell of the crystal. The preceding argument shows that the energy eigenstates can be taken to be the eigenstates of the translation operator as well.

It is common to choose a basis and represent all $\mathbf{T}_g$'s in terms of matrices. Then one gets a **matrix representation** of the group G.

> matrix representations

Example 24.1.5 Consider the action of the 2D rotation group $SO(2)$ (rotation about the z-axis) on $\mathbb{R}^3$:

$$\mathbf{r}' = R_z(\theta)\mathbf{r} \quad \Rightarrow \quad \begin{aligned} x' &= x\cos\theta - y\sin\theta, \\ y' &= x\sin\theta + y\cos\theta, \\ z' &= z. \end{aligned}$$

For a Hilbert space, also choose $\mathbb{R}^3$. Define the homomorphism $T : G \to GL(\mathcal{H})$ to be the identity map, so that $T(R_z(\theta)) \equiv \mathbf{T}_\theta = R_z(\theta)$. The operator $\mathbf{T}_\theta$ transforms the standard basis vectors of $\mathcal{H}$ as

$$\mathbf{T}_\theta \hat{\mathbf{e}}_1 = \mathbf{T}_\theta(1, 0, 0) = (\cos\theta, \sin\theta, 0) = \cos\theta\hat{\mathbf{e}}_1 + \sin\theta\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3,$$

$$\mathbf{T}_\theta \hat{\mathbf{e}}_2 = \mathbf{T}_\theta(0, 1, 0) = (-\sin\theta, \cos\theta, 0) = -\sin\theta\hat{\mathbf{e}}_1 + \cos\theta\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3,$$

$$\mathbf{T}_\theta \hat{\mathbf{e}}_3 = \mathbf{T}_\theta(0, 0, 1) = (0, 0, 1) = 0\hat{\mathbf{e}}_1 + 0\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$$

It follows that the *matrix representation* of $SO(2)$ in the standard basis of $\mathcal{H}$ is

$$\mathbf{T}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $SO(2)$ is an *infinite* group; its cardinality is determined by the "number" of θ's.

Example 24.1.6 Let S_3 act on $\mathbb{R}^3$ on the right by shuffling components:

$$(x_1, x_2, x_3) \cdot \pi = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}), \quad \pi \in S_3.$$

For the carrier space, choose $\mathbb{R}^3$ as well. Let $T : S_3 \to GL(\mathbb{R}^3)$ be given as follows: $T(\pi)$ is the matrix that takes the column vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to $\begin{pmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ x_{\pi(3)} \end{pmatrix}$. As a specific illustration, consider $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and write $\mathbf{T}_\pi$ for $T(\pi)$. Then

$$\mathbf{T}_\pi(\hat{\mathbf{e}}_1) = \mathbf{T}_\pi(1, 0, 0) = (1, 0, 0) \cdot \pi = (0, 1, 0) = \hat{\mathbf{e}}_2,$$

$$\mathbf{T}_\pi(\hat{\mathbf{e}}_2) = \mathbf{T}_\pi(0, 1, 0) = (0, 1, 0) \cdot \pi = (0, 0, 1) = \hat{\mathbf{e}}_3,$$

$$\mathbf{T}_\pi(\hat{\mathbf{e}}_3) = \mathbf{T}_\pi(0, 0, 1) = (0, 0, 1) \cdot \pi = (1, 0, 0) = \hat{\mathbf{e}}_1,$$

which give rise to the matrix

$$\mathbf{T}_\pi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The reader may construct the other five matrices of this representation and verify directly that it is indeed a (faithful) representation: Products and inverses of permutations are mapped onto products and inverses of the corresponding matrices.

24.2   Irreducible Representations

The utility of a representation lies in our comfort with the structure of vector spaces. The climax of such comfort is the spectral decomposition theorems of (normal) operators on vector spaces of finite (Chap. 6) and infinite (Chap. 17) dimensions. The operators $\mathbf{T}_g$, relevant to our present discussion, are, in general, neither normal nor simultaneously commuting. Therefore, the complete diagonalizability of all $\mathbf{T}_g$'s is out of the question (unless the group happens to be abelian).

The best thing next to complete diagonalization is to see whether there are common *invariant subspaces* of the vector space $\mathcal{H}$ carrying the representation. We already know how to construct (minimal) "invariant" subsets of $\mathcal{H}$: these are precisely the *orbits* of the action of the group G on $\mathcal{H}$. The linearity of $\mathbf{T}_g$'s guarantees that the span of each orbit is actually an invariant subspace, and that such subspaces are the smallest invariant subspaces

containing a given vector. Our aim is to find those *minimal* invariant subspaces whose orthogonal complements are also invariant. We encountered the same situation in Chap. 6 for a single operator.

Definition 24.2.1 A representation $T : G \to GL(\mathcal{H})$ is called **reducible** if there exist subspaces $\mathcal{U}$ and $\mathcal{W}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus \mathcal{W}$ and both $\mathcal{U}$ and $\mathcal{W}$ are invariant under all $\mathbf{T}_g$'s. If no such subspaces exist, $\mathcal{H}$ is said to be **irreducible**.

reducible and irreducible representations

In most cases of physical interest, where $\mathcal{H}$ is a Hilbert space, $\mathcal{W} = \mathcal{U}^\perp$. Then, in the language of Definition 6.1.4, a representation is reducible if a proper subspace of $\mathcal{H}$ reduces all $\mathbf{T}_g$'s.

Example 24.2.2 Let S_3 act on $\mathbb{R}^3$ as in Example 24.1.6. For the carrier space $\mathcal{H}$, choose the space of functions on $\mathbb{R}^3$, and for T, the homomorphism $T : G \to GL(\mathcal{H})$, given by $\mathbf{T}_g \psi(\mathbf{x}) = \psi(\mathbf{x} \cdot g)$, for $\psi \in \mathcal{H}$. Any ψ that is symmetric in x, y, z, such as xyz, $x + y + z$, or $x^2 + y^2 + z^2$, defines a one-dimensional invariant subspace of $\mathcal{H}$. To obtain another invariant subspace, consider $\psi_1(x, y, z) \equiv xy$ and let $\{\pi_i\}_{i=1}^6$ be as given in Example 23.4.1. Then, denoting $\mathbf{T}_{\pi_i}$ by $\mathbf{T}_i$, the reader may check that

$$[\mathbf{T}_1 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_1\big) = \psi_1(x, y, z) = xy = \psi_1(x, y, z),$$

$$[\mathbf{T}_2 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_2\big) = \psi_1(y, x, z) = yx = \psi_1(x, y, z),$$

$$[\mathbf{T}_3 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_3\big) = \psi_1(z, y, x) = zy \equiv \psi_2(x, y, z),$$

$$[\mathbf{T}_4 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_4\big) = \psi_1(x, z, y) = xz \equiv \psi_3(x, y, z),$$

$$[\mathbf{T}_5 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_5\big) = \psi_1(z, x, y) = zx = \psi_3(x, y, z),$$

$$[\mathbf{T}_6 \psi_1](x, y, z) = \psi_1\big((x, y, z) \cdot \pi_6\big) = \psi_1(y, z, x) = yz = \psi_2(x, y, z).$$

This is clearly a three-dimensional invariant subspace of $\mathcal{H}$ with ψ_1, ψ_2, and ψ_3 as a convenient basis, in which the first three permutations are represented by

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{T}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is instructive for the reader to verify these relations and to find the three remaining matrices.

Example 24.2.3 Let S_3 act on $\mathbb{R}^3$ as in Example 24.1.6. For the carrier space of representation, choose the subspace $\mathcal{V}$ of the $\mathcal{H}$ of Example 24.2.2 spanned by the six functions x, y, z, xy, xz, and yz. For T, choose the same homomorphism as in Example 24.2.2 restricted to $\mathcal{V}$. It is clear that the subspaces $\mathcal{U}$ and $\mathcal{W}$ spanned, respectively, by the first three and the last three functions are invariant under S_3, and that $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$. It follows that

the representation is reducible. The matrix form of this representation is found to be of the general form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where B is one of the 6 matrices of Example 24.2.2. The matrix A, corresponding to the three functions x, y, and z, can be found similarly.

Let $\mathcal{H}$ be a carrier space, finite- or infinite-dimensional. For any vector $|a\rangle$, the reader may check that the span of $\{\mathbf{T}_g|a\rangle\}_{g\in G}$ is an invariant subspace of $\mathcal{H}$. If G is finite, this subspace is clearly finite-dimensional. The irreducible subspace containing $|a\rangle$, a subspace of the span of $\{\mathbf{T}_g|a\rangle\}_{g\in G}$, will also be finite-dimensional. Because of the arbitrariness of $|a\rangle$, it follows that every vector of $\mathcal{H}$ lies in an irreducible subspace, and that

> **Box 24.2.4** *All irreducible representations of a finite group are finite-dimensional.*

Due to the importance and convenience of unitary operators (for example, the fact that they leave the inner product invariant), it is desirable to be able to construct a unitary representation—or a representation that is equivalent to one—of groups. The following theorem ensures that this desire can be realized for finite groups.

All representations are equivalent to unitary representations.

> **Theorem 24.2.5** *Every representation of a finite group G is equivalent to some unitary representation.*

Proof We present the proof because of its simplicity and elegance. Let T be a representation of G. Consider the positive hermitian operator $\mathbf{T} \equiv \sum_{x\in G}\mathbf{T}_x^\dagger\mathbf{T}_x$ and note that

$$
\begin{aligned}
\mathbf{T}_g^\dagger\mathbf{T}\mathbf{T}_g &= \sum_{x\in G}\left[T(g)\right]^\dagger\left[T(x)\right]^\dagger T(x)T(g) \\
&= \sum_{x\in G}\left[T(xg)\right]^\dagger T(xg) = \sum_{y\in G}\left[T(y)\right]^\dagger T(y) = \mathbf{T},
\end{aligned} \tag{24.2}
$$

where we have used the fact that the sum over x and $y \equiv xg$ sweep through the entire group. Now let $\mathbf{S} = \sqrt{\mathbf{T}}$, and multiply both sides of Eq. (24.2)—with $\mathbf{S}^2$ replacing $\mathbf{T}$—by $\mathbf{S}^{-1}$ on the left and by $\mathbf{T}_g^{-1}\mathbf{S}^{-1}$ on the right to obtain

$$
\mathbf{S}^{-1}\mathbf{T}_g^\dagger\mathbf{S} = \mathbf{S}\mathbf{T}_g^{-1}\mathbf{S}^{-1} \quad \Rightarrow \quad \left(\mathbf{S}\mathbf{T}_g\mathbf{S}^{-1}\right)^\dagger = \left(\mathbf{S}\mathbf{T}_g\mathbf{S}^{-1}\right)^{-1} \quad \forall g \in G.
$$

This shows that the representation T' defined by $\mathbf{T}_g' \equiv \mathbf{S}\mathbf{T}_g\mathbf{S}^{-1}$ for all $g \in G$ is unitary.   □

There is another convenience afforded by unitary representations:

Theorem 24.2.6 *Let $T : G \to GL(\mathcal{H})$ be a unitary representation and $\mathcal{W}$ an invariant subspace of $\mathcal{H}$. Then, $\mathcal{W}^\perp$ is also invariant.*

Proof Suppose $|a\rangle \in \mathcal{W}^\perp$. We need to show that $\mathbf{T}_g|a\rangle \in \mathcal{W}^\perp$ for all $g \in G$. To this end, let $|b\rangle \in \mathcal{W}$. Then

$$\langle b|\mathbf{T}_g|a\rangle = \left(\langle a|\mathbf{T}_g^\dagger|b\rangle\right)^* = \left(\langle a|\mathbf{T}_g^{-1}|b\rangle\right)^* = \left(\langle a|\mathbf{T}_{g^{-1}}|b\rangle\right)^* = 0,$$

because $\mathbf{T}_{g^{-1}}|b\rangle \in \mathcal{W}$. It follows from this equality that $\mathbf{T}_g|a\rangle \in \mathcal{W}^\perp$ for all $g \in G$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The carrier space $\mathcal{H}$ of a unitary representation is either irreducible or has an invariant subspace $\mathcal{W}$, in which case we have $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$, where, by Theorem 24.2.6, $\mathcal{W}^\perp$ is also invariant. If $\mathcal{W}$ and $\mathcal{W}^\perp$ are not irreducible, then they too can be written as direct sums of invariant subspaces. Continuing this process, we can decompose $\mathcal{H}$ into irreducible invariant subspaces $\mathcal{W}^{(k)}$ such that

$$\mathcal{H} = \mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)} \oplus \mathcal{W}^{(3)} \oplus \cdots .$$

If the carrier space is finite-dimensional, which we assume from now on and for which we use the notation $\mathcal{V}$, then the above direct sum is finite and we write

$$\mathcal{V} = \mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)} \oplus \cdots \oplus \mathcal{W}^{(p)} \equiv \bigoplus_{k=1}^{p} \mathcal{W}^{(k)}. \qquad (24.3)$$

One can think of $\mathcal{W}^{(k)}$ as the carrier space of an (irreducible) representation. The homomorphism $T^{(k)} : G \to GL(\mathcal{W}^{(k)})$ is simply the restriction of T to the subspace $\mathcal{W}^{(k)}$, and we write

$$\mathbf{T}_g = \mathbf{T}_g^{(1)} \oplus \mathbf{T}_g^{(2)} \oplus \cdots \oplus \mathbf{T}_g^{(r)} \equiv \bigoplus_{k=1}^{r} \mathbf{T}_g^{(k)}.$$

If we identify all equivalent irreducible representations and collect them together, we may rewrite the last equation as

$$\mathbf{T}_g = m_1\mathbf{T}_g^{(1)} \oplus m_2\mathbf{T}_g^{(2)} \oplus \cdots \oplus m_\rho\mathbf{T}_g^{(\rho)} \equiv \bigoplus_{\alpha=1}^{\rho} m_\alpha\mathbf{T}_g^{(\alpha)}, \qquad (24.4)$$

where ρ is the number of *inequivalent* irreducible representations and m_α are positive integers giving the number of times an irreducible representation $\mathbf{T}_g^{(\alpha)}$ and all its equivalents occur in a given representation.

