25 Representations of the Symmetric Group

The symmetric (permutation) group is an important prototype of finite groups. In fact, **Cayley's theorem** (see [Rotm 84, p. 46] for a proof) states that any finite group of order n is isomorphic to a subgroup of S_n. Moreover, the representation of S_n leads directly to the representation of many of the Lie groups encountered in physical applications. It is, therefore worthwhile to devote some time to the analysis of the representations of S_n.

Cayley's theorem

25.1 Analytic Construction

The starting point of the construction of representations of the symmetric group is Eq. (24.33), which is valid for any finite group. There is one simple character that every group has, namely, the character of the one-dimensional symmetric representation in which all elements of the group are mapped to $1 \in \mathbb{R}$. Setting $\xi_j^{(\kappa)} = 1$ in (24.33), and noting that $\sum_j d_j = d_i$, we obtain

$$\psi_i^H \equiv \frac{|G| d_i}{|H| c_i}, \tag{25.1}$$

where $\{\psi_i^H\}$ are the components of a compound character of G.

Frobenius has shown that by a clever choice of H, one can completely solve the problem of the construction of the irreducible representations of S_n. The interested reader may refer to [Hame 89, pp. 189–192] for details. We are really interested in the simple characters of S_n, and Frobenius came up with a powerful method of calculating them. Since there is a one-to-one correspondence between the irreducible representations and conjugacy classes, and another one between conjugacy classes of S_n and partitions of n, we shall label the simple characters of S_n by partitions of n. Thus, instead of our common notation $\chi_i^{(\alpha)}$, we use $\chi_{(l)}^{(\lambda)}$, where (λ) denotes a partition of n, and (l) a cycle structure of S_n.

Suppose we want to find the irreducible characters corresponding to the cycle structure $(l) = (1^\alpha, 2^\beta, 3^\gamma, \dots)$. These form a *column* under the class (l) in a character table. To calculate the irreducible characters, form two polynomials in $(x_1, x_2, \dots, x_n)$ as follows. The first one, which is com-

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_25,

pletely symmetric in all variables, is

$$s_{(l)} \equiv \left(\sum_{i=1}^{n} x_i\right)^{\alpha} \left(\sum_{i=1}^{n} x_i^2\right)^{\beta} \left(\sum_{i=1}^{n} x_i^3\right)^{\gamma} \cdots . \tag{25.2}$$

The second one is completely antisymmetric, and can be written as

$$D(x_1, \ldots, x_n) \equiv \prod_{i<j} (x_i - x_j) = \sum_{\pi} \epsilon_\pi x_{\pi(1)}^{n-1} x_{\pi(2)}^{n-2} \cdots x_{\pi(n-1)} x_{\pi(n)}^{0}. \tag{25.3}$$

It can be shown that the simple characters of S_n are coefficients of certain terms of the product of these polynomials. To be exact, we have

$$\begin{aligned} &s_{(l)} D(x_1, \ldots, x_n) \\ &\quad = \sum_{(\lambda)} \chi_{(l)}^{(\lambda)} \sum_{\pi} \epsilon_\pi x_{\pi(1)}^{\lambda_1+n-1} x_{\pi(2)}^{\lambda_2+n-2} \ldots x_{\pi(n-1)}^{\lambda_{n-1}+1} x_{\pi(n)}^{\lambda_n}. \end{aligned} \tag{25.4}$$

The outer sum goes over all partitions of n, the inner sum over all permutations of S_n. The procedure for finding the simple characters of S_n should now be clear from (25.4):

Proposition 25.1.1 *To find the simple character $\chi_{(1^\alpha 2^\beta \ldots)}^{(\lambda_1 \ldots \lambda_n)}$, construct the corresponding symmetric and antisymmetric polynomials of* (25.2) *and* (25.3), *multiply them together, collect all terms of the form*

$$x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_{n-1}^{\lambda_{n-1}+1} x_n^{\lambda_n}.$$

The coefficient of such a term is the desired character.

Example 25.1.2 The best way to understand the procedure described above is to go through an example in detail. We calculate the characters of S_3 using the above method. Label the rows of the character table with the partitions of 3. These are (3), (2, 1), and (1, 1, 1). Similarly, label the columns with the conjugacy classes, or cycle structures: (1^3), $(1, 2)$, and (3). The first cycle structure has $\alpha = 3$, $\beta = 0 = \gamma$. Therefore,

$$\begin{aligned} s_{(1^3)} &= (x_1 + x_2 + x_3)^3 = x_1^3 + x_2^3 + x_3^3 \\ &\quad + 3\big(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2\big) + 6 x_1 x_2 x_3 \end{aligned} \tag{25.5}$$

and

$$\begin{aligned} D(x_1, x_2, x_3) &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ &= x_1^2 x_2 - x_1^2 x_3 - x_2^2 x_1 + x_2^2 x_3 - x_3^2 x_2 + x_3^2 x_1. \end{aligned} \tag{25.6}$$

