14 Second-Order Linear Differential Equations

The discussion of Chap. 13 has clearly singled out ODEs, especially those of second order, as objects requiring special attention because most common PDEs of mathematical physics can be separated into ODEs (of second order). This is really an oversimplification of the situation. Many PDEs of physics, both at the fundamental theoretical level (as in the general theory of relativity) and from a practical standpoint (weather forecast) are nonlinear, and the method of the separation of variables does not work. Since no general analytic solutions for such nonlinear systems have been found, we shall confine ourselves to the linear systems, especially those that admit a separated solution.

With the exception of the infinite power series, no systematic method of solving DEs existed during the first half of the nineteenth century. The majority of solutions were completely ad hoc and obtained by trial and error, causing frustration and anxiety among mathematicians. It was to overcome this frustration that Sophus Lie, motivated by the newly developed concept of group, took up the systematic study of DEs in the second half of the nineteenth century. This study not only gave a handle on the disarrayed area of DEs, but also gave birth to one of the most beautiful and fundamental branches of mathematical physics, Lie group theory. We shall come back to a thorough treatment of this theory in Parts VII and IX.

Our main task in this chapter is to study the second-order linear differential equations (SOLDEs). However, to understand SOLDEs, we need some basic understanding of differential equations in general. The next section outlines some essential properties of general DEs. Section 2 is a very brief introduction to first-order DEs, and the remainder of the chapter deals with SOLDEs.

14.1 General Properties of ODEs

The most general ODE can be expressed as

$$G\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0, \tag{14.1}$$

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_14,

in which $G : \mathbb{R}^{n+2} \to \mathbb{R}$ is a real-valued function of $n + 2$ real variables. When G depends explicitly and nontrivially on $d^n y/dx^n$, Eq. (14.1) is called an *nth-order ODE*. An ODE is said to be **linear** if the part of the function G that includes y and all its derivatives is linear in y. The most general nth-order *linear* ODE is

$$p_0(x)y + p_1(x)\frac{dy}{dx} + \cdots + p_n(x)\frac{d^n y}{dx^n} = q(x) \quad \text{for } p_n(x) \neq 0, \qquad (14.2)$$

homogeneous and inhomogeneous ODEs

where $\{p_i\}_{i=0}^n$ and q are functions of the independent variable x. Equation (14.2) is said to be **homogeneous** if $q = 0$; otherwise, it is said to be **inhomogeneous** and $q(x)$ is called the *inhomogeneous term*. It is customary, and convenient, to define a linear differential operator $\mathbf{L}$ by[1]

$$\mathbf{L} \equiv p_0(x) + p_1(x)\frac{d}{dx} + \cdots + p_n(x)\frac{d^n}{dx^n}, \quad p_n(x) \neq 0, \qquad (14.3)$$

and write Eq. (14.2) as

$$\mathbf{L}[y] = q(x). \qquad (14.4)$$

A **solution** of Eq. (14.1) or (14.4) is a single-variable function $f : \mathbb{R} \to \mathbb{R}$ such that $G(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$, or $\mathbf{L}[f] = q(x)$, for all x in the domain of definition of f. The solution of a differential equation may not exist if we put too many restrictions on it. For instance, if we demand that $f : \mathbb{R} \to \mathbb{R}$ be differentiable too many times, we may not be able to find a solution, as the following example shows.

Example 14.1.1 The most general solution of $dy/dx = |x|$ that vanishes at $x = 0$ is

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0, \\ -\frac{1}{2}x^2 & \text{if } x \leq 0. \end{cases}$$

This function is continuous and has first derivative $f'(x) = |x|$, which is also continuous at $x = 0$. However, if we demand that its second derivative also be continuous at $x = 0$, we cannot find a solution, because

$$f''(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

If we want $f'''(x)$ to exist at $x = 0$, then we have to expand the notion of a function to include distributions, or generalized functions.

Overrestricting a solution for a differential equation results in its absence, but underrestricting it allows multiple solutions. To strike a balance between these two extremes, we agree to make a solution as many times differentiable as plausible and to satisfy certain **initial conditions**. For an nth-order

[1]Do not confuse this linear differential operator with the angular momentum (vector) operator $\vec{\mathbf{L}}$.

DE such initial conditions are commonly equivalent (but not restricted) to a specification of the function and of its first $n-1$ derivatives. This sort of specification is made feasible by the following theorem.

Theorem 14.1.2 (Implicit function theorem) *Let $G:\mathbb{R}^{n+1}\to\mathbb{R}$ have continuous partial derivatives up to the kth order in some neighborhood of a point $P_0=(r_1,r_2,\dots,r_{n+1})$ in $\mathbb{R}^{n+1}$. Let $(\partial G/\partial x_{n+1})|_{P_0}\neq 0$. Then there exists a unique function $F:\mathbb{R}^n\to\mathbb{R}$ that is continuously differentiable k times at (some smaller) neighborhood of P_0 such that*

implicit function theorem

$$x_{n+1}=F(x_1,x_2,\dots,x_n)$$

for all points $P=(x_1,x_2,\dots,x_{n+1})$ in a neighborhood of P_0 and

$$G\big(x_1,x_2,\dots,x_n,F(x_1,x_2,\dots,x_n)\big)=0.$$

Theorem 14.1.2 simply asserts that under certain (mild) conditions we can "solve" for one of the independent variables in $G(x_1,x_2,\dots,x_{n+1})=0$ in terms of the others. A proof of this theorem can be found in advanced calculus books.

Application of this theorem to Eq. (14.1) leads to

$$\frac{d^n y}{dx^n}=F\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\dots,\frac{d^{n-1}y}{dx^{n-1}}\right),$$

provided that G satisfies the conditions of the theorem. If we know the solution $y=f(x)$ and its derivatives up to order $n-1$, we can evaluate its nth derivative using this equation. In addition, we can calculate the derivatives of all orders (assuming they exist) by differentiating this equation. This allows us to expand the solution in a Taylor series. Thus—for solutions that have derivatives of all orders—knowledge of the value of a solution and its first $n-1$ derivatives at a point x_0 determines that solution at a neighboring point x.

We shall not study the general ODE of Eq. (14.1) or even its simpler linear version (14.2). We will only briefly study ODEs of the first order in the next section, and then concentrate on linear ODEs of the second order for the rest of this chapter.

14.2 Existence/Uniqueness for First-Order DEs

A general first-order DE (FODE) is of the form $G(x,y,y')=0$. We can find y' (the derivative of y) in terms of a function of x and y if the function $G(x_1,x_2,x_3)$ is differentiable with respect to its third argument and $\partial G/\partial x_3\neq 0$. In that case we have

the most general FODE in normal form

$$y'\equiv\frac{dy}{dx}=F(x,y),\tag{14.5}$$

which is said to be a **normal** FODE. If $F(x, y)$ is a *linear* function of y, then Eq. (14.5) becomes a first-order linear DE (FOLDE), which can generally be written as

$$p_1(x)\frac{dy}{dx} + p_0(x)y = q(x). \tag{14.6}$$

It can be shown that the general FOLDE has an explicit solution: (see [Hass 08])

explicit solution to a general first-order linear differential equation

Theorem 14.2.1 *Any first order linear DE of the form* $p_1(x)y' + p_0(x)y = q(x)$, *in which* p_0, p_1, *and* q *are continuous functions in some interval* (a, b), *has a general solution*

$$y = f(x) = \frac{1}{\mu(x)p_1(x)}\left[C + \int_{x_1}^{x} \mu(t)q(t)\,dt\right], \tag{14.7}$$

where C *is an arbitrary constant and*

$$\mu(x) = \frac{1}{p_1(x)} \exp\left[\int_{x_0}^{x} \frac{p_0(t)}{p_1(t)}\,dt\right], \tag{14.8}$$

where x_0 *and* x_1 *are arbitrary points in the interval* (a, b).

