Representation of Lie Groups and Lie Algebras 30

The representation of Lie groups is closely related to the representation of their Lie *algebras*, and we shall discuss them later in this chapter. In the case of *compact* groups, however, there is a well developed representation theory, which we shall consider in the first section. Before discussing compact groups, let us state a definition and a proposition that hold for *all* Lie groups.

Definition 30.0.1 A **representation** of a Lie group G on a Hilbert space $\mathcal{H}$ is a Lie group homomorphism $T : G \to GL(\mathcal{H})$. Similarly, a representation of the Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{T} : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H})$.

The proposition we have in mind is the important **Schur's lemma** which we state without proof (for a proof see [Baru 86, pp. 143–144]).

Proposition 30.0.2 (**Schur's lemma**) *A unitary representation $T : G \to GL(\mathcal{H})$ of a Lie group G is irreducible if and only if the only operators commuting with all the $\mathbf{T}_g$ are scalar multiples of the unit operator.* Schur's lemma

30.1 Representation of Compact Lie Groups

In this section, we shall consider the representation of compact Lie *groups*, because for such groups, many of the ideas developed for finite groups hold.

Example 30.1.1 (Compactness of $U(n)$, $O(n)$, $SU(n)$, and $SO(n)$) Identify $GL(n, \mathbb{C})$ with $\mathbb{R}^{2n^2}$ via components. The map

$$f : GL(n, \mathbb{C}) \to GL(n, \mathbb{C}) \quad \text{given by } f(\mathsf{A}) = \mathsf{A}\mathsf{A}^\dagger$$

is continuous because it is simply the products of elements of matrices. It follows that $f^{-1}(\mathbf{1})$ is closed, because the matrix $\mathbf{1}$ is a single point in $\mathbb{R}^{2n^2}$, which is therefore closed. $f^{-1}(\mathbf{1})$ is also bounded, because

$$\mathsf{A}\mathsf{A}^\dagger = \mathbf{1} \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij} a_{kj}^* = \delta_{ik} \quad \Rightarrow \quad \sum_{i,j=1}^{n} |a_{ij}|^2 = n.$$

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Thus, $f^{-1}(1)$ is a $(2n^2-1)$-dimensional sphere of radius $\sqrt{n}$ in $\mathbb{R}^{2n^2}$, which is clearly bounded. The BWHB theorem (of Chap. 17) now implies that $f^{-1}(1)$ is compact. Now note that $f^{-1}(1)$ consists of all matrices that have their hermitian adjoints for an inverse; but these are precisely the set $U(n)$ of unitary matrices.

Now consider the map $\det : U(n) \to \mathbb{C}$. This map is also continuous, implying that $\det^{-1}(1)$ is a closed subset of $U(n)$. The boundedness of $U(n)$ implies that $\det^{-1}(1)$ is also bounded. Invoking the BWHB theorem again, we conclude that $\det^{-1}(1) = SU(n)$, being closed and bounded, is compact.

If instead of complex numbers, we restrict ourselves to the reals, $O(n)$ and $SO(n)$ will replace $U(n)$ and $SU(n)$, respectively.

The result of the example above can be summarized:

Box 30.1.2 *The unitary $U(n)$, orthogonal $O(n)$, special unitary $SU(n)$, and special orthogonal $SO(n)$ groups are all compact.*

We now start our study of the representations of compact Lie groups. We first show that we can always assume that the representation is unitary.

All representations of compact groups can be made unitary.

Theorem 30.1.3 *Let $T : G \to GL(\mathcal{H})$ be any representation of the compact group G. There exists a new inner product in $\mathcal{H}$ relative to which T is unitary.*

Proof Let $\langle|\rangle$ be the initial inner product. Define a new inner product $(|)$ by

$$(u|v) \equiv \int_G \langle \mathbf{T}_g u|\mathbf{T}_g v\rangle d\mu_g$$

where $d\mu_g$ is the Haar measure, which is both left- and right-invariant. The reader may check that this is indeed an inner product. For every $h \in G$, we have

$$\begin{aligned}
(\mathbf{T}_h u|\mathbf{T}_h v) &= \int_G \langle \mathbf{T}_g \mathbf{T}_h u|\mathbf{T}_g \mathbf{T}_h v\rangle d\boldsymbol{\mu}_g \\
&= \int_G \langle \mathbf{T}_{gh} u|\mathbf{T}_{gh} v\rangle d\boldsymbol{\mu}_g \quad \text{(because } T \text{ is a representation)} \\
&= \int_G \langle \mathbf{T}_{gh} u|\mathbf{T}_{gh} v\rangle d\boldsymbol{\mu}_{gh} \quad \text{(because } \boldsymbol{\mu}_g \text{ is right invariant)} \\
&= (u|v).
\end{aligned}$$

This shows that $\mathbf{T}_h$ is unitary for all $h \in G$. □

From now on, we shall restrict our discussion to unitary representations of compact groups.

The study of representations of compact groups is facilitated by the following construction:

Definition 30.1.4 Let $T : G \to GL(\mathcal{H})$ be a unitary representation of the compact group G and $|u\rangle \in \mathcal{H}$ a fixed vector. The **Weyl operator** $\mathbf{K}_u$ associated with $|u\rangle$ is defined as

Weyl operator for a compact Lie group

$$\mathbf{K}_u = \int_G |\mathbf{T}_g u\rangle\langle \mathbf{T}_g u|\, d\boldsymbol{\mu}_g. \tag{30.1}$$

The essential properties of the Weyl operator are summarized in the following:

Proposition 30.1.5 *Let $T : G \to GL(\mathcal{H})$ be a unitary representation of the compact group G. Then the Weyl operator has the following properties*

1. $\mathbf{K}_u$ *is hermitian.*
2. $\mathbf{K}_u\mathbf{T}_g = \mathbf{T}_g\mathbf{K}_u$ *for all $g \in G$. Therefore, any eigenspace of $\mathbf{K}_u$ is an invariant subspace of all $\mathbf{T}_g$'s.*
3. $\mathbf{K}_u$ *is a Hilbert-Schmidt operator.*

Proof Statement (1), in the form $\langle w|\mathbf{K}_u|v\rangle^* = \langle v|\mathbf{K}_u|w\rangle$, follows directly from the definition.

(2) From $\mathbf{T}_g \int_G |\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|\, d\boldsymbol{\mu}_x = \int_G |\mathbf{T}_g\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|\, d\boldsymbol{\mu}_x$, the fact that T is a representation (therefore, $\mathbf{T}_g\mathbf{T}_x = \mathbf{T}_{gx}$), and redefining the integration variable to $y = gx$, we get

$$\mathbf{T}_g\mathbf{K}_u = \int_G |\mathbf{T}_y u\rangle\langle \mathbf{T}_{g^{-1}y} u|\, \underbrace{d\boldsymbol{\mu}_{g^{-1}y}}_{=d\boldsymbol{\mu}_y} = \int_G |\mathbf{T}_y u\rangle\langle \mathbf{T}_{g^{-1}}\mathbf{T}_y u|\, d\boldsymbol{\mu}_y,$$

where we used the left invariance of $\boldsymbol{\mu}$ and the fact that T is a representation. Unitarity of T now gives

$$\mathbf{T}_g\mathbf{K}_u = \int_G |\mathbf{T}_y u\rangle\langle \mathbf{T}_g^\dagger\mathbf{T}_y u|\, d\boldsymbol{\mu}_y = \int_G |\mathbf{T}_y u\rangle\langle \mathbf{T}_y u|\mathbf{T}_g\, d\boldsymbol{\mu}_y = \mathbf{K}_u\mathbf{T}_g.$$

(3) Recall that an operator $\mathbf{A} \in \mathcal{L}(\mathcal{H})$ is Hilbert-Schmidt if $\sum_{i=1}^\infty \|\mathbf{A}|e_i\rangle\|^2$ is finite for any orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{H}$. In the present case, we have

$$\mathbf{K}_u|e_i\rangle = \int_G |\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|e_i\rangle\, d\boldsymbol{\mu}_x.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^\infty \big\|\mathbf{K}_u|e_i\rangle\big\|^2 &= \sum_{i=1}^\infty \biggl(\int_G \langle e_i|\mathbf{T}_y u\rangle\langle \mathbf{T}_y u|\, d\boldsymbol{\mu}_y\biggr)\biggl(\int_G |\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|e_i\rangle\, d\boldsymbol{\mu}_x\biggr) \\ &= \sum_{i=1}^\infty \int_G\int_G \langle e_i|\mathbf{T}_y u\rangle\langle \mathbf{T}_y u|\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|e_i\rangle\, d\boldsymbol{\mu}_x\, d\boldsymbol{\mu}_y.\end{aligned}$$

If we switch the order of summation and integration and use

$$\sum_{i=1}^\infty \langle \mathbf{T}_x u|e_i\rangle\langle e_i|\mathbf{T}_y u\rangle = \langle \mathbf{T}_x u|\mathbf{T}_y u\rangle,$$

we obtain

$$\sum_{i=1}^{\infty} \big\| \mathbf{K}_u |e_i\rangle \big\|^2 = \int_G \int_G \big| \langle \mathbf{T}_y u | \mathbf{T}_x u \rangle \big|^2 d\boldsymbol{\mu}_x \, d\boldsymbol{\mu}_y,$$

and using the Schwarz inequality in the integral yields

$$\begin{aligned} \sum_{i=1}^{\infty} \big\| \mathbf{K}_u |e_i\rangle \big\|^2 &\le \int_G \int_G \langle \mathbf{T}_x u | \mathbf{T}_x u \rangle \langle \mathbf{T}_y u | \mathbf{T}_y u \rangle d\boldsymbol{\mu}_x \, d\boldsymbol{\mu}_y \\ &= \int_G \int_G \langle u|u\rangle \langle u|u\rangle d\boldsymbol{\mu}_x \, d\boldsymbol{\mu}_y \quad \text{(because rep. is unitary)} \\ &= \|u\|^4 \int_G d\boldsymbol{\mu}_x \int_G d\boldsymbol{\mu}_y = \|u\|^4 V_G^2 < \infty, \end{aligned}$$

where V_G is the *finite* volume of G. □

Historical Notes

Hermann Klaus Hugo Weyl 1885–1955

Hermann Klaus Hugo Weyl (1885–1955) attended the gymnasium at Altona and, on the recommendation of the headmaster of his gymnasium, who was a cousin of Hilbert, decided at the age of eighteen to enter the University of Göttingen. Except for one year at Munich he remained at Göttingen, as a student and later as Privatdozent, until 1913, when he became professor at the University of Zurich. After Klein's retirement in 1913, Weyl declined an offer to be his successor at Göttingen but accepted a second offer in 1930, after Hilbert had retired. In 1933 he decided he could no longer remain in Nazi Germany and accepted a position at the Institute for Advanced Study at Princeton, where he worked until his retirement in 1951. In the last years of his life he divided his time between Zurich and Princeton.

Weyl undoubtedly was the most gifted of Hilbert's students. Hilbert's thought dominated the first part of his mathematical career; and although later he sharply diverged from his master, particularly on questions related to foundations of mathematics, Weyl always shared his convictions that the value of abstract theories lies in their success in solving classical problems and that the proper way to approach a question is through a deep analysis of the concepts it involves rather than by blind computations.

Weyl arrived at Göttingen during the period when Hilbert was creating the spectral theory of self-adjoint operators, and spectral theory and harmonic analysis were central in Weyl's mathematical research throughout his life. Very soon, however, he considerably broadened the range of his interests, including areas of mathematics into which Hilbert had never penetrated, such as the theory of Lie groups and the analytic theory of numbers, thereby becoming one of the most universal mathematicians of his generation. He also had an important role in the development of mathematical physics, the field to which his most famous books, *Raum, Zeit und Materie* (1918), on the theory of relativity, and *Gruppentheorie und Quantenmechanik* (1928), are devoted.

Weyl's versatility is illustrated in a particularly striking way by the fact that immediately after some original advances in number theory (which he obtained in 1914), he spent more than ten years as a geometer—a geometer in the most modern sense of the word, uniting in his methods topology, algebra, analysis, and geometry in a display of dazzling virtuosity and uncommon depth reminiscent of Riemann. Drawn by war mobilization into the German army, Weyl did not resume his interrupted work when he was allowed to return to civilian life in 1916. At Zurich he had worked with Einstein for one year, and he became keenly interested in the general theory of relativity, which had just been published; with his characteristic enthusiasm he devoted most of the next five years to exploring the mathematical framework of the theory. In these investigations Weyl introduced the concept of what is now called a *linear connection*, linked not to the Lorentz group of orthogonal transformations, but to the enlarged group of conformal transformations; he even thought for a time that this would give him a unified theory of gravitation and electromagnetism, the forerunner of what is now called *gauge theories*.

Weyl's use of tensor calculus in his work on relativity led him to reexamine the basic methods of that calculus and, more generally, of classical invariant theory that had been its forerunner but had fallen into near oblivion after Hilbert's work of 1890. On the other hand, his semiphilosophical, semimathematical ideas on the general concept of "space" in connection with Einstein's theory had directed his investigations to generalizations of Helmholtz's problem of characterizing Euclidean geometry by properties of "free mobility." From these two directions Weyl was brought into contact with the theory of linear representations of Lie groups; his papers on the subject (1925–1927) certainly represent his masterpiece and must be counted among the most influential works in twentieth-century mathematics.
Based on the early 1900s works of Frobenius, I. Schur, and A. Young, Weyl inaugurated a new approach for the representation of continuous groups by focusing his attention on Lie groups, rather than Lie algebras.
Very few of Weyl's 150 published books and papers—even those chiefly of an expository character—lack an original idea or a fresh viewpoint. The influence of his works and of his teaching was considerable: He proved by his example that an "abstract" approach to mathematics is perfectly compatible with "hard" analysis and, in fact, can be one of the most powerful tools when properly applied.
Weyl was one of that rare breed of modern mathematician whose contribution to physics was also substantial. In an interview with a reporter in 1929, Dirac is asked the following question: "... I want to ask you something more: They tell me that you and Einstein are the only two real sure-enough high-brows and the only ones who can really understand each other. I won't ask you if this is straight stuff, for I know you are too modest to admit it. But I want to know this—Do you ever run across a fellow that even you can't understand?" To this Dirac replies one word: "Weyl."
Weyl had a lifelong interest in philosophy and metaphysics, and his mathematical activity was seldom free from philosophical undertones or afterthoughts. At the height of the controversy over the foundations of mathematics, between the formalist school of Hilbert and the intuitionist school of Brouwer, he actively fought on Brouwer's side. His own comment, stated somewhat jokingly, sums up his personality: "My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful."