Now we note that for $(\lambda) = (3)$, $\lambda_1 = 3$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Therefore, the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n} = x_1^5 x_2$ gives $\chi_{(1^3)}^{(3)}$. Similarly, for $(\lambda) = (2, 1, 0)$, $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, and the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n} = x_1^4 x_2^2$ gives $\chi_{(1^3)}^{(2,1)}$. Finally, for $(\lambda) = (1, 1, 1)$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and the coefficient of $x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n} = x_1^3 x_2^2 x_3$ gives $\chi_{(1^3)}^{(1,1,1)}$. These coefficients can be read off by scanning through Eq. (25.5) while multiplying its terms by those of Eq. (25.6) and keeping track of the coefficients of the products of the relevant powers of x_1, x_2, and x_3. The reader may verify that there is only one term of the form $x_1^5 x_2$, whose coefficient is 1, giving $\chi_{(1^3)}^{(3)} = 1$; there are two terms of the form $x_1^4 x_2^2$, whose coefficients are -1 and 3, giving $\chi_{(1^3)}^{(2,1)} = 2$; and there are four terms of the form $x_1^3 x_2^2 x_3$, whose coefficients are $+1$, -3, -3, and $+6$, giving $\chi_{(1^3)}^{(1,1,1)} = 1$. Therefore, the first column of the character table of S_3 is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

To obtain the second column, we consider the second conjugacy class, $(1, 2)$, with $\alpha = 1 = \beta$ and $\gamma = 0$. The corresponding symmetric polynomial is

$$\begin{aligned} s_{(1,2)} &= (x_1 + x_2 + x_3)\left(x_1^2 + x_2^2 + x_3^2\right) \\ &= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2 x_3^2 + x_1 x_3^2 + x_2 x_3^2. \end{aligned} \tag{25.7}$$

$D(x_1, x_2, x_3)$ is the same as before. Multiplying and keeping track of the coefficients of $x_1^5 x_2$, $x_1^4 x_2^2$, and $x_1^3 x_2^2 x_3$, we obtain $\chi_{(1,2)}^{(3)} = 1$, $\chi_{(1,2)}^{(2,1)} = 0$, and $\chi_{(1,2)}^{(1,1,1)} = -1$. It follows that the second column of the character table of S_3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

The last column is obtained similarly. We note that $\alpha = 0 = \beta$, and $\gamma = 1$. Therefore, the symmetric polynomial is

$$s_{(3)} = x_1^3 + x_2^3 + x_3^3,$$

and the antisymmetric polynomial is the same as before. Multiplying these two polynomials and extracting the coefficients as before, we get $\chi_{(3)}^{(3)} = 1$, $\chi_{(3)}^{(2,1)} = -1$, and $\chi_{(3)}^{(1,1,1)} = 1$. It follows that the third column of the character table of S_3 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Collecting all the data obtained above, we can reconstruct the character table of S_3. This is shown in Table 25.1. The irreducible representations are labeled by the three possible partitions of 3, and the conjugacy classes by the three cycle structures.

Table 25.1 The character table for S_3. Each column corresponds to a conjugacy class, each row to a partition of 3. The last two rows have been switched compared to Table 24.4

	(1^3)	$(1,2)$	(3)
$T^{(3)}$	1	1	1
$T^{(2,1)}$	2	0	−1
$T^{(1,1,1)}$	1	−1	1

25.2 Graphical Construction

The analytic construction of the previous subsection can be handled using graphical techniques that are considerably simpler. To begin with, let us find the character of the identity element of S_n. The cycle structure is (1^n), i.e., all cycles consist of a single element. Thus, $\alpha = n$, and β, γ, etc. are all zero. It follows that the LHS of Eq. (25.4) is $(\sum x_i)^n D(x_j)$. We calculate this product one power of $\sum x_i$ at a time. For the same reason as in the example above, $\chi^{(\lambda)}_{(1^n)}$ will be the coefficient of

$$x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \dots x_{n-1}^{\lambda_{n-1}+1} x_n^{\lambda_n}.$$

Historical Notes

Ferdinand Georg Frobenius 1849–1917

Ferdinand Georg Frobenius (1849–1917), the son of a parson, was born in Berlin and began his mathematical studies at Göttingen in 1867. He received his doctorate in Berlin three years later. Four years later, on the basis of his mathematical research, he was appointed assistant professor at the University of Berlin. He achieved the rank of full professor at the *Eidgenössische Polytechnikum Zürich* before returning to Berlin as a professor of mathematics in 1892. During the early years of Frobenius's career, modern group theory was in its infancy. He combined its three main branches of study—the theory of solutions to algebraic equations (permutation groups and the work of Galois), geometry (transformation and Lie groups), and number theory—to produce the concept of the abstract group. He collaborated with Issai Schur in representation and character theory of groups.

His paper *Über die Gruppencharactere* is of fundamental importance. It was presented to the Berlin Academy on 16 July 1896 and it contains work that Frobenius had done in the preceding few months. In a series of letters to Dedekind, the first on 12 April 1896, his ideas on group characters quickly develop, and Frobenius is able to construct a complete set of representations by complex numbers. In a letter to Dedekind on 26 April 1896 Frobenius finds the irreducible characters for the alternating group, and the symmetric groups.

In 1897 Frobenius reformulated the work of Molien—the Latvian student of Klein, who, in his thesis, classified the semi-simple algebras using the method of group rings—in terms of matrices and then showed that his characters are the traces of the irreducible representations. Frobenius's character theory found important applications in quantum mechanics and was used with great effect by Burnside, who wrote it up in the 1911 edition of his *Theory of Groups of Finite Order*.

Frobenius is also remembered as the originator of a series method for solving ordinary differential equations. Despite the clearly greater importance of his work in group theory, this *method of Frobenius* serves admirably to perpetuate his name.