No such explicit solution exists for nonlinear first-order DEs. Nevertheless, it is reassuring to know that a solution of such a DE always exists and under some mild conditions, this solution is unique. We summarize some of the ideas involved in the proof of the existence and uniqueness of the solutions to FODEs. (For proofs, see the excellent book by Birkhoff and Rota [Birk 78].) We first state an existence theorem due to Peano:

Peano existence theorem

Theorem 14.2.2 (Peano existence theorem) *If the function* $F(x, y)$ *is continuous for the points on and within the rectangle defined by* $|y - c| \le K$ *and* $|x - a| \le N$, *and if* $|F(x, y)| \le M$ *there, then the differential equation* $y' = F(x, y)$ *has* at least *one solution*, $y = f(x)$, *defined for* $|x - a| \le \min(N, K/M)$ *and satisfying the initial condition* $f(a) = c$.

This theorem guarantees only the existence of solutions. To ensure uniqueness, the function F needs to have some additional properties. An important property is stated in the following definition.

Lipschitz condition

Definition 14.2.3 A function $F(x, y)$ satisfies a **Lipschitz condition** in a domain $D \subset \mathbb{R}^2$ if for some finite constant L *(Lipschitz constant)*, it satisfies the inequality

$$\left|F(x, y_1) - F(x, y_2)\right| \le L|y_1 - y_2|$$

for all points (x, y_1) and (x, y_2) in D.

uniqueness theorem

Theorem 14.2.4 (Uniqueness) *Let* $f(x)$ *and* $g(x)$ *be any two solutions of* *the FODE $y' = F(x, y)$ in a domain D, where F satisfies a Lipschitz condition with Lipschitz constant L. Then*

$$\bigl|f(x) - g(x)\bigr| \le e^{L|x-a|}\bigl|f(a) - g(a)\bigr|.$$

In particular, the FODE has at most one solution curve passing through the point $(a, c) \in D$.

The final conclusion of this theorem is an easy consequence of the assumed differentiability of F and the requirement $f(a) = g(a) = c$. The theorem says that if there is a solution $y = f(x)$ to the DE $y' = F(x, y)$ satisfying $f(a) = c$, then it is *the* solution.

The requirements of the Peano existence theorem are too broad to yield solutions that have some nice properties. For instance, the interval of definition of the solutions may depend on their initial values. The following example illustrates this point.

Example 14.2.5 Consider the DE $dy/dx = e^y$. The general solution of this DE can be obtained by direct integration:

$$e^{-y}dy = dx \quad \Rightarrow \quad -e^{-y} = x + C.$$

If $y = b$ when $x = 0$, then $C = -e^{-b}$, and

$$e^{-y} = -x + e^{-b} \quad \Rightarrow \quad y = -\ln\bigl(e^{-b} - x\bigr).$$

Thus, the solution is defined for $-\infty < x < e^{-b}$, i.e., the interval of definition of a solution changes with its initial value.

To avoid situations illustrated in the example above, one demands not just the continuity of F—as does the Peano existence theorem—but a Lipschitz condition for it. Then one ensures not only the existence, but also the uniqueness:

local existence and uniqueness theorem

Theorem 14.2.6 (Local existence and uniqueness) *Suppose that the function $F(x, y)$ is defined and continuous in the rectangle*

$$|y - c| \le K, \qquad |x - a| \le N$$

and satisfies a Lipschitz condition there. Let $M = \max|F(x, y)|$ in this rectangle. Then the differential equation $y' = F(x, y)$ has a unique solution $y = f(x)$ satisfying $f(a) = c$ and defined on the interval $|x - a| \le$ $\min(N, K/M)$.

14.3 General Properties of SOLDEs

The most general SOLDE is

$$p_2(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = p_3(x). \tag{14.9}$$

Dividing by $p_2(x)$ and writing p for p_1/p_2, q for p_0/p_2, and r for p_3/p_2 reduces this to the **normal form**

normal form of a SOLDE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x). \tag{14.10}$$

Equation (14.10) is equivalent to (14.9) if $p_2(x) \neq 0$. The points at which $p_2(x)$ vanishes are called the **singular points** of the differential equation.

singular points of a SOLDE

There is a crucial difference between the singular points of linear differential equations and those of nonlinear differential equations. For a nonlinear differential equation such as $(x^2 - y)y' = x^2 + y^2$, the curve $y = x^2$ is the collection of singular points. This makes it impossible to construct solutions $y = f(x)$ that are defined on an interval $I = [a, b]$ of the x-axis because for any $x \in I$, there is a y for which the differential equation is undefined. *Linear* differential equations do not have this problem, because the coefficients of the derivatives are functions of x only. Therefore, all the singular "curves" are vertical. Thus, we have the following:

regular SOLDE

Definition 14.3.1 The normal form of a SOLDE, Eq. (14.10), is **regular** on an interval $[a, b]$ of the x-axis if $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$. A solution of a normal SOLDE is a *twice-differentiable* function $y = f(x)$ that satisfies the SOLDE at every point of $[a, b]$.

Any function that satisfies (14.10) or (14.9) must necessarily be twice differentiable, and that is all that is demanded of the solutions. Any higher-order differentiability requirement may be too restrictive, as was pointed out in Example 14.1.1. Most solutions to a normal SOLDE, however, automatically have derivatives of order higher than two.

We write Eq. (14.9) in the operator form as

$$\mathbf{L}[y] = p_3, \quad \text{where } \mathbf{L} \equiv p_2\frac{d^2}{dx^2} + p_1\frac{d}{dx} + p_0. \tag{14.11}$$

It is clear that $\mathbf{L}$ is a *linear* operator because d/dx is linear, as are all powers of it. Thus, for constants α and β,

$$\mathbf{L}[\alpha y_1 + \beta y_2] = \alpha\mathbf{L}[y_1] + \beta\mathbf{L}[y_2].$$

In particular, if y_1 and y_2 are two solutions of Eq. (14.11), then $\mathbf{L}[y_1 - y_2] = 0$. That is, the difference between any two solutions of a SOLDE is a solution of the **homogeneous equation** obtained by setting $p_3 = 0$.[2]

An immediate consequence of the linearity of $\mathbf{L}$ is the following:

Lemma 14.3.2 *If* $\mathbf{L}[u] = r(x)$, $\mathbf{L}[v] = s(x)$, α *and* β *are constants, and* $w = \alpha u + \beta y$, *then* $\mathbf{L}[w] = \alpha r(x) + \beta s(x)$.

The proof of this lemma is trivial, but the result describes the fundamental property of linear operators: When $r = s = 0$, that is, in dealing with

[2] This conclusion is, of course, not limited to the SOLDE; it holds for all linear DEs.

homogeneous equations, the lemma says that any linear combination of solutions of the homogeneous SOLDE (HSOLDE) is also a solution. This is called the **superposition principle**.

superposition principle

Based on physical intuition, we expect to be able to predict the behavior of a physical system if we know the differential equation obeyed by that system, and, equally importantly, the initial data. Physical intuition also tells us that if the initial conditions are changed by an infinitesimal amount, then the solutions will be changed infinitesimally. Thus, the solutions of linear differential equations are said to be continuous functions of the initial conditions.

The rise and fall of chaos

Remark 14.3.1 Nonlinear differential equations can have completely different solutions for two initial conditions that are infinitesimally close. Since initial conditions cannot be specified with mathematical precision in practice, nonlinear differential equations lead to unpredictable solutions, or **chaos**. Chaos was a hot topic in the late 1980s and early 1990s. Some enthusiasts called it the third pillar of modern physics on a par with relativity and quantum physics. The enthusiasm has waned, however, because chaos, driven entirely by the availability of computers and their superb graphic capabilities, has produced absolutely no fundamental results comparable with relativity and quantum theory.