In terms of matrices, $\mathbf{T}_g$ will be represented in a block-diagonal form as

$$\mathbf{T}_g = \begin{pmatrix} \mathsf{T}_g^{(1)} & 0 & \dots & 0 \\ 0 & \mathsf{T}_g^{(2)} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \mathsf{T}_g^{(r)} \end{pmatrix},$$

or

$$
\mathsf{T}_g = \begin{pmatrix} [\mathsf{T}_g^{(1)}]_{m_1} & 0 & \dots & 0 \\ 0 & [\mathsf{T}_g^{(2)}]_{m_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & [\mathsf{T}_g^{(\rho)}]_{m_\rho} \end{pmatrix},
$$

where, in the first matrix some of the $\mathsf{T}_g^{(k)}$ may be equivalent, and in the second matrix, $[\mathsf{T}_g^{(\alpha)}]_{m_\alpha}$ is a block-diagonal matrix consisting of m_α copies of the matrix $\mathsf{T}_g^{(\alpha)}$.

Example 24.2.7 A one-dimensional (and therefore irreducible) representation, defined for all groups, is the **trivial** (symmetric) representation $T : G \to \mathbb{C}$ given by $T(g) = 1$ for all $g \in G$. For the permutation group S_n, one can define another one-dimensional (thus irreducible) representation $T : S_n \to \mathbb{C}$, called the **antisymmetric** representation, given by $T(\pi) = +1$ if π is even, and $T(\pi) = -1$ if π is odd.

antisymmetric representation of a permutation group

Given any (matrix) representation T of G, one can form the transpose inverse matrices $(\mathsf{T}_g^t)^{-1}$, and complex conjugate matrices T_g^*. The reader may check that each set of these matrices forms a representation of G.

adjoint and complex conjugate representations

Definition 24.2.8 The set of matrices $(\mathsf{T}_g^t)^{-1}$ and T_g^* are called, respectively, the **adjoint representation**, denoted by $\overline{T}$, and the **complex conjugate representation**, denoted by T^*.

24.3   Orthogonality Properties

Homomorphisms preserve group structures. By studying a group that is more attuned to concrete manipulations, we gain insight into the structure of groups that are homomorphic to it. The group of invertible operators on a vector space, especially in their matrix representation, are particularly suited for such a study because of our familiarity with matrices and operators. The last section reduced this study to inequivalent irreducible representations. This section is devoted to a detailed study of such representations.

Schur's lemma

Lemma 24.3.1 (Schur's lemma) *Let $T : G \to GL(\mathcal{V})$ and $T' : G \to GL(\mathcal{V}')$ be irreducible representations of G. If $\mathbf{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ is such that*

$$
\mathbf{A}\mathsf{T}_g = \mathsf{T}_g'\mathbf{A} \quad \forall g \in G, \tag{24.5}
$$

then either $\mathbf{A}$ is an isomorphism (i.e., T is equivalent to T'), or $\mathbf{A} = 0$.

Proof Let $|a\rangle \in \ker\mathbf{A}$. Then

$$
\mathbf{A}\mathsf{T}_g|a\rangle = \mathsf{T}_g'\underbrace{\mathbf{A}|a\rangle}_{=0} = 0 \quad \Rightarrow \quad \mathsf{T}_g|a\rangle \in \ker\mathbf{A} \quad \forall g \in G.
$$

It follows that $\ker \mathbf{A}$, a subspace of $\mathcal{V}$, is invariant under T. Irreducibility of T implies that either $\ker \mathbf{A} = \mathcal{V}$, or $\ker \mathbf{A} = 0$. The first case asserts that $\mathbf{A}$ is the zero linear transformation; the second case implies that $\mathbf{A}$ is injective.

Similarly, let $|b\rangle \in \mathbf{A}(\mathcal{V})$. Then $|b\rangle = \mathbf{A}|x\rangle$ for some $|x\rangle \in \mathcal{V}$:

$$\mathbf{T}'_g|b\rangle = \mathbf{T}'_g\mathbf{A}|x\rangle = \underbrace{\mathbf{A}\mathbf{T}_g|x\rangle}_{\in \mathbf{A}(\mathcal{V})} \quad \Rightarrow \quad \mathbf{T}'_g|b\rangle \in \mathbf{A}(\mathcal{V}) \quad \forall g \in G.$$

It follows that $\mathbf{A}(\mathcal{V})$, a subspace of $\mathcal{V}'$, is invariant under T'. Irreducibility of T' implies that either $\mathbf{A}(\mathcal{V}) = 0$, or $\mathbf{A}(\mathcal{V}) = \mathcal{V}'$. The first case is consistent with the first conclusion drawn above: $\ker \mathbf{A} = \mathcal{V}$. The second case asserts that $\mathbf{A}$ is surjective. Combining the two results, we conclude that $\mathbf{A}$ is either the zero operator or an isomorphism. □

Lemma 24.3.1 becomes extremely useful when we concentrate on a single irreducible representation, i.e., when $T' = T$.

> **Lemma 24.3.2** *Let $T : G \rightarrow GL(\mathcal{V})$ be an irreducible representation of G. If $\mathbf{A} \in \mathcal{L}(\mathcal{V})$ is such that $\mathbf{A}\mathbf{T}_g = \mathbf{T}_g\mathbf{A}$ for all $g \in G$, then $\mathbf{A} = \lambda \mathbf{1}$.*

Proof Replacing $\mathcal{V}'$ with $\mathcal{V}$ in Lemma 24.3.1, we conclude that $\mathbf{A} = \mathbf{0}$ or $\mathbf{A}$ is an isomorphism of $\mathcal{V}$. In the first case, $\lambda = 0$. In the second case, $\mathbf{A}$ must have a nonzero eigenvalue λ and at least one eigenvector (see Theorem 6.2.5). It follows that the operator $\mathbf{A} - \lambda \mathbf{1}$ commutes with all $\mathbf{T}_g$'s and it is not an isomorphism (why not?). Therefore, it must be the zero operator. □

We can immediately put this lemma to good use. If G is abelian, all operators $\{\mathbf{T}_x\}_{x \in G}$ commute with one another. Focusing on one of these operators, say $\mathbf{T}_g$, noting that it commutes with all operators of the representation, and using Lemma 24.3.2, we conclude that $\mathbf{T}_g = \lambda \mathbf{1}$. It follows that when $\mathbf{T}_g$ acts on a vector, it gives a multiple of that vector. Therefore, it leaves any one-dimensional subspace of the carrier space invariant. Since this is true for all $g \in G$, we have the following result.

Theorem 24.3.3 *All irreducible representations of an abelian group are one-dimensional.*

This theorem is an immediate consequence of Schur's lemma, and is independent of the order of G. In particular, it holds for infinite groups, if Schur's lemma holds for those groups. One important class of infinite groups for which Schur's lemma holds is the Lie groups (to be discussed in Part IX). Thus, all abelian Lie groups have 1-dimensional irreducible representations. We shall see later that the converse of Theorem 24.3.3 is also true for finite groups.

Issai Schur (1875–1941) was one of the most brilliant mathematicians active in Germany during the first third of the twentieth century. He attended the Gymnasium in Libau (now Liepaja, Latvia) and then the University of Berlin, where he spent most of his scientific career from 1911 until 1916. When he returned to Berlin, he was an assistant professor at Bonn. He became full professor at Berlin in 1919. Schur was forced to retire by the Nazi authorities in 1935 but was able to emigrate to Palestine in 1939. He died there of a heart ailment several years later. Schur had been a member of the Prussian Academy of Sciences before the Nazi purges. He married and had a son and daughter.

Schur's principal field was the representation theory of groups, founded a little before 1900 by his teacher Frobenius. Schur seems to have completed this field shortly before World War I, but he returned to the subject after 1925, when it became important for physics. Further developed by his student Richard Brauer, it is in our time experiencing an extraordinary growth through the opening of new questions. Schur's dissertation (1901) became fundamental to the representation theory of the general linear group; in fact, English mathematicians have named certain of the functions appearing in the work "S-functions" in Schur's honor. In 1905 Schur reestablished the theory of group characters—the keystone of representation theory. The most important tool involved is "Schur's lemma." Along with the representation of groups by integral linear substitutions, Schur was also the first to study representation by linear fractional substitutions, treating this more difficult problem almost completely in two works (1904, 1907). In 1906 Schur considered the fundamental problems that appear when an algebraic number field is taken as the domain; a number appearing in this connection is now called the Schur index. His works written after 1925 include a complete description of the rational and of the continuous representations of the general linear group; the foundations of this work were in his dissertation.

A lively interchange with many colleagues led Schur to contribute important memoirs to other areas of mathematics. Some of these were published as collaborations with other authors, although publications with dual authorship were almost unheard of at that time. Here we simply indicate the areas: pure group theory, matrices, algebraic equations, number theory, divergent series, integral equations, and function theory.

Issai Schur 1875–1941

All vectors of each irreducible subspace of the representation of a symmetry of the hamiltonian are eigenstates of the hamiltonian corresponding to a single eigenvalue.

Example 24.3.4 Suppose that the Hamiltonian **H** of a quantum mechanical system with Hilbert space $\mathcal{H}$ has a group of symmetry with a representation $T : G \to GL(\mathcal{H})$. Then $\mathbf{H}\mathbf{T}_g = \mathbf{T}_g\mathbf{H}$ for all $g \in G$. It follows that $\mathbf{H} = \lambda\mathbf{1}$ if the representation is irreducible. Therefore, all vectors of each invariant irreducible subspace are eigenstates of the hamiltonian corresponding to the same eigenvalue, i.e., they all have the same energy. Therefore, the degeneracy of that energy state is at least as large as the dimension of the carrier space.

It is helpful to arrive at the statement above from a different perspective. Consider a vector $|x\rangle$ in the eigenspace $\mathcal{M}_i$ corresponding to the energy eigenvalue E_i. Since $\mathbf{T}_g$ and $\mathbf{H}$ commute, $\mathbf{T}_g|x\rangle$ is also in $\mathcal{M}_i$. Therefore, an eigenspace of a Hamiltonian with a group of symmetry is invariant under all $\mathbf{T}_g$ for any representation T of that group. If T is one of the irreducible representations of G, say $T^{(\alpha)}$ with dimension n_α, then $\dim \mathcal{M}_i \geq n_\alpha$.