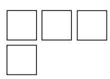
If we multiply $D(x_j)$ by $\sum x_i$ one x at a time, we increase the power of one of the x_i's by one. If at any stage, two of the exponents become equal, the term must vanish, due to the antisymmetry of $(\sum x_i)D(x_j)$. Therefore, as we raise the degree of the polynomial by one at each stage, the power of x_1 must be raised at least as fast as x_2, and the power of x_2 must be raised at least as fast as x_3, etc. Our goal is to raise the power of x_1 by λ_1, that of x_2 by λ_2, and, in general, the power of x_i by λ_i, making sure that at each stage, the number of multiplications by x_1 is greater than or equal to the number of multiplications by x_2, etc. The total number of ways by which we can reach this goal will be $\chi^{(\lambda)}_{(1^n)}$, which is also the dimension of the irreducible representation (λ) by Eq. (24.9).

To see the argument more clearly, suppose that we are interested in the dimension of the irreducible representation of S_4 corresponding to $(3, 1)$. Then we must raise the power of x_1 by 3 and the power of x_2 by 1; x_3 and x_4 will remain intact, and therefore will not enter in the following discussion. It follows that $D(x_j)$ is to be multiplied by $x_1^3 x_2$, one x-factor at a time, the number of x_1-factors always exceeding the number of x_2-factors. The possible ways of doing this are

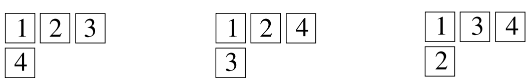
$$x_1^3 x_2, \quad x_1^2 x_2 x_1, \quad x_1 x_2 x_1^2. \tag{25.8}$$

Note that as we count the factors from left to right, the number of x_1's is always greater than or equal to the number of x_2's. Thus $x_2 x_1^3$ is absent because x_2 occurs without x_1 occurring first. It follows that the dimension of the irreducible representation $(3, 1)$ is 3.

A graphical way to arrive at the same result is to draw $\lambda_1 = 3$ boxes on top and $\lambda_2 = 1$ box below it:



The next step is to fill in the boxes with numbers corresponding to the position of x_1 (filling up the first row) and x_2 factors (filling up the second row) in Eq. (25.8). Since in the first term of (25.8), the x_1's occupy the first, second, and third positions, we enter 1, 2, and 3 in the first row, and 4 in the second row corresponding to the last position occupied by x_2. Similarly, in the second term of (25.8), the x_1's occupy the first, second, and fourth positions; therefore, we enter 1, 2, and 4 in the first row, and 3 in the second row corresponding to the position occupied by x_2. Finally, in the last term of (25.8), the x_1's occupy the first, third, and fourth positions; therefore, we enter 1, 3, and 4 in the first row, and 2 in the second row corresponding to the position occupied by x_2. The result is the graph shown below:



Young frame defined

Definition 25.2.1 Let $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of n. The **Young frame** (or the **Young pattern**) associated with (λ) is a collection of rows of boxes (squares) aligned at the left such that the first row has λ_1 boxes, the second row λ_2 boxes, etc. Since $\lambda_i \geq \lambda_{i+1}$, the length of the rows decreases as one goes to the bottom of the frame.

The Young frame associated with (λ) represents $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$, which multiplies the antisymmetric polynomial $D(x_j)$. To find the dimension of the irreducible representation $T^{(\lambda)}$, we have to count the number of ways in which the x-factors can be permuted among themselves such that as we scan the product, the number of x_i's is never less than number of x_j's if $j > i$. This leads to

standard Young tableaux defined

Definition 25.2.2 A **standard Young tableau** (or diagram, or graph) is a Young frame filled with numbers 1 through n such that

1. the numbers are placed consecutively left to right on the rows starting with 1 in the far-left box of the first row;
2. no box of any row is to be filled unless all boxes to its left are already filled;
3. at each stage, the number of boxes filled in any row is never less than the number of boxes filled in the rows below it.

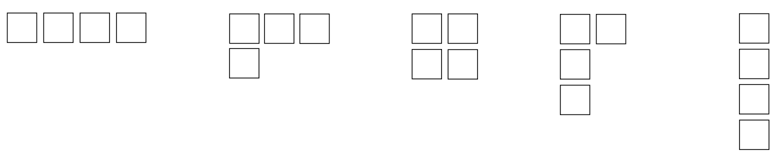
regular graphs

Tableaux satisfying the last condition are called **regular graphs**.

It follows that in a Young tableau, the number 1 is always in the upper left-hand box, and that going down in a column, the numbers must increase.

Theorem 25.2.3 *Let* (λ) *be a partition of* n. *Then the dimension of the irreducible representation* $T^{(\lambda)}$ *is equal to the number of standard Young tableaux associated with* (λ).

Example 25.2.4 We wish to calculate the dimension of each irreducible representation of S_4. The partitions are (4), (3, 1), (2, 2), (2, 1, 1), and (1, 1, 1, 1) whose associated Young frames are shown below:



(4) (3,1) (2,2) (2,1,1) (1,1,1,1)

The number of standard Young tableaux associated with $(\lambda) = (4)$ is 1, because there is only one way to place the numbers 1 through 4 in the four boxes. Thus, the dimension of $T^{(4)}$ is 1. For $(\lambda) = (3, 1)$, we can place 2 either to the right of 1 or below it. The first choice gives rise to two possibilities for the placement of 3: Either to the right of 2 or below 1. The second choice gives rise to only one possibility for 3, namely to the right of 1. With 1, 2, and 3 in place, the position of 4 is predetermined. Thus, we have 3 possibilities for $(\lambda) = (3, 1)$, and the dimension of $T^{(3,1)}$ is 3. For $(\lambda) = (2, 2)$, we can place 2 either to the right of 1 or below it. Both choices give rise to only one possibility for 3: In the first case, 3 can only go under 1; in the