A prediction is not a prediction unless it is unique. This expectation for linear equations is borne out in the language of mathematics in the form of an existence theorem and a uniqueness theorem. We consider the latter next. But first, we need a lemma.

Lemma 14.3.3 *The only solution $g(x)$ of the homogeneous differential equation $y'' + py' + qy = 0$ defined on the interval $[a, b]$ that satisfies $g(a) = 0 = g'(a)$ is the trivial solution $g = 0$.*

Proof Introduce the nonnegative function $u(x) \equiv [g(x)]^2 + [g'(x)]^2$ and differentiate it to get

$$\begin{aligned} u'(x) &= 2g'g + 2g'g'' = 2g'\bigl(g + g''\bigr) = 2g'\bigl(g - pg' - qg\bigr) \\ &= -2p\bigl(g'\bigr)^2 + 2(1-q)gg'. \end{aligned}$$

Since $(g \pm g')^2 \geq 0$, it follows that $2|gg'| \leq g^2 + g'^2$. Thus,

$$\begin{aligned} 2(1-q)gg' &\leq 2\bigl|(1-q)gg'\bigr| = 2\bigl|(1-q)\bigr|\bigl|gg'\bigr| \\ &\leq \bigl|(1-q)\bigr|\bigl(g^2 + g'^2\bigr) \leq \bigl(1 + |q|\bigr)\bigl(g^2 + g'^2\bigr), \end{aligned}$$

and therefore,

$$\begin{aligned} u'(x) &\leq \bigl|u'(x)\bigr| = \bigl|-2pg'^2 + 2(1-q)gg'\bigr| \\ &\leq 2|p|g'^2 + \bigl(1 + |q|\bigr)\bigl(g^2 + g'^2\bigr) \\ &= \bigl[1 + \bigl|q(x)\bigr|\bigr]g^2 + \bigl[1 + \bigl|q(x)\bigr| + 2\bigl|p(x)\bigr|\bigr]g'^2. \end{aligned}$$

Now let $K = 1 + \max[|q(x)| + 2|p(x)|]$, where the maximum is taken over $[a, b]$. Then we obtain

$$u'(x) \le K\left(g^2 + g'^2\right) = Ku(x) \quad \forall x \in [a, b].$$

Using the result of Problem 14.1 yields $u(x) \le u(a)e^{K(x-a)}$ for all $x \in [a, b]$. This equation, plus $u(a) = 0$, as well as the fact that $u(x) \ge 0$ imply that $u(x) = g^2(x) + g'^2(x) = 0$. It follows that $g(x) = 0 = g'(x)$ for all $x \in [a, b]$. □

uniqueness of solutions to SOLDE

Theorem 14.3.4 (Uniqueness) *If p and q are continuous on $[a, b]$, then at most one solution $y = f(x)$ of the DE $y'' + p(x)y' + q(x)y = 0$ can satisfy the initial conditions $f(a) = c_1$ and $f'(a) = c_2$, where c_1 and c_2 are arbitrary constants.*

Proof Let f_1 and f_2 be two solutions satisfying the given initial conditions. Then their difference, $g \equiv f_1 - f_2$, satisfies the homogeneous equation [with $r(x) = 0$]. The initial condition that $g(x)$ satisfies is clearly $g(a) = 0 = g'(a)$. By Lemma 14.3.3, $g = 0$ or $f_1 = f_2$. □

Theorem 14.3.4 can be applied to any *homogeneous* SOLDE to find the latter's most general solution. In particular, let $f_1(x)$ and $f_2(x)$ be any two solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{14.12}$$

defined on the interval $[a, b]$. Assume that the two vectors $\mathbf{v}_1 = (f_1(a), f_1'(a))$ and $\mathbf{v}_2 = (f_2(a), f_2'(a))$ in $\mathbb{R}^2$ are linearly independent.[3] Let $g(x)$ be another solution. The vector $(g(a), g'(a))$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, giving the two equations

$$\begin{aligned} g(a) &= c_1 f_1(a) + c_2 f_2(a), \\ g'(a) &= c_1 f_1'(a) + c_2 f_2'(a). \end{aligned}$$

Now consider the function $u(x) \equiv g(x) - c_1 f_1(x) - c_2 f_2(x)$, which satisfies Eq. (14.12) and the initial conditions $u(a) = u'(a) = 0$. By Lemma 14.3.3, we must have $u(x) = 0$ or $g(x) = c_1 f_1(x) + c_2 f_2(x)$. We have proved the following:

Theorem 14.3.5 *Let f_1 and f_2 be two solutions of the HSOLDE*

$$y'' + py' + qy = 0,$$

[3] If they are not, then one must choose a different initial point for the interval.

where p and q are continuous functions defined on the interval $[a, b]$. If

$$\bigl(f_1(a), f_1'(a)\bigr) \quad \textit{and} \quad \bigl(f_2(a), f_2'(a)\bigr)$$

are linearly independent vectors in $\mathbb{R}^2$, then every solution $g(x)$ of this HSOLDE is equal to some linear combination $g(x) = c_1 f_1(x) + c_2 f_2(x)$ of f_1 and f_2 with constant coefficients c_1 and c_2.

14.4 The Wronskian

The two solutions $f_1(x)$ and $f_2(x)$ in Theorem 14.3.5 have the property that any other solution $g(x)$ can be expressed as a linear combination of them. We call f_1 and f_2 a **basis of solutions** of the HSOLDE. To form a basis of solutions, f_1 and f_2 must be linearly independent.[4]

basis of solutions

Definition 14.4.1 The **Wronskian** of any two differentiable functions $f_1(x)$ and $f_2(x)$ is

Wronskian defined

$$W(f_1, f_2; x) = f_1(x) f_2'(x) - f_2(x) f_1'(x) = \det \begin{pmatrix} f_1(x) & f_1'(x) \\ f_2(x) & f_2'(x) \end{pmatrix}.$$

Proposition 14.4.2 *The Wronskian of any two solutions of Eq. (14.12) satisfies*

$$W(f_1, f_2; x) = W(f_1, f_2; c) e^{-\int_c^x p(t)\,dt},$$

where c is any number in the interval $[a, b]$.

Proof Differentiating both sides of the definition of Wronskian and substituting from Eq. (14.12) yields a FOLDE for $W(f_1, f_2; x)$, which can be easily solved. The details are left as a problem. □

An important consequence of Proposition 14.4.2 is that the Wronskian of any two solutions of Eq. (14.12) does not change sign in $[a, b]$. In particular, if the Wronskian vanishes at one point in $[a, b]$, it vanishes at all points in $[a, b]$.

The real importance of the Wronskian is contained in the following theorem, whose straightforward proof is left as an exercise for the reader.

Theorem 14.4.3 *Two differentiable functions f_1 and f_2, which are nonzero in the interval $[a, b]$, are linearly dependent if and only if their Wronskian vanishes.*

Historical Notes

Josef Hoëné de Wronski (1778–1853) was born Josef Hoëné, but he adopted the name Wronski around 1810 just after he married. He had moved to France and become a French

[4] The linear dependence or independence of a number of functions $\{f_i\}_{i=1}^n : [a, b] \to \mathbb{R}$ is a concept that must hold for all $x \in [a, b]$.

Josef Hoëné de Wronski
1778–1853

citizen in 1800 and moved to Paris in 1810, the same year he published his first memoir on the foundations of mathematics, which received less than favorable reviews from Lacroix and Lagrange. His other interests included the design of caterpillar vehicles to compete with the railways. However, they were never manufactured.