Consider two irreducible representations $T^{(\alpha)}$ and $T^{(\beta)}$ of a group G with carrier spaces $\mathcal{W}^{(\alpha)}$ and $\mathcal{W}^{(\beta)}$, respectively. Let **X** be any operator in $\mathcal{L}(\mathcal{W}^{(\alpha)}, \mathcal{W}^{(\beta)})$, and define

$$\mathbf{A} \equiv \sum_{x \in G} \mathbf{T}_x^{(\alpha)} \mathbf{X} \mathbf{T}_{x^{-1}}^{(\beta)} = \sum_{x \in G} T^{(\alpha)}(x) \mathbf{X} T^{(\beta)}(x^{-1}).$$

Then, we have

$$
\begin{aligned}
\mathbf{T}_g^{(\alpha)}\mathbf{A} &= \sum_{x \in G} T^{(\alpha)}(g) T^{(\alpha)}(x) \mathbf{X} T^{(\beta)}\big(x^{-1}\big) T^{(\beta)}\big(g^{-1}\big) T^{(\beta)}(g) \\
&= \underbrace{\sum_{x \in G} T^{(\alpha)}(gx) \mathbf{X} T^{(\beta)}\big((gx)^{-1}\big)}_{=\mathbf{A} \text{ because this sum also covers all } G} T^{(\beta)}(g) = \mathbf{A}\mathbf{T}_g^{(\beta)}.
\end{aligned}
$$

We are interested in the two cases where $T^{(\alpha)} = T^{(\beta)}$, and where $T^{(\alpha)}$ is not equivalent to $T^{(\beta)}$. In the first case, Lemma 24.3.2 gives $\mathbf{A} = \lambda\mathbf{1}$; in the second case, Lemma 24.3.1 gives $\mathbf{A} = 0$. Combining these two results and labeling the constant multiplying the unit operator by X, we can write

$$
\sum_{g \in G} T^{(\alpha)}(g) \mathbf{X} T^{(\beta)}\big(g^{-1}\big) = \lambda_X \delta_{\alpha\beta}\mathbf{1}. \tag{24.6}
$$

The presence of the completely arbitrary operator $\mathbf{X}$ indicates that Eq. (24.6) is a powerful statement about—and a severe restriction on—the operators $T^{(\alpha)}(g)$. This becomes more transparent if we select a basis, represent all operators by matrices, and for X, the matrix representation of $\mathbf{X}$, choose a matrix whose only nonzero element is 1 and occurs at the lth row and mth column. Then Eq. (24.6) becomes

$$
\sum_{g \in G} T_{il}^{(\alpha)}(g) T_{mj}^{(\beta)}\big(g^{-1}\big) = \lambda_{lm} \delta_{\alpha\beta} \delta_{ij},
$$

where λ_{lm} is a constant that can be evaluated by setting $j = i$, $\alpha = \beta$, and summing over i. The RHS will give $\lambda_{lm} \sum_i \delta_{ii} = \lambda_{lm} n_\alpha$, where n_α is the dimension of the carrier space of $T^{(\alpha)}$. For the LHS we get

$$
\begin{aligned}
\text{LHS} &= \sum_{g \in G} \sum_i \mathsf{T}_{il}^{(\alpha)}(g) \mathsf{T}_{mi}^{(\alpha)}\big(g^{-1}\big) = \sum_{g \in G} \big(T^{(\alpha)}\big(g^{-1}\big) T^{(\alpha)}(g)\big)_{ml} \\
&= \sum_{g \in G} T_{ml}^{(\alpha)}\big(g^{-1}g\big) = \sum_{g \in G} \underbrace{T_{ml}^{(\alpha)}(e)}_{=(1)_{ml}} = |G|\delta_{ml},
\end{aligned}
$$

where $|G|$ is the order of the group. Putting everything together, we obtain

$$
\sum_{g \in G} T_{il}^{(\alpha)}(g) T_{mj}^{(\beta)}\big(g^{-1}\big) = \frac{|G|}{n_\alpha} \delta_{ml} \delta_{\alpha\beta} \delta_{ij}. \tag{24.7}
$$

If the representation is unitary, then

$$
\sum_{g \in G} T_{il}^{(\alpha)}(g) T_{jm}^{(\beta)*}(g) = \frac{|G|}{n_\alpha} \delta_{ml} \delta_{\alpha\beta} \delta_{ij}. \tag{24.8}
$$

Equations (24.7) and (24.8) depend on the basis chosen in which to express matrices. To eliminate this dependence, we first introduce the important concept of character.

Definition 24.3.5 Let $T : G \to GL(\mathcal{V})$ be a representation of the group G. The **character** of this representation is the map $\chi : G \to \mathbb{C}$ given by

$$\chi(g) \equiv \operatorname{tr} \mathbf{T}_g = \sum_i \mathsf{T}_{ii}(g),$$

where $\mathsf{T}(g)$ is the matrix representation of $\mathbf{T}_g$ in any basis of $\mathcal{V}$. If T is irreducible, the character is called **simple**; otherwise, it is called **compound**.

The character of the identity element in any representation can be calculated immediately. Since a homomorphism maps identity onto identity, $\mathbf{T}_e = \mathbf{1}$. Therefore,

$$\chi(e) = \operatorname{tr}(\mathbf{1}) = \dim \mathcal{V}. \tag{24.9}$$

Recall that two elements $x, y \in G$ belong to the same conjugacy class if there exist $g \in G$ such that $x = gyg^{-1}$. This same relation holds for the operators representing the elements: $\mathbf{T}_x = \mathbf{T}_g \mathbf{T}_y \mathbf{T}_{g^{-1}}$. Taking the trace of both sides, and noting that $\mathbf{T}_{g^{-1}} = \mathbf{T}_g^{-1}$, one obtains

> **Box 24.3.6** *All elements of a group belonging to the same conjugacy class have the same character.*

Setting $i = l$ and $j = m$ in (24.7) and summing over i and j, we obtain

$$\sum_{g \in G} \chi^{(\alpha)}(g) \chi^{(\beta)}\big(g^{-1}\big)$$

$$= \frac{|G|}{n_\alpha} \delta_{\alpha\beta} \sum_{i,j} \delta_{ji} \delta_{ij} = \frac{|G|}{n_\alpha} \delta_{\alpha\beta} \underbrace{\sum_j \delta_{jj}}_{=n_\alpha} = |G| \delta_{\alpha\beta}. \tag{24.10}$$

If the representation is unitary, then (24.8) gives

$$\sum_{g \in G} \chi^{(\alpha)}(g) \chi^{(\beta)*}(g) = |G| \delta_{\alpha\beta}. \tag{24.11}$$

This equation suggests a useful interpretation: Characters can be thought of as vectors in a $|G|$-dimensional inner product space. According to Eq. (24.11), the characters of inequivalent irreducible representations are orthogonal. In particular, since there cannot be more orthogonal vectors than the dimension of a vector space, we conclude that the number of irreducible inequivalent representations of a group cannot be more that the cardinality of that group. Actually, we can do better. Restricting ourselves to unitary representations and collecting all elements belonging to the same conjugacy class together, we write

$$\sum_{i=1}^r c_i \chi_i^{(\alpha)} \chi_i^{(\beta)*} = |G| \delta_{\alpha\beta} \quad \Rightarrow \quad \big\langle \chi^{(\beta)} \big| \chi^{(\alpha)} \big\rangle = |G| \delta_{\alpha\beta}, \tag{24.12}$$

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where i labels conjugacy classes, c_i is the number of elements in the ith class, r is the number of classes in G, and $|\chi^{(\alpha)}\rangle \in \mathbb{C}^r$ is an r-dimensional vector with components $\{c_i^{1/2}\chi_i^{(\alpha)}\}_{i=1}^r$. Equation (24.12) shows that vectors belonging to different irreducible representations are orthogonal. Since there cannot be more orthogonal vectors than the dimension of a vector space, we conclude that

Proposition 24.3.7 *The number of inequivalent irreducible representations of a group cannot be more that the number of conjugacy classes of the group, i.e.,* $\rho \leq r$.

The characters of the adjoint representation are obtained from

$$\bar{\chi}(g) = \chi\left(g^{-1}\right) \quad \Rightarrow \quad \bar{\chi}_i = \chi_{i'},$$

where $K_{i'}$ is the class consisting of all elements inverse to those of the class K_i. The equations involving characters of inverses of group elements can be written in terms of the characters of the adjoint representation. For example, Eq. (24.10) becomes

$$\sum_{g \in G} \chi^{(\alpha)}(g)\bar{\chi}^{(\beta)}(g) = |G|\delta_{\alpha\beta} \quad \Rightarrow \quad \sum_{i=1}^r c_i \chi_i^{(\alpha)} \bar{\chi}_i^{(\beta)} = |G|\delta_{\alpha\beta}. \quad (24.13)$$

Other relations can be obtained similarly.

24.4   Analysis of Representations

We can use the results obtained in the last section to gain insight into a given representation. Take the trace of both sides of Eq. (24.4) and write the result as

$$\chi(g) = m_1\chi^{(1)}(g) + \cdots + m_\rho\chi^{(\rho)}(g) \equiv \sum_{\alpha=1}^\rho m_\alpha\chi^{(\alpha)}(g); \qquad (24.14)$$

i.e., a compound character is a linear combination of simple characters *with nonnegative integer coefficients*. Furthermore, the orthogonality of simple characters gives

$$m_\alpha = \frac{1}{|G|}\sum_{g \in G} \chi(g)\chi^{(\alpha)*}(g), \qquad (24.15)$$

yielding the number of times the irreducible representation $T^{(\alpha)}$ occurs in the representation T.

Another useful relation is obtained if we multiply Eq. (24.14) by its complex conjugate and sum over g; the result is

$$\sum_{g \in G} \left| \chi(g) \right|^2 = \sum_{g \in G} \chi(g) \chi^*(g) = \sum_{g \in G} \sum_{\alpha} m_\alpha \chi^{(\alpha)}(g) \sum_{\beta} m_\beta \chi^{(\beta)*}(g)$$

$$= \sum_{\alpha, \beta} m_\alpha m_\beta \underbrace{\sum_{g \in G} \chi^{(\alpha)}(g) \chi^{(\beta)*}(g)}_{= |G| \delta_{\alpha\beta}} = |G| \sum_{\alpha} m_\alpha^2. \quad (24.16)$$

In particular, if T is irreducible, all m_α are zero except for one, which is

criterion for irreducibility   unity. We therefore obtain the **criterion for irreducibility**:

$$\sum_{g \in G} \left| \chi(g) \right|^2 = \sum_{i=1}^{r} c_i |\chi_i|^2 = |G| \quad \text{if } T \text{ is irreducible.} \quad (24.17)$$

For groups of low order and representations of small dimensions, Eq. (24.16) becomes a powerful tool for testing the irreducibility of the representation.

Example 24.4.1 Let $G = S_3$ and consider the representation of Example 24.2.2. The characters of the first three elements of this representation are easily calculated:

$$\chi_1 = \text{tr}\,\mathbf{T}_1 = 3, \qquad \chi_2 = \text{tr}\,\mathbf{T}_2 = 1, \qquad \chi_3 = \text{tr}\,\mathbf{T}_3 = 1.$$

Similarly, one can obtain $\chi_4 = 1$, $\chi_5 = 0$, and $\chi_6 = 0$. Substituting this in Eq. (24.16) yields

$$\sum_{g \in G} \left| \chi(g) \right|^2 = \sum_{j=1}^{6} |\chi_j|^2 = 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 = 12.$$

Comparing this with the RHS of (24.16) with $|G| = 6$ yields $\sum_\alpha m_\alpha^2 = 2$. This restricts the nonzero α's to two, say $\alpha = 1$ and $\alpha = 2$. Moreover, m_1 and m_2 can be only 1. Thus, the representation of Example 24.2.2 is reducible, and there are precisely two inequivalent irreducible representations in it, each occurring once.

We can actually find the invariant subspaces corresponding to the two irreducible representations revealed above. The first is easy to guess. Just taking the sum of the three functions ψ_1, ψ_2, and ψ_3 gives a one-dimensional invariant subspace; so, let $\phi_1 \equiv \psi_1 + \psi_2 + \psi_3$, and note that the space $\mathcal{W}_1$ spanned by ϕ_1 is invariant. The second is harder to discover. However, if we *assume* that ψ_1, ψ_2, and ψ_3 are orthonormal, then using the Gram–Schmidt process, we can find the other two functions orthogonal to ϕ_1 (but not orthogonal to each other!). These are

$$\phi_2 = -\psi_1 + 2\psi_2 - \psi_3, \qquad \phi_3 = -\psi_1 - \psi_2 + 2\psi_3.$$

The reader is urged to convince himself/herself that the subspace $\mathcal{W}^{(2)}$ spanned by ϕ_2 and ϕ_3 is the complement of $\mathcal{W}^{(1)}$ [i.e., $\mathcal{V} = \mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)}$] and that it is invariant under all $\mathbf{T}_g$'s.

A very useful representation can be constructed as follows. Let $G = \{g_j\}_{j=1}^m$, and recall that left multiplication of elements of G by a fixed element g_i is a permutation of $(g_1, g_2, \ldots, g_m)$. Denote this permutation by π_i. Now define a representation $R : G \to GL(\mathbb{R}^m)$, called the **regular repre-**   regular representation
sentation, by

$$\mathbf{R}_{g_i}(x_1, x_2, \ldots, x_m) = (x_{\pi_i(1)}, x_{\pi_i(2)}, \ldots, x_{\pi_i(m)}).$$

That this is indeed a representation is left as a problem for the reader. One can obtain a matrix representation of R by choosing the standard basis $\{\hat{\mathbf{e}}_j\}_{j=1}^m$ of $\mathbb{R}^m$ and noting that $\mathbf{R}_{g_i}\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_{\pi_i^{-1}(j)}$. From such a matrix representation it follows that all characters χ^R of the regular representations are zero except for the identity, whose character is $\chi^R(e) = m$ [see Eq. (24.9)]. Now use Eq. (24.14) for $g = e$ and for the regular representation to obtain $m = \sum_{\alpha=1}^\rho m_\alpha n_\alpha$ where n_α is the dimension of the α-th irreducible representation. We can find m_α by using Eq. (24.15) and noting that only $g = e$ contributes to the sum:

$$m_\alpha = \frac{1}{|G|} \sum_{g \in G} \chi^R(g) \chi^{(\alpha)*}(g) = \frac{1}{m} \chi^R(e) \underbrace{\chi^{(\alpha)*}(e)}_{n_\alpha} = n_\alpha.$$

In words,

> **Box 24.4.2** *The number of times an irreducible representation occurs in the regular representation is equal to the dimension of that irreducible representation.*

We therefore obtain the important relations

$$\chi_i^R = |G|\delta_{i1} = \sum_{\alpha=1}^\rho n_\alpha \chi_i^{(\alpha)} \quad \text{and} \quad |G| = \sum_{\alpha=1}^\rho n_\alpha^2, \qquad (24.18)$$

where we have assumed that the first conjugacy class is that of the identity. For finite groups of small order, the second equation can be very useful in obtaining the dimensions of irreducible representations.