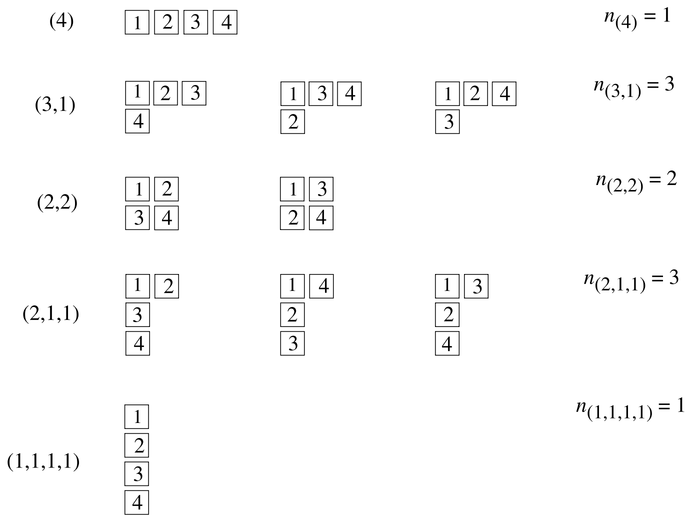


Fig. 25.1 The standard Young tableaux, and the dimensions of irreducible representations of S_4

second case to its right. With 1, 2, and 3 in place, the position of 4 is again predetermined. Thus, we have 2 possibilities for $(\lambda) = (2, 2)$, and the dimension of $T^{(3,1)}$ is 2. The reader may check that the dimension of $T^{(2,1,1)}$ is 3, and that of $T^{(1,1,1,1)}$ is 1. Figure 25.1 summarizes these findings. We note that the dimensions satisfy $1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24$, the second equation of (24.18).

25.3 Graphical Construction of Characters

The product of the symmetric polynomial $s_{(l)}$ and the antisymmetric polynomial $D(x_j)$ contains all the information regarding the representations of S_n. We can extract the simple characters by looking at the coefficients of appropriate products of the x-factors. This can also be done graphically. Without going into the combinatorics of the derivation of the results, we state the rules for calculating the simple characters, and examine one particular case in detail to elucidate the procedure, whose statement can be very confusing.

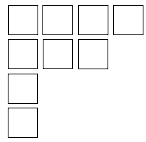
As before, we label the irreducible representations with the partitions of n. However, we separate out the common factors in a cyclic structure, labeling the cycles by l_1, l_2, etc. For example, $(2, 1^2)$ has $l_1 = 2$, $l_2 = 1$, and $l_3 = 1$. So, $(2, 1^2)$ becomes $(2, 1, 1)$, and in general, we write (l) as $(l_1, l_2, \ldots, l_m)$.

Definition 25.3.1 A **regular application** of r identical symbols to a Young frame is the placement of those symbols in the boxes of the frame as follows. Add the symbols to any given row, starting with the first (farthest to the left) unoccupied cell, until the symbols are all used or the number of filled boxes exceeds that of the preceding line by one. In the latter case, go to the

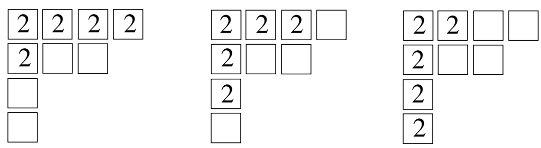
regular application

preceding line and repeat the procedure, making sure that the final result of adding all r symbols will be a regular graph. If the number of rows occupied by the symbols is odd (even) the application is **positive** (**negative**).

As an illustration, consider the regular application of five 2's to the blank Young frame shown below.



We cannot start on the first row because it does not have enough boxes for the five 2's. We can start on the second row and put one 2 in the first box. This brings the number of 2's in the second row to one more than in the first row; therefore, we should now go to the first row and put the rest of the symbols there. We could start at the third row, put one 2 in the first box, put a second 2 in the first box of the second row, and the rest in the first row. Altogether we will have 3 regular applications of the five 2's. These are shown in the diagram below.



Of these the first and the last tableaux are negative applications, and the middle one is positive.

Theorem 25.3.2 *The character of the irreducible representation $T^{(\lambda)}$ of the class $(l) = (l_1, l_2, \ldots, l_m)$ is obtained by successive regular applications of l_1 identical symbols (usually taken to be 1's), then l_2 identical symbols of a different kind (usually taken to be 2's), etc. The character $\chi_{(l)}^{(\lambda)}$ is then equal to the number of ways of building positive applications minus the number of ways of building negative applications.*

The order in which the l_i's are applied is irrelevant. However, it is usually convenient to start with the largest cycle.

Detailed analysis of the construction of the character table for S_4

The best way to understand the procedure is to construct a character table. Let us do this for S_4. As usual, the rows are labeled by the various partitions (λ) of 4. We choose the order (4), (3, 1), $(2, 2) = (2^2)$, $(2, 1, 1) = (2, 1^2)$, $(1, 1, 1, 1) = (1^4)$. The columns are labeled by classes (l) in the following order: (1^4), $(2, 1^2)$, (2^2), (3, 1), (4), where, for example, $(2, 1^2)$ means that $l_1 = 2$, $l_2 = 1$, and $l_3 = 1$. Example 25.2.4 gives us the first column of the character table. Similarly, the first row has 1 in all places, because it is the trivial representation. Our task is therefore to fill in the rest of the table one row at a time. The second row, with $(\lambda) = (3, 1)$, has a Young frame that looks like

$$\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array}$$

and for each class (column) labeled $(l_1, \ldots, l_m)$, we need to fill this in with l_1 identical symbols (1's), l_2 identical symbols of a different kind (2's), etc.