Wronski was interested mainly in applying philosophy to mathematics, the philosophy taking precedence over rigorous mathematical proofs. He criticised Lagrange's use of infinite series and introduced his own ideas for series expansions of a function. The coefficients in this series are determinants now known as **Wronskians** [so named by Thomas Muir (1844–1934), a Glasgow High School science master who became an authority on determinants by devoting most of his life to writing a five-volume treatise on the history of determinants].

For many years Wronski's work was dismissed as rubbish. However, a closer examination of the work in more recent times shows that although some is wrong and he has an incredibly high opinion of himself and his ideas, there are also some mathematical insights of great depth and brilliance hidden within the papers.

14.4.1 A Second Solution to the HSOLDE

If we know one solution to Eq. (14.12), say f_1, then by differentiating both sides of

$$f_1(x)f_2'(x) - f_2(x)f_1'(x) = W(x) = W(c)e^{-\int_c^x p(t)\,dt},$$

dividing the result by f_1^2, and noting that the LHS will be the derivative of f_2/f_1, we can solve for f_2 in terms of f_1. The result is

$$f_2(x) = f_1(x)\left\{C + K\int_\alpha^x \frac{1}{f_1^2(s)}\exp\left[-\int_c^s p(t)\,dt\right]ds\right\},$$

where $K \equiv W(c)$ is another arbitrary (nonzero) constant; we do not have to know $W(x)$ (this would require knowledge of f_2, which we are trying to calculate!) to obtain $W(c)$. In fact, the reader is urged to check directly that $f_2(x)$ satisfies the DE of (14.12) for arbitrary C and K. Whenever possible—and convenient—it is customary to set $C = 0$, because its presence simply gives a term that is proportional to the known solution $f_1(x)$.

Theorem 14.4.4 *Let f_1 be a solution of $y'' + p(x)y' + q(x)y = 0$. Then*

$$f_2(x) = f_1(x)\int_\alpha^x \frac{1}{f_1^2(s)}\exp\left[-\int_c^s p(t)\,dt\right]ds,$$

is another solution and $\{f_1, f_2\}$ forms a basis of solutions of the DE.

Example 14.4.5 Here are some examples of finding the second solution from the first:

(a) A solution to the SOLDE $y'' - k^2 y = 0$ is e^{kx}. To find a second solution, we let $C = 0$ and $K = 1$ in Theorem 14.4.4. Since $p(x) = 0$, we have

$$f_2(x) = e^{kx}\left(0 + \int_\alpha^x \frac{ds}{e^{2ks}}\right) = -\frac{1}{2k}e^{-kx} + \frac{e^{-2k\alpha}}{2k}e^{kx},$$

which, ignoring the second term (which is proportional to the first solution), leads directly to the choice of e^{-kx} as a second solution.

(b) The differential equation $y'' + k^2 y = 0$ has $\sin kx$ as a solution. With $C = 0$, $\alpha = \pi/(2k)$, and $K = 1$, we get

$$f_2(x) = \sin kx\left(0 + \int_{\pi/2k}^x \frac{ds}{\sin^2 ks}\right) = -\sin kx \cot ks|_{\pi/2k}^x = -\cos kx.$$

(c) For the solutions in part (a),

$$W(x) = \det\begin{pmatrix} e^{kx} & ke^{kx} \\ e^{-kx} & -ke^{-kx}\end{pmatrix} = -2k,$$

and for those in part (b),

$$W(x) = \det\begin{pmatrix} \sin kx & k\cos kx \\ \cos kx & -k\sin kx\end{pmatrix} = -k.$$

Both Wronskians are constant. In general, the Wronskian of any two linearly independent solutions of $y'' + q(x)y = 0$ is constant.

Most special functions used in mathematical physics are solutions of SOLDEs. The behavior of these functions at certain special points is determined by the physics of the particular problem. In most situations physical expectation leads to a preference for one particular solution over the other. For example, although there are two linearly independent solutions to the Legendre DE

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right] + n(n+1)y = 0,$$

the solution that is most frequently encountered is the Legendre polynomial $P_n(x)$ discussed in Chap. 8. The other solution can be obtained by solving the Legendre equation or by using Theorem 14.4.4, as done in the following example.

Example 14.4.6 The Legendre equation can be reexpressed as

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{n(n+1)}{1-x^2}y = 0.$$

This is an HSOLDE with

$$p(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{n(n+1)}{1-x^2}.$$

One solution of this HSOLDE is the well-known Legendre polynomial $P_n(x)$. Using this as our input and employing Theorem 14.4.4, we can generate another set of solutions.

Let $Q_n(x)$ stand for the linearly independent "partner" of $P_n(x)$. Then

$$\begin{aligned} Q_n(x) &= P_n(x)\int_\alpha^x \frac{1}{P_n^2(s)}\exp\left[\int_0^s \frac{2t}{1-t^2}\,dt\right]ds \\ &= P_n(x)\int_\alpha^x \frac{1}{P_n^2(s)}\left[\frac{1}{1-s^2}\right]ds = A_n P_n(x)\int_\alpha^x \frac{ds}{(1-s^2)P_n^2(s)}, \end{aligned}$$

where A_n is an arbitrary constant determined by standardization, and α is an arbitrary point in the interval $[-1,+1]$. For instance, for $n=0$, we have $P_0 = 1$, and we obtain

$$Q_0(x) = A_0\int_\alpha^x \frac{ds}{1-s^2} = A_0\left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\ln\left|\frac{1+\alpha}{1-\alpha}\right|\right].$$

The standard form of $Q_0(x)$ is obtained by setting $A_0 = 1$ and $\alpha = 0$:

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad \text{for } |x| < 1.$$

Similarly, since $P_1(x) = x$,

$$Q_1(x) = A_1 x\int_\alpha^x \frac{ds}{s^2(1-s^2)} = Ax + Bx\ln\left|\frac{1+x}{1-x}\right| + C \quad \text{for } |x| < 1.$$

Here standardization is $A = 0$, $B = \frac{1}{2}$, and $C = -1$. Thus,

$$Q_1(x) = \frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right| - 1.$$

14.4.2 The General Solution to an ISOLDE

Inhomogeneous SOLDEs (ISOLDEs) can be most elegantly discussed in terms of Green's functions, the subject of Chap. 20, which automatically incorporate the boundary conditions. However, the most general solution of an ISOLDE, with no boundary specification, can be discussed at this point.

Let $g(x)$ be a particular solution of

$$\mathbf{L}[y] = y'' + py' + qy = r(x) \tag{14.13}$$

and let $h(x)$ be any other solution of this equation. Then $h(x) - g(x)$ satisfies Eq. (14.12) and can be written as a linear combination of a basis of solutions $f_1(x)$ and $f_2(x)$, leading to the following equation:

$$h(x) = c_1 f_1(x) + c_2 f_2(x) + g(x). \tag{14.14}$$

Thus, if we have a *particular* solution of the ISOLDE of Eq. (14.13) and two basis solutions of the HSOLDE, then the *most general* solution of (14.13) can be expressed as the sum of a linear combination of the two basis solutions and the particular solution.