Example 24.4.3 A group of order 2 or 3 has only one-dimensional inequivalent irreducible representations, because the only way that Eq. (24.18) can be satisfied for $|G| = 2$ or 3 is for all n_α's to be 1. A group of order 4 can have either 4 one-dimensional or one 2-dimensional inequivalent irreducible representations. The symmetric group S_3, being of order 6, can have 6 one-dimensional, or 2 one-dimensional and one 2-dimensional inequivalent irreducible representations. We shall see later that if all inequivalent irreducible representations of a group are one-dimensional, then the group must be abelian. Thus, the first possibility for S_3 must be excluded.

24.5   Group Algebra

Think of group elements as (linearly independent) vectors. In fact, given any set, one can generate a vector space by taking linear combinations of the elements of the set assumed to form a basis. In the case of groups one gets a bonus: The product already defined on the basis (group elements) can be extended by linearity to all elements of the vector space to turn it into an

group algebra defined   algebra called the **group algebra**. For $G = \{g_j\}_{j=1}^m$, a typical element of the group algebra is $\mathbf{a} = \sum_{i=1}^m a_i g_i$. One can add two vectors as usual. But the product of two vectors is also defined:

$$\mathbf{ab} = \left(\sum_{i=1}^m a_i g_i\right)\left(\sum_{j=1}^m b_j g_j\right) = \sum_{i=1}^m \sum_{j=1}^m a_i b_j \underbrace{g_i g_j}_{g_k} \equiv \sum_{k=1}^m c_k g_k,$$

where c_k is a sum involving a_i and b_j. The best way to learn this is to see an example.

Example 24.5.1 Let $G = S_3$ and consider $\mathbf{a} = 2\pi_1 - 3\pi_3 + \pi_5$ and $\mathbf{b} = \pi_2 - 2\pi_4 + 3\pi_6$. Then, using Table 23.1, we obtain

$$\begin{aligned}
\mathbf{ab} &= (2\pi_1 - 3\pi_3 + \pi_5)(\pi_2 - 2\pi_4 + 3\pi_6) \\
&= 2\pi_1\pi_2 - 4\pi_1\pi_4 + 6\pi_1\pi_6 - 3\pi_3\pi_2 + 6\pi_3\pi_4 \\
&\quad - 9\pi_3\pi_6 + \pi_5\pi_2 - 2\pi_5\pi_4 + 3\pi_5\pi_6 \\
&= 2\pi_2 - 4\pi_4 + 6\pi_6 - 3\pi_6 + 6\pi_5 - 9\pi_2 + \pi_4 - 2\pi_3 + 3\pi_1 \\
&= 3\pi_1 - 7\pi_2 - 2\pi_3 - 3\pi_4 + 6\pi_5 + 3\pi_6.
\end{aligned}$$

24.5.1 Group Algebra and Representations

Group algebra is very useful for the construction and analysis of representations of groups. In fact, we have already used a similar approach in the construction of the regular representation. Instead of $\mathbb{R}^m$ used before, use the m-dimensional vector space $\mathcal{A}$, the group algebra. Then left-multiplication by a group element g can be identified with $\mathbf{T}_g^{(R)}$, the operators of the regular representation, and the invariant subspaces of $\mathcal{A}$ become the left ideals of $\mathcal{A}$, and we can write

$$\mathcal{A} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_r.$$

Moreover, since the identity element of the group is the identity element of

resolution of the identity   the algebra as well, we have

$$e = e_1 + \cdots + e_r, \qquad e_i^2 = e_i, \qquad e_i e_j = 0 \quad \text{for } i \neq j. \qquad (24.19)$$

It is clear that if $\mathbf{a}^2 = \alpha \mathbf{a}$, then $\mathbf{a}/\alpha$ will be idempotent. So, we can essentially ignore the constant α, which is why $\mathbf{a}$ is called **essentially idempotent**. Now consider the element of the group algebra

essentially idempotent elements

$$P = \sum_{x \in G} x \qquad (24.20)$$

and note that $g P = \sum_{x \in G} gx = P$. It follows that

$$P^2 = \sum_{g \in G} g \sum_{x \in G} x = \sum_{g \in G} \sum_{x \in G} gx = \sum_{g \in G} P = |G| P.$$

So, P is essentially idempotent. Furthermore, the reader may verify that the ideal generated by P is one-dimensional.

Let us now apply the notion of the group algebra to derive further relations among characters. Denote the elements of the ith class K_i of G by $\{x_l^{(i)}\}_{l=1}^{c_i}$ and construct the element of the group algebra $\kappa_i \equiv \sum_{l=1}^{c_i} x_l^{(i)}$. If in the product of two such quantities

$$\kappa_i \kappa_j = \sum_{l=1}^{c_i} \sum_{m=1}^{c_j} x_l^{(i)} x_m^{(j)}, \qquad (24.21)$$

$x_l^{(i)} x_m^{(j)} \equiv y \in G$, is in a certain conjugacy class, then the rest of that class can be obtained by taking all conjugates of y, i.e., elements of G that can be written as

$$g y g^{-1} = g x_l^{(i)} x_m^{(j)} g^{-1} = \underbrace{g x_l^{(i)} g^{-1}}_{\in K_i} \underbrace{g x_m^{(j)} g^{-1}}_{\in K_j}.$$

It follows that if one member of a class appears in the double sum of Eq. (24.21), all members will appear there. The reader may check that if y occurs k times in the double sum, then all members of the class of y occur k times as well. Collecting all such members together, we can write

$$\kappa_i \kappa_j = \sum_{l=1}^{r} c_{ijl} \kappa_l, \qquad (24.22)$$

where c_{ijl} are positive integers.

Now consider the αth irreducible representation, and add all operators corresponding to a given class:

$$\mathbf{T}_i^{(\alpha)} \equiv \sum_{g \in K_i} \mathbf{T}_g^{(\alpha)} \quad \Rightarrow \quad \mathbf{T}_i^{(\alpha)} \mathbf{T}_j^{(\alpha)} = \sum_{l=1}^{r} c_{ijl} \mathbf{T}_l^{(\alpha)}, \qquad (24.23)$$

where the second equation follows from the same sort of argument used above to establish Eq. (24.22). One can show that $\mathbf{T}_i^{(\alpha)}$ commutes with all $\mathbf{T}_g^{(\alpha)}$. Therefore, by Schur's lemma, $\mathbf{T}_i^{(\alpha)} = \lambda_i^{(\alpha)} \mathbf{1}$, and the second equation in (24.23) becomes

$$\lambda_i^{(\alpha)} \lambda_j^{(\alpha)} = \sum_{l=1}^{r} c_{ijl} \lambda_l^{(\alpha)}. \qquad (24.24)$$

Taking the characters of both sides of $\mathbf{T}_i^{(\alpha)} = \lambda_i^{(\alpha)} \mathbf{1}$ and using the first equation in (24.23), noting that all elements of a class have the same character, we get

$$c_i \chi_i^{(\alpha)} = \lambda_i^{(\alpha)} n_\alpha \quad \Rightarrow \quad \lambda_i^{(\alpha)} = \frac{c_i \chi_i^{(\alpha)}}{n_\alpha}.$$

Substituting this in Eq. (24.24), we obtain

$$c_i c_j \chi_i^{(\alpha)} \chi_j^{(\alpha)} = n_\alpha \sum_{l=1}^{r} c_{ijl} c_l \chi_l^{(\alpha)}. \qquad (24.25)$$

This is another equation that is useful for computing characters. Note that this equation connects the purely group properties (c_i's and c_{ijl}'s) with the properties of the representation ($\chi_i^{(\alpha)}$'s and n_α). Summing Eq. (24.25) over α and using the first equation in (24.18), we get

$$c_i c_j \sum_{\alpha=1}^{\rho} \chi_i^{(\alpha)} \chi_j^{(\alpha)} = \sum_{l=1}^{r} c_{ijl} c_l \underbrace{\sum_{\alpha=1}^{\rho} n_\alpha \chi_l^{(\alpha)}}_{=|G|\delta_{l1} \text{ by (24.18)}} = c_{ij1}|G|$$

because $c_1 = 1$ (there is only one element in the class of the identity). Problem 24.12 shows that $c_{ij1} = c_i \delta_{i'j}$ where $K_{i'}$ is the class consisting of inverses of elements of K_i. It then follows that

$$\sum_{\alpha=1}^{\rho} \chi_i^{(\alpha)} \chi_j^{(\alpha)} = \frac{|G|}{c_j} \delta_{i'j}. \qquad (24.26)$$

For a unitary representation, $\chi_{i'}^{(\alpha)} = \chi_i^{(\alpha)*}$, so Eq. (24.26) becomes

$$\sum_{\alpha=1}^{\rho} \chi_i^{(\alpha)} \chi_j^{(\alpha)*} = \frac{|G|}{c_j} \delta_{ij} \quad \Rightarrow \quad \langle \chi_j | \chi_i \rangle = \frac{|G|}{c_j} \delta_{ij}, \qquad (24.27)$$

where $|\chi_i\rangle \in \mathbb{C}^\rho$ is a ρ-dimensional vector with components $\{\chi_i^{(\alpha)}\}_{\alpha=1}^{\rho}$. This equation can also be written in terms of group elements rather than classes. Since $\chi_i^{(\alpha)} = \chi^{(\alpha)}(x)$ for any $x \in K_i$, we have

$$\sum_{\alpha=1}^{\rho} \chi^{(\alpha)}(x) \chi^{(\alpha)*}(y) = \frac{|G|}{|K_x^G|} \delta(K_x^G, K_y^G), \qquad (24.28)$$

where K_x^G is the conjugacy class of G containing x, $|K_x^G|$ is the number of its elements, and

$$\delta(K_x^G, K_y^G) = \begin{cases} 1 & \text{if } K_x^G = K_y^G, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (24.27) shows that the r vectors $\chi_i^{(\alpha)}$ are mutually orthogonal; therefore, $r \leq \rho$. Combining this with Proposition 24.3.7, we obtain the following:

Table 24.1 A typical character table

	$^{c_1}K_1$	$^{c_2}K_2$	$\cdots$	$^{c_i}K_i$	$\cdots$	$^{c_r}K_r$
$T^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_i^{(1)}$	$\cdots$	$\chi_r^{(1)}$
$T^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_i^{(2)}$	$\cdots$	$\chi_r^{(2)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$T^{(\alpha)}$	$\chi_1^{(\alpha)}$	$\chi_2^{(\alpha)}$	$\cdots$	$\chi_i^{(\alpha)}$	$\cdots$	$\chi_r^{(\alpha)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$T^{(r)}$	$\chi_1^{(r)}$	$\chi_2^{(r)}$	$\cdots$	$\chi_i^{(r)}$	$\cdots$	$\chi_r^{(r)}$

Theorem 24.5.2 *The number of inequivalent irreducible representations of a finite group is equal to the number of conjugacy classes in the group.*

It is convenient to summarize our result in a square table with rows labeled by the irreducible representation and columns labeled by the conjugacy classes of G. Then on the αth row and ith column we list $\chi_i^{(\alpha)}$, and we get Table 24.1, called the **character table** of G. Note that c_i, the order of K_i, is written as a left superscript. Character tables have the property that any two of their rows are orthogonal in the sense of Eq. (24.12), and any two of their columns are orthogonal in the sense of Eq. (24.27).

character table of a finite group

If all inequivalent irreducible representations of a group G have dimension one, then there will be $|G|$ of them [by Eq. (24.18)]. Hence, there will be $|G|$ conjugacy classes; i.e., each class consists of a single element. By Problem 23.16, the group must be abelian. Combining this with Theorem 24.3.3, we have the following theorem.