The second column has $l_1 = 2$, $l_2 = 1 = l_3$. So we have two 1's, one 2, and one 3. If we start with the first row, the two 1's can be placed in its first two boxes. If we start with the second row, the two 1's must be placed vertically on top of each other. In the first case, we have two choices for the 2: Either on the first row next to the two 1's, or on the second line. In the second case, we have only one choice for the 2: in the first row next to 1. With 1's and 2 in place, the position of 3 is determined. The three possibilities are shown below:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array}$$

The first two are positive applications, the third is negative because the 1's occupy an even number of rows. We therefore have

$$\chi^{(3,1)}_{(2,1^2)} = +1 + 1 - 1 = +1.$$

The third column has $l_1 = 2 = l_2$. So we have two 1's and two 2's. We place the 1's as before. When the two 1's are placed vertically, we can put the 2's on the first row and we are done. When the 1's are initially placed in the first row, we have no way of placing the 2's by regular application. We cannot start on the first row because there is only one spot available (remember, we cannot go down once we start at a row). We cannot start on the second row because once we place the first 2, we are blocked, and the number of symbols in the second row does not exceed that of the first row by one. So, there is only one possibility:

$$\underset{\text{Not allowed}}{\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}} \qquad \underset{\text{Allowed}}{\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}}$$

The only allowed diagram is obtained by a negative application of 1's. Therefore, $\chi^{(3,1)}_{(2^2)} = -1$.

The fourth column has $l_1 = 3$ and $l_2 = 1$. So we have three 1's and one 2. There are two ways to place the 1's: all on the first row, or starting on the second row and working our way up until all boxes are filled except the last box of the first row. The placement of 2 will be then predetermined. The result is the two diagrams shown below:

Table 25.2 Character table for S_4. The rows and columns are labeled by the partitions of 4 and cycle structures, respectively

	(1^4)	$(2, 1^2)$	(2^2)	$(3, 1)$	(4)
$T^{(4)}$	1	1	1	1	1
$T^{(3,1)}$	3	1	−1	0	−1
$T^{(2^2)}$	2	0	2	−1	0
$T^{(2,1^2)}$	3	−1	−1	0	1
$T^{(1^4)}$	1	−1	1	1	−1

$$\begin{array}{lll} \boxed{1}\boxed{1}\boxed{1} & \qquad & \boxed{1}\boxed{1}\boxed{2} \\ \boxed{2} & & \boxed{1} \end{array}$$

The first diagram is obtained by a positive application of 1's, the second by a negative application. Therefore,

$$\chi^{(3,1)}_{(3,1)} = +1 - 1 = 0.$$

Finally, for the last column, $l_1 = 4$. There is only one way to put all the 1's in the frame, and that is a negative application. Thus, $\chi^{(3,1)}_{(4)} = -1$.

Rather than going through the rest of the table in the same gory detail, we shall point out some of the trickier calculations, and leave the rest of the table for the reader to fill in. One confusion may arise in the calculation of $\chi^{(2^2)}_{(2^2)}$. The frame looks like this,

$$\begin{array}{l} \boxed{}\boxed{} \\ \boxed{}\boxed{} \end{array}$$

and we need to fill this with two 1's and two 2's. The 1's can go into the first row or the first column. The 2's then can be placed in the second row or the second column. The result is

$$\begin{array}{lll} \boxed{1}\boxed{1} & \qquad & \boxed{1}\boxed{2} \\ \boxed{2}\boxed{2} & & \boxed{1}\boxed{2} \end{array}$$

The first diagram has no negative application. The second has *two* negative applications, one for the 1's, and one for the 2's. Therefore, the overall sign for the second diagram is positive. It follows that $\chi^{(2^2)}_{(2^2)} = +1 + 1 = +2$.

The calculation of $\chi^{(2^2)}_{(4)}$ may also be confusing. We need to place four 1's in the frame. If we start on the first row, we are stuck, because there is room for only two 1's. If we start in the second row, then we can only go up: Putting the first 1 in the second row causes that row to have one extra 1 in comparison with the preceding row. However, once we go up, we have room for only two 1's (we cannot go back down). So, there is no way we can place the four 1's in the (2^2) frame, and $\chi^{(2^2)}_{(4)} = 0$.

The character table for S_4 is shown in Table 25.2 (see Problem 24.15 as well). The reader is urged to verify all entries not calculated above. The

Table 25.3 Character table for S_5. The rows and columns are labeled by the partitions of 5 and cycle structures, respectively

	(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,2)$	$(3,1^2)$	$(4,1)$	(5)
$T^{(5)}$	1	1	1	1	1	1	1
$T^{(4,1)}$	4	2	0	−1	1	0	−1
$T^{(3,2)}$	5	1	1	1	−1	−1	0
$T^{(3,1^2)}$	6	0	−2	0	0	0	1
$T^{(2^2,1)}$	5	−1	1	−1	−1	1	0
$T^{(2,1^3)}$	4	−2	0	1	1	0	−1
$T^{(1^5)}$	1	−1	1	−1	1	−1	1

character table for S_5 can also be calculated with only minor tedium. We quote the result here in Table 25.3 and let the reader check the entries of the table.