We now come to the most fundamental theorem of representation theory of compact Lie groups. Before stating and proving this theorem, we need the following lemma:

Lemma 30.1.6 *Let $T : G \to GL(\mathcal{H})$ be an irreducible unitary representation of a compact Lie group G. For any nonzero $|u\rangle, |v\rangle \in \mathcal{H}$, we have*

$$\frac{1}{\|u\|^2\|v\|^2}\int_G \left|\langle v|\mathbf{T}_x|u\rangle\right|^2 d\boldsymbol{\mu}_x = c, \tag{30.2}$$

where $c > 0$ is a constant independent of $|u\rangle$ and $|v\rangle$.

Proof By Schur's lemma and (2) of Proposition 30.1.5, $\mathbf{K}_u = \lambda(u)\mathbf{1}$. Therefore, on the one hand,

$$\langle v|\mathbf{K}_u|v\rangle = \lambda(u)\|v\|^2. \tag{30.3}$$

On the other hand,

$$\langle v|\mathbf{K}_u|v\rangle = \int_G \langle v|\mathbf{T}_x u\rangle\langle \mathbf{T}_x u|v\rangle\, d\boldsymbol{\mu}_x = \int_G \left|\langle v|\mathbf{T}_x|u\rangle\right|^2 d\boldsymbol{\mu}_x. \tag{30.4}$$

Moreover, if we use $d\boldsymbol{\mu}_g = d\boldsymbol{\mu}_{g^{-1}}$ (see Problem 29.12), then

$$\begin{aligned}\langle v|\mathbf{K}_u|v\rangle &= \int_G \langle v|\mathbf{T}_x|u\rangle\langle u|\mathbf{T}_x^\dagger|v\rangle\, d\boldsymbol{\mu}_x = \int_G \langle v|\mathbf{T}_x|u\rangle\langle u|\mathbf{T}_x^{-1}|v\rangle\, d\boldsymbol{\mu}_x \\ &= \int_G \langle u|\mathbf{T}_{x^{-1}}|v\rangle\langle v|\mathbf{T}_x|u\rangle\, d\boldsymbol{\mu}_x = \int_G \langle u|\mathbf{T}_y|v\rangle\langle v|\mathbf{T}_{y^{-1}}|u\rangle\, d\boldsymbol{\mu}_{y^{-1}} \\ &= \int_G \langle u|\mathbf{T}_y|v\rangle \underbrace{\langle v|\mathbf{T}^\dagger|u\rangle}_{=\langle\mathbf{T}_y v|u\rangle} \underbrace{d\boldsymbol{\mu}_{y^{-1}}}_{d\boldsymbol{\mu}_y} = \langle u|\mathbf{K}_v|u\rangle.\end{aligned}$$

This equality plus Eq. (30.3) gives

$$\lambda(u)\|v\|^2 = \lambda(v)\|u\|^2 \quad \Rightarrow \quad \frac{\lambda(v)}{\|v\|^2} = \frac{\lambda(u)}{\|u\|^2}.$$

Since $|u\rangle$ and $|v\rangle$ are arbitrary, we conclude that $\lambda(u) = c\|u\|^2$ for all $|u\rangle \in \mathcal{H}$, where c is a constant. Equations (30.3) and (30.4) now yield Eq. (30.2). If we let $|u\rangle = |v\rangle$ in Eq. (30.4) and use (30.3), we obtain

$$\int_G \bigl|\langle u|\mathbf{T}_x|u\rangle\bigr|^2\, d\boldsymbol{\mu}_x = \lambda(u)\|u\|^2 = c\|u\|^4.$$

That $c > 0$ follows from the fact that the LHS is a nonnegative continuous function that has at least one strictly positive value in its integration range, namely at $x = e$, the identity. □

Theorem 30.1.7 *Every irreducible unitary representation of a compact Lie group is finite-dimensional.*

Proof Let $\{|e_i\rangle\}_{i=1}^n$ be any set of orthonormal vectors in $\mathcal{H}$. Then, unitarity of $\mathbf{T}_g$ implies that $\{\mathbf{T}_g|e_i\rangle\}_{i=1}^n$ is also an orthonormal set. Applying Lemma 30.1.6 to $|e_j\rangle$ and $|e_1\rangle$, we obtain

$$\int_G \bigl|\langle e_1|\mathbf{T}_x|e_j\rangle\bigr|^2\, d\boldsymbol{\mu}_x = c.$$

Now sum over j to get

$$\begin{aligned}nc &= \sum_{j=1}^n \int_G \bigl|\langle e_1|\mathbf{T}_x|e_j\rangle\bigr|^2\, d\boldsymbol{\mu}_x = \int_G \sum_{j=1}^n \bigl|\langle e_1|\mathbf{T}_x e_j\rangle\bigr|^2\, d\boldsymbol{\mu}_x \\ &\le \int_G \langle e_1|e_1\rangle\, d\boldsymbol{\mu}_x = V_G,\end{aligned}$$

where we used the Parseval inequality [Eq. (7.3)] as applied to the vector $|e_1\rangle$ and the orthonormal set $\{\mathbf{T}_g|e_i\rangle\}_{i=1}^n$. Since both V_G and c are finite, n must be finite as well. Thus, $\mathcal{H}$ cannot have an infinite set of orthonormal vectors. □

So far, we have discussed irreducible representations. What can we say about arbitrary representations? We recall that in the case of finite groups, every representation can be written as a direct sum of irreducible representations. Is this also true for compact Lie groups?

Firstly, we note that the Weyl operator, being Hilbert-Schmidt, is necessarily compact. It is also hermitian. Therefore, by the spectral theorem, its eigenspaces span the carrier space $\mathcal{H}$. Specifically, we can write $\mathcal{H} = \mathcal{M}_0 \oplus \sum_{j=1}^{N} \oplus \mathcal{M}_j$, where $\mathcal{M}_0$ is the eigenspace corresponding to the zero eigenvalue of $\mathbf{K}_u$, and N could be infinity.

Secondly, from the relation $\langle v|\mathbf{K}_u|v\rangle = c\|u\|^2\|v\|^2$ and the fact that $c \neq 0$ and $|u\rangle \neq 0$, we conclude that $\mathbf{K}_u$ cannot have any nonzero eigenvector for its zero eigenvalue. It follows that $\mathcal{M}_0$ contains only the zero vector. Therefore, if $\mathcal{H}$ is infinite-dimensional, then $N = \infty$.

Thirdly, consider any representation T of G. Because $\mathbf{K}_u$ commutes with all $\mathbf{T}_g$, each eigenspace of $\mathbf{K}_u$ is an invariant subspace under T. If a subspace $\mathcal{U}$ is invariant under T, then $\mathcal{U} \cap \mathcal{M}_j$, a subspace of $\mathcal{M}_j$, is also invariant (reader, please verify!). Thus, all invariant subspaces of G are reducible to invariant subspaces of eigenspaces of $\mathbf{K}_u$. In particular, all irreducible invariant subspaces of T are subspaces of eigenspaces of $\mathbf{K}_u$.

Lastly, since all $\mathcal{M}_j$ are finite-dimensional, we can use the procedure used in the case of finite groups and decompose $\mathcal{M}_j$ into irreducible invariant subspaces of T. We have just shown the following result:

Theorem 30.1.8 *Every unitary representation T of a compact Lie group G is a direct sum of irreducible finite-dimensional unitary representations.*

By choosing a basis for the finite-dimensional invariant subspaces of T, we can represent each $\mathbf{T}_g$ by a matrix. Therefore,

Box 30.1.9 *Compact Lie groups can be represented by matrices.*

As in the case of finite groups, one can work with matrix elements and characters of representations. The only difference is that summations are replaced with integration and order of the group $|G|$ is replaced with V_G, which we take to be unity.[1] For example, Eq. (24.6) becomes

$$\int_G T^{(\alpha)}(g)\mathbf{X}T^{(\beta)}\left(g^{-1}\right) d\boldsymbol{\mu}_g = \lambda_X \delta_{\alpha\beta}\mathbf{1}, \tag{30.5}$$

and the analogue of Eq. (24.8) is

$$\int_G T_{il}^{(\alpha)}(g)T_{jm}^{(\beta)*}(g)\, d\boldsymbol{\mu}_g = \frac{1}{n_\alpha}\delta_{ml}\delta_{\alpha\beta}\delta_{ij}. \tag{30.6}$$

[1] This can always be done by rescaling the volume element.

Characters satisfy similar relations: Eq. (24.11) becomes

$$\int_G \chi^{(\alpha)}(g)\chi^{(\beta)*}(g)\,d\boldsymbol{\mu}_g = \delta_{\alpha\beta}, \tag{30.7}$$

and the useful Eq. (24.16) turns into

$$\int_G \left|\chi(g)\right|^2 d\boldsymbol{\mu}_g = \sum_\alpha m_\alpha^2. \tag{30.8}$$

This formula can be used to test for irreducibility of a representation: If the integral is unity, the representation is irreducible; otherwise, it is reducible.

Peter-Weyl theorem

Finally, we state the celebrated **Peter-Weyl** theorem (for a proof, see [Baru 86, pp. 172–173])

Theorem 30.1.10 (Peter-Weyl theorem) *The functions*

$$\sqrt{n_\alpha}T_{ij}^{(\alpha)}(g), \quad \forall\alpha \quad \text{and} \quad 1 \le i, j \le n_\alpha,$$

form a complete set of functions in $\mathcal{L}^2(G)$, the Hilbert space of square-integrable functions on G.

If $u \in \mathcal{L}^2(G)$, we can write

$$u(g) = \sum_\alpha \sum_{i,j}^{n_\alpha} b_{ij}^\alpha T_{ij}^{(\alpha)}(g) \quad \text{where} \quad b_{ij}^\alpha = n_\alpha \int_G u(g)T_{ij}^{(\alpha)*}(g)\,d\boldsymbol{\mu}_g. \tag{30.9}$$

The Peter-Weyl theorem is the generalization of the Fourier series expansion of periodic functions.

Example 30.1.11 Equation (30.9) is the generalization of the Fourier series expansion of functions. The connection with Fourier series becomes more transparent if we consider a particular compact group. The unit circle S^1 is a one-dimensional abelian compact 1-parameter Lie group. In fact, fixing an "origin" on the circle, any other point can be described by the parameter θ, the angular distance from the point to the origin. S^1 is obviously abelian; it is also compact, because it is a bounded closed region of $\mathbb{R}^2$ (BWHB theorem). By Theorem 24.3.3, which holds for all Lie groups, all irreducible representations of S^1 are 1-dimensional. So $T_{ij}^{(\alpha)}(g) \to T^{(\alpha)}(\theta)$. Furthermore, $T^{(\alpha)}(\theta)T^{(\alpha)}(\theta') = T^{(\alpha)}(\theta+\theta')$. Differentiating both sides with respect to θ' at $\theta'=0$ yields the differential equation

$$T^{(\alpha)}(\theta)\underbrace{\left.\frac{dT^{(\alpha)}}{d\theta'}\right|_{\theta'=0}}_{\equiv a} = \left.\frac{dT^{(\alpha)}}{dy}\right|_{y=\theta} \equiv \frac{dT^{(\alpha)}}{d\theta}, \quad y \equiv \theta + \theta'.$$

The solution to this DE is $Ae^{a\theta}$. Since $T^{(\alpha)}$ are unitary, and since a 1-dimensional unitary matrix must look like $e^{i\varphi}$, we must have $A=1$. Furthermore, θ and $\theta+2\pi$ are identified on the unit circle; therefore, we must conclude that a is i times an integer n, which determines the irreducible representation. We label the irreducible representation by n and write

$$T^{(n)}(\theta) = e^{in\theta}, \quad n = 0 \pm 1 \pm 2 \dots .$$

The Peter-Weyl theorem now becomes the rule of Fourier series expansion of *periodic functions*. This last property follows from the fact that any function $u : S^1 \to \mathbb{R}$ is necessarily periodic.

There are many occasions in physics where the state functions describing physical quantities transform irreducibly under the action of a Lie group (which we assume to be compact). Often this Lie group also acts on the underlying space-time manifold. So we have a situation in which a Lie group G acts on a Euclidean space $\mathbb{R}^n$ as well as on the space of (square-integrable) functions $\mathcal{L}(\mathbb{R}^n)$. Therefore, the functions $\{\phi_i^{(\alpha)}(\mathbf{x})\}$, belonging to the αth irreducible representation transform among themselves not only because of the index i, but also because of the argument $\mathbf{x} \in \mathbb{R}^n$.

To see the connection between physics and representation theory, consider the transformation of the simplest case, a scalar function. As a concrete example, choose temperature. To observer O at the corner of a room 8 meters long, 6 meters wide, and 3 meters high, the temperature of the center of the room is given by $\theta(4, 3, 1.5)$ where $\theta(x, y, z)$ is a function that gives O the temperature of various points of the room. Observer O' is sitting in the middle of the floor, so that the center of the room has coordinates $(0, 0, 1.5)$. O' also has a function that gives her the temperature at various points. But this function must necessarily be different from θ because of the different coordinates the same points have for O and O'. Calling this function θ', we have $\theta'(0, 0, 1.5) = \theta(4, 3, 1.5)$, and in general,

$$\theta'\bigl(x', y', z'\bigr) = \theta(x, y, z),$$

where (x', y', z') describes the same point for O' that (x, y, z) describes for O.

In the context of representation theory, we can think of (x', y', z') as the transformed coordinates obtained as a result of the action of some group: $(x', y', z') = g \cdot (x, y, z)$, or $\mathbf{x}' = g \cdot \mathbf{x}$. So, the equation above can be written as

$$\theta'\bigl(\mathbf{x}'\bigr) = \theta(\mathbf{x}) = \theta\bigl(g^{-1} \cdot \mathbf{x}'\bigr) \quad \text{or} \quad \theta'(\mathbf{x}) = \theta\bigl(g^{-1} \cdot \mathbf{x}\bigr).$$

It is natural to call θ' the transform of θ under the action of g and write $\theta' = \mathbf{T}_g\theta$. This is one way of constructing a representation [see the comments after Eq. (24.1)]. Instead of g^{-1} on the left, one could act with g on the right.