We know how to find a second solution to the HSOLDE once we know one solution. We now show that knowing one such solution will also allow us to find a particular solution to the ISOLDE. The method we use is called the **variation of constants**. This method can also be used to find a second solution to the HSOLDE.

method of variation of constants

Let f_1 and f_2 be the two (known) solutions of the HSOLDE and $g(x)$ the sought-after solution to Eq. (14.13). Write g as $g(x) = f_1(x)v(x)$ and substitute it in (14.13) to get a SOLDE for $v(x)$:

$$v'' + \left(p + \frac{2f_1'}{f_1}\right)v' = \frac{r}{f_1}.$$

This is a *first* order linear DE in v', which has a solution of the form

$$v' = \frac{W(x)}{f_1^2(x)}\left[C + \int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt\right],$$

where $W(x)$ is the (known) Wronskian of Eq. (14.13). Substituting

$$\frac{W(x)}{f_1^2(x)} = \frac{f_1(x)f_2'(x) - f_2(x)f_1'(x)}{f_1^2(x)} = \frac{d}{dx}\left(\frac{f_2}{f_1}\right)$$

in the above expression for v' and setting $C = 0$ (we are interested in a *particular* solution), we get

$$\begin{aligned}\frac{dv}{dx} &= \frac{d}{dx}\left(\frac{f_2}{f_1}\right)\int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt \\ &= \frac{d}{dx}\left[\frac{f_2(x)}{f_1(x)}\int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt\right] - \frac{f_2(x)}{f_1(x)}\underbrace{\frac{d}{dx}\int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt}_{=f_1(x)r(x)/W(x)}\end{aligned}$$

and

$$v(x) = \frac{f_2(x)}{f_1(x)}\int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt - \int_a^x \frac{f_2(t)r(t)}{W(t)}\,dt.$$

This leads to the particular solution

$$g(x) = f_1(x)v(x) = f_2(x)\int_a^x \frac{f_1(t)r(t)}{W(t)}\,dt - f_1(x)\int_a^x \frac{f_2(t)r(t)}{W(t)}\,dt. \tag{14.15}$$

We just proved the following result:

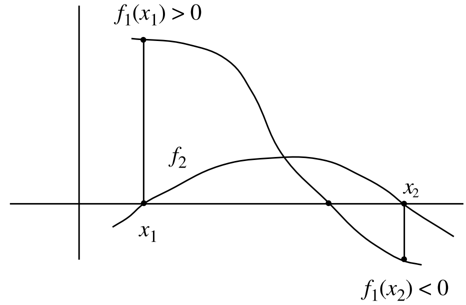

Fig. 14.1 If $f_2'(x_1) > 0 > f_2'(x_2)$, then (assuming that the Wronskian is positive) $f_1(x_1) > 0 > f_1(x_2)$

Proposition 14.4.7 *Given a single solution $f_1(x)$ of the HSOLDE corresponding to an ISOLDE, one can use Theorem 14.4.4 to find a second solution $f_2(x)$ of the HSOLDE and Eq. (14.15) to find a particular solution $g(x)$of the ISOLDE. The most general solution h will then be*

$$h(x) = c_1 f_1(x) + c_2 f_2(x) + g(x).$$

14.4.3 Separation and Comparison Theorems

The Wronskian can be used to derive some properties of the graphs of solutions of HSOLDEs. One such property concerns the relative position of the zeros of two linearly independent solutions of an HSOLDE.

the separation theorem

Theorem 14.4.8 (Separation) *The zeros of two linearly independent solutions of an HSOLDE occur alternately.*

Proof Let $f_1(x)$ and $f_2(x)$ be two independent solutions of Eq. (14.12). We have to show that a zero of f_1 exists between any two zeros of f_2. The linear independence of f_1 and f_2 implies that $W(f_1, f_2; x) \neq 0$ for any $x \in [a, b]$. Let $x_i \in [a, b]$ be a zero of f_2. Then

$$0 \neq W(f_1, f_2; x_i) = f_1(x_i) f_2'(x_i) - f_2(x_i) f_1'(x_i) = f_1(x_i) f_2'(x_i).$$

Thus, $f_1(x_i) \neq 0$ and $f_2'(x_i) \neq 0$. Suppose that x_1 and x_2—where $x_2 > x_1$—are two successive zeros of f_2. Since f_2 is continuous in $[a, b]$ and $f_2'(x_1) \neq 0$, f_2 has to be either increasing $[f_2'(x_1) > 0]$ or decreasing $[f_2'(x_1) < 0]$ at x_1. For f_2 to be zero at x_2, the next point, $f_2'(x_2)$ must have the *opposite* sign from $f_2'(x_1)$ (see Fig. 14.1). We proved earlier that the sign of the Wronskian does not change in $[a, b]$ (see Proposition 14.4.2 and comments after it). The above equation then says that $f_1(x_1)$ and $f_1(x_2)$ also have opposite signs. The continuity of f_1 then implies that f_1 must cross the x-axis somewhere between x_1 and x_2. A similar argument shows that there exists one zero of f_2 between any two zeros of f_1. □

Example 14.4.9 Two linearly independent solutions of $y'' + y = 0$ are $\sin x$ and $\cos x$. The separation theorem suggests that the zeros of $\sin x$ and $\cos x$ must alternate, a fact known from elementary trigonometry: The zeros of $\cos x$ occur at odd multiples of $\pi/2$, and those of $\sin x$ occur at even multiples of $\pi/2$.

A second useful result is known as the comparison theorem (for a proof, see [Birk 78, p. 38]).

Theorem 14.4.10 (Comparison) *Let f and g be nontrivial solutions of $u'' + p(x)u = 0$ and $v'' + q(x)v = 0$, respectively, where $p(x) \geq q(x)$ for all $x \in [a, b]$. Then f vanishes at least once between any two zeros of g, unless $p = q$ and f is a constant multiple of g.*

the comparison theorem

The form of the differential equations used in the comparison theorem is not restrictive because any HSOLDE can be cast in this form, as the following proposition shows.

Proposition 14.4.11 *If $y'' + p(x)y' + q(x)y = 0$, then*

$$u = y \exp\left[\frac{1}{2}\int_{\alpha}^{x} p(t)\,dt\right]$$

satisfies $u'' + S(x)u = 0$, where $S(x) = q - \frac{1}{4}p^2 - \frac{1}{2}p'$.

Proof Define $w(x)$ by $y = wu$, and substitute in the HSOLDE to obtain

$$\left(u'w + w'u\right)' + p\left(u'w + w'u\right) + quw = 0,$$

or

$$wu'' + \left(2w' + pw\right)u' + \left(qw + pw' + w''\right)u = 0. \tag{14.16}$$

If we demand that the coefficient of u' be zero, we obtain the DE $2w' + pw = 0$, whose solution is

$$w(x) = C \exp\left[-\frac{1}{2}\int_{\alpha}^{x} p(t)\,dt\right].$$

Dividing (14.16) by this w and substituting for w yields

$$u'' + S(x)u = 0, \quad \text{where} \quad S(x) = q + p\frac{w'}{w} + \frac{w''}{w} = q - \frac{1}{4}p^2 - \frac{1}{2}p'.$$

□

A useful special case of the comparison theorem is given as the following corollary whose straightforward but instructive proof is left as a problem.

Corollary 14.4.12 *If $q(x) \le 0$ for all $x \in [a, b]$, then no nontrivial solution of the differential equation $v'' + q(x)v = 0$ can have more than one zero.*

Example 14.4.13 It should be clear from the preceding discussion that the oscillations of the solutions of $v'' + q(x)v = 0$ are mostly determined by the sign and magnitude of $q(x)$. For $q(x) \le 0$ there is no oscillation; that is, there is no solution that changes sign more than once. Now suppose that $q(x) \ge k^2 > 0$ for some real k. Then, by Theorem 14.4.10, any solution of $v'' + q(x)v = 0$ must have at least one zero between any two successive zeros of the solution $\sin kx$ of $u'' + k^2 u = 0$. This means that any solution of $v'' + q(x)v = 0$ has a zero in any interval of length π/k if $q(x) \ge k^2 > 0$.