25.4 Young Operators

The group algebra techniques of Sect. 24.5—which we used in our discussion of representation theory in a very limited way—provide a powerful and elegant tool for unraveling the representations of finite groups. These techniques have been particularly useful in the analysis of the representations of the symmetric group. Our emphasis on the symmetric group is not merely due to the importance of S_n as a paradigm of all finite groups. It has also to do with the unexpected usefulness of the representations of S_n in studying the representations of $GL(\mathcal{V})$, the paradigm of all (classical) *continuous* groups. We shall come back to this observation later when we discuss representations of Lie groups in Chap. 30.

To begin with, consider the element of the S_n group algebra as defined in Eq. (24.20). Since multiplying P (on the left) by a group element does not change P, the ideal generated by P is not only one-dimensional, but all elements of S_n are represented by the number 1. Therefore, the ideal $\mathcal{A}P$ corresponds to the (irreducible) identity representation.

For S_n, there is another group algebra element that has similar properties. This is

$$Q = \sum_{i=1}^{n!} \epsilon_{\pi_i} \pi_i, \qquad \pi_i \in S_n. \tag{25.9}$$

The reader may check that

$$\pi_j Q = \epsilon_{\pi_j} Q \quad \text{and} \quad Q^2 = n! Q.$$

As in the case of P, Q generates a one-dimensional ideal, but a left multiplication may introduce a minus sign (when the permutation is odd). Thus, the ideal generated by Q must correspond to the antisymmetric (or alternating) representation.

All the irreducible representations, including the special one-dimensional cases above, can be obtained using this group-algebraic method. We shall not give the proofs here, and we refer the reader to the classic book [Boer 63, pp. 102–125]. The starting point is the Young frame corresponding to the partition $(\lambda) = (\lambda_1, \ldots, \lambda_m)$. One puts the numbers 1 through n in the frame in *any* order, consistent with tableau construction, so that the end product is a Young tableau. Let p be any permutation of a Young tableau that permutes only the elements of each row among themselves. Such a p is called a **horizontal permutation**. Similarly, let q be a **vertical permutation** of the Young tableau.

horizontal and vertical permutations

Definition 25.4.1 Consider the kth Young tableau corresponding to the partition (λ). Let the **Young symmetrizer** $P_k^{(\lambda)}$ and **Young antisymmetrizer** $Q_k^{(\lambda)}$ be the elements of the group algebra of S_n defined as

Young operators defined

$$P_k^{(\lambda)} = \sum_p p, \qquad Q_k^{(\lambda)} = \sum_q \epsilon_q q.$$

Then, the **Young operator** $Y_k^{(\lambda)}$ of this tableau, another element of the group algebra, is given by $Y_k^{(\lambda)} = Q_k^{(\lambda)} P_k^{(\lambda)}$.

It can be shown that the following holds.

Theorem 25.4.2 *The Young operator $Y_k^{(\lambda)}$ is essentially idempotent, and generates a minimal left ideal, hence an irreducible representation for S_n. Representations thus obtained from different frames are inequivalent. Different tableaux with the same frame give equivalent irreducible representations.*

In practice, one usually chooses the standard Young tableaux and applies the foregoing procedure to them to obtain the entire collection of inequivalent irreducible representations of S_n. We have already seen how to calculate characters of S_n employing both analytical and graphical methods. Theorem 25.4.2 gives yet another approach to analyzing representations of S_n. For low values of n this technique may actually be used to determine the characters, but as n grows, it becomes unwieldy, and the graphical method becomes more manageable.

Example 25.4.3 Let us apply this method to S_3. The partitions are (3), (2, 1), and (1^3). There is only one standard Young tableau associated with (3) and (1^3). Thus,

$$Y^{(3)} = P^{(3)} = \frac{1}{3!}\sum_{j=1}^{6} \pi_j = \frac{1}{6}(e + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6),$$

$$Y^{(1^3)} = Q^{(1^3)} = \frac{1}{3!}\sum_{j=1}^{6} \epsilon_{\pi_j}\pi_j = \frac{1}{6}(e - \pi_2 - \pi_3 - \pi_4 + \pi_5 + \pi_6),$$

where we have divided these Young operators by 6 to make them idempotent; we have also used the notation of Example 23.4.1. One can show directly that $Y^{(3)}Y^{(1^3)} = 0$. In fact, one can prove this for general S_n (see Problem 25.6).

For the partition (2, 1), there are two Young tableaux. The first one has the numbers 1 and 2 in the first row and 3 in the second. In the second tableau the numbers 2 and 3 are switched. Therefore, using the multiplication table for S_3 as given in Example 23.4.1, we have

$$Y_1^{(2,1)} = Q_1^{(2,1)} P_1^{(2,1)} = (e - \pi_3)(e + \pi_2) = e + \pi_2 - \pi_3 - \pi_6,$$
$$Y_2^{(2,1)} = Q_2^{(2,1)} P_2^{(2,1)} = (e - \pi_2)(e + \pi_3) = e - \pi_2 + \pi_3 - \pi_5.$$

The reader may verify that the product of any two Young operators corresponding to different Young tableaux is zero and that

$$Y_1^{(2,1)}Y_1^{(2,1)} = 3Y_1^{(2,1)}, \qquad Y_2^{(2,1)}Y_2^{(2,1)} = 3Y_2^{(2,1)}.$$