When the physical quantity is not a scalar, it is natural to group together the smallest set of functions that transform into one another. This leads to the set of functions that transform according to a row of an irreducible representation of the group. In some sense, this situation is a combination of

(24.1) and (24.35). The reader may verify that

$$\mathbf{T}_g \phi_i^{(\alpha)}(\mathbf{x}) = \sum_{j=1}^{n_\alpha} T_{ji}^{(\alpha)}(g) \phi_j^{(\alpha)}\big(\mathbf{x} \cdot g^{-1}\big) \tag{30.10}$$

defines a representation of G.

We now use Box 29.1.31 to construct an irreducible representation of the Lie algebra of G from Eq. (30.10). By the definition of the infinitesimal action, we let $g = \exp(\boldsymbol{\xi} t)$ and differentiate both sides with respect to t at $t = 0$. This yields

$$\begin{aligned}
\underbrace{\frac{d}{dt} \mathbf{T}_{\exp(\boldsymbol{\xi} t)} \phi_i^{(\alpha)}(\mathbf{x}) \bigg|_{t=0}}_{\equiv \sum_j \mathfrak{D}_{ji}(\boldsymbol{\xi}) \phi_j^{(\alpha)}(\mathbf{x})} &= \sum_{j=1}^{n_\alpha} \frac{d}{dt} \Big\{ T_{ji}^{(\alpha)}\big(\exp(\boldsymbol{\xi} t)\big) \phi_j^{(\alpha)}\big(\mathbf{x} \cdot \exp(-\boldsymbol{\xi} t)\big) \Big\} \bigg|_{t=0} \\
&= \sum_{j=1}^{n_\alpha} \frac{d}{dt} T_{ji}^{(\alpha)}\big(\exp(\boldsymbol{\xi} t)\big) \bigg|_{t=0} \phi_j^{(\alpha)}\big(\underbrace{\mathbf{x} \cdot \exp(-\boldsymbol{\xi} 0)}_{=\mathbf{x}}\big) \\
&\quad + \sum_{j=1}^{n_\alpha} \underbrace{T_{ji}^{(\alpha)}\big(\exp(\boldsymbol{\xi} 0)\big)}_{=\delta_{ji}} \frac{d}{dt} \phi_j^{(\alpha)}\big(\mathbf{x} \cdot \exp(-\boldsymbol{\xi} t)\big) \bigg|_{t=0},
\end{aligned}$$

where we have defined the matrices $\mathfrak{D}_{ji}(\boldsymbol{\xi})$ for the LHS. The derivative in the first sum is simply $\mathfrak{T}_{ji}^{(\alpha)}(\boldsymbol{\xi})$ the representation of the generator $\boldsymbol{\xi}$ of the 1-parameter group of transformations in the space of functions $\{\phi_i^{(\alpha)}\}$. The derivative in the second sum can be found by writing $\mathbf{x}'(t) = \mathbf{x} \cdot \exp(-\boldsymbol{\xi} t)$ and differentiating as follows:

$$\begin{aligned}
\frac{d}{dt} \phi_j^{(\alpha)}\big(\mathbf{x}'(t)\big) \bigg|_{t=0} &\equiv \frac{d}{dt} \phi_j^{(\alpha)}\big(x'^1(t), \dots, x'^n(t)\big) \bigg|_{t=0} = \partial_k \phi_j^{(\alpha)} \frac{d}{dt} \big(x'^k(t)\big) \bigg|_{t=0} \\
&= \partial_k \phi_j^{(\alpha)} u_\mu^k(\mathbf{x}) \frac{da^\mu}{dt} \equiv \partial_k \phi_j^{(\alpha)} X^k(\mathbf{x}; \boldsymbol{\xi}) \equiv \partial_\nu \phi_j^{(\alpha)} X^\nu(\mathbf{x}; \boldsymbol{\xi}),
\end{aligned}$$

where we used Eq. (29.23) and defined $X^k(\mathbf{x}; \boldsymbol{\xi})$ by the last equality. We also changed the coordinate index to Greek to avoid confusing it with the index of the functions. Collecting everything together, we obtain

$$\sum_{j=1}^{n_\alpha} \mathfrak{D}_{ij}(\boldsymbol{\xi}) \phi_j^{(\alpha)}(\mathbf{x}) = \sum_{j=1}^{n_\alpha} \mathfrak{T}_{ij}^{(\alpha)}(\boldsymbol{\xi}) \phi_j^{(\alpha)}(\mathbf{x}) + \sum_{j=1}^{n_\alpha} \delta_{ij} X^\nu(\mathbf{x}; \boldsymbol{\xi}) \frac{\partial \phi_j^{(\alpha)}}{\partial x^\nu},$$

or, since $\phi_j^{(\alpha)} = \sum_k \phi_k^{(\alpha)} (\partial / \partial \phi_k^{(\alpha)}) \phi_j^{(\alpha)}$,

$$\mathfrak{D}_{ij}(\boldsymbol{\xi}) = \mathfrak{T}_{ij}^{(\alpha)}(\boldsymbol{\xi}) \phi_k^{(\alpha)}(\mathbf{x}) \frac{\partial}{\partial \phi_k^{(\alpha)}} + \delta_{ij} X^\nu(\mathbf{x}; \boldsymbol{\xi}) \frac{\partial}{\partial x^\nu}, \tag{30.11}$$

where $X^\nu(\mathbf{x}; \boldsymbol{\xi})$ is the νth component of the infinitesimal generator of the action induced by $\boldsymbol{\xi} \in \mathfrak{g}$. We shall put Eq. (30.11) to good use when we discuss symmetries and conservation laws in Chap. 33. The derivative with respect to the functions, although meaningless at this point, will be necessary when we discuss conservation laws.

30.2 Representation of the General Linear Group

$GL(\mathcal{V})$ is not a compact group, but we can use the experience we gained in the analysis of the symmetric group to find the irreducible representations of $GL(\mathcal{V})$. The key is to construct tensor product spaces of $\mathcal{V}$—which, as the reader may verify, is a carrier space of $GL(\mathcal{V})$—and look for its irreducible subspaces. In fact, if r is an arbitrary positive integer, $T : G \to GL(\mathcal{V})$ is a representation, and

$$\mathcal{V}^{\otimes r} \equiv \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{r \text{ times}},$$

then $T^{\otimes r} : G \to GL(\mathcal{V}^{\otimes r})$, given by

$$\big[T^{\otimes r}(g)\big](\mathbf{v}_1, \ldots, \mathbf{v}_r) \equiv \mathbf{T}_g^{\otimes r}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \mathbf{T}_g(\mathbf{v}_1) \otimes \cdots \otimes \mathbf{T}_g(\mathbf{v}_r),$$

is also a representation. In particular, considering $\mathcal{V}$ as the (natural) carrier space for $GL(\mathcal{V})$, we conclude that $T^{\otimes r} : GL(\mathcal{V}) \to GL(\mathcal{V}^{\otimes r})$ is a representation.

This tensor product representation is reducible, because as is evident from its definition, $\mathbf{T}_g^{\otimes r}$ preserves any symmetry of the tensor it acts on. For example, the subspace of the full n^r-dimensional tensor product space—with n being the dimension of $\mathcal{V}$—consisting of the completely symmetric tensors of the type

$$\mathbf{t}_s \equiv \sum_{\pi \in S_r} \mathbf{v}_{\pi(1)} \otimes \mathbf{v}_{\pi(2)} \otimes \cdots \otimes \mathbf{v}_{\pi(r)}$$

is invariant. Similarly, the subspace consisting of the completely antisymmetric tensor products—the r-fold wedge products—is invariant.

To reduce $\mathcal{V}^{\otimes r}$, we choose a basis $\{\mathbf{e}_k\}_{k=1}^n$ for $\mathcal{V}$. Then the collection of n^r tensor products $\{\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}\}$, where each k_i runs from 1 to n, is a basis for $\mathcal{V}^{\otimes r}$. An invariant subspace of $\mathcal{V}^{\otimes r}$ is a span of linear combinations of certain of these basis vectors. Since the only thing that distinguishes among $\{\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}\}$ is a permutation of the r labels, we start to see the connection between the reduction of $\mathcal{V}^{\otimes r}$ and S_r. This connection becomes more evident if we recall that the left multiplication of the group algebra of S_r by its elements provides the regular representation, which is reducible. The irreducible representations are the minimal ideals of the algebra generated by the Young operators.

The same idea works here as well: Certain linear combination of the basis vectors of $\mathcal{V}^{\otimes r}$ obtained by permutations can serve as the basis vectors for irreducible representations of $GL(\mathcal{V})$. Let us elaborate on this. Recall that a Young operator of S_r is written in the form $Y = QP$ where Q and P are linear combinations of permutations in S_r. Y has the property that if one operates on it (via left multiplication) with all permutations of S_r, one

connection between the Young tableaux and irreducible representations of $GL(\mathcal{V})$

generates a minimal ideal, i.e., an *irreducible representation* of S_r. Now let Y be a Young operator that acts on the indices $(k_1, \dots, k_r)$, giving linear combinations of the basis vectors of $\mathcal{V}^{\otimes r}$. From the minimality of the ideal generated by Y and the fact that operators in $GL(\mathcal{V})$ permute the factors in $\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}$ in all possible ways, it should now be clear that if we choose any *single* basis vector $\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}$, then $\mathsf{Y}(\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r})$ generates an irreducible representation of $GL(\mathcal{V})$. We therefore have the following:

Theorem 30.2.1 *Let $\{\mathbf{e}_k\}_{k=1}^n$ be any basis for $\mathcal{V}$. Let $\mathsf{Y} = \mathsf{QP}$ be the Young operator of S_r that permutes (and takes linear combinations of) the basis vectors $\{\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}\}$. Then for any given such basis vector, the vectors*

$$\left\{\mathsf{T}_g^{\otimes r}\mathsf{Y}(\mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_r}) \mid g \in GL(\mathcal{V})\right\}$$

span an irreducible subspace of $\mathcal{V}^{\otimes r}$.

A basis of such an irreducible representation can be obtained by taking into account all the Young tableaux associated with the irreducible representation. But which of the symmetry types will be realized for given values of n and r? Clearly, the Young tableau should not contain more than n rows, because then one of the symbols will be repeated in a column, and the Young operator will vanish due to the antisymmetry in its column indices. We can therefore restrict the partition (λ) to

$$(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_n), \qquad \lambda_1 + \cdots + \lambda_n = r, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Let us consider an example for clarification.

Example 30.2.2 First, let $n = r = 2$. The tensor product space has $2^2 = 4$ dimensions. To reduce it, we consider the Young operators, which correspond to $e + (k_1, k_2)$ and $e - (k_1, k_2)$. Let us denote these operators by Y_1 and Y_2, respectively. By applying each one to a generic basis vector $\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2}$, we can generate all the irreducible representations. The first operator gives

$$\mathsf{Y}_1(\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2}) = \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} + \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1},$$

where k_1 and k_2 can be 1 or 2. For $k_1 = k_2 = 1$, we get $2\mathbf{e}_1 \otimes \mathbf{e}_1$. For $k_1 = 1$, $k_2 = 2$, or $k_1 = 2$, $k_2 = 1$, we get $\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$. Finally, for $k_1 = k_2 = 2$, we get $2\mathbf{e}_2 \otimes \mathbf{e}_2$. Altogether, we obtain 3 linearly independent vectors that are completely symmetric.

When the second operator acts on a generic basis vector, it gives

$$\mathsf{Y}_2(\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2}) = \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} - \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1}.$$

The only time that this is not zero is when k_1 and k_2 are different. In either case, we get $\pm(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)$. This subspace is therefore one-dimensional.

The reduction of the tensor product space can therefore be written as

$$\begin{aligned}\mathcal{V}^{\otimes 2} &= \underbrace{\mathrm{Span}\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2\}}_{\text{3D symmetric subspace}} \\ &\quad \oplus \underbrace{\mathrm{Span}\{\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1\}}_{\text{1D antisymmetric subspace}}.\end{aligned}$$

Next, let us consider the case of $n = 2$, $r = 3$. The tensor product space has $2^3 = 8$ dimensions. To reduce it, we need to consider all Young operators of S_3. There are four of these, corresponding to the following tableaux:

$$\boxed{\mathrm{k}_1}\boxed{\mathrm{k}_2}\boxed{\mathrm{k}_3} \qquad \begin{array}{ll}\boxed{\mathrm{k}_1} & \boxed{\mathrm{k}_2} \\ \boxed{\mathrm{k}_3} & \end{array} \qquad \begin{array}{ll}\boxed{\mathrm{k}_1} & \boxed{\mathrm{k}_3} \\ \boxed{\mathrm{k}_2} & \end{array} \qquad \begin{array}{l}\boxed{\mathrm{k}_1} \\ \boxed{\mathrm{k}_2} \\ \boxed{\mathrm{k}_3}\end{array}$$

Let us denote these operators by $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, and $\mathbf{Y}_4$, respectively. By applying each one to a generic basis vector $\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3}$, we can generate all the irreducible representations. The first operator gives

$$\begin{aligned}\mathbf{Y}_1(\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3}) &= \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3} + \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_2} + \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_3} \\ &\quad + \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_1} + \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \\ &\quad + \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1},\end{aligned}$$

where k_1, k_2, and k_2 can be 1 or 2. For $k_1 = k_2 = k_3 = 1$, we get $6\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1$. For the case where two of the k_i's are 1 and the third is 2, we get

$$2(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1).$$

For the case where two of the k_i's are 2 and the third is 1, we get

$$2(\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1).$$

Finally, for $k_1 = k_2 = k_3 = 2$, we get $6\mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2$. Altogether, we obtain 4 linearly independent vectors that are completely symmetric.

When the second operator acts on a generic basis vector, it gives[2]

$$\begin{aligned}\mathbf{Y}_2(\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3}) &= \big[e - (k_1, k_3)\big]\big[e + (k_1, k_2)\big](\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3}) \\ &= \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3} + \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_3} \\ &\quad - \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_1} - \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3} \otimes \mathbf{e}_{k_1}.\end{aligned}$$

If all three indices are the same, we get zero. Suppose $k_1 = 1$. Then k_2 can be 1 or 2. For $k_2 = 1$, we must set $k_3 = 2$ to get $\mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1$. For $k_2 = 2$, we must set $k_3 = 1$ to obtain $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2$. If we

[2]When a symmetric group is considered as an abstract group—as opposed to a group of transformations—we may multiply permutations (keep track of how each number is repeatedly transformed) from left to right. However, since the permutations here act on vectors on their right, it is more natural to calculate their products from right to left.

start with $k_1 = 2$, we will not produce any new vectors, as the reader is urged to verify. Therefore, the dimension of the irreducible subspace spanned by the second Young tableau is 2.