Theorem 24.5.3 *A finite group is abelian if and only if all its inequivalent irreducible representations are one-dimensional.*

24.6 Relationship of Characters to Those of a Subgroup

Let H be a subgroup of G. Denote by K_h^H and K_g^G the H-class containing $h \in H$ and the G-class containing g, respectively. Let d_j and c_i be the number of elements in the jth H-class and ith G-class, respectively. Any representation of G defines a representation of H by restriction. An irreducible representation of G may be reducible as a representation of H. This is because although the subspace $\mathcal{W}^{(\alpha)}$ of the carrier space that is irreducible under G is the smallest such subspace containing a given vector, it is possible to generate a smaller subspace by applying a subset of the operators $\mathbf{T}_g$ corresponding to those g's that belong to H. It follows that

$$T^{(\alpha)}(h) = \sum_{\sigma} m_{\alpha\sigma} t^{(\sigma)}(h), \quad h \in H, \qquad (24.29)$$

where $m_{\alpha\sigma}$ are nonnegative integers as in Eq. (24.14) and $t^{(\sigma)}$ are irreducible representations of H. If $\chi^{(\alpha)}$ and $\xi^{(\sigma)}$ denote the characters of irreducible representations of G and H, respectively, then the equivalent equation for the characters is

$$\chi^{(\alpha)}(h) = \sum_{\sigma} m_{\alpha\sigma} \xi^{(\sigma)}(h), \quad h \in H. \tag{24.30}$$

Multiply both sides by $\xi^{(\kappa)*}(h)$, sum over $h \in H$, and take the complex conjugate at the end. Then by the orthogonality relation (24.11), applied to H, we obtain

$$m_{\alpha\kappa} = \frac{1}{|H|} \sum_{h \in H} \chi^{(\alpha)*}(h) \xi^{(\kappa)}(h). \tag{24.31}$$

Now multiply both sides of Eq. (24.31) by $\chi^{(\alpha)}(g)$, sum over α, and use Eq. (24.28) to obtain

$$\sum_{\alpha} m_{\alpha\kappa} \chi^{(\alpha)}(g) = \frac{|G|}{|H||K_g^G|} \sum_{h \in H} \delta\left(K_h^G, K_g^G\right) \xi^{(\kappa)}(h). \tag{24.32}$$

The sum on the right can be transformed into a sum over conjugacy classes of H. Then Eq. (24.32) becomes

$$\sum_{\alpha} m_{\alpha\kappa} \chi_i^{(\alpha)} = \frac{|G|}{|H| c_i} \sum_{j} d_j \xi_j^{(\kappa)}, \quad i = 1, 2, \ldots, r, \tag{24.33}$$

where the sum on the LHS is over irreducible representations of G, and on the RHS it is over those H-classes j that lie in the ith G-class. Note that the coefficients $|G| d_j / (|H| c_i)$ are integers by Problem 23.17.

Equations (24.32) and (24.33) are useful for obtaining characters of G when those of a subgroup H are known. The general procedure is to note that the RHS of these equations are completely determined by the structure of the group G and the characters of H. Varying i, the RHS of (24.33) determines the r components of a (compound) character $|\psi\rangle$, which, by the LHS, can be written as a linear combination of characters of G:

$$|\psi\rangle \equiv \sum_{\alpha=1}^{r} m_{\alpha} |\chi^{(\alpha)}\rangle, \tag{24.34}$$

where we have suppressed the irrelevant subscript κ. If we know some of the $|\chi^{(\alpha)}\rangle$'s, we may be able to determine the rest by taking successive inner products to find the integers m_{α}, and subtracting each irreducible factor of the sum from the LHS. We illustrate this procedure for S_n in the following example.

Example 24.6.1 Let $K_1 = (1^2)$ and $K_2 = (2)$ for S_2 (see Sect. 23.4 for notation). Example 24.2.7 showed that we can construct two irreducible representations for any S_n, the symmetric and the antisymmetric representations. The reader may verify that these two representations are inequivalent. Since

Table 24.2 Character table for S_2

	$^1K_1$	$^1K_2$
$T^{(1)}$	1	1
$T^{(2)}$	1	-1

Table 24.3 Partially filled character table for S_3

	$^1K_1$	$^3K_2$	$^2K_3$
$T^{(1)}$	1	1	1
$T^{(2)}$	1	-1	1
$T^{(3)}$	?	?	?

the number of inequivalent irreducible representations is equal to the number of classes in a group, we have all the information needed to construct the character table for S_2. Table 24.2 shows this character table. We want to use the S_2 character table to construct the character table for S_3. With our knowledge of the symmetric and the antisymmetric representations, we can partially fill in the S_3 character table. Let $K_1 = (1^3)$, $K_2 = (2, 1)$, and $K_3 = (3)$ and note that $c_1 = 1$, $c_2 = 3$, and $c_3 = 2$. Then we obtain Table 24.3. To complete the table, we start with $\kappa = 1$, and write the RHS of Eq. (24.33) as

$$\psi_i = \frac{6}{2c_i} \sum_j d_j \xi_j^{(1)} = \frac{3}{c_i} \sum_j \xi_j^{(1)}$$

because $d_j = 1$ for the two classes of S_2. The sum on the RHS is over S_2-classes that are inside the ith S_3-class. For $i = 1$, only the first S_2-class contributes. Noting that $\xi_j^{(\kappa)}$ are the entries of Table 24.2, we get

$$\psi_1 = \frac{3}{c_1} \xi_1^{(1)} = \frac{3}{1} \cdot 1 = 3.$$

Similarly,

$$\psi_2 = \frac{3}{c_2} \xi_2^{(1)} = \frac{3}{3} \cdot 1 = 1 \quad \text{and} \quad \psi_3 = \frac{3}{c_3} \cdot 0 = 0.$$

The second equation follows from the fact that there are no classes of S_2 inside the third class of S_3. Equation (24.34) now gives

$$|\psi\rangle = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \sum_{\alpha=1}^{r} m_\alpha |\chi^{(\alpha)}\rangle.$$

We can find the number of times $|\chi^{(1)}\rangle$ occurs in this compound character by taking the inner product:

$$\langle \chi^{(1)} | \psi \rangle = \sum_{\alpha=1}^{r} m_\alpha \langle \chi^{(1)} | \chi^{(\alpha)} \rangle = m_1 |G| = 6m_1.$$

Table 24.4 Complete character table for S_3

	$^1K_1$	$^3K_2$	$^2K_3$
$T^{(1)}$	1	1	1
$T^{(2)}$	1	-1	1
$T^{(3)}$	2	0	-1

But

$$\langle \chi^{(1)} | \psi \rangle = \sum_{i=1}^{r} c_i \chi_i^{(1)} \psi_i = 1 \cdot 1 \cdot 3 + 3 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 0 = 6.$$

These two equations show that $m_1 = 1$. So,

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + m_2 |\chi^{(2)}\rangle + m_3 |\chi^{(3)}\rangle.$$

Subtracting the column vectors, we get a new character:

$$|\psi'\rangle \equiv \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = m_2 |\chi^{(2)}\rangle + m_3 |\chi^{(3)}\rangle.$$

Taking the inner product with $|\chi^{(2)}\rangle$ yields $m_2 = 0$. It follows that $|\psi'\rangle$ is a simple character. In fact,

$$\sum_i c_i |\psi_i'|^2 = 1 \cdot 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2 = 6,$$

and the criterion of irreducibility, Eq. (24.17), is satisfied.

We can now finish up Table 24.3 to obtain Table 24.4, which is the complete character table for S_3.

24.7   Irreducible Basis Functions

We have studied the operators $\mathbf{T}_g$ and their characters representing group elements in rather extensive detail. Let us now turn our attention to the carrier space itself. In particular, we want to concentrate on the basis functions of the irreducible representations. We choose "functions," rather than vectors, because of their use in quantum mechanics as discussed at the beginning of this chapter.

Let $\{|\psi_i^{(\alpha)}\rangle\}_{i=1}^{n_\alpha}$ be a set of basis functions for $\mathcal{W}^{(\alpha)}$, the αth invariant irreducible subspace. Invariance of $\mathcal{W}^{(\alpha)}$ implies that

$$\mathbf{T}_g |\psi_i^{(\alpha)}\rangle = \sum_{j=1}^{n_\alpha} T_{ji}^{(\alpha)}(g) |\psi_j^{(\alpha)}\rangle,$$

where $T_{ji}^{(\alpha)}(g)$ are elements of the matrix $\mathsf{T}_g^{(\alpha)}$ representing $g \in G$.

Definition 24.7.1 A function (or vector) $|\phi_i^{(\alpha)}\rangle$ is said to **belong to the** **ith row of the αth irreducible representation** (or to transform according to the ith row of the αth irreducible representation) if there exists a basis $\{|\psi_i^{(\alpha)}\rangle\}_{i=1}^{n_\alpha}$ of the αth irreducible representation of G with matrices $(T_{ji}^{(\alpha)}(g))$ and $n_\alpha - 1$ other functions $\{|\phi_j^{(\alpha)}\rangle\}$ such that

$$\mathbf{T}_g|\phi_i^{(\alpha)}\rangle = \sum_{j=1}^{n_\alpha} T_{ji}^{(\alpha)}(g)|\phi_j^{(\alpha)}\rangle. \tag{24.35}$$

functions belonging to the ith row of the αth irreducible representation

Functions that belong to rows of irreducible representations have some remarkable properties. Let $|\psi_i^{(\alpha)}\rangle$ and $|\phi_j^{(\beta)}\rangle$ transform according to the ith and jth rows of the αth and βth irreducible representations, respectively. Choose an inner product for the carrier space such that all representations are unitary. Then we have

$$\langle\psi_i^{(\alpha)}|\phi_j^{(\beta)}\rangle = \langle\mathbf{T}_g\psi_i^{(\alpha)}|\mathbf{T}_g\phi_j^{(\beta)}\rangle$$
$$= \sum_{l=1}^{n_\alpha}\sum_{m=1}^{n_\beta} T_{li}^{(\alpha)*}(g)T_{mj}^{(\beta)}(g)\langle\psi_l^{(\alpha)}|\phi_m^{(\beta)}\rangle.$$

Summing this equation over g yields $|G|\langle\psi_i^{(\alpha)}|\phi_j^{(\beta)}\rangle$ for the LHS, while

$$\text{RHS} = \sum_{l=1}^{n_\alpha}\sum_{m=1}^{n_\beta}\overbrace{\sum_{g\in G} T_{li}^{(\alpha)*}(g)T_{mj}^{(\beta)}(g)}^{(|G|/n_\alpha)\delta_{\alpha\beta}\delta_{lm}\delta_{ij}}\langle\psi_l^{(\alpha)}|\phi_m^{(\beta)}\rangle$$
$$= \frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{ij}\sum_{l=1}^{n_\alpha}\langle\psi_l^{(\alpha)}|\phi_l^{(\beta)}\rangle,$$

where we have made use of Eq. (24.8). Therefore,

$$\langle\psi_i^{(\alpha)}|\phi_j^{(\beta)}\rangle = \frac{1}{n_\alpha}\delta_{\alpha\beta}\delta_{ij}\sum_{l=1}^{n_\alpha}\langle\psi_l^{(\alpha)}|\phi_l^{(\alpha)}\rangle. \tag{24.36}$$

This shows that functions belonging to different irreducible representations are orthogonal. We should expect this, because in our construction of invariant irreducible subspaces, we kept dividing the whole space into orthogonal complements. What is surprising is that *functions transforming according to different rows of an irreducible representation are orthogonal*. We had no control over this property! It is a consequence of Eq. (24.35). Another surprise is the independence of the inner product from i: If we let $i = j$ and $\alpha = \beta$ on both sides of (24.36), we obtain

$$\langle\psi_i^{(\alpha)}|\phi_i^{(\alpha)}\rangle = \frac{1}{n_\alpha}\sum_{l=1}^{n_\alpha}\langle\psi_l^{(\alpha)}|\phi_l^{(\alpha)}\rangle, \tag{24.37}$$

which indicates that the inner product on the LHS is independent of i.

symmetry and the
quantum mechanical
perturbation theory;
lifting of degeneracy

Example 24.7.2 The quantum-mechanical perturbation theory starts with a known Hamiltonian $\mathbf{H}_0$ with eigenvalues E_i and the corresponding eigenstates $|E_i\rangle$. Subsequently, a (small) perturbing "potential" $\mathbf{V}$ is added to the Hamiltonian, and the eigenvalues and eigenstates of the new Hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ are sought. One can draw important conclusions about the eigenvalues and eigenstates of the total Hamiltonian by symmetry arguments.

Suppose the symmetry group of $\mathbf{H}_0$ is G, and that of $\mathbf{H}$ is H, which has to be a subgroup of G. In most cases, the eigenspaces of $\mathbf{H}_0$ are irreducible carrier spaces of G, i.e., their basis vectors transform according to the rows of irreducible representations of G. If H is a proper subgroup of G, then the eigenspaces of $\mathbf{H}_0$ will split according to Eq. (24.29). We say that some of the *degeneracy is lifted* because of the perturbation $\mathbf{V}$. The nature of the split, i.e., the number and the dimensionality of the vector spaces into which a given eigenspace splits, can be obtained by the characters of G and H and Eq. (24.30). The original eigenspaces are represented on an *energy diagram* with a line corresponding to each eigenspace. The split of the eigenspace into k new subspaces is then indicated by the branching of the old line into k new lines.

To the lowest approximation—first-order perturbation theory—the magnitude of the split, i.e., the difference between the eigenvalues of $\mathbf{H}_0$ and those of $\mathbf{H}$, is given by [see Eq. (21.57)] the expectation value $\langle \phi_i^{(\alpha)} | \mathbf{V} | \phi_j^{(\alpha)} \rangle$, where $|\phi_i^{(\alpha)}\rangle$ belongs to the ith row of the αth irreducible representation, and $|\phi_j^{(\alpha)}\rangle$ to its jth row ($i \neq j$). Only if this expectation value is nonzero will a split occur. This, in turn, depends on the symmetry of $\mathbf{V}$: If $\mathbf{V}$ is *at least* as symmetric as $\mathbf{H}_0$ (corresponding to $G = H$), then $\langle \phi_i^{(\alpha)} | \mathbf{V} | \phi_j^{(\alpha)} \rangle = 0$, and no splitting occurs (Problem 24.17). If, on the other hand, $\mathbf{V}$ is less symmetric than $\mathbf{H}_0$ (corresponding to $H \subset G$), then $\mathbf{V} | \phi_j^{(\alpha)} \rangle$ will not belong to the jth row of the αth irreducible representation, and in general, $\langle \phi_i^{(\alpha)} | \mathbf{V} | \phi_j^{(\alpha)} \rangle \neq 0$.