Let us calculate the left ideal generated by these four Young operators. We already know from our discussion at the beginning of this subsection that $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(1^3)}$, the ideals generated by $Y^{(3)}$ and $Y^{(1^3)}$, are one-dimensional. Let us find $\mathcal{L}_1^{(2,1)}$, the ideal generated by $Y_1^{(2,1)}$. This is the span of all vectors obtained by multiplying $Y_1^{(2,1)}$ on the left by elements of the group algebra. It is sufficient to multiply $Y_1^{(2,1)}$ by the basis of the algebra, namely the group elements:

$$\begin{aligned}
eY_1^{(2,1)} &= Y_1^{(2,1)},\\
\pi_2 Y_1^{(2,1)} &= \pi_2 + e - \pi_5 - \pi_4 \equiv X_1^{(2,1)},\\
\pi_3 Y_1^{(2,1)} &= \pi_3 + \pi_6 - e - \pi_2 = -Y_1^{(2,1)},\\
\pi_4 Y_1^{(2,1)} &= \pi_4 + \pi_5 - \pi_6 - \pi_3 = -X_1^{(2,1)} + Y_1^{(2,1)},\\
\pi_5 Y_1^{(2,1)} &= \pi_5 + \pi_4 - \pi_2 - e = -X_1^{(2,1)},\\
\pi_6 Y_1^{(2,1)} &= \pi_6 + \pi_3 - \pi_4 - \pi_5 = X_1^{(2,1)} - Y_1^{(2,1)}.
\end{aligned}$$

It follows from the above calculation that $\mathcal{L}_1^{(2,1)}$, as a vector space, is spanned by $\{Y_1^{(2,1)}, X_1^{(2,1)}\}$, and since these two vectors are linearly independent, $\mathcal{L}_1^{(2,1)}$ is a two-dimensional minimal ideal corresponding to a two-dimensional irreducible representation of S_3. One can use this basis to find representation matrices and the simple characters of S_3.

The other two-dimensional irreducible representation of S_3, *equivalent* to the one above, is obtained by constructing the ideal $\mathcal{L}_2^{(2,1)}$ generated by $Y_2^{(2,1)}$. This construction is left for the reader, who is also asked to verify its dimensionality.

The resolution of the identity is easily verified:

$$e = \underbrace{\frac{1}{6}Y^{(3)}}_{\equiv e_1} + \underbrace{\frac{1}{6}Y^{(1^3)}}_{\equiv e_2} + \underbrace{\frac{1}{3}Y_1^{(2,1)}}_{\equiv e_3} + \underbrace{\frac{1}{3}Y_2^{(2,1)}}_{\equiv e_4}.$$

The e_i's are idempotents that satisfy $e_i e_j = 0$ for $i \neq j$.

25.5 Products of Representations of S_n

In the quantum theory of systems of many *identical* particles, the wave function must have a particular symmetry under exchange of the particles: If the particles are all fermions (bosons), the overall wave function must be completely antisymmetric (symmetric). Since the space of functions of several variables can provide a carrier space for the representation of any group, we can, in the case of S_n, think of the antisymmetric (symmetric) functions as basis functions for the one-dimensional irreducible identity (alternating) representation. To obtain these basis functions, we apply the Young operator $Y^{(1^n)}$ (or $Y^{(n)}$) to the arguments of any given function of n variables to obtain the completely antisymmetric (or symmetric) wave function.[1]

Under certain conditions, we may require mixed symmetries. For instance, in the presence of spin, the product of the total spin wave function and the total space wave function must be completely antisymmetric for Fermions. Thus, the space part (or the spin part) of the wave functions will, in general, have mixed symmetry. Such a mixed symmetry corresponds to some other Young operator, and the wave function is obtained by applying that Young operator to the arguments of the wave function.

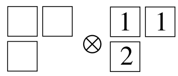
Now suppose that we have two separate systems consisting of n_1 and n_2 particles, respectively, which are all assumed to be identical. As long as the two systems are not interacting, each will consist of states that are classified according to the irreducible representations of its symmetric group. When the two systems interact, we should classify the states of the total system according to the irreducible representations of all $n_1 + n_2$ particles. We have already encountered the mathematical procedure for such classification: It is the Clebsch-Gordan decomposition of the direct product of the states of the two systems. Since the initial states correspond to Young tableaux, and since we are interested in the inequivalent irreducible representations, we need to examine the decomposition of the direct product of Young frames into a sum of Young frames. We first state (without proof) the procedure for such a decomposition, and then give an example to illustrate it.