The action of the third operator on a generic basis vector yields

$$\begin{aligned}\mathbf{Y}_3(\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3}) &= \big[e-(k_1,k_2)\big]\big[e+(k_1,k_3)\big](\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3})\\ &= \mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3}+\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_3}\\ &\quad -\mathbf{e}_{k_3}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_1}-\mathbf{e}_{k_3}\otimes\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}.\end{aligned}$$

The reader may check that we obtain a two-dimensional irreducible representation spanned by $\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_1$ and $\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_1$.

The fourth Young operator gives zero because it is completely antisymmetric in three slots and we have only two indices. The reduction of the tensor product space can therefore be written as

$$\begin{aligned}\mathcal{V}^{\otimes 3} &= \underbrace{\mathrm{Span}\big\{\mathbf{Y}_1(\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3})\big\}}_{\text{dim=4}}\oplus\underbrace{\mathrm{Span}\big\{\mathbf{Y}_2(\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3})\big\}}_{\text{dim=2}}\\ &\quad\oplus\underbrace{\mathrm{Span}\big\{\mathbf{Y}_3(\mathbf{e}_{k_1}\otimes\mathbf{e}_{k_2}\otimes\mathbf{e}_{k_3})\big\}}_{\text{dim=2}}.\end{aligned}$$

We note that the total dimensions on both sides match.

There is a remarkable formula that gives the dimension of all irreducible representations of $GL(\mathcal{V})$ (see [Hame 89, pp. 384–387] for a derivation):

Theorem 30.2.3 *Let $\mathcal{V}$ be an n-dimensional vector space, and $\mathcal{V}^{(\lambda)}$, the irreducible subspace of tensors with symmetry associated with the partition $(\lambda) = (\lambda_1, \dots, \lambda_n)$. Then*

$$\dim\mathcal{V}^{(\lambda)} = \frac{D(l_1,\dots,l_n)}{D(n-1,n-2,\dots,0)},$$

where $l_j \equiv \lambda_j + n - j$ and $D(x_1,\dots,x_n)$ is as given in Eq. (25.3).

30.3 Representation of Lie Algebras

The diffeomorphism established by the exponential map (Theorem 29.1.21) reduces the *local* study of a Lie group to that of its Lie algebra.[3] In this book, we are exclusively interested in the local properties of Lie groups, and we

[3] We use the word "local" to mean the collection of all points that can be connected to the identity by a curve in the Lie group G. If this collection exhausts G, then we say that G is **connected**. If, furthermore, all closed curves (loops) in G can be shrunk to a point, we say that G is **simply connected**. The word "local" can be replaced by "simply connected" in what follows.

shall therefore confine ourselves to Lie algebras to study the structure of Lie groups. Recall that any Lie group homomorphism leads to a corresponding Lie algebra homomorphism [Eq. (29.7)]. Conversely, a homomorphism of Lie algebras can, through the identification of the neighborhoods of their identities with their Lie algebras, be "exponentiated" to a (local) homomorphism of their Lie groups. This leads to the following theorem (see [Fult 91, pp. 108 and 119] for a proof).

Theorem 30.3.1 *Let G be a Lie group with algebra $\mathfrak{g}$. A representation $T : G \to GL(\mathcal{H})$ determines a Lie algebra representation $T_* : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H})$. Conversely, a Lie algebra representation $\mathfrak{T} : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H})$ determines a Lie group representation.*

It follows from this theorem that all (local) Lie group representations result from corresponding Lie algebra representations. Therefore, we shall restrict ourselves to the representations of Lie algebras.

30.3.1 Representation of Subgroups of $GL(\mathcal{V})$

Let $\mathfrak{g}$ be any Lie algebra with basis vectors $\{\mathbf{X}_i\}$. Let a representation T map these vectors to $\{\mathsf{T}_i\} \in \mathfrak{gl}(\mathcal{H})$ for some carrier space $\mathcal{H}$. Then, a general element $\mathbf{X} = \sum_i \alpha_i \mathbf{X}_i$ of $\mathfrak{g}$ will be mapped to $\mathsf{T} = \sum_i \alpha_i \mathsf{T}_i$. Now suppose that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Then the restriction of T to $\mathfrak{h}$ provides a representation of $\mathfrak{h}$. This restriction may be reducible. If it is, then there is an invariant subspace $\mathcal{H}_1$ of $\mathcal{H}$. It follows that

$$\langle b|\mathsf{T}_\mathbf{X}|a\rangle = 0 \quad \forall \mathbf{X} \in \mathfrak{h} \quad \text{whenever } |a\rangle \in \mathcal{H}_1 \text{ and } |b\rangle \in \mathcal{H}_1^\perp,$$

where $\mathsf{T}_\mathbf{X} \equiv T(\mathbf{X})$. If we write $\mathsf{T}_\mathbf{X} = \sum_i \alpha_i^{(\mathbf{X})} \mathsf{T}_i$, then in terms of T_i, the equation above can be written as

$$\sum_{i=1}^{\dim \mathfrak{g}} \alpha_i^{(\mathbf{X})} \langle b|\mathsf{T}_i|a\rangle \equiv \sum_{i=1}^{\dim \mathfrak{g}} \alpha_i^{(\mathbf{X})} \tau_i^{(ba)} = 0 \quad \forall \mathbf{X} \in \mathfrak{h}, \tag{30.12}$$

where $\tau_i^{(ba)} \equiv \langle b|\mathsf{T}_i|a\rangle$ are complex numbers. Equation (30.12) states that

Box 30.3.2 *If T, as a representation of $\mathfrak{h}$ (a Lie subalgebra of $\mathfrak{g}$), is reducible, then there exist a number of equations that $\alpha_i^{(\mathbf{X})}$ must satisfy whenever $\mathbf{X} \in \mathfrak{h}$. If T, as a representation of $\mathfrak{g}$, is irreducible, then no relation such as given in (30.12) will exist when $\mathbf{X}$ runs over all of $\mathfrak{g}$.*

This last statement will be used to analyze certain subgroups of $GL(\mathcal{V})$.

Let us first identify $GL(\mathcal{V})$ with $GL(n, \mathbb{C})$. Next, consider $GL(n, \mathbb{R})$, which is a subgroup of $GL(n, \mathbb{C})$, and transfer the discussion to their respective algebras. If $\{\mathbf{X}_i\}$ is a basis of $\mathfrak{gl}(n, \mathbb{C})$, then an arbitrary element can be written as $\sum_i \alpha_i \mathbf{X}_i$. The difference between $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{R})$ is that the α_i's are *real* in the latter case; i.e., for *all* real values of $\{\alpha_i\}$, the sum belongs to $\mathfrak{gl}(n, \mathbb{R})$. Now suppose that T is an irreducible representation of $\mathfrak{gl}(n, \mathbb{C})$ that is reducible when restricted to $\mathfrak{gl}(n, \mathbb{R})$. Equation (30.12) states that the function

$$f(z_1, \ldots, z_{n^2}) \equiv \sum_{i=1}^{n^2} z_i \tau_i^{(ba)}$$

vanishes for *all* real values of the z_i's. Since this function is obviously entire, it must vanish for all *complex* values of z_i's by analytic continuation (see Theorem 12.3.1). But this is impossible because T is irreducible for $\mathfrak{gl}(n, \mathbb{C})$. We have to conclude that T is irreducible as a representation of $\mathfrak{gl}(n, \mathbb{R})$.

The next subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ we consider is the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of the special linear group. The only restriction on the elements of $\mathfrak{sl}(n, \mathbb{C})$ is for them to have a vanishing trace. Denoting $\operatorname{tr} \mathbf{X}_i$ by t_i, we conclude that $\mathbf{X} = \sum_i \alpha_i \mathbf{X}_i$ belongs to $\mathfrak{sl}(n, \mathbb{C})$ if and only if $\sum_i \alpha_i t_i = 0$. Let $(t_1^*, \ldots, t_{n^2}^*) \equiv |t\rangle \in \mathbb{C}^{n^2}$. Then $\mathfrak{sl}(n, \mathbb{C})$ can be characterized as the subspace consisting of vectors $|a\rangle \in \mathbb{C}^{n^2}$ such that $\langle a|t\rangle = 0$. Such a subspace has $n^2 - 1$ dimensions. If any irreducible representation of $\mathfrak{gl}(n, \mathbb{C})$ is reducible for $\mathfrak{sl}(n, \mathbb{C})$, then the set of complex numbers $\{\alpha_i\}$ must, in addition, satisfy Eq. (30.12). This amounts to the condition that $|a\rangle$ be orthogonal to $|\tau^{(ba)}\rangle$ as well. But this is impossible, because then the set $\{|a\rangle, |t\rangle, |\tau^{(ba)}\rangle\}$ would constitute a subspace of $\mathbb{C}^{n^2}$ whose dimension is at least $n^2 + 1$: There are $n^2 - 1$ of $|a\rangle$'s, one $|t\rangle$, and at least one $|\tau^{(ba)}\rangle$. Therefore, all irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$ are also irreducible representations of $\mathfrak{sl}(n, \mathbb{C})$.

The last subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ we consider is the Lie algebra $\mathfrak{u}(n)$ of the unitary group. To study this algebra, we start with the Weyl basis of Eq. (29.32) for $\mathfrak{gl}(n, \mathbb{C})$, and construct a new *hermitian* basis $\{\mathsf{X}_{kj}\}$ defined as

$$\begin{aligned} \mathsf{X}_{jj} &\equiv \mathsf{e}_{jj} && \text{for all } j = 1, 2, \ldots, n, \\ \mathsf{X}_{kj} &\equiv i\bigl(\mathsf{e}_{kj} - \mathsf{e}_{kj}^t\bigr) && \text{if } k \neq j. \end{aligned}$$

A typical element of $\mathfrak{gl}(n, \mathbb{C})$ is of the form $\sum_{kj} \alpha_{kj} \mathsf{X}_{kj}$, where α_{kj} are complex numbers. If we restrict ourselves to *real* values of α_{kj}, then we obtain the subalgebra of hermitian matrices whose Lie group is the unitary group $U(n)$. The fact that the irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$ will not reduce under $\mathfrak{u}(n)$ follows immediately from our discussion concerning $\mathfrak{gl}(n, \mathbb{R})$. We summarize our findings in the following:

Theorem 30.3.3 *The irreducible representations of $GL(n, \mathbb{C})$ are also irreducible representations of $GL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $U(n)$, and $SU(n)$.*

The case of $SU(n)$ follows from the same argument given earlier that connected $GL(n, \mathbb{C})$ to $SL(n, \mathbb{C})$.

30.3.2 Casimir Operators

In the general representation theory of Lie algebras, it is desirable to label each irreducible representation with a quantity made out of the basis vectors of the Lie algebra. An example is the labeling of the energy states of a quantum mechanical system with angular momentum. Each value of the total angular momentum labels an irreducible subspace whose vectors are further labeled by the third component of angular momentum (see Chap. 13). This subsection is devoted to the generalization of this concept to an arbitrary Lie algebra.

Casimir operator defined

Definition 30.3.4 Let $\mathfrak{T} : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H})$ be a representation of the Lie algebra $\mathfrak{g}$. A **Casimir operator $\mathbf{C}$** for this representation is an operator that commutes with all $\mathbf{T}_{\mathbf{X}}$ of the representation.

If the representation is irreducible, then by Schur's lemma, $\mathbf{C}$ is a multiple of the unit operator. Therefore, all vectors of an irreducible invariant subspace of the carrier space $\mathcal{H}$ are eigenvectors of $\mathbf{C}$ corresponding to the same eigenvalue. That Casimir operators actually determine the irreducible representations of a semisimple Lie algebra is the content of the following theorem (for a proof, see [Vara 84, pp. 333–337]).

Chevalley's theorem

Theorem 30.3.5 (Chevalley) *For every semisimple Lie algebra $\mathfrak{g}$ of rank*[4] *r with a basis $\{\mathbf{X}_i\}$, there exists a set of r Casimir operators in the form of polynomials in $\mathbf{T}_{\mathbf{X}_i}$ whose eigenvalues characterize the irreducible representations of $\mathfrak{g}$.*

From now on, we shall use the notation $\mathbf{X}_i$ for $\mathbf{T}_{\mathbf{X}_i}$. It follows from Theorem 30.3.5 that all irreducible invariant vector subspaces of the carrier space can be labeled by the eigenvalues of the r Casimir operators. This means that each invariant irreducible subspace has a basis all of whose vectors carry a set of r labels corresponding to the eigenvalues of the r Casimir operators.

One Casimir operator—in the form of a polynomial of degree two—which works only for semisimple Lie algebras, is obtained easily:

$$\mathbf{C} = \sum_{i,j} g^{ij} \mathbf{X}_i \mathbf{X}_j, \tag{30.13}$$

where g^{ij} is the inverse of the Cartan metric tensor. In fact, with the summation convention in place, we have

[4] Recall that the rank of $\mathfrak{g}$ is the dimension of the Cartan subalgebra of $\mathfrak{g}$.

$$\begin{aligned}[\mathbf{C}, \mathbf{X}_k] &= g^{ij}[\mathbf{X}_i\mathbf{X}_j, \mathbf{X}_k] = g^{ij}\left\{\mathbf{X}_i[\mathbf{X}_j, \mathbf{X}_k] + [\mathbf{X}_i, \mathbf{X}_k]\mathbf{X}_j\right\} \\ &= g^{ij}\left\{c^r_{jk}\mathbf{X}_i\mathbf{X}_r + c^r_{ik}\mathbf{X}_r\mathbf{X}_j\right\} \\ &= g^{ij}c^r_{ik}(\mathbf{X}_j\mathbf{X}_r + \mathbf{X}_r\mathbf{X}_j) \quad \text{(because } g^{ij} \text{ is symmetric)} \\ &= g^{ij}g^{sr}c_{iks}(\mathbf{X}_j\mathbf{X}_r + \mathbf{X}_r\mathbf{X}_j) \\ &= 0 \quad \text{(because } g^{ij}g^{sr}c_{iks} \text{ is antisymmetric in } j, r).\end{aligned}$$

The last equality follows from the fact that g^{ij} and g^{sr} are symmetric, c_{iks} is completely antisymmetric [see the discussion following Eq. (29.50)], and there is a sum over the dummy index s.