We have decomposed the carrier space $\mathcal{V}$ of a representation into invariant irreducible subspaces $\mathcal{W}^{(\alpha)}$. The argument above shows that each $\mathcal{W}^{(\alpha)}$ has a basis consisting of the "rows" of the irreducible representations. Corresponding to such a basis, there is a set of projection operators $\mathbf{P}_i^{(\alpha)}$ with the property $\sum_{\alpha,i} \mathbf{P}_i^{(\alpha)} = \mathbf{1}$ (Chap. 6). Our aim is to find an expression for these operators, which have the defining property $\mathbf{P}_i^{(\alpha)} |\psi_i^{(\alpha)}\rangle = |\psi_i^{(\alpha)}\rangle$. We start with Eq. (24.35), multiply both sides of it by $T_{lm}^{(\beta)*}(g)$, sum over $g \in G$, and use Eq. (24.8) to obtain

$$\sum_{g \in G} T_{lm}^{(\beta)*}(g) \mathbf{T}_g |\psi_i^{(\alpha)}\rangle = \sum_{j=1}^{n_\alpha} |\psi_j^{(\alpha)}\rangle \sum_{g \in G} T_{lm}^{(\beta)*}(g) T_{ji}^{(\alpha)}(g)$$

$$= \frac{|G|}{n_\alpha} \sum_{j=1}^{n_\alpha} |\psi_j^{(\alpha)}\rangle \delta_{lj} \delta_{mi} \delta_{\alpha\beta} = \frac{|G|}{n_\alpha} |\psi_l^{(\alpha)}\rangle \delta_{mi} \delta_{\alpha\beta}.$$

Let $\beta = \alpha$, $m = l = i$, and multiply both sides by $n_\alpha/|G|$. Then this equation becomes

$$\frac{n_\alpha}{|G|} \sum_{g \in G} T_{ii}^{(\alpha)*}(g) \mathbf{T}_g |\psi_i^{(\alpha)}\rangle = |\psi_i^{(\alpha)}\rangle,$$

which suggests the identification

$$\mathbf{P}_i^{(\alpha)} = \frac{n_\alpha}{|G|} \sum_{g \in G} T_{ii}^{(\alpha)*}(g) \mathbf{T}_g \qquad (24.38)$$

projection operator onto the ith row of the αth irreducible representation

with the properties

$$\mathbf{P}_i^{(\alpha)} |\psi_j^{(\beta)}\rangle = |\psi_i^{(\alpha)}\rangle \delta_{ij} \delta_{\alpha\beta}, \qquad \mathbf{P}_i^{(\alpha)} |\phi\rangle = |\phi_i^{(\alpha)}\rangle, \qquad (24.39)$$

where $|\phi_i^{(\alpha)}\rangle$ is the projection of $|\phi\rangle$ along the ith row of the αth irreducible representation.

We are also interested in the projection operator that projects onto the irreducible subspace $\mathcal{W}^{(\alpha)}$. Such an operator is obtained by summing $\mathbf{P}_i^{(\alpha)}$ over i. We thus obtain

$$\mathbf{P}^{(\alpha)} = \frac{n_\alpha}{|G|} \sum_{g \in G} \underbrace{\sum_{i=1}^{n_\alpha} T_{ii}^{(\alpha)*}(g)}_{=\chi^{(\alpha)*}(g)} \mathbf{T}_g = \frac{n_\alpha}{|G|} \sum_{g \in G} \chi^{(\alpha)*}(g) \mathbf{T}_g \qquad (24.40)$$

projection operator onto the αth irreducible representation

and

$$\mathbf{P}^{(\alpha)} |\psi^{(\beta)}\rangle = |\psi^{(\alpha)}\rangle \delta_{\alpha\beta}, \qquad \mathbf{P}^{(\alpha)} |\phi\rangle = |\phi^{(\alpha)}\rangle, \qquad (24.41)$$

where $|\phi^{(\alpha)}\rangle$ is the projection of $|\phi\rangle$ onto the αth irreducible invariant subspace. These formulas are extremely useful in identifying the irreducible subspaces of a given carrier space:

> **Box 24.7.3** *Start with a basis $\{|a_i\rangle\}$ of the carrier space, apply $\mathbf{P}^{(\alpha)}$ of Eq. (24.40) to all basis vectors, and collect all the linearly independent vectors of the form $\mathbf{P}^{(\alpha)}|a_i\rangle$. These vectors form a basis of the αth irreducible representation.*

The following example illustrates this point.

Example 24.7.4 Consider the representation of S_3 given in Example 24.2.2, where the carrier space is the span of the three functions $|\psi_1\rangle = xy$, $|\psi_2\rangle = yz$, and $|\psi_3\rangle = xz$.

We refer to the character table for S_3 (Table 24.4) and use Eq. (24.40) to obtain

$$\mathbf{P}^{(1)} = \frac{1}{6}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 + \mathbf{T}_5 + \mathbf{T}_6),$$

$$\mathbf{P}^{(2)} = \frac{1}{6}(\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3 - \mathbf{T}_4 + \mathbf{T}_5 + \mathbf{T}_6),$$

$$\mathbf{P}^{(3)} = \frac{2}{6}(2\mathbf{T}_1 - \mathbf{T}_5 - \mathbf{T}_6),$$

where, as in Example 24.2.2, we have used the notation $\mathbf{T}_i$ for $\mathbf{T}_{\pi_i}$, and the result $n_1 = n_2 = 1$ and $n_3 = 2$ obtained from Eq. (24.18), Theorem 24.5.3, and the fact that S_3 is nonabelian.

To get the first irreducible subspace of this representation, we apply $\mathbf{P}^{(1)}$ to $|\psi_1\rangle$. Since this subspace is one-dimensional, the procedure will give a basis for it if the vector so obtained is nonzero:

$$\begin{aligned}
\mathbf{P}^{(1)}|\psi_1\rangle &= \frac{1}{6}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 + \mathbf{T}_5 + \mathbf{T}_6)|\psi_1\rangle \\
&= \frac{1}{6}\big(|\psi_1\rangle + |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle + |\psi_3\rangle + |\psi_2\rangle\big) \\
&= \frac{1}{3}\big(|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle\big).
\end{aligned}$$

This is a basis for the carrier space of the irreducible identity representation.

For the second irreducible representation, we get

$$\mathbf{P}^{(2)}|\psi_1\rangle = \frac{1}{6}\big(|\psi_1\rangle - |\psi_1\rangle - |\psi_2\rangle - |\psi_3\rangle + |\psi_3\rangle + |\psi_2\rangle\big) = 0.$$

Similarly, $\mathbf{P}^{(2)}|\psi_2\rangle = 0$ and $\mathbf{P}^{(2)}|\psi_3\rangle = 0$. This means that $T^{(2)}$ is not included in the representation we are working with. We should have expected this, because if this one-dimensional irreducible representation were included, it would force the last irreducible representation to be one-dimensional as well [see Eq. (24.18)], and, by Theorem 24.5.3, the group S_3 to be abelian!

The last irreducible representation is obtained similarly. We have

$$\mathbf{P}^{(3)}|\psi_1\rangle = \frac{1}{3}(2\mathbf{T}_1 - \mathbf{T}_5 - \mathbf{T}_6)|\psi_1\rangle = \frac{1}{3}\big(2|\psi_1\rangle - |\psi_3\rangle - |\psi_2\rangle\big),$$

$$\mathbf{P}^{(3)}|\psi_2\rangle = \frac{1}{3}(2\mathbf{T}_1 - \mathbf{T}_5 - \mathbf{T}_6)|\psi_2\rangle = \frac{1}{3}\big(2|\psi_2\rangle - |\psi_1\rangle - |\psi_3\rangle\big).$$

These two vectors are linearly independent. Therefore, they form a basis for the last irreducible representation. The reader may check that $\mathbf{P}^{(3)}|\psi_3\rangle$ is a linear combination of $\mathbf{P}^{(3)}|\psi_1\rangle$ and $\mathbf{P}^{(3)}|\psi_2\rangle$.

24.8   Tensor Product of Representations

A simple quantum mechanical system possessing a group of symmetry is described by vectors that transform irreducibly (or according to a row of an irreducible representation). For example, a rotationally invariant system can be described by an eigenstate of angular momentum, the generator of rotation.[3] These eigenstates transform as rows of irreducible representations

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[3]Chapter 29 will make explicit the connection between groups and their generators.

of the rotation group. At a more fundamental level, the very concept of a particle or field is thought of as states that transform irreducibly under the fundamental group of spacetime, the Poincaré group.

Often these irreducible states are "combined" to form new states. For example, the state of two (noninteracting) particles is described by a two-particle state, labeled by the combined eigenvalues of the two sets of operators that describe each particle separately. In the case of angular momentum, the single-particle states may be labeled as $|l_i, m_i\rangle$ for $i = 1, 2$. Then the combined state will be labeled as $|l_1, m_1; l_2, m_2\rangle$, and one can define an action of the rotation group on the vector space spanned by these combined states to construct a representation. We now describe the way in which this is done.

Let $T : G \to GL(\mathcal{V})$ and $S : G \to GL(\mathcal{W})$ be two representations of a group G. Define an action of the group G on $\mathcal{V} \otimes \mathcal{W}$, the tensor product of $\mathcal{V}$ and $\mathcal{W}$, via the representation $T \otimes S : G \to GL(\mathcal{V} \otimes \mathcal{W})$ given by

$$(T \otimes S)(g)\big(|v\rangle, |w\rangle\big) = \big(T(g)|v\rangle, S(g)|w\rangle\big).$$

We note that

$$
\begin{aligned}
(T \otimes S)&(g_1 g_2)\big(|v\rangle, |w\rangle\big) \\
&= \big(T(g_1 g_2)|v\rangle, S(g_1 g_2)|w\rangle\big) = \big(T(g_1)T(g_2)|v\rangle, S(g_1)S(g_2)|w\rangle\big) \\
&= (T \otimes S)(g_1)\big(T(g_2)|v\rangle, S(g_2)|w\rangle\big) \\
&= \big[(T \otimes S)(g_1)(T \otimes S)(g_2)\big]\big(|v\rangle, |w\rangle\big).
\end{aligned}
$$

It follows that $T \otimes S$ is indeed a representation, called the **tensor product** or **direct product** or **Kronecker product** representation. It is common, especially in the physics literature, to write $|v, w\rangle$, or simply $|vw\rangle$ for $(|v\rangle, |w\rangle)$, and TS for $T \otimes S$. If we choose the orthonormal bases $\{|v_i\rangle\}$ for $\mathcal{V}$ and $\{|w_a\rangle\}$ for $\mathcal{W}$, and define an inner product on $\mathcal{V} \otimes \mathcal{W}$ by

$$\langle v, w | v', w' \rangle \equiv \langle v|v'\rangle\langle w|w'\rangle,$$

we obtain a matrix representation of the group with matrix elements given by

$$
\begin{aligned}
(T \otimes S)_{ia, jb}(g) &\equiv \langle v_i, w_a | T \otimes S(g) | v_j, w_b \rangle \\
&= \langle v_i | T(g) | v_j \rangle \langle w_a | S(g) | w_b \rangle \equiv T_{ij}(g) S_{ab}(g).
\end{aligned}
$$

Note that the rows and columns of this matrix are distinguished by double indices. If the matrix T is $m \times m$ and S is $n \times n$, then the matrix $\mathsf{T} \otimes \mathsf{S}$ is $(mn) \times (mn)$. The character of the tensor product representation is

character of a product representation is a product of characters

$$
\chi^{T \otimes S}(g) = \sum_{i,a} (T \otimes S)_{ia,ia}(g) = \sum_{i,a} T_{ii}(g) S_{aa}(g) = \sum_{i} T_{ii}(g) \sum_{a} S_{aa}(g)
$$

$$
= \chi^T(g)\chi^S(g) \quad \Rightarrow \quad \chi_i^{T \otimes S} = \chi_i^T \cdot \chi_i^S. \tag{24.42}
$$

So the character of the tensor product is the product of the individual characters.