Let us apply this to the **Bessel DE**,

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0.$$

By Proposition 14.4.11, we can eliminate the y' term by substituting $v/\sqrt{x}$ for y.[5] This transforms the Bessel DE into

$$v'' + \left(1 - \frac{4n^2 - 1}{4x^2}\right) v = 0.$$

oscillation of the Bessel function of order zero

We compare this, for $n = 0$, with $u'' + u = 0$, which has a solution $u = \sin x$, and conclude that each interval of length π of the positive x-axis contains at least one zero of any solution of order zero ($n = 0$) of the Bessel equation. Thus, in particular, the zeroth Bessel function, denoted by $J_0(x)$, has a zero in each interval of length π of the x-axis.

On the other hand, for $4n^2 - 1 > 0$, or $n > \frac{1}{2}$, we have $1 > [1 - (4n^2 - 1)/4x^2]$. This implies that $\sin x$ has *at least* one zero between any two successive zeros of the Bessel functions of order greater than $\frac{1}{2}$. It follows that such a Bessel function can have *at most* one zero between any two successive zeros of $\sin x$ (or in each interval of length π on the positive x-axis).

Example 14.4.14 Let us apply Corollary 14.4.12 to $v'' - v = 0$ in which $q(x) = -1 < 0$. According to the corollary, the most general solution, $c_1 e^x + c_2 e^{-x}$, can have at most one zero. Indeed,

$$c_1 e^x + c_2 e^{-x} = 0 \quad \Rightarrow \quad x = \frac{1}{2} \ln\left|-\frac{c_2}{c_1}\right|,$$

and this (real) x (if it exists) is the only possible solution, as predicted by the corollary.

[5] Because of the square root in the denominator, the range of x will have to be restricted to positive values.

14.5 Adjoint Differential Operators

We discussed adjoint operators in detail in the context of finite-dimensional vector spaces in Chap. 4. In particular, the importance of self-adjoint, or hermitian, operators was clearly spelled out by the spectral decomposition theorem of Chap. 6. A consequence of that theorem is the completeness of the eigenvectors of a hermitian operator, the fact that an arbitrary vector can be expressed as a linear combination of the (orthonormal) eigenvectors of a hermitian operator.

Self-adjoint differential operators are equally important because their "eigenfunctions" also form complete orthogonal sets, as we shall see later. This section generalizes the concept of the adjoint to the case of a differential operator (of second degree).

Definition 14.5.1 The HSOLDE

$$\mathbf{L}[y] \equiv p_2(x)y'' + p_1(x)y' + p_0(x)y = 0 \tag{14.17}$$

is said to be **exact** if

exact HSOLDE

$$\mathbf{L}[f] \equiv p_2(x)f'' + p_1(x)f' + p_0(x)f = \frac{d}{dx}\big[A(x)f' + B(x)f\big] \tag{14.18}$$

for all $f \in \mathcal{C}^2[a,b]$ and for some $A, B \in \mathcal{C}^1[a,b]$. An **integrating factor** for $\mathbf{L}[y]$ is a function $\mu(x)$ such that $\mu(x)\mathbf{L}[y]$ is exact.

integrating factor for HSOLDE

If the HSOLDE (14.17) is exact, then Eq. (14.18) gives

$$\frac{d}{dx}\big[A(x)y' + B(x)y\big] = 0 \quad \Rightarrow \quad A(x)y' + B(x)y = C,$$

a FOLDE with a constant inhomogeneous term.

If (14.17) has an integrating factor, then even the ISOLDE corresponding to it can be solved, because

$$\begin{aligned} \mu(x)\mathbf{L}[y] = \mu(x)r(x) \quad &\Rightarrow \quad \frac{d}{dx}\big[A(x)y' + B(x)y\big] = \mu(x)r(x) \\ &\Rightarrow \quad A(x)y' + B(x)y = \int_\alpha^x \mu(t)r(t)\,dt, \end{aligned}$$

which is a general FOLDE. Thus, the existence of an integrating factor completely solves a SOLDE. It is therefore important to know whether or not a SOLDE admits an integrating factor. First let us give a criterion for the *exactness* of a SOLDE.

Proposition 14.5.2 *The HSOLDE of Eq.* (14.17) *is exact if and only if* $p_2'' - p_1' + p_0 = 0$.

Proof If the HSOLDE is exact, then Eq. (14.18) holds for all f, implying that $p_2 = A$, $p_1 = A' + B$, and $p_0 = B'$. It follows that $p_2'' = A''$, $p_1' = A'' + B'$, and $p_0 = B'$, which in turn give $p_2'' - p_1' + p_0 = 0$.

Conversely if $p_2'' - p_1' + p_0 = 0$, then, substituting $p_0 = -p_2'' + p_1'$ in the LHS of Eq. (14.17), we obtain

$$\begin{aligned} p_2 y'' + p_1 y' + p_0 y &= p_2 y'' + p_1 y' + \left(-p_2'' + p_1'\right) y \\ &= p_2 y'' - p_2'' y + (p_1 y)' = \left(p_2 y' - p_2' y\right)' + (p_1 y)' \\ &= \frac{d}{dx}\left(p_2 y' - p_2' y + p_1 y\right), \end{aligned}$$

and the DE is exact. □

A general HSOLDE is clearly not exact. Can we make it exact by multiplying it by an integrating factor? The following proposition contains the answer.

Proposition 14.5.3 *A function μ is an integrating factor of the HSOLDE of Eq.* (14.17) *if and only if it is a solution of the HSOLDE*

$$\mathbf{M}[\mu] \equiv (p_2\mu)'' - (p_1\mu)' + p_0\mu = 0. \tag{14.19}$$

Proof This is an immediate consequence of Proposition 14.5.2. □

We can expand Eq. (14.19) to obtain the equivalent equation

$$p_2\mu'' + \left(2p_2' - p_1\right)\mu' + \left(p_2'' - p_1' + p_0\right)\mu = 0. \tag{14.20}$$

adjoint of a second-order linear differential operator

The operator $\mathbf{M}$ given by

$$\mathbf{M} \equiv p_2\frac{d^2}{dx^2} + \left(2p_2' - p_1\right)\frac{d}{dx} + \left(p_2'' - p_1' + p_0\right) \tag{14.21}$$

is called the **adjoint** of the operator $\mathbf{L}$ and denoted by $\mathbf{M} \equiv \mathbf{L}^\dagger$. The reason for the use of the word "adjoint" will be made clear below.

Proposition 14.5.3 confirms the existence of an integrating factor. However, the latter can be obtained only by solving Eq. (14.20), which is at least as difficult as solving the original differential equation! In contrast, the integrating factor for a FOLDE can be obtained by a mere integration [$\mu(x)$ given in Eq. (14.8) is an integrating factor of the FOLDE (14.6), as the reader can verify].