Example 30.3.6 The rotation group $SO(3)$ in $\mathbb{R}^3$ is a compact 3-parameter Lie group. The infinitesimal generators are the three components of the angular momentum operator (see Example 29.1.35). From the commutation relations of the angular momentum operators developed in Chap. 13, we conclude that $c^k_{ij} = i\epsilon_{ijk}$. It follows that the Cartan metric tensor is

$$g_{ij} = c^r_{is}c^s_{jr} = (i\epsilon_{isr})(i\epsilon_{jrs}) = +\epsilon_{isr}\epsilon_{jsr} = 2\delta_{ij}.$$

Ignoring the factor of 2 and denoting the angular momentum operators by $\mathbf{L}_i$, we conclude that

irreducible representations of the rotation group and spherical harmonics

$$\mathbf{L}^2 \equiv \mathbf{L}_1^2 + \mathbf{L}_2^2 + \mathbf{L}_3^2$$

is a Casimir operator. But this is precisely the operator discussed in detail in Chap. 13. We found there that the eigenvalues of $\mathbf{L}^2$ were labeled by j, where j was either an integer or a half odd integer. In the context of our present discussion, we note that the Lie algebra $\mathfrak{so}(3)$ has rank one, because there is no higher dimensional subalgebra of $\mathfrak{so}(3)$ all of whose vectors commute with one another. It follows from Theorem 30.3.5 that $\mathbf{L}^2$ is the only Casimir operator, and that all irreducible representations $T^{(j)}$ of $\mathfrak{so}(3)$ are distinguished by their label j. Furthermore, the construction of Chap. 13 showed explicitly that the dimension of $T^{(j)}$ is $2j+1$.

The connection between the representation of Lie algebras and Lie groups permits us to conclude that the irreducible representations of the rotation group are labeled by the (half) integers j, and the jth irreducible representation has dimension $2j+1$. When j is an integer l and the carrier space is $\mathcal{L}^2(S^2)$, the square-integrable functions on the unit sphere, then $\mathbf{L}^2$ becomes a *differential operator*, and the spherical harmonics $Y_{lm}(\theta, \varphi)$, with a fixed value of l, provide a basis for the lth irreducible invariant subspace.

The last sentence of Example 30.3.6 is at the heart of the connection between symmetry, Lie groups, and the equations of mathematical physics. A symmetry operation of mathematical physics is expressed in terms of the action of a Lie group on an underlying manifold M, i.e., as a group of transformations of M. The Lie algebra of such a Lie group consists of the infinitesimal generators of the corresponding transformation. These generators can be expressed as first-order differential operators as in Eq. (29.25).

Connection between Casimir operators and the PDEs of mathematical physics

It is therefore natural to choose as the carrier space of a representation the Hilbert space $\mathcal{L}^2(M)$ of the square-integrable functions on M, which, through the local identification of M with $\mathbb{R}^m$ ($m = \dim M$), can be identified with functions on $\mathbb{R}^m$. Then the infinitesimal generators act directly on the functions of $\mathcal{L}^2(M)$ as first-order differential operators.

The Casimir operators $\{\mathbf{C}_\alpha\}_{\alpha=1}^r$, where r is the rank of the Lie algebra, are polynomials in the infinitesimal generators, i.e., differential operators of higher order. On the irreducible invariant subspaces of $\mathcal{L}^2(M)$, each $\mathbf{C}_\alpha$ acts as a multiple of the identity, so if $f(\mathbf{r})$ belongs to such an invariant subspace, we have

$$\mathbf{C}_\alpha f(\mathbf{r}) = \lambda(\alpha) f(\mathbf{r}), \quad \alpha = 1, 2, \ldots, r. \tag{30.14}$$

This is a set of differential equations that are invariant under the symmetry of the physical system, i.e., its solutions transform among themselves under the action of the group of symmetries.

It is a stunning reality and a fact of profound significance that many of the differential equations of mathematical physics are, as in Eq. (30.14), expressions of the invariance of the Casimir operators of some Lie algebra in a particular representation. Moreover, all the standard functions of mathematical physics, such as Bessel, hypergeometric, and confluent hypergeometric functions, are related to matrix elements in the representations of a few of the simplest Lie groups (see [Mill 68] for a thorough discussion of this topic).

Historical Notes

Claude Chevalley
1909–1984

Claude Chevalley (1909–1984) was the only son of Abel and Marguerite Chevalley who were the authors of the Oxford Concise French Dictionary. He studied under Emile Picard at the Ecole Normale Supérieur in Paris, graduating in 1929 and becoming the youngest of the mathematicians of the Bourbaki school.

After graduation, Chevalley went to Germany to continue his studies under Artin at Hamburg during the session 1931–1932. He then went to the University of Marburg to work with Hasse, who had been appointed to fill Hensel's chair there in 1930. He was awarded his doctorate in 1937. A year later Chevalley went to the Institute for Advanced Study at Princeton, where he also served on the faculty of Princeton University. From July 1949 until June 1957 he served as professor of mathematics at Columbia University, afterwards returning to the University of Paris.

Chevalley had a major influence on the development of several areas of mathematics. His papers of 1936 and 1941 led to major advances in class field theory and also in algebraic geometry. He did pioneering work in the theory of local rings in 1943, developing the ideas of Krull into a theorem bearing his name. Chevalley's theorem was important in applications made in 1954 to quasi-algebraically closed fields and the following year to algebraic groups. Chevalley groups play a central role in the classification of finite simple groups. His name is also attached to Chevalley decompositions and to a Chevalley type of semi-simple algebraic group.

Many of his texts have become classics. He wrote *Theory of Lie Groups* in three volumes which appeared in 1946, 1951, and 1955. He also published *Theory of Distributions* (1951), *Introduction to the Theory of Algebraic Functions of one Variable* (1951), *The Algebraic Theory of Spinors* (1954), *Class Field Theory* (1954), *Fundamental Concepts of Algebra* (1956), and *Foundations of Algebraic Geometry* (1958).

Chevalley was awarded many honors for his work. Among these was the Cole Prize of the American Mathematical Society. He was elected a member of the London Mathematical Society in 1967.

30.3.3 Representation of $\mathfrak{so}(3)$ and $\mathfrak{so}(3,1)$

Because of their importance in physical applications, we study the representations of $\mathfrak{so}(3)$, the rotation, and $\mathfrak{so}(3,1)$, the Lorentz, algebras. For rotations, we define $\mathbf{J}_1 \equiv -i\mathbf{M}_{23}$, $\mathbf{J}_2 \equiv i\mathbf{M}_{13}$, and $\mathbf{J}_3 \equiv -i\mathbf{M}_{12}$,[5] and note that the $\mathbf{J}_i$'s satisfy exactly the same commutation relations as the angular momentum operators of Chap. 13. Therefore, the irreducible representations of $\mathfrak{so}(3)$ are labeled by j, which can be an integer or a half-odd integer (see also Example 30.3.6). These representations are finite-dimensional because $SO(3)$ is a compact group (Example 30.1.1 and Theorem 30.1.7). The dimension of the irreducible representation of $\mathfrak{so}(3)$ labeled by j is $2j+1$.

Because of local isomorphism of Lie groups and their Lie algebras, the same irreducible spaces found for Lie algebras can be used to represent the Lie groups. In particular, the states $\{|jm\rangle\}_{m=-j}^{j}$, where m is the eigenvalue of $\mathbf{J}_z$, can also be used as a basis of the j-th irreducible representation.

The flow of each infinitesimal generator of $\mathfrak{so}(3)$ is a one-parameter subgroup of $SO(3)$. For example, $\exp(\mathbf{M}_{12}\varphi)$ is a rotation of angle φ about the z-axis. Using Euler angles, we can write a general rotation as

$$\mathbf{R}(\psi,\theta,\varphi) = \exp(\mathbf{M}_{12}\psi)\exp(\mathbf{M}_{31}\theta)\exp(\mathbf{M}_{12}\varphi).$$

The corresponding rotation *operator* acting on the vectors of the carrier space is

$$\mathbf{R}(\psi,\theta,\varphi) = \exp(\mathbf{M}_{12}\psi)\exp(\mathbf{M}_{31}\theta)\exp(\mathbf{M}_{12}\varphi) = e^{i\mathbf{J}_z\psi}e^{i\mathbf{J}_y\theta}e^{i\mathbf{J}_z\varphi}.$$

rotation matrix

The **rotation matrix** corresponding to the above operator is obtained by sandwiching $\mathbf{R}(\psi,\theta,\varphi)$ between basis vectors of a given irreducible representation:

$$\begin{aligned} D^{(j)}_{m'm}(\psi,\theta,\varphi) &\equiv \langle jm'|\mathbf{R}(\psi,\theta,\varphi)|jm\rangle = \langle jm'|e^{i\mathbf{J}_z\psi}e^{i\mathbf{J}_y\theta}e^{i\mathbf{J}_z\varphi}|jm\rangle \\ &= e^{im'\psi}e^{im\varphi}\langle jm'|e^{i\mathbf{J}_y\theta}|jm\rangle = e^{i(m'\psi+m\varphi)}d^{(j)}_{m'm}(\theta). \end{aligned} \tag{30.15}$$

Wigner formula for rotation matrices

Thus, the calculation of rotation matrices is reduced to finding $d^{(j)}_{m'm}(\theta)$. These are given by the **Wigner formula** (see [Hame 89, pp. 348–357]):

$$d^{(j)}_{m'm}(\theta) = \sum_\mu \phi(j,m,m';\mu)\left(\cos\frac{\theta}{2}\right)^{2(j-\mu)+m-m'}\left(\sin\frac{\theta}{2}\right)^{2\mu+m'-m} \tag{30.16}$$

where

$$\phi(j,m,m';\mu) \equiv (-1)^\mu \frac{[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}}{(j+m-\mu)!\mu!(j-m'-\mu)!(m'-m+\mu)!}$$

and the summation extends over all integral values of μ for which the factorials have a meaning. The number of terms in the summation is equal to $1+\tau$, where τ is the smallest of the four integers $j\pm m$, $j\pm m'$.

[5]Sometimes we use $\mathbf{J}_x$, $\mathbf{J}_y$, and $\mathbf{J}_z$ instead of $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$.

From the rotation matrices, we can obtain the characters of the rotation group. However, an easier way is to use Euler's theorem (Theorem 6.6.15), Example 23.2.19, and Box 24.3.6 to conclude that the character of a rotation depends only on the angle of rotation, and not on the direction of the rotation axis. Choosing the z-axis as our only axis of rotation, we obtain

$$\begin{aligned}\chi^{(j)}(\varphi) &= \sum_{m=-j}^{j} \langle jm|e^{i\mathbf{J}_z\varphi}|jm\rangle = \sum_{m=-j}^{j} e^{im\varphi} = e^{-ij\varphi}\sum_{m=-j}^{j} e^{i(j+m)\varphi} \\ &= e^{-ij\varphi}\sum_{k=0}^{2j} e^{ik\varphi} = e^{-ij\varphi}\frac{e^{i(2j+1)\varphi}-1}{e^{i\varphi}-1} \\ &= \frac{e^{i(j+1)\varphi}-e^{-ij\varphi}}{e^{i\varphi}-1} = \frac{\sin(j+\frac{1}{2})}{\sin(\varphi/2)}. \end{aligned} \tag{30.17}$$

Equation (30.17) can be used to obtain the celebrated **addition theorem** for angular momenta. Suppose that initially we have two physical systems corresponding to angular momenta j_1 and j_2. When these systems are made to interact with one another, the total system will be described by the tensor product states. These states are vectors in the tensor product of the irreducible representations $T^{(j_1)}$ and $T^{(j_2)}$ of the rotation group, as discussed in Sect. 24.8. This product is reducible. To find the factors into which it reduces, we consider its character corresponding to angle φ. Using Eq. (24.42), we have

$$\begin{aligned}\chi^{(j_1\times j_2)}(\varphi) &= \chi^{(j_1)}(\varphi)\cdot\chi^{(j_2)}(\varphi) = \sum_{m_1=-j_1}^{j_1} e^{im_1\varphi}\sum_{m_2=-j_2}^{j_2} e^{im_2\varphi} \\ &= \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} e^{i(m_1+m_2)\varphi} \\ &= \sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{M=-J}^{J} e^{iM\varphi} = \sum_{J=|j_1-j_2|}^{j_1+j_2}\chi^{(J)}(\varphi),\end{aligned}$$

where the double sum on the third line is an equivalent way of writing the double summation of the second line, as the reader may verify. From this equation we read off the Clebsch-Gordan decomposition of the tensor product:

addition theorem for angular momenta

$$T^{(j_1)}\otimes T^{(j_2)} = \sum_{J=|j_1-j_2|}^{j_1+j_2} T^{(J)}, \tag{30.18}$$

which is also known as the addition theorem for angular momenta. Equation (30.18) shows that (see page 753).

Box 30.3.7 *The rotation group is simply reducible.*

The RHS of Eq. (30.18) tells us which irreducible representations result from multiplying $T^{(j_1)}$ and $T^{(j_2)}$. In particular, if $j_1 = j_2 \equiv l$, the RHS includes the $J = 0$ representation, i.e., a scalar. In terms of the states, this says that we can combine two states with angular momentum l to obtain a scalar state. Let us find this combination. We use Eq. (24.46) in the form

$$|JM\rangle = \sum_{m_1,m_2} C(j_1 j_2; J|m_1 m_2; M)|j_1, m_1; j_2, m_2\rangle, \quad m_1 + m_2 = M. \tag{30.19}$$

In the case under investigation, $J = 0 = M$, so (30.19) becomes

$$|00\rangle = \sum_{m=-l}^{l} C(ll; 0|m, -m; 0)|lm; l, -m\rangle.$$

Problem 30.9 shows that $C(ll; 0|m, -m; 0) = (-1)^{l-m}/\sqrt{2l+1}$, so that

$$|00\rangle = \sum_{m=-l}^{l} \frac{(-1)^{l-m}}{\sqrt{2l+1}}|lm; l, -m\rangle.$$

Take the "inner product" of this with $\langle\theta, \varphi; \theta', \varphi'|$ to obtain

$$\begin{aligned}\langle\theta, \varphi; \theta', \varphi'|00\rangle &= \sum_{m=-l}^{l} \frac{(-1)^{l-m}}{\sqrt{2l+1}}\langle\theta, \varphi; \theta', \varphi'|lm; l, -m\rangle \\ &= \sum_{m=-l}^{l} \frac{(-1)^{l-m}}{\sqrt{2l+1}} \underbrace{\langle\theta, \varphi|lm\rangle}_{Y_{lm}(\theta,\varphi)} \underbrace{\langle\theta', \varphi'|l, -m\rangle}_{Y_{l,-m}(\theta',\varphi')},\end{aligned} \tag{30.20}$$

where we have used $\langle\theta, \varphi; \theta', \varphi'| = \langle\theta, \varphi|\langle\theta', \varphi'|$ and contracted each bra with a ket. We can evaluate the LHS of (30.20) by noting that since it is a scalar, the choice of orientation of coordinates is immaterial. So, let $\theta = 0$ to get $\theta' = \gamma$, the angle between the two directions. Then using the facts

$$Y_{lm}(0, \varphi) = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}} \quad \text{and} \quad Y_{l0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

on the RHS of (30.20), we obtain

$$\langle\theta, \varphi; \theta', \varphi'|00\rangle = \frac{(-1)^l}{4\pi}\sqrt{2l+1}\,P_l(\cos\gamma).$$

Substituting this in the LHS of Eq. (30.20), we get

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}(-1)^m Y_{lm}(\theta, \varphi)Y_{l,-m}\big(\theta', \varphi'\big),$$

which is the addition theorem for spherical harmonics discussed in Chap. 13.