Theorem 25.5.1 *To find the components of Young frames in the product of two Young frames, draw one of the frames. In the other frame, assign the same symbol, say* 1, *to all boxes in the first row, the same symbol* 2 *to all*

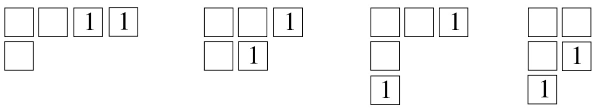
[1] We must make the additional assumption that the permuted functions are all independent.

boxes in the second row, etc. Now attach the first row to the first frame, and enlarge in all possible ways subject to the restriction that no two 1*'s appear in the same column, and that the resultant graph be regular. Repeat with the* 2*'s, etc., making sure in each step that as we read from right to left and top to bottom, no symbol is counted fewer times than the symbol that came after it. The product is the sum of all diagrams so obtained.*

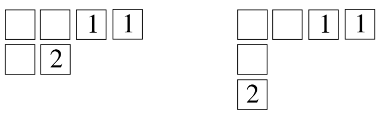
To illustrate the procedure, consider the product



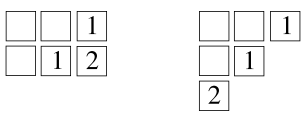
We have put two 1's in the first row and one 2 in the second row of the frame to the right. Now apply the first row to the frame on the left. The result is



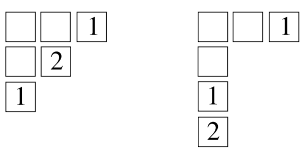
Now we apply the 2 to each of these graphs separately. We cannot put a 2 to the right of the 1's, because in that case, as we count from right to left, we would start with a 2 without having counted any 1's. The allowed graphs obtained from the first diagram are



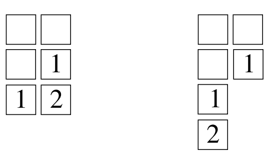
Applying the 2 to the second graph, we obtain



and to the third graph gives



Finally the last graph yields



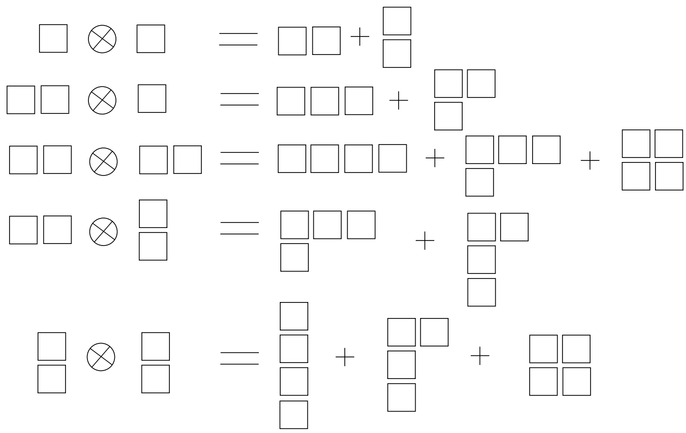
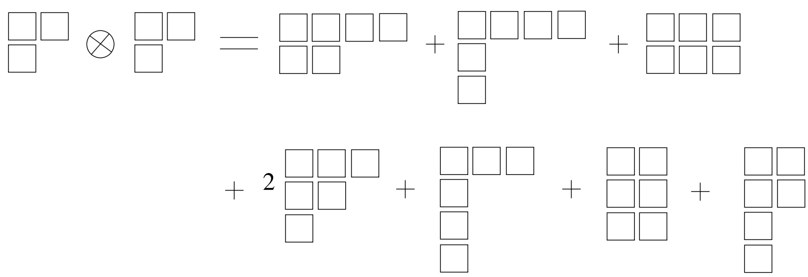


Fig. 25.2 Some products of Young frames for small values of n

The entire process described above is written in terms of frames as



Some simple products, some of which will be used later, are given in Fig. 25.2.

25.6 Problems

25.1 Construct the character table of S_4 using the analytical method and Eq. (25.4).

25.2 Find all the standard Young tableaux for S_5. Thus, determine the dimension of each irreducible representations of S_5. Check that the dimensions satisfy the second equation of (24.18).

25.3 Verify the remaining entries of Table 25.2.

25.4 Construct the character table of S_5.

25.5 Suppose that Q, an element of the group algebra of S_n, is given by

$$Q = \sum_{i=1}^{n!} \epsilon_{\pi_i} \pi_i, \qquad \pi_i \in S_n.$$

Show that

$$\pi_j Q = \epsilon_{\pi_j} Q \quad \text{and} \quad Q^2 = n!Q.$$

25.6 Show that $Y^{(n)}Y^{(1^n)} = 0$. Hint: There are as many even permutations in S_n as there are odd permutations.

25.7 Show that the product of any two Young operators of S_3 corresponding to different Young tableaux is zero and that

$$Y_1^{(2,1)}Y_1^{(2,1)} = 3Y_1^{(2,1)}, \qquad Y_2^{(2,1)}Y_2^{(2,1)} = 3Y_2^{(2,1)}.$$

25.8 Construct the ideal $\mathcal{L}_2^{(2,1)}$ generated by $Y_2^{(2,1)}$ and verify that it is two dimensional.

25.9 Using the ideal $\mathcal{L}_1^{(2,1)}$ generated by $Y_1^{(2,1)}$, find the matrices of the irreducible representation $T^{(2,1)}$. From these matrices calculate the simple characters of S_3 and compare your result with Table 24.4. Show that the ideal $\mathcal{L}_2^{(2,1)}$ generated by $Y_2^{(2,1)}$ gives the same set of characters.

25.10 Find all the Young operators for S_4 corresponding to the first entry of each row of Fig. 25.1. Find the ideals $\mathcal{L}_1^{(3,1)}$ and $\mathcal{L}_1^{(2^2)}$ generated by the Young operators $Y_1^{(3,1)}$ and $Y_1^{(2^2)}$ corresponding to the second and third rows of the table. Show that $\mathcal{L}_1^{(3,1)}$ and $\mathcal{L}_1^{(2^2)}$ have 3 and 2 dimensions, respectively.

25.11 Verify the products of the Young frames of Fig. 25.2.