Although integrating factors for SOLDEs are not as useful as their counterparts for FOLDEs, they can facilitate the study of SOLDEs. Let us first note that the adjoint of the adjoint of a differential operator is the original operator: $(\mathbf{L}^\dagger)^\dagger = \mathbf{L}$ (see Problem 14.13). This suggests that if v is an integrating factor of $\mathbf{L}[u]$, then u will be an integrating factor of $\mathbf{M}[v] \equiv \mathbf{L}^\dagger[v]$. In particular, multiplying the first one by v and the second one by u and subtracting the results, we obtain [see Equations (14.17) and (14.19)] $v\mathbf{L}[u] - u\mathbf{M}[v] = (vp_2)u'' - u(p_2 v)'' + (vp_1)u' + u(p_1 v)'$, which can be simplified to

$$v\mathbf{L}[u] - u\mathbf{M}[v] = \frac{d}{dx}\left[p_2 vu' - (p_2 v)' u + p_1 uv\right]. \tag{14.22}$$

Integrating this from a to b yields

$$\int_a^b \big(v\mathbf{L}[u] - u\mathbf{M}[v]\big)\,dx = \big[p_2vu' - (p_2v)'u + p_1uv\big]\big|_a^b. \qquad (14.23)$$

Equations (14.22) and (14.23) are called the **Lagrange identities**. Equation (14.23) embodies the reason for calling $\mathbf{M}$ the adjoint of $\mathbf{L}$: If we consider u and v as abstract vectors $|u\rangle$ and $|v\rangle$, $\mathbf{L}$ and $\mathbf{M}$ as operators in a Hilbert space with the inner product $\langle u|v\rangle = \int_a^b u^*(x)v(x)\,dx$, then Eq. (14.23) can be written as

Lagrange identities

$$\langle v|\mathbf{L}|u\rangle - \langle u|\mathbf{M}|v\rangle = \langle u|\mathbf{L}^\dagger|v\rangle^* - \langle u|\mathbf{M}|v\rangle = \big[p_2vu' - (p_2v)'u + p_1uv\big]\big|_a^b.$$

If the RHS is zero, then $\langle u|\mathbf{L}^\dagger|v\rangle^* = \langle u|\mathbf{M}|v\rangle$ for all $|u\rangle, |v\rangle$, and since all these operators and functions are real, $\mathbf{L}^\dagger = \mathbf{M}$.

As in the case of finite-dimensional vector spaces, a self-adjoint differential operator merits special consideration. For $\mathbf{M}[v] \equiv \mathbf{L}^\dagger[v]$ to be equal to $\mathbf{L}$, we must have [see Eqs. (14.17) and (14.20)] $2p_2' - p_1 = p_1$ and $p_2'' - p_1' + p_0 = p_0$. The first equation gives $p_2' = p_1$, which also solves the second equation. If this condition holds, then we can write Eq. (14.17) as $\mathbf{L}[y] = p_2y'' + p_2'y' + p_0y$, or

$$\mathbf{L}[y] = \frac{d}{dx}\left[p_2(x)\frac{dy}{dx}\right] + p_0(x)y = 0.$$

Can we make all SOLDEs self-adjoint? Let us multiply both sides of Eq. (14.17) by a function $h(x)$, to be determined later. We get the new DE

$$h(x)p_2(x)y'' + h(x)p_1(x)y' + h(x)p_0(x)y = 0,$$

which we desire to be self-adjoint. This will be accomplished if we choose $h(x)$ such that $hp_1 = (hp_2)'$, or $p_2h' + h(p_2' - p_1) = 0$, which can be readily integrated to give

$$h(x) = \frac{1}{p_2}\exp\left[\int^x \frac{p_1(t)}{p_2(t)}\,dt\right].$$

We have just proved the following:

Theorem 14.5.4 *The SOLDE of Eq.* (14.17) *is self-adjoint if and only if* $p_2' = p_1$, *in which case the DE has the form*

all SOLDEs can be made self-adjoint

$$\frac{d}{dx}\left[p_2(x)\frac{dy}{dx}\right] + p_0(x)y = 0.$$

If it is not self-adjoint, it can be made so by multiplying it through by

$$h(x) = \frac{1}{p_2}\exp\left[\int^x \frac{p_1(t)}{p_2(t)}\,dt\right].$$

Example 14.5.5 (a) The Legendre equation in normal form,

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0,$$

is not self-adjoint. However, if we multiply through by $h(x) = 1 - x^2$, we get

$$\left(1 - x^2\right)y'' - 2xy' + \lambda y = 0,$$

which can be rewritten as $[(1 - x^2)y']' + \lambda y = 0$, which is self-adjoint.

(b) Similarly, the normal form of the Bessel equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

is not self-adjoint, but multiplying through by $h(x) = x$ yields

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(x - \frac{n^2}{x}\right)y = 0,$$

which is clearly self-adjoint.

14.6 Power-Series Solutions of SOLDEs

Analysis is one of the richest branches of mathematics, focusing on the endless variety of objects we call functions. The simplest kind of function is a polynomial, which is obtained by performing the simple algebraic operations of addition and multiplication on the independent variable x. The next in complexity are the trigonometric functions, which are obtained by taking ratios of geometric objects. If we demand a simplistic, intuitive approach to functions, the list ends there. It was only with the advent of derivatives, integrals, and differential equations that a vastly rich variety of functions exploded into existence in the eighteenth and nineteenth centuries. For instance, e^x, nonexistent before the invention of calculus, can be thought of as the function that solves $dy/dx = y$.

Although the definition of a function in terms of DEs and integrals seems a bit artificial, for most applications it is the only way to define a function. For instance, the **error function**, used in statistics, is defined as

$$\operatorname{erf}(x) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-t^2}\,dt.$$

Such a function cannot be expressed in terms of elementary functions. Similarly, functions (of x) such as

$$\int_x^{\infty} \frac{\sin t}{t}\,dt, \qquad \int_0^{\pi/2} \sqrt{1 - x^2\sin^2 t}\,dt, \qquad \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2\sin^2 t}},$$

and so on are encountered frequently in applications. None of these functions can be expressed in terms of other well-known functions.

An effective way of studying such functions is to study the differential equations they satisfy. In fact, the majority of functions encountered in mathematical physics obey the HSOLDE of Eq. (14.17) in which the $p_i(x)$ are elementary functions, mostly ratios of polynomials (of degree at most 2). Of course, to specify functions completely, appropriate boundary conditions are necessary. For instance, the error function mentioned above satisfies the HSOLDE $y'' + 2xy' = 0$ with the boundary conditions $y(0) = \frac{1}{2}$ and $y'(0) = 1/\sqrt{\pi}$.

The natural tendency to resist the idea of a function as a solution of a SOLDE is mostly due to the abstract nature of differential equations. After all, it is easier to imagine constructing functions by simple multiplications or with simple geometric figures that have been around for centuries. The following beautiful example (see [Birk 78, pp. 85–87]) should overcome this resistance and convince the skeptic that differential equations contain all the information about a function.

This example illustrates that all information about sine and cosine is hidden in their differential equation.

Example 14.6.1 (Trigonometric functions as solutions of DEs) We can show that the solutions to $y'' + y = 0$ have all the properties we expect of $\sin x$ and $\cos x$. Let us denote the two linearly independent solutions of this equation by $C(x)$ and $S(x)$. To specify these functions completely, we set $C(0) = S'(0) = 1$, and $C'(0) = S(0) = 0$. We claim that this information is enough to identify $C(x)$ and $S(x)$ as $\cos x$ and $\sin x$, respectively.

First, let us show that the solutions exist and are well-behaved functions. With $C(0)$ and $C'(0)$ given, the equation $y'' + y = 0$ can generate all derivatives of $C(x)$ at zero: $C''(0) = -C(0) = -1$, $C'''(0) = -C'(0) = 0$, $C^{(4)}(0) = -C''(0) = +1$, and, in general,

$$C^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^k & \text{if } n = 2k \text{ where } k = 0, 1, 2, \dots. \end{cases}$$

Thus, the Taylor expansion of $C(x)$ is

$$C(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \tag{14.24}$$

Similarly,

$$S(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \tag{14.25}$$

A simple ratio test on the series representation of $C(x)$ yields

$$\begin{aligned} \lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{(-1)^{k+1} x^{2(k+1)}/(2k+2)!}{(-1)^k x^{2k}/(2k)!} \\ &= \lim_{k\to\infty} \frac{-x^2}{(2k+2)(2k+1)} = 0, \end{aligned}$$

which shows that the series for $C(x)$ converges for all values of x. Similarly, the series for $S(x)$ is also convergent. Thus, we are dealing with well-defined finite-valued functions.

Let us now enumerate and prove some properties of $C(x)$ and $S(x)$.