Let us now turn to $\mathfrak{so}(3, 1)$. We collect the generators in two categories

$$\mathbf{M} \equiv (M_1, M_2, M_3) \equiv (\mathbf{M}_{23}, \mathbf{M}_{31}, \mathbf{M}_{12}),$$

$$\mathbf{N} \equiv (N_1, N_2, N_3) \equiv (\mathbf{M}_{01}, \mathbf{M}_{02}, \mathbf{M}_{03}),$$

and verify that

$$[M_i, M_j] = -\epsilon_{ijk} M_k, \qquad [N_i, N_j] = \epsilon_{ijk} M_k, \qquad [M_i, N_j] = -\epsilon_{ijk} N_k,$$

and that there are two Casimir operators: $\mathbf{M}^2 - \mathbf{N}^2$ and $\mathbf{M} \cdot \mathbf{N}$. It follows that the irreducible representations of $\mathfrak{so}(3, 1)$ are labeled by two numbers. To find these numbers, define the generators

$$\mathbf{J} \equiv \frac{1}{2i}(\mathbf{M} + i\mathbf{N}), \qquad \mathbf{K} \equiv \frac{1}{2i}(\mathbf{M} - i\mathbf{N}),$$

and show that

$$[J_i, J_m] = \epsilon_{imk} J_k, \qquad [K_i, K_j] = \epsilon_{ijm} K_m, \qquad [J_i, K_j] = 0.$$

It follows that the J's and the K's generate two completely independent Lie algebras isomorphic to the angular momentum algebras and that $\mathfrak{so}(3, 1)$ is a direct sum of these algebras. Since each one requires a (half-odd) integer to designate its irreducible representations, we can choose these two numbers as the eigenvalues of the Casimir operators needed to label the irreducible representations of $\mathfrak{so}(3, 1)$. Thus, the irreducible representations of $\mathfrak{so}(3, 1)$ are of the form $T^{(jj')}$, where j and j' can each be an integer or a half-odd integer.

30.3.4 Representation of the Poincaré Algebra

The Poincaré algebra $\mathfrak{p}(p, n - p)$, introduced in Sect. 29.2.1, is the generalization of the Lie algebra of the invariance group of the special theory of relativity. It contains the Lorentz, the rotation, and the translation groups as its proper subgroups. Its irreducible representations are of direct physical significance, and we shall study them here.

As the first step in the construction of representations of $\mathfrak{p}(p, n - p)$, we shall try to find its Casimir operators. Eq. (30.13) suggests one, but it works only for semisimple Lie algebras, and the Poincaré algebra is not semisimple. Nevertheless, let us try to find an operator based on that construction. From the commutation relations for $\mathfrak{p}(p, n - p)$, as given in Eq. (29.45), and the double-indexed structure constants defined by,[6]

$$[\mathsf{M}_{ij}, \mathsf{M}_{kl}] = c_{ij,kl}^{mn} \mathsf{M}_{mn}, \qquad [\mathsf{M}_{ij}, \mathsf{P}_k] = c_{ij,k}^{m} \mathsf{P}_m,$$

we obtain

$$\begin{aligned} c_{ij,kl}^{mn} &= \delta_j^m \delta_l^n \eta_{ik} - \delta_j^m \delta_k^n \eta_{il} + \delta_i^m \delta_k^n \eta_{jl} - \delta_i^m \delta_l^n \eta_{jk}, \\ c_{ij,k}^{m} &= \delta_j^m \eta_{ik} - \delta_i^m \eta_{jk}. \end{aligned} \tag{30.21}$$

[6] Please make sure to differentiate between the pair $(\mathsf{M}_{ij}, \mathsf{P}_k)$ (which acts on $\mathbf{p}$) and the pair $(\mathbf{M}_{ij}, \mathbf{P}_k)$, which acts on the state vectors in the Hilbert space of representation.

From these structure constants, we can construct a double indexed "metric"

$$g_{ij,kl} = c^{rs}_{ij,mn} c^{mn}_{kl,rs} + c^{r}_{ij,m} c^{m}_{kl,r},$$

which the reader may verify to be equal to

$$g_{ij,kl} = 2(n-1)(\eta_{jk}\eta_{il} - \eta_{ik}\eta_{jl}).$$

There is no natural way of constructing a single-indexed metric. Therefore, we can only contract the M's. In doing so, it is understood that the indices are raised and lowered by η_{ij}. So, the first candidate for a Casimir operator is

$$\mathbf{M}^2 \equiv g_{ij,kl}\mathbf{M}^{ij}\mathbf{M}^{kl} = 2(n-1)(\eta_{jk}\eta_{il} - \eta_{ik}\eta_{jl})\mathbf{M}^{ij}\mathbf{M}^{kl} = -4(n-1)\mathbf{M}^{ij}\mathbf{M}_{ij}$$

The reader may verify that $\mathbf{M}^2$ commutes with all the $\mathbf{M}^{ij}$'s but not with the $\mathbf{P}^i$'s. This is to be expected because $\mathbf{M}^2$, the total "angular momentum" operator[7] is a scalar and should commute with all its components. But commutation with the $\mathbf{P}^i$'s is not guaranteed.

The construction above, although a failure, gives us a clue for a successful construction. We can make another scalar out of the $\mathbf{P}$'s. The reader may check that $\mathbf{P}^2 \equiv \eta^{ij}\mathbf{P}_i\mathbf{P}_j$ indeed commutes with all elements of the Poincaré algebra. We have thus found one Casimir operator. Can we find more? We have exhausted the polynomials of degree two. The only third-degree polynomials that we can construct are $\mathbf{M}^{ij}\mathbf{P}_i\mathbf{P}_j$ and $\eta_{il}\mathbf{M}^{ij}\mathbf{M}_{jk}\mathbf{M}^{kl}$. The first one is identically zero (why?), and the second one will not commute with the $\mathbf{P}$'s.

To find higher-order polynomials in the infinitesimal generators, we build new tensors out of them and contract these tensors with one another. For example, consider the vector

$$\mathbf{C}_i \equiv \mathbf{M}_{ij}\mathbf{P}^j = \eta^{kj}\mathbf{M}_{ij}\mathbf{P}_k. \tag{30.22}$$

Then $\mathbf{C}^i\mathbf{C}_i$, a fourth-degree polynomial in the generators, is a scalar, and therefore, it commutes with the $\mathbf{M}_{ij}$'s, but unfortunately, not with $\mathbf{P}_i$'s.

Another common way to construct tensors is to contract various numbers of the generators with the Levi-Civita tensor. For example,

$$\mathbf{W}^{i_1\ldots i_{n-3}} \equiv \epsilon^{i_1\ldots i_{n-3}jkl}\mathbf{M}_{jk}\mathbf{P}_l \tag{30.23}$$

is a contravariant tensor of rank $n-3$. Let us contract $\mathbf{W}$ with itself to find a scalar (which we expect to commute with all the $\mathbf{M}_{ij}$'s):

$$\begin{aligned}
\mathbf{W}^2 &\equiv \mathbf{W}^{i_1\ldots i_{n-3}}\mathbf{W}_{i_1\ldots i_{n-3}} \\
&= \epsilon^{i_1\ldots i_{n-3}jkl}\mathbf{M}_{jk}\mathbf{P}_l\epsilon_{i_1\ldots i_{n-3}rst}\mathbf{M}^{rs}\mathbf{P}^t \\
&= (-1)^{n-}\sum_{\pi}\epsilon_\pi\delta^{i_1}_{\pi(i_1)}\delta^{i_2}_{\pi(i_2)}\cdots\delta^{i_{n-3}}_{\pi(i_{n-3})}\delta^{j}_{\pi(r)}\delta^{k}_{\pi(s)}\delta^{l}_{\pi(t)}\mathbf{M}_{jk}\mathbf{P}_l\mathbf{M}^{rs}\mathbf{P}^t \\
&= (-1)^p(n-3)!\sum_{\pi}\epsilon_\pi\delta^{j}_{\pi(r)}\delta^{k}_{\pi(s)}\delta^{l}_{\pi(t)}\mathbf{M}_{jk}\mathbf{P}_l\mathbf{M}^{rs}\mathbf{P}^t,
\end{aligned}$$

[7] This "angular momentum" includes ordinary rotations as well as the Lorentz boosts.

where we used Eq. (26.45). The sum above can be carried out, with the final result

$$\begin{aligned}\mathbf{W}^2 &= 2(-1)^p(n-3)!\big(\mathbf{M}_{ij}\mathbf{M}^{ij}\mathbf{P}^2 - 2\mathbf{C}_i\mathbf{C}^i\big)\\ &= 2(-1)^p(n-3)!\big(\mathbf{M}^2\mathbf{P}^2 - 2\mathbf{C}^2\big),\end{aligned} \tag{30.24}$$

where $\mathbf{C}_i$ was defined in Eq. (30.22). We have already seen that $\mathbf{M}^2$, $\mathbf{P}^2$, and $\mathbf{C}^2$ all commute with the $\mathbf{M}_{jk}$'s. The reader may check that $\mathbf{W}^2$ commutes with the $\mathbf{P}_j$'s as well. In fact, $\mathbf{W}^{i_1\dots i_{n-3}}$ itself commutes with all the $\mathbf{P}_j$'s. Other tensors and Casimir operators can be constructed in a similar fashion.

We now want to construct the irreducible vector spaces that are labeled by the eigenvalues of the Casimir operators. We take advantage of the fact that the Poincaré algebra has a commutative subalgebra, the translation generators. Since the $\mathbf{P}_k$'s commute among themselves and with $\mathbf{P}^2$ and $\mathbf{W}^2$, we can choose simultaneous eigenvectors of $\{\mathbf{P}_k\}_{k=1}^n$, $\mathbf{P}^2$, and $\mathbf{W}^2$. In particular, we can label the vectors of an irreducible invariant subspace by the eigenvalues of these operators. The $\mathbf{P}^2$ and $\mathbf{W}^2$ labels will be the same for all vectors in each irreducible invariant subspace, while the $\mathbf{P}_k$'s will label different vectors of the same invariant subspace.

Let us concentrate on the momentum labels and let $|\psi_{\mathbf{p}}^{\mu}\rangle$ be a vector in an irreducible representation of $\mathfrak{p}(p, n-p)$, where $\mathbf{p}$ labels momenta and μ distinguishes among all different vectors that have the same momentum label. We thus have

$$\mathbf{P}_k\big|\psi_{\mathbf{p}}^{\mu}\big\rangle = p_k\big|\psi_{\mathbf{p}}^{\mu}\big\rangle \quad \text{for } k = 1, 2, \dots, n, \tag{30.25}$$

where p_k is the eigenvalue of $\mathbf{P}_k$. We also need to know how the "rotation" operators act on $|\psi_{\mathbf{p}}^{\mu}\rangle$. Instead of the full operator $e^{\mathbf{M}_{ij}\theta^{ij}}$, we apply its small-angle approximation $\mathbf{1} + \mathbf{M}_{ij}\theta^{ij}$. Since all states are labeled by momentum, we expect the rotated state to have a new momentum label, i.e., to be an eigenstate of $\mathbf{P}_k$. We want to show that $(\mathbf{1} + \mathbf{M}_{ij}\theta^{ij})|\psi_{\mathbf{p}}^{\mu}\rangle$ is an eigenvector of $\mathbf{P}_k$. Let the eigenvalue be $\mathbf{p}'$, which should be slightly different from $\mathbf{p}$. Then, the problem reduces to determining $\delta\mathbf{p}' \equiv \mathbf{p}' - \mathbf{p}$. Ignoring the index μ for a moment, we have

$$\mathbf{P}_k|\psi_{\mathbf{p}'}\rangle = p'_k|\psi_{\mathbf{p}'}\rangle = (p_k + \delta p_k)\big(\mathbf{1} + \mathbf{M}_{ij}\theta^{ij}\big)|\psi_{\mathbf{p}}\rangle.$$

Using the commutation relations between $\mathbf{P}_k$ and $\mathbf{M}_{ij}$, we can write the LHS as

$$\text{LHS} = \mathbf{P}_k\big(\mathbf{1} + \mathbf{M}_{ij}\theta^{ij}\big)|\psi_{\mathbf{p}}\rangle = \big[p_k + \theta^{ij}(\mathbf{M}_{ij}p_k + \eta_{jk}p_i - \eta_{ik}p_j)\big]|\psi_{\mathbf{p}}\rangle.$$

The RHS, to first order in infinitesimal quantities, can be expressed as

$$\text{RHS} = \big(p_k + \delta p_k + p_k\theta^{ij}\mathbf{M}_{ij}\big)|\psi_{\mathbf{p}}\rangle.$$

Comparison of the last two equations shows that

$$\delta p_k = \theta^{ij}(\eta_{jk}p_i - \eta_{ik}p_j) = \theta^{ij}(\eta_{jk}\eta_{il} - \eta_{ik}\eta_{jl})p^l = \theta^{ij}(\mathsf{M}_{ij})_{kl}p^l,$$

where we used Eq. (29.42). It follows that

$$\mathbf{p}' = \mathbf{p} + \delta\mathbf{p} = \left(\mathbf{1} + \theta^{ij}\mathsf{M}_{ij}\right)\mathbf{p},$$

stating that the rotation operator of the carrier Hilbert space rotates the momentum label of the state. Note that since "rotations" do not change the length (induced by $\boldsymbol{\eta}$), $\mathbf{p}'$ and $\mathbf{p}$ have the same length.