An important special case is the tensor product of a representation with itself. For such a representation, the matrix elements satisfy the symmetry relation

$$(T \otimes T)_{ia,jb}(g) = (T \otimes T)_{ai,bj}(g).$$

This symmetry can be used to decompose the tensor product space into two subspaces that are separately invariant under the action of the group. To do this, take the span of all the symmetric vectors of the form $(|v_i w_j\rangle + |v_j w_i\rangle) \in \mathcal{V} \otimes \mathcal{V}$ and denote it by $(\mathcal{V} \otimes \mathcal{V})_s$. Similarly, take the span of all the antisymmetric vectors of the form $(|v_i w_j\rangle - |v_j w_i\rangle) \in \mathcal{V} \otimes \mathcal{V}$ and denote it by $(\mathcal{V} \otimes \mathcal{V})_a$. Next note that

$$|v_i w_j\rangle = \frac{1}{2}\big(|v_i w_j\rangle + |v_j w_i\rangle\big) + \frac{1}{2}\big(|v_i w_j\rangle - |v_j w_i\rangle\big).$$

It follows that every vector of the product space can be written as the sum of a symmetric and an antisymmetric vector. Furthermore, the only vector that is both symmetric and antisymmetric is the zero vector. Therefore,

$$\mathcal{V} \otimes \mathcal{V} = (\mathcal{V} \otimes \mathcal{V})_s \oplus (\mathcal{V} \otimes \mathcal{V})_a.$$

Now consider the action of the group on each of these subspaces separately. From the relation

$$(T \otimes T)(g)|v_i w_j\rangle \equiv (T \otimes T)(g)\big(|v_i\rangle, |w_j\rangle\big)$$

$$= \left(\sum_k T_{ki}(g)|v_k\rangle, \sum_l T_{lj}(g)|w_l\rangle \right)$$

$$= \sum_{k,l} T_{ki} T_{lj}(g)(g)\big(|v_k\rangle, |w_l\rangle\big)$$

$$= \sum_{k,l} (T \otimes T)_{kl,ij}(g)|v_k w_l\rangle$$

we obtain

$$(T \otimes T)(g)\big(|v_i w_j\rangle \pm |v_j w_i\rangle\big)$$

$$= \sum_{k,l} \big[(T \otimes T)_{kl,ij}(g) \pm (T \otimes T)_{kl,ji}(g)\big]|v_k w_l\rangle. \qquad (24.43)$$

Kronecker product reduces to the symmetric and the antisymmetric representations

Problem 24.21 shows that the RHS can be written as a sum over the symmetric (for the plus sign) or antisymmetric (for the minus sign) vectors alone. It follows that

> **Box 24.8.1** *The Kronecker product of a representation with itself is always reducible into two representations, the **symmetrized product** and the **antisymmetrized product** representations.*

24.8.1 Clebsch-Gordan Decomposition

A common situation in quantum mechanics is to combine two simple systems into a composite system and see which properties of the original simple systems the composite system retains. For example, combining the angular momenta of two particles gives a new total angular momentum operator. The question of what single-particle angular momentum states are included in the states of the total angular momentum operator is the content of **selection rules** and is of great physical interest: A quark and an antiquark (two fermions) with spin $\frac{1}{2}$ always combine to form a meson (a boson), because the resulting composite state has no projection onto the subspace spanned by half-integer-spin particles. In this section, we study the mathematical foundation of this situation. The tensor product of two irreducible representations $T^{(\alpha)}$ and $T^{(\beta)}$ of G is denoted by $T^{(\alpha \times \beta)}$, and it is, in general, a reducible representation. The characters, generally compound, are denoted by $\chi^{(\alpha \times \beta)}$. Equation (24.14), combined with Eq. (24.42), tells us what irreducible representations are present in the tensor product, and therefore onto which irreducible representations the product representation has nonzero projection:

selection rules

$$\chi_i^{(\alpha \times \beta)} = \chi_i^{(\alpha)} \cdot \chi_i^{(\beta)} = \sum_{\sigma=1}^{r} m_\sigma^{\alpha\beta} \chi_i^{(\sigma)},$$

where $m_\sigma^{\alpha\beta}$ are nonnegative integers. We rewrite this more conveniently in terms of vectors as

$$\left| \chi^{(\alpha \times \beta)} \right\rangle = \sum_{\sigma=1}^{r} m_\sigma^{\alpha\beta} \left| \chi^{(\sigma)} \right\rangle,$$

$$m_\sigma^{\alpha\beta} = \frac{1}{|G|} \left\langle \chi^{(\sigma)} \middle| \chi^{(\alpha \times \beta)} \right\rangle = \frac{1}{|G|} \sum_{i=1}^{r} c_i \bar{\chi}_i^{(\sigma)} \chi_i^{(\alpha)} \chi_i^{(\beta)}. \tag{24.44}$$

A group for which $m_\sigma^{\alpha\beta} = 0, 1$ is called **simply reducible**.

simply reducible group

Historical Notes

Rudolph Friedrich Alfred Clebsch (1833–1872) studied mathematics in the shadow of Jacobi at the University of Königsberg, two of his teachers having been students of Jacobi. After graduation he held a number of positions in Germany, including positions at the universities of Berlin, Giessen, and finally Göttingen, where he remained until his death. He and Carl Neumann, the son of one of the aforementioned teachers who were students of Jacobi, founded the *Mathematische Annalen*.

Clebsch began his career in mathematical physics, producing a doctoral thesis on hydrodynamics and a book on elasticity in which he treated the elastic vibrations of rods and plates. These works were primarily mathematical, however, and he soon turned his attention more to pure mathematics. His links to Jacobi gave rise to his first work in that vein, concerning problems in variational calculus and partial differential equations, in which he surpassed the results of Jacobi's work.

Clebsch first achieved significant recognition for his work in projective invariants and algebraic geometry. He was intrigued by the interplay between algebra and geometry, and, since many results in the theory of invariants have geometric interpretations, the two fields seemed natural choices.

Rudolph Friedrich Alfred Clebsch 1833–1872

Example 24.8.2 Referring to Table 24.5 of Problem 24.15, and using Eq. (24.42), we can construct the compound character $|\chi^{(4\times5)}\rangle$ with components $9, -1, 1, 0, -1$. Then, we have

$$\left|\chi^{(4\times5)}\right\rangle = \begin{pmatrix} 9 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \sum_{\sigma=1}^{5} m_\sigma^{45}\left|\chi^{(\sigma)}\right\rangle, \qquad m_\sigma^{45} = \frac{1}{24}\left\langle\chi^{(\sigma)}\middle|\chi^{(4\times5)}\right\rangle.$$

For the first irreducible representation, we get

$$m_1^{45} = \frac{1}{24}\left\langle\chi^{(1)}\middle|\chi^{(4\times5)}\right\rangle = \frac{1}{24}\sum_{i=1}^{5} c_i\,\chi_i^{(1)*}\,\chi_i^{(4\times5)}$$

$$= \frac{1}{24}\big[1\cdot1\cdot9 + 6\cdot1\cdot(-1) + 3\cdot1\cdot1 + 8\cdot1\cdot0 + 6\cdot1\cdot(-1)\big] = 0.$$

For the second irreducible representation, we get

$$m_2^{45} = \frac{1}{24}\left\langle\chi^{(2)}\middle|\chi^{(4\times5)}\right\rangle$$

$$= \frac{1}{24}\big[1\cdot1\cdot9 + 6\cdot(-1)\cdot(-1) + 3\cdot1\cdot1 + 8\cdot(-1)\cdot0$$

$$+ 6\cdot(-1)\cdot(-1)\big] = 1.$$

Similarly, $m_3^{45} = 1$, $m_4^{45} = 1$, and $m_5^{45} = 1$. We thus see that the identity representation is not included in the direct product of irreducible representations 4 and 5; all other irreducible representations of S_4 occur once in $T^{(4\times5)}$.

Clebsch-Gordan series

In terms of representations themselves, we have the so-called **Clebsch-Gordan series**

$$T^{(\alpha\times\beta)}(g) = \sum_{\sigma=1}^{r} m_\sigma^{\alpha\beta}\,T^{(\sigma)}(g), \qquad m_\sigma^{\alpha\beta} = \frac{1}{|G|}\sum_{i=1}^{r} c_i\,\bar{\chi}_i^{(\sigma)}\,\chi_i^{(\alpha)}\,\chi_i^{(\beta)},$$

(24.45)

where we have used Eq. (24.13)

How to obtain invariants from the product of representations

The one-dimensional identity representation plays a special role in the application of group theory to physics because any vector (function) in its carrier space is invariant under the action of the group, and invariant vectors often describe special states of the quantum mechanical systems. For example, the ground state of an atomic system with rotational invariance has zero orbital angular momentum, corresponding to a spherically symmetric state.

Another example comes from particle physics. Quarks are usually placed in the states of an irreducible representation of a group $[SU(n)$, where n is the number of "flavors" such as up, down, charm], and antiquarks in its adjoint. A question of great importance is what combination of quarks and antiquarks leads to particles—called singlets—that are an invariant of the

group. For the case of quark-antiquark combination, the answer comes in the analysis of the tensor product of one irreducible representation, say $T^{(\alpha)}$, and one adjoint representation, say $\bar{T}^{(\beta)}$. In fact, using Eq. (24.45), we have

$$m_1^{\alpha\beta} = \frac{1}{|G|} \sum_{i=1}^{r} c_i \, \bar{\chi}_i^{(1)} \, \chi_i^{(\alpha)} \, \bar{\chi}_i^{(\beta)} = \frac{1}{|G|} \sum_{i=1}^{r} c_i \, \chi_i^{(\alpha)} \, \bar{\chi}_i^{(\beta)} = \delta_{\alpha\beta},$$

where we used Eq. (24.13) and the fact that all characters of the identity representation are unity. Thus

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Box 24.8.3 *To construct an invariant state, we need to combine a representation with* its *adjoint, in which case we obtain the identity representation only once.*

Historical Notes

Paul Albert Gordan (1837–1912), the son of David Gordan, a merchant, attended gymnasium and business school, then worked for several years in banks. His early interest in mathematics was encouraged by the private tutoring he received from a professor at the *Friedrich Wilhelm Gymnasium*. He attended Ernst Kummer's lectures in number theory at the University of Berlin in 1855, then studied at the universities of Breslau, Königsberg, and Berlin. At Königsberg he came under the influence of Karl Jacobi's school, and at Berlin his interest in algebraic equations was aroused. His dissertation (1862), which concerned geodesics on spheroids, received a prize offered by the philosophy faculty of the University of Breslau. The techniques that Gordan employed in it were those of Lagrange and Jacobi.

Paul Albert Gordan
1837–1912

Gordan's interest in function theory led him to visit G.F.B. Riemann in Göttingen in 1862, but Riemann was ailing, and their association was brief. The following year, Gordan was invited to Giessen by Clebsch, thus beginning the fruitful collaboration most physicists recognize. Together they produced work on the theory of Abelian functions, based on Riemann's fundamental paper on that topic, and several of Clebsch's papers are considered important steps toward establishing for Riemann's theories a firm foundation in terms of pure algebraic geometry. Of course, the Clebsch-Gordan collaboration also produced the famous coefficients that bear their names, so indispensable to the theory of angular momentum coupling found in almost every area of modern physics.

In 1874 Gordan became a professor at Erlangen, where he remained until his retirement in 1910. He married Sophie Deuer, the daughter of a Giessen professor of Roman law, in 1869. In 1868 Clebsch introduced Gordan to the theory of invariants, which originated in an observation of George Boole's in 1841 and was further developed by Arthur Cayley in 1846. Following the work of these two Englishmen, a German branch of the theory was developed by S.H. Aronhold and Clebsch, the latter elaborating the former's symbolic methods of characterizing algebraic forms and their invariants. Invariant theory was Gordan's main interest for the rest of his mathematical career; he became known as the greatest expert in the field, developing many techniques for representing and generating forms and their invariants.

Gordan made important contributions to algebra and solutions of algebraic equations, and gave simplified proofs of the transcendence of e and π. The overall style of Gordan's mathematical work was algorithmic. He shied away from presenting his ideas in informal literary forms. He derived his results computationally, working directly toward the desired goal without offering explanations of the concepts that motivated his work.

Gordan's only doctoral student, Emmy Noether, was one of the first women to receive a doctorate in Germany. She carried on his work in invariant theory for a while, but under the stimulus of Hilbert's school at Göttingen her interests shifted and she became one of the primary contributors to modern algebra.