construction and properties of the little group

To obtain all the vectors of an irreducible representation of $\mathfrak{p}(p, n-p)$, we must apply the rotation operators to vectors such as $|\psi^{\mu}_{\mathbf{p}}\rangle$. But not all rotations will change the label $\mathbf{p}$; for example, in three dimensions, the vector $\mathbf{p}$ will not be affected[8] by a rotation about $\mathbf{p}$. This motivates the following definition.

little group and little algebra

Definition 30.3.8 Let $\mathbf{p}_0$ be any given eigenvalue of the translation generators. The set $\mathcal{R}_{\mathbf{p}_0}$ of all rotations $\mathsf{A}^{\mathbf{p}_0}$ that do not change $\mathbf{p}_0$, is a subgroup of the rotation group $O(p, n-p)$, called the **little group** corresponding to $\mathbf{p}_0$. The **little algebra** consists of the generators $\mathsf{M}^{\mathbf{p}_0}_{ij}$ satisfying

$$\mathsf{M}^{\mathbf{p}_0}_{ij}\mathbf{p}_0 = 0.$$

The significance of the little group resides in the fact that a representation of $\mathcal{R}_{\mathbf{p}_0}$ induces a representation of the whole Poincaré group. We shall only sketch the proof in the following and refer the reader to Mackey [Mack 68] for a full and rigorous discussion of **induced representations**.

induced representations

Suppose we have found an irreducible representation of $\mathcal{R}_{\mathbf{p}_0}$ with operators $\mathbf{A}^{\mathbf{p}_0}$. Let $\mathsf{A}^{\mathbf{p}\mathbf{p}_0}$ be the rotation that carries[9] $\mathbf{p}_0$ to $\mathbf{p}$, i.e., $\mathbf{p} = \mathsf{A}^{\mathbf{p}\mathbf{p}_0}\mathbf{p}_0$. Consider any rotation A and let $\mathbf{p}'$ be the momentum obtained when A acts on $\mathbf{p}$, i.e., $\mathsf{A}\mathbf{p} \equiv \mathbf{p}'$. Then

$$\mathsf{A}\,\underbrace{\mathsf{A}^{\mathbf{p}\mathbf{p}_0}\mathbf{p}_0}_{=\mathbf{p}} = \underbrace{\mathsf{A}^{\mathbf{p}'\mathbf{p}_0}\mathbf{p}_0}_{\equiv\mathbf{p}'} \quad\Rightarrow\quad \left(\mathsf{A}^{\mathbf{p}'\mathbf{p}_0}\right)^{-1}\mathsf{A}\mathsf{A}^{\mathbf{p}\mathbf{p}_0}\mathbf{p}_0 = \mathbf{p}_0.$$

This shows that $(\mathsf{A}^{\mathbf{p}'\mathbf{p}_0})^{-1}\mathsf{A}\mathsf{A}^{\mathbf{p}\mathbf{p}_0}$ belongs to the little group. So,

$$\left(\mathsf{A}^{\mathbf{p}'\mathbf{p}_0}\right)^{-1}\mathsf{A}\mathsf{A}^{\mathbf{p}\mathbf{p}_0} = \mathsf{A}^{\mathbf{p}_0}$$

for some $\mathsf{A}^{\mathbf{p}_0} \in \mathcal{R}_{\mathbf{p}_0}$. Thus, $\mathsf{A} = \mathsf{A}^{\mathbf{p}'\mathbf{p}_0}\mathsf{A}^{\mathbf{p}_0}(\mathsf{A}^{\mathbf{p}\mathbf{p}_0})^{-1}$, and

$$\begin{aligned}
T(\mathsf{A})\left|\psi^{\mu}_{\mathbf{p}}\right\rangle &\equiv \mathbf{A}\left|\psi^{\mu}_{\mathbf{p}}\right\rangle = \mathbf{A}^{\mathbf{p}'\mathbf{p}_0}\mathbf{A}^{\mathbf{p}_0}\left(\mathbf{A}^{\mathbf{p}\mathbf{p}_0}\right)^{-1}\left|\psi^{\mu}_{\mathbf{p}}\right\rangle = \mathbf{A}^{\mathbf{p}'\mathbf{p}_0}\mathbf{A}^{\mathbf{p}_0}\left|\psi^{\mu}_{\mathbf{p}_0}\right\rangle \\
&= \mathbf{A}^{\mathbf{p}'\mathbf{p}_0}\sum_{\nu} T_{\nu\mu}\left(\mathsf{A}^{\mathbf{p}_0}\right)\left|\psi^{\nu}_{\mathbf{p}_0}\right\rangle = \sum_{\nu} T_{\nu\mu}\left(\mathsf{A}^{\mathbf{p}_0}\right)\mathbf{A}^{\mathbf{p}'\mathbf{p}_0}\left|\psi^{\nu}_{\mathbf{p}_0}\right\rangle \\
&= \sum_{\nu} T_{\nu\mu}\left(\mathsf{A}^{\mathbf{p}_0}\right)\left|\psi^{\nu}_{\mathbf{p}'}\right\rangle = \sum_{\nu} T_{\nu\mu}\left(\mathsf{A}^{\mathbf{p}_0}\right)\left|\psi^{\nu}_{\mathsf{A}\mathbf{p}}\right\rangle.
\end{aligned}$$

[8]The reader should be warned that although such a rotation does not change $\mathbf{p}$, the rotation *operator* may change the state $|\psi_{\mathbf{p}}\rangle$. However, the resulting state will be an eigenstate of the $\mathbf{P}_k$'s with eigenvalue $\mathbf{p}$.

[9]We are using the fact that $O(p, n-p)$ is transitive (see Problem 30.15).

Note how the matrix elements of the representation of *the little group alone* have entered in the last line. We therefore consider

$$T(\mathsf{A})\big|\psi_{\mathbf{p}}^{\mu}\big\rangle \equiv \sum_{\nu} R_{\nu\mu}\big(\mathsf{A}^{\mathbf{p}_0}\big)\big|\psi_{\mathsf{A}\mathbf{p}}^{\nu}\big\rangle, \quad \text{where} \quad \begin{cases} \mathsf{A} = \mathsf{A}^{\mathbf{p}'\mathbf{p}_0}\mathsf{A}^{\mathbf{p}_0}\big(\mathsf{A}^{\mathbf{p}\mathbf{p}_0}\big)^{-1}, \\ \mathbf{p}' \equiv \mathsf{A}\mathbf{p}. \end{cases} \tag{30.26}$$

To avoid confusion, we have used R for the representation of the little group. We claim that Eq. (30.26) defines a (matrix) representation of the whole group. In fact,

$$\begin{aligned} T(\mathsf{A}_1)T(\mathsf{A}_2)\big|\psi_{\mathbf{p}}^{\mu}\big\rangle &= T(\mathsf{A}_1)\sum_{\nu} R_{\nu\mu}\big(\mathsf{A}_2^{\mathbf{p}_0}\big)\big|\psi_{\mathsf{A}_2\mathbf{p}}^{\nu}\big\rangle \\ &= \sum_{\nu} R_{\nu\mu}\big(\mathsf{A}_2^{\mathbf{p}_0}\big)\sum_{\rho} R_{\rho\nu}\big(\mathsf{A}_1^{\mathbf{p}_0}\big)\big|\psi_{\mathsf{A}_1\mathsf{A}_2\mathbf{p}}^{\rho}\big\rangle \\ &= \sum_{\rho}\underbrace{\bigg(\sum_{\nu} R_{\rho\nu}\big(\mathsf{A}_1^{\mathbf{p}_0}\big)R_{\nu\mu}\big(\mathsf{A}_2^{\mathbf{p}_0}\big)\bigg)}_{=R_{\rho\mu}(\mathsf{A}_1^{\mathbf{p}_0}\mathsf{A}_2^{\mathbf{p}_0})\text{ since } R \text{ is a rep.}}\big|\psi_{\mathsf{A}_1\mathsf{A}_2\mathbf{p}}^{\rho}\big\rangle. \end{aligned}$$

The reader may check that $\mathsf{A}_1^{\mathbf{p}_0}\mathsf{A}_2^{\mathbf{p}_0} \equiv (\mathsf{A}_1\mathsf{A}_2)^{\mathbf{p}_0}$. Therefore,

$$T(\mathsf{A}_1)T(\mathsf{A}_2)\big|\psi_{\mathbf{p}}^{\mu}\big\rangle = \sum_{\rho} R_{\rho\mu}\big((\mathsf{A}_1\mathsf{A}_2)^{\mathbf{p}_0}\big)\big|\psi_{\mathsf{A}_1\mathsf{A}_2\mathbf{p}}^{\rho}\big\rangle \equiv T(\mathsf{A}_1\mathsf{A}_2)\big|\psi_{\mathbf{p}}^{\mu}\big\rangle,$$

and T is indeed a representation. It turns out that if R is irreducible, then so is T. The discussion above shows that the irreducible representations of the Poincaré group are entirely determined by those of the little group and Eq. (30.25). The recipe for the construction of the irreducible representations of $\mathfrak{p}(p, n-p)$ is now clear:

Theorem 30.3.9 *Choose any simultaneous eigenvector* $\mathbf{p}_0$ *of the* P_k*'s. Find the little algebra* $\mathfrak{R}_{\mathbf{p}_0}$ *at* $\mathbf{p}_0$ *by finding all* M_{ij}*'s satisfying* $\mathsf{M}_{ij}\mathbf{p}_0 = 0$. *Find all irreducible representations of* $\mathfrak{R}_{\mathbf{p}_0}$. *The same eigenvalues that label the irreducible representations of* $\mathfrak{R}_{\mathbf{p}_0}$ *can be used, in addition to those of* $\mathbf{P}^2$ *and* $\mathbf{W}^2$, *to label the irreducible representations of* $\mathfrak{p}(p, n-p)$.

We are particularly interested in $\mathfrak{p}(3, 1)$, the symmetry group of the special theory of relativity. In applying the formalism developed above, we need to make contact with the physical world. This always involves interpretations. Borrowing from the angular momentum theory, in which a physical system was given the attribute of angular momentum, the label of the irreducible representation of the rotation group, we attribute the labels of an irreducible representation of the Poincaré group, i.e., the eigenvalues of the four translation generators and the two Casimir operators, to a physical system. Since the four translation generators are identified as the three components of momentum and energy, and their specification implies their constancy over time, we have to come to the conclusion that

Box 30.3.10 *An irreducible representation of the Poincaré group specifies a free relativistic particle.*

There may be some internal interactions between constituents of a (composite) particle, e.g. between quarks inside a proton, but as a whole, the composite will be interpreted as a single particle. To construct the little group, we have to specify a 4-momentum $\mathbf{p}_0$. We shall consider two cases: In the first case, $\mathbf{p}_0 \cdot \mathbf{p}_0 \neq 0$, whereby the particle is deduced to be massive and we can choose[10] $\mathbf{p}_0 = (0, 0, 0, m)$. In the second case, $\mathbf{p}_0 \cdot \mathbf{p}_0 = 0$, in which case the particle is massless, and we can choose $\mathbf{p}_0 = (p, 0, 0, p)$. We consider these two cases separately.

The little group (really, the little Lie algebra) for $\mathbf{p}_0 = (0, 0, 0, m)$ is obtained by searching for those rotations that leave $\mathbf{p}_0$ fixed. This is equivalent to searching for M_{ij}'s that annihilate $(0, 0, 0, m)$, namely, the solutions to

$$(\mathsf{M}_{ij}\mathbf{p}_0)_l = (\mathsf{M}_{ij})_{lr}(\mathbf{p}_0)^r = (\mathsf{M}_{ij})_{l0}m = 0 \quad \Rightarrow \quad (\mathsf{M}_{ij})_{l0} = 0.$$

Since $(\mathsf{M}_{ij})_{l0} = \eta_{i0}\eta_{jl} - \eta_{j0}\eta_{il}$, we conclude that $(\mathsf{M}_{ij})_{l0} = 0$ if and only if $i \neq 0$ and $j \neq 0$. Thus the little group is generated by $(\mathsf{M}_{23}, \mathsf{M}_{31}, \mathsf{M}_{12})$ which are the components of angular momentum. The reader may also verify directly that when the 4-momentum has only a time component, the Casimir operator $\mathbf{W}^2$ reduces essentially to the total angular momentum operator. Since we are dealing with a single particle, the total angular momentum can only be spin. Therefore, we have the following theorem.

Theorem 30.3.11 *In the absence of any interactions, a massive relativistic particle is specified by its mass m and its spin s, the former being any positive number, the latter taking on integer or half-odd-integer values.*

The case of the massless particle can be handled in the same way. We seek those M_{ij}'s that annihilate $(p, 0, 0, p)$, namely, the solutions to

$$(\mathsf{M}_{ij}\mathbf{p}_0)_k = (\mathsf{M}_{ij})_{kr}(\mathbf{p}_0)^r = (\mathsf{M}_{ij})_{k0}p + (\mathsf{M}_{ij})_{k1}p = 0.$$

The reader may check that

$$\begin{aligned}
&(\mathsf{M}_{01}\mathbf{p}_0)_k = \eta_{1k}p - \eta_{0k}p, \qquad &&(\mathsf{M}_{02}\mathbf{p}_0)_k = \eta_{2k}p, \qquad &&(\mathsf{M}_{03}\mathbf{p}_0)_k = \eta_{3k}p, \\
&(\mathsf{M}_{23}\mathbf{p}_0)_k = 0, &&(\mathsf{M}_{12}\mathbf{p}_0)_k = \eta_{2k}p, &&(\mathsf{M}_{13}\mathbf{p}_0)_k = \eta_{3k}p.
\end{aligned}$$

[10] We use units in which $c = 1$.

Clearly, M_{23} is one of the generators of the little group. Subtracting the middle terms and the last terms of each line, we see that $\mathsf{M}_{02} - \mathsf{M}_{12}$ and $\mathsf{M}_{03} - \mathsf{M}_{13}$ are the other two generators. These happen to be the components of W. In fact, it is easily verified that

$$\begin{aligned} &W^0 = W^1 = \mathsf{M}_{23} p, \qquad W^2 = 2p(\mathsf{M}_{13} - \mathsf{M}_{03}), \\ &W^3 = 2p(\mathsf{M}_{02} - \mathsf{M}_{12}). \end{aligned} \tag{30.27}$$

Therefore, the little group is generated by all the components of W. Furthermore, $\mathbf{W}^2$ has zero eigenvalue for $|\psi_{\mathbf{p}_0}\rangle$ when $\mathbf{p}_0 = (p, 0, 0, p)$. Since both Casimir operators annihilate the state $|\psi_{\mathbf{p}_0}\rangle$, we need to come up with another way of labeling the states.

Historical Notes

Eugene Paul Wigner
1902–1995

Eugene Paul Wigner (1902–1995) was the second of three children born to Hungarian Jewish parents in Budapest. His father operated a large leather tannery and hoped that his son would follow him in that vocation, but the younger Wigner soon discovered both a taste and an aptitude for mathematics and physics. Although Wigner tried hard to accommodate his father's wishes, he clearly heard his calling, and the world of physics is fortunate that he did.

Wigner began his education in what he said "may have been the finest high school in the world." He later studied chemical engineering and returned to Budapest to apply that training in his father's tannery. He kept track of the seminal papers during the early years of quantum theory and, when the lure of physics became too strong, returned to Berlin to work in a crystallography lab. He lectured briefly at the University of Göttingen before moving to America to escape the Nazis.

Wigner accepted a visiting professorship to Princeton in 1930. When the appointment was not made permanent, the disappointed young professor moved to the University of Wisconsin, where he served happily until his new wife died suddenly of cancer only a few months after their marriage. As Wigner prepared, quite understandably, to leave Wisconsin, Princeton corrected its earlier mistake and offered him a permanent position. Except for occasional visiting appointments in America and abroad, he remained at Princeton until his death.

Wigner's contributions to mathematical physics began during his studies in Berlin, where his supervisor suggested a problem dealing with the symmetry of atoms in a crystal. John von Neumann, a fellow Hungarian physicist, pointed out the relevance of papers by Frobenius and Schur on representation theory. Wigner soon became enamored with the group theory inherent in the problem and began to apply that approach to quantum mechanical problems. Largely at the urging of Leo Szilard (another Hungarian physicist and Wigner's best friend), Wigner collected many of his results into the classic textbook *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*.

The decades that followed were filled with important contributions to mathematical physics, with applications of group theory comprising a large share: angular momentum; nuclear physics and $SU(4)$ or "supermultiplet" theory; parity; and studies of the Lorentz group and Wigner's classic definition of an elementary particle. Other work included early efforts in many-body theory and a paper on level spacings derived from the properties of Hermitian matrices that later proved useful to workers in quantum chaos.

As with most famous figures, Wigner's personality became as well known as his professional accomplishments. His insistence on "reasonable" behavior, for instance, made him refuse to pay a relative's hospital bill until after the patient was released—it was obviously unreasonable to hold a sick person hostage. His gentleness is exemplified in an anecdote in which on getting into an argument about a tip with a New York City cab driver, Wigner loses his patience, stamps his foot, and says, "Oh, go to hell, ...*please!*"

He held others' feelings in such high regard that it was said to be impossible to follow Wigner through a door. He was light-hearted and fun-loving, but also devoted to his family and concerned about the future of the planet. This combination of exceptional skill and laudable humanity ensures Wigner's place among the most highly regarded of his field.

(Taken from E. Vogt, *Phys. Today* **48** (12) (1995) 40–44.)

Define the new quantities

$$H_{\pm} \equiv \frac{1}{2}(W_1 \pm i W_2), \qquad H_0 \equiv \frac{1}{2p} W_0$$

and the corresponding operators acting on the carrier space. From Eq. (30.27), it follows that $[\mathbf{W}_1, \mathbf{W}_2] = 0$, $\mathbf{W}^2 = -4\mathbf{H}_+\mathbf{H}_-$, and that

$$[\mathbf{H}_+, \mathbf{H}_0] = -\mathbf{H}_+, \qquad [\mathbf{H}_+, \mathbf{H}_0] = \mathbf{H}_-, \qquad [\mathbf{H}_+, \mathbf{H}_-] = 0.$$

Denote the eigenstates of $\mathbf{W}^2$ and $\mathbf{H}_0$ by $|\alpha, \beta\rangle$:

$$\mathbf{W}^2|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle, \qquad \mathbf{H}_0|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle.$$

Then the reader may check that $\mathbf{H}_\pm|\alpha, \beta\rangle$ has eigenvalues α and $\beta \pm 1$. By applying $\mathbf{H}_\pm$ repeatedly, we can generate all eigenvalues of $\mathbf{H}_0$ and note that they are of the form

$$\beta = r + n, \quad \text{where} \quad n = 0, \pm 1, \pm 2, \ldots \text{ and } 1 > r \geq 0.$$

Since $\mathbf{H}_0 = \mathbf{M}_{23}$, $\mathbf{H}_0$ is recognized as an angular momentum operator whose eigenvalues are integer (for bosons) and half-odd integer (for fermions). Therefore, $r = 0$ for bosons and $r = \frac{1}{2}$ for fermions.

Now, within an irreducible representation, only those $|\alpha, \beta\rangle$'s can occur that have the same α. Therefore, if we relabel the β values by integers, then

$$\langle \alpha, n|\mathbf{H}_0|\alpha, m\rangle = (r + n)\delta_{nm}.$$

Similarly,

$$\langle \alpha, n|\mathbf{H}_+|\alpha, m\rangle = a_n \delta_{n,m+1},$$
$$\langle \alpha, n|\mathbf{H}_-|\alpha, m\rangle = b_n \delta_{n,m-1},$$

where a_n and b_n are some constants. It follows that

$$\begin{aligned}\alpha = \langle \alpha, n|\mathbf{W}^2|\alpha, n\rangle &= \langle \alpha, n|\mathbf{H}_+\mathbf{H}_-|\alpha, n\rangle \\ &= \langle \alpha, n|\mathbf{H}_+|\alpha, n-1\rangle\langle \alpha, n-1|\mathbf{H}_-|\alpha, n\rangle \\ &= a_n b_n.\end{aligned}$$

If we assume that the representation is unitary, then all $\mathbf{W}_j$'s will be hermitian, $(\mathbf{H}_+)^\dagger = \mathbf{H}_-$, so $a_n = b_n^*$ and $\alpha = |a_n|^2 \geq 0$.

If $\alpha = 0$, then $a_n = 0$ and $b_n = 0$ for all n. Consequently, $\mathbf{H}_+ = 0 = \mathbf{H}_-$, i.e., there are no raising or lowering operators. It follows that there are only two spin states, corresponding to the maximum and the minimum eigenvalues of $\mathbf{H}_0$. A natural axis for the projection of spin is the direction of motion of the particle. Then the projection of spin is called **helicity**. We summarize our discussion in the following theorem.

helicity of massless particles

Theorem 30.3.12 *In the absence of any interactions, a massless relativistic particle is specified by its spin and its helicity. The former taking on integer or half-odd-integer values s, the latter having values $+s$ and $-s$.*

Theorems 30.3.11 and 30.3.12 are beautiful examples of the fruitfulness of the interplay between mathematics and physics. Physics has provided mathematics with a group, the Poincaré group, and mathematics, through its theory of group representation, has provided physics with the deep result that all particles must have a spin that takes on a specific value, and none other; that massive particles are allowed to have $2s + 1$ different values for the projection of their spin; and that massless particles are allowed to have only two values for their spin projection. Such far-reaching results that are both universal and specific makes physics unique among all other sciences. It also provides impetus for the development of mathematics as the only dialect through which nature seems to communicate to us her deepest secrets.

If $\alpha > 0$, then the resulting representations will have continuous spin variables. Such representations do not correspond to particles found in nature; therefore, we shall not pursue them any further.

30.4 Problems

30.1 Show that the operation on a compact group defined by

$$(u|v) \equiv \int_G \langle \mathbf{T}_g u | \mathbf{T}_g v \rangle d\mu_g$$

is an inner product.

30.2 Show that the Weyl operator $\mathbf{K}_u$ is hermitian.

30.3 Derive Eqs. (30.5) and (30.6). Hint: Follow the finite-group analogy.

30.4 Suppose that a Lie group G acts on a Euclidean space $\mathbb{R}^n$ as well as on the space of (square-integrable) functions $\mathcal{L}(\mathbb{R}^n)$. Let $\phi_i^{(\alpha)}$ transform as the ith row of the αth irreducible representation. Verify that the relation

$$\mathbf{T}_g \phi_i^{(\alpha)}(\mathbf{x}) = \sum_{j=1}^{n_\alpha} T_{ji}^{(\alpha)}(g) \phi_j^{(\alpha)}\left(\mathbf{x} \cdot g^{-1}\right)$$

defines a representation of G.

30.5 Show that $GL(\mathcal{V})$ is not a compact group. Hint: Find a continuous function $GL(\mathcal{V}) \to \mathbb{C}$ whose image is not compact.

30.6 Suppose that $T : G \to GL(\mathcal{V})$ is a representation, and let

$$\mathcal{V}^{\otimes r} \equiv \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{r \text{ times}}$$

be the r-fold tensor product of $\mathcal{V}$. Show that $T^{\otimes r} : G \to GL(\mathcal{V}^{\otimes r})$, given by

$$\mathbf{T}_g^{\otimes r}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \mathbf{T}_g(\mathbf{v}_1) \otimes \cdots \otimes \mathbf{T}_g(\mathbf{v}_r),$$

is also a representation.

30.7 Suppose that in Example 30.2.2, we set $k_1 = 2$ for our treatment of $n = 2, r = 3$. Show that $\mathbf{Y}_2(\mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \mathbf{e}_{k_3})$ does not produce any new vector beyond what we obtained for $k_1 = 1$.

30.8 Show that $g^{ij} g^{sr} c_{iks}$ is antisymmetric in j and r.

30.9 Operate $\mathbf{L}_+$ on

$$|00\rangle = \sum_{m=-l}^{l} C(ll;0|m,-m;0)|lm;l,-m\rangle$$

and use $\mathbf{L}_+|00\rangle = 0$ to find a recursive relation among $C(ll;0|m,-m;0)$. Use normalization and the convention that $C(ll;0|m,-m;0) > 0$ to show that

$$C(ll;0|m,-m;0) = (-1)^{l-m}/\sqrt{2l+1}$$

(see Sect. 13.3).

30.10 Show that the generators of $\mathfrak{so}(3,1)$,

$$\mathbf{M} \equiv (M_1, M_2, M_3) \equiv (\mathsf{M}_{23}, \mathsf{M}_{31}, \mathsf{M}_{12}),$$

$$\mathbf{N} \equiv (N_1, N_2, N_3) \equiv (\mathsf{M}_{01}, \mathsf{M}_{02}, \mathsf{M}_{03}),$$

satisfy the commutation relations

$$[M_i, M_j] = -\epsilon_{ijk} M_k, \qquad [N_i, N_j] = \epsilon_{ijk} M_k, \qquad [M_i, N_j] = -\epsilon_{ijk} N_k,$$

and that $\mathbf{M}^2 - \mathbf{N}^2$ and $\mathbf{M} \cdot \mathbf{N}$ commute with all the M's and the N's.

30.11 Let the double-indexed "metric" of the Poincaré algebra be defined as

$$g_{ij,kl} = c_{ij,mn}^{rs} c_{kl,rs}^{mn} + c_{ij,m}^{r} c_{kl,r}^{m},$$

where the structure constants are given in Eq. (30.21). Show that

$$g_{ij,kl} = 2(n-1)(\eta_{jk}\eta_{il} - \eta_{ik}\eta_{jl}).$$

30.12 Show that $[\mathbf{M}^2, \mathbf{M}^{ij}] = 0$, and

$$\big[\mathbf{M}^2, \mathbf{P}_k\big] = 4\mathbf{M}_{kj}\mathbf{P}^j + 2(n-1)\mathbf{P}_k.$$

30.13 Show that the vector operator

$$\mathbf{C}_i \equiv \mathbf{M}_{ij}\mathbf{P}^j = \eta^{kj}\mathbf{M}_{ij}\mathbf{P}_k$$

satisfies the following commutation relations:

$$[\mathbf{C}_i, \mathbf{P}_j] = \eta_{ij}\mathbf{P}^2 - \mathbf{P}_i\mathbf{P}_j, \qquad [\mathbf{C}_i, \mathbf{M}_{jk}] = \eta_{ik}\mathbf{C}_j - \eta_{ij}\mathbf{C}_k,$$
$$[\mathbf{C}_i, \mathbf{C}_j] = \mathbf{M}_{ij}\mathbf{P}^2.$$

Show also that $[\mathbf{C}^2, \mathbf{M}_{jk}] = 0$, $\mathbf{C}^i\mathbf{P}_i = 0$, and

$$\mathbf{P}^i\mathbf{C}_i = -(n-1)\mathbf{P}^2, \qquad \big[\mathbf{C}^2, \mathbf{P}_i\big] = \big\{2\mathbf{C}_i + (n-1)\mathbf{P}_i\big\}\mathbf{P}^2.$$

30.14 Derive Eq. (30.24) and show that $\mathbf{W}^{i_1 \dots i_{n-3}}$ commutes with all the $\mathbf{P}_j$'s.

30.15 Let $\hat{\mathbf{e}}_x \equiv (x_1, \dots, x_n)$ be any unit vector in $\mathbb{R}^n$.

(a) Show that a matrix is $\boldsymbol{\eta}$-orthogonal, i.e., it satisfies Eq. (29.38), if and only if its columns are $\boldsymbol{\eta}$-orthogonal.

(b) Show that there exists an $\mathsf{A} \in O(p, n-p)$ such that $\hat{\mathbf{e}}_x = \mathsf{A}\hat{\mathbf{e}}_1$ where $\hat{\mathbf{e}}_1 = (1, 0, \dots, 0)$. Hint: Find the first column of A and use (a).

(c) Conclude that $O(p, n-p)$ is transitive in its action on the collection of all vectors of the same length.

30.16 Verify directly that when the 4-momentum has only a time component, the Casimir operator $\mathbf{W}^2 = \mathbf{W} \cdot \mathbf{W}$ reduces essentially to the total angular momentum operator.

30.17 Verify that for the case of a massless particle, when $\mathbf{p}_0 = (p, 0, 0, p)$,

$$W_0 = W_1 = \mathsf{M}_{23}, \qquad W_2 = 2p(\mathsf{M}_{13} - \mathsf{M}_{03}), \qquad W_3 = 2p(\mathsf{M}_{02} - \mathsf{M}_{12}),$$

and that $\mathbf{W}^2 = \mathbf{W} \cdot \mathbf{W}$ annihilates $|\psi_{\mathbf{p}_0}\rangle$.