So far, we have concentrated on the reduction of the operators and carrier spaces into irreducible components. Let us now direct our attention to the vectors themselves. Given two irreducible representations $T^{(\alpha)}$ and $T^{(\beta)}$ with carrier spaces spanned by vectors $\{|\phi_i^{(\alpha)}\rangle\}_{i=1}^{n_\alpha}$ and $\{|\psi_j^{(\beta)}\rangle\}_{j=1}^{n_\beta}$, we form the direct product representation $T^{(\alpha \times \beta)}$ with the carrier space spanned by vectors $\{|\phi_i^{(\alpha)}\psi_j^{(\beta)}\rangle\}$. We know that $T^{(\alpha \times \beta)}$ is reducible, and Eq. (24.45) tells us how many times each irreducible factor occurs in $T^{(\alpha \times \beta)}$. This means that the span of $\{|\phi_i^{(\alpha)}\psi_j^{(\beta)}\rangle\}$ can be decomposed into invariant irreducible subspaces; i.e., there must exist a basis of the carrier of the product space the vectors of which belong to irreducible representations of G. More specifically, we should be able to form the linear combinations

Clebsch-Gordan coefficients

$$\left|\Psi_k^{(\sigma),q}\right\rangle = \sum_{ij} C(\alpha\beta; \sigma, q|ij; k)\left|\phi_i^{(\alpha)}\psi_j^{(\beta)}\right\rangle, \tag{24.46}$$

which transform according to the rows of the σth irreducible representation. Here the subscript k refers to the row of the σth representation, and q distinguishes among functions that have the same σ and k, corresponding to the case where $m_\sigma^{\alpha\beta} \geq 2$. For simply reducible groups, the label q is unnecessary. The coefficients $C(\alpha\beta; \sigma, q|ij; k)$ are called the **Clebsch-Gordan coefficients** for G. These coefficients are normalized such that

$$\sum_{ij} C^*(\alpha\beta; \sigma, q|ij; k)C(\alpha\beta; \sigma', q'|ij; k') = \delta_{\sigma\sigma'}\delta_{qq'}\delta_{kk'},$$

$$\sum_{\sigma qk} C^*(\alpha\beta; \sigma, q|ij; k)C(\alpha\beta; \sigma, q|i'j'; k) = \delta_{ii'}\delta_{jj'}.$$

This will guarantee that $|\Psi_k^{(\sigma),q}\rangle$ are orthonormal if the product vectors form an orthonormal set. Using these relations, we can write the inverse of Eq. (24.46) as

$$\left|\phi_i^{(\alpha)}\psi_j^{(\beta)}\right\rangle = \sum_{\sigma qk} C^*(\alpha\beta; \sigma, q|ij; k)\left|\Psi_k^{(\sigma),q}\right\rangle. \tag{24.47}$$

24.8.2 Irreducible Tensor Operators

An operator **A** acting in the carrier space of the representation of a group G is transformed into another operator, $\mathbf{A} \mapsto \mathbf{T}_g\mathbf{A}\mathbf{T}_g^{-1}$, by the action of the group. Just as in the case of vector spaces, one can thus construct a set of operators that transform among themselves by such action and lump these operators in irreducible sets.

irreducible set of operators

Definition 24.8.4 An operator $\mathbf{A}_i^{(\alpha)}$ is said to **transform according to the ith row of the αth irreducible representation** if there exist $n_\alpha - 1$ other operators $\{\mathbf{A}_j^{(\alpha)}\}$ and a basis $\{|\psi_i^{(\alpha)}\rangle\}$ such that

$$\mathbf{T}_g\mathbf{A}_i^{(\alpha)}\mathbf{T}_g^{-1} = \sum_{j=1}^{n_\alpha} T_{ji}^{(\alpha)}(g)\mathbf{A}_j^{(\alpha)}, \qquad (24.48)$$

where $(T_{ji}^{(\alpha)}(g))$ is the matrix representation of g. The set of such operators is called an **irreducible set of operators** (or irreducible tensorial set).

In particular, if $T_{ij}^{(\alpha)}(g) = \delta_{ij}$, i.e., if the representation is the identity representation, then $\mathbf{A} = \mathbf{T}_g\mathbf{A}\mathbf{T}_g^{-1}$, and $\mathbf{A}$ is called a **scalar operator**. The term "scalar" refers to the fact that $\mathbf{A}$ has only one "component," in contrast to the other operators of Eq. (24.48), which may possess several components. scalar operator

Consider the set of vectors (functions) defined by $|\psi_{ij}^{(\alpha\beta)}\rangle \equiv \mathbf{A}_i^{(\alpha)}|\phi_j^{(\beta)}\rangle$, where $|\phi_j^{(\beta)}\rangle$ transform according to the βth irreducible representation. These vectors transform according to

$$\mathbf{T}_g\big|\psi_{ij}^{(\alpha\beta)}\big\rangle = \mathbf{T}_g\mathbf{A}_i^{(\alpha)}\mathbf{T}_g^{-1}\mathbf{T}_g\big|\phi_j^{(\beta)}\big\rangle = \sum_{k=1}^{n_\alpha} T_{ki}^{(\alpha)}(g)\mathbf{A}_k^{(\alpha)}\sum_{l=1}^{n_\beta} T_{lj}^{(\beta)}(g)\big|\phi_l^{(\beta)}\big\rangle$$

$$= \sum_{k,l} T_{ki}^{(\alpha)}(g)T_{lj}^{(\beta)}(g)\mathbf{A}_k^{(\alpha)}\big|\phi_l^{(\beta)}\big\rangle = \sum_{k,l} T_{kl,ij}^{(\alpha\times\beta)}(g)\big|\psi_{kl}^{(\alpha\beta)}\big\rangle, \qquad (24.49)$$

i.e., according to the representation $T^{(\alpha\times\beta)}$. This means that the vectors $|\psi_{ij}^{(\alpha\beta)}\rangle$ have the same transformation properties as the tensor product vectors $|\phi_i^{(\alpha)}\psi_j^{(\beta)}\rangle$. Therefore, using Eq. (24.47), we can write

$$\mathbf{A}_i^{(\alpha)}\big|\phi_j^{(\beta)}\big\rangle = \sum_{\sigma q k} C^*(\alpha\beta;\sigma,q|ij;k)\big|\Psi_k^{(\sigma),q}\big\rangle,$$

and more importantly,

$$\big\langle\phi_m^{(\gamma)}\big|\mathbf{A}_i^{(\alpha)}\big|\phi_j^{(\beta)}\big\rangle = \sum_{\sigma q k} C^*(\alpha\beta;\sigma,q|ij;k)\underbrace{\big\langle\phi_m^{(\gamma)}\big|\Psi_k^{(\sigma),q}\big\rangle}_{\text{use Eq. (24.36) here}}$$

$$= \sum_q C^*(\alpha\beta;\gamma,q|ij;m)\big\langle\phi_m^{(\gamma)}\big|\Psi_m^{(\gamma),q}\big\rangle. \qquad (24.50)$$

It follows that the matrix element of the operator $\mathbf{A}_i^{(\alpha)}$ will vanish unless the irreducible representation $T^{(\gamma)}$ occurs in the reduction of the tensor product $T^{(\alpha)} \otimes T^{(\beta)}$, and this can be decided from the character tables and the Clebsch-Gordan series, Eq. (24.45).

There is another remarkable property of Eq. (24.50) that has significant physical consequences. Notice how the dependence on i and j is contained entirely in the Clebsch-Gordan coefficients. Moreover, Eq. (24.37) implies that $\langle\phi_m^{(\gamma)}|\Psi_m^{(\gamma),q}\rangle$ is independent of m. Therefore, this dependence must *also* be contained entirely in Clebsch-Gordan coefficients. One therefore Wigner–Eckart theorem and reduced matrix elements

writes (24.50) as

$$\langle\phi_m^{(\gamma)}|\mathbf{A}_i^{(\alpha)}|\phi_j^{(\beta)}\rangle \equiv \sum_q C^*(\alpha\beta;\gamma,q|ij;m)\underbrace{\langle\phi^{(\gamma)}\|\mathbf{A}^{(\alpha)}\|\phi^{(\beta)}\rangle_q}_{\text{reduced matrix element}}. \quad (24.51)$$

This equation is known as the **Wigner-Eckart theorem**, and the numbers multiplying the Clebsch-Gordan coefficients are known as the **reduced matrix elements**.

From the point of view of physics, Eq. (24.51) can be very useful in calculating matrix elements (expectation values and transition between states), once we know the transformation properties of the physical operator. For example, for a **scalar operator S**, which, by definition, transforms according to the identity representation, (24.51) becomes

$$\langle\phi_m^{(\gamma)}|\mathbf{A}|\phi_j^{(\beta)}\rangle = \langle\phi^{(\gamma)}\|\mathbf{A}^{(\alpha)}\|\phi^{(\beta)}\rangle\delta_{\gamma\beta}\delta_{mj};$$

i.e., scalar operators have no matrix elements between different irreducible representations of a group, and within an irreducible representation, they are multiples of the identity matrix. This result is also a consequence of Schur's lemma.

24.9 Problems

24.1 Show that the action of a group G on the space of functions ψ given by $\mathbf{T}_g\psi(\mathbf{x}) = \psi(g^{-1}\cdot\mathbf{x})$ is a representation of G.

24.2 Complete Example 24.1.6.

24.3 Let the vector space carrying the representation of S_3 be the space of functions. Choose $\psi_1(x,y,z) \equiv xy$ and find the matrix representation of S_3 in the minimal invariant subspace containing ψ_1. Hint: See Example 24.2.2.

24.4 Let the vector space carrying the representation of S_3 be the space of functions. Choose (a) $\psi_1(x,y,z) \equiv x$ and (b) $\psi_1(x,y,z) \equiv x^2$, and in each case, find the matrix representation of S_3 in the minimal invariant subspace containing ψ_1.

24.5 Show that the representations T, $\overline{T}$, and T^* are either all reducible or all irreducible.

24.6 Use the hermitian conjugate of Eq. (24.5) to show that $\mathbf{S} \equiv \mathbf{A}^\dagger\mathbf{A}$ commutes with all $\mathbf{T}_g$'s. This result is used to prove Schur's lemmas in infinite dimensions.

24.7 Show that elements of a group belonging to the same conjugacy class have the same characters.

24.8 Show that the regular representation is indeed a representation, i.e., that $R: G \to GL(m,\mathbb{R})$ is a homomorphism.

Table 24.5 Character table for S_4

	$^1K_1$	$^6K_2$	$^3K_3$	$^8K_4$	$^6K_5$
$T^{(1)}$	1	1	1	1	1
$T^{(2)}$	1	-1	1	1	-1
$T^{(3)}$	2	0	2	-1	0
$T^{(4)}$	3	1	-1	0	-1
$T^{(5)}$	3	-1	-1	0	1

24.9 Prove **Maschke's Theorem**: The group algebra is semi-simple.

24.10 Let G be a finite group. Define the element $P = \sum_{x \in G} x$ of the group algebra and show that the left ideal generated by P is one-dimensional.

24.11 Show that $\mathbf{T}_i^{(\alpha)}$ defined in Eq. (24.23) commutes with all operators $\mathbf{T}_g^{(\alpha)}$. Hint: Consider $\mathbf{T}_g^{(\alpha)}\mathbf{T}_i^{(\alpha)}(\mathbf{T}_g^{(\alpha)})^{-1}$.

24.12 Let $K_{i'}$ denote the set of inverses of a conjugacy class K_i with c_i elements.

(a)   Show that $K_{i'}$ is also a class with c_i elements.
(b)   Show that identity occurs exactly c_i times in the product $\kappa_i \kappa_{i'}$, and none in the product $\kappa_i \kappa_j$ if $j \neq i'$ [see Eq. (24.21)].
(c)   Conclude that

$$c_{ij1} = \begin{cases} 0 & \text{if } j \neq i', \\ c_i & \text{if } j = i'. \end{cases}$$

24.13 Show that the coefficients $|G|d_j/|H|c_i$ of Eq. (24.33) are integers.

24.14 Show that the symmetric and the antisymmetric representations of S_n are inequivalent.

24.15 Construct the character table for S_4 from that of S_3 (given as Table 24.4), and verify that it is given by Table 24.5.

24.16 Show that all functions transforming according to a given row of an irreducible representation have the same norm.

24.17 Show that if the group of symmetry of $\mathbf{V}$ contains that of $\mathbf{H}_0$ and $|\phi_j^{(\alpha)}\rangle$ belongs to the jth column of the αth irreducible representation, then so does $\mathbf{V}|\phi_j^{(\alpha)}\rangle$. Conclude that $\langle \phi_i^{(\alpha)}|\mathbf{V}|\phi_j^{(\alpha)}\rangle = 0$ for $i \neq j$.

24.18 Find the irreducible components of the representation of Example 24.1.6.

24.19 Show that $\mathbf{P}^{(3)}|\psi_3\rangle$ of Example 24.7.4 is a linear combination of $\mathbf{P}^{(3)}|\psi_1\rangle$ and $\mathbf{P}^{(3)}|\psi_2\rangle$.

24.20 Show that the tensor product of two unitary representations is unitary.

24.21 Switch the dummy indices of the double sum in (24.43), add (subtract) the two sums, and use $(T \otimes T)_{ia,jb}(g) = (T \otimes T)_{ai,bj}(g)$ to show that the double sum can be written as a sum over the *symmetric (antisymmetric)* vectors alone.

24.22 Show that the characters $\chi^S(g)$ and $\chi^A(g)$ of the symmetrized and antisymmetrized product representations are given, respectively, by

$$\chi^S(g) = \frac{1}{2}\left[\left(\chi(g)\right)^2 + \chi\left(g^2\right)\right] \quad \text{and} \quad \chi^A(g) = \frac{1}{2}\left[\left(\chi(g)\right)^2 - \chi\left(g^2\right)\right].$$

24.23 Suppose that $\mathbf{A}_i^{(\alpha)}$ transforms according to $T^{(\alpha)}$, and $\mathbf{A}_j^{(\beta)}$ according to $T^{(\beta)}$. Show that $\mathbf{A}_i^{(\alpha)}\mathbf{A}_j^{(\beta)}$ transforms according to $T^{(\alpha\times\beta)}$.

24.24 Show that

$$\frac{1}{m_\gamma^{\alpha\beta}}\sum_{ij}|\langle\phi_m^{(\gamma)}|\mathbf{A}_i^{(\alpha)}|\phi_j^{(\beta)}\rangle|^2 = |\langle\phi^{(\gamma)}\|\mathbf{A}^{(\alpha)}\|\phi^{(\beta)}\rangle_q|^2.$$

One can interpret this as the statement that the square of the reduced matrix element is proportional to the average (over i and j) of the square of the full matrix elements.