(a) $C'(x) = -S(x)$.
We prove this relation by differentiating $C''(x) + C(x) = 0$ and writing the result as $[C'(x)]'' + C'(x) = 0$ to make evident the fact that $C'(x)$ is also a solution. Since $C'(0) = 0$ and $[C'(0)]' = C''(0) = -1$, and since $-S(x)$ satisfies the same initial conditions, the uniqueness theorem implies that $C'(x) = -S(x)$. Similarly, $S'(x) = C(x)$.

(b) $C^2(x) + S^2(x) = 1$.
Since the $p(x)$ term is absent from the SOLDE, Proposition 14.4.2 implies that the Wronskian of $C(x)$ and $S(x)$ is constant. On the other hand,

$$\begin{aligned} W(C, S; x) &= C(x)S'(x) - C'(x)S(x) = C^2(x) + S^2(x) \\ &= W(C, S; 0) = C^2(0) + S^2(0) = 1. \end{aligned}$$

(c) $S(a + x) = S(a)C(x) + C(a)S(x)$.
The use of the chain rule easily shows that $S(a + x)$ is a solution of the equation $y'' + y = 0$. Thus, it can be written as a linear combination of $C(x)$ and $S(x)$ [which are linearly independent because their Wronskian is nonzero by (b)]:

$$S(a + x) = AS(x) + BC(x). \tag{14.26}$$

This is a functional identity, which for $x = 0$ gives $S(a) = BC(0) = B$. If we differentiate both sides of Eq. (14.26), we get

$$C(a + x) = AS'(x) + BC'(x) = AC(x) - BS(x),$$

which for $x = 0$ gives $C(a) = A$. Substituting the values of A and B in Eq. (14.26) yields the desired identity. A similar argument leads to

$$C(a + x) = C(a)C(x) - S(a)S(x).$$

(d) Periodicity of $C(x)$ and $S(x)$.
Let x_0 be the smallest positive real number such that $S(x_0) = C(x_0)$. Then property (b) implies that $C(x_0) = S(x_0) = 1/\sqrt{2}$. On the other hand,

$$\begin{aligned} S(x_0 + x) &= S(x_0)C(x) + C(x_0)S(x) = C(x_0)C(x) + S(x_0)S(x) \\ &= C(x_0)C(x) - S(x_0)S(-x) = C(x_0 - x). \end{aligned}$$

The third equality follows because by Eq. (14.25), $S(x)$ is an odd function of x. This is true for all x; in particular, for $x = x_0$ it yields $S(2x_0) = C(0) = 1$, and by property (b), $C(2x_0) = 0$. Using property (c) once more, we get

$$\begin{aligned} S(2x_0 + x) &= S(2x_0)C(x) + C(2x_0)S(x) = C(x), \\ C(2x_0 + x) &= C(2x_0)C(x) - S(2x_0)S(x) = -S(x). \end{aligned}$$

Substituting $x = 2x_0$ yields $S(4x_0) = C(2x_0) = 0$ and $C(4x_0) = -S(2x_0) = -1$. Continuing in this manner, we can easily obtain

$$S(8x_0 + x) = S(x), \qquad C(8x_0 + x) = C(x),$$

which prove the periodicity of $S(x)$ and $C(x)$ and show that their period is $8x_0$. It is even possible to determine x_0. This determination is left as a problem, but the result is

$$x_0 = \int_0^{1/\sqrt{2}} \frac{dt}{\sqrt{1-t^2}}.$$

A numerical calculation will show that this is $\pi/4$.

14.6.1 Frobenius Method of Undetermined Coefficients

A proper treatment of SOLDEs requires the medium of complex analysis and will be undertaken in the next chapter. At this point, however, we are seeking a *formal* infinite series solution to the SOLDE

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are real and analytic. This means that $p(x)$ and $q(x)$ can be represented by convergent power series in some interval (a, b). [The interesting case where $p(x)$ and $q(x)$ may have singularities will be treated in the context of complex solutions.]

The general procedure is to write the expansions[6]

$$p(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad q(x) = \sum_{k=0}^{\infty} b_k x^k, \qquad y = \sum_{k=0}^{\infty} c_k x^k \tag{14.27}$$

for the coefficient functions p and q and the solution y, substitute them in the SOLDE, and equate the coefficient of each power of x to zero. For this purpose, we need expansions for derivatives of y:

$$y' = \sum_{k=1}^{\infty} kc_k x^{k-1} = \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k,$$

$$y'' = \sum_{k=1}^{\infty} (k+1)kc_{k+1} x^{k-1} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k.$$

[6] Here we are expanding about the origin. If such an expansion is impossible or inconvenient, one can expand about another point, say x_0. One would then replace all powers of x in all expressions below with powers of $x - x_0$. These expansions assume that p, q, and y have no singularity at $x = 0$. In general, this assumption is not valid, and a different approach, in which the whole series is multiplied by a (not necessarily positive integer) power of x, ought to be taken. Details are provided in Chap. 15.

Thus

$$p(x)y' = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_m x^m (k+1)c_{k+1}x^k = \sum_{k,m}(k+1)a_m c_{k+1}x^{k+m}.$$

Let $k+m \equiv n$ and sum over n. Then the other sum, say m, cannot exceed n. Thus,

$$p(x)y' = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m+1)a_m c_{n-m+1}x^n.$$

Similarly, $q(x)y = \sum_{n=0}^{\infty}\sum_{m=0}^{n} b_m c_{n-m}x^n$. Substituting these sums and the series for y'' in the SOLDE, we obtain

$$\sum_{n=0}^{\infty}\left\{(n+1)(n+2)c_{n+2} + \sum_{m=0}^{n}\left[(n-m+1)a_m c_{n-m+1} + b_m c_{n-m}\right]\right\}x^n = 0.$$

For this to be true for all x, the coefficient of each power of x must vanish:

$$(n+1)(n+2)c_{n+2} = -\sum_{m=0}^{n}\left[(n-m+1)a_m c_{n-m+1} + b_m c_{n-m}\right] \quad \text{for } n \ge 0,$$

or

$$n(n+1)c_{n+1} = -\sum_{m=0}^{n-1}\left[(n-m)a_m c_{n-m} + b_m c_{n-m-1}\right] \quad \text{for } n \ge 1. \quad (14.28)$$

If we know c_0 and c_1 (for instance from boundary conditions), we can uniquely determine c_n for $n \ge 2$ from Eq. (14.28). This, in turn, gives a unique power-series expansion for y, and we have the following theorem.

existence theorem for SOLDE

Theorem 14.6.2 (Existence) *For any SOLDE of the form* $y'' + p(x)y' + q(x)y = 0$ *with analytic coefficient functions given by the first two equations of* (14.27), *there exists a unique power series, given by the third equation of* (14.27) *that formally satisfies the SOLDE for each choice of* c_0 *and* c_1.

This theorem merely states the existence of a formal power series and says nothing about its convergence. The following example will demonstrate that convergence is not necessarily guaranteed.

Example 14.6.3 The formal power-series solution for $x^2y' - y + x = 0$ can be obtained by letting $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$, and substitution in the DE gives

$$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+2} - \sum_{n=0}^{\infty}c_n x^n + x = 0,$$

or

$$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+2} - c_0 - c_1 x - \sum_{n=2}^{\infty}c_n x^n + x = 0.$$

We see that $c_0 = 0$, $c_1 = 1$, and $(n+1)c_{n+1} = c_{n+2}$ for $n \geq 0$. Thus, we have the recursion relation $nc_n = c_{n+1}$ for $n \geq 1$ whose unique solution is $c_n = (n-1)!$, which generates the following solution for the DE:

$$y = x + x^2 + (2!)x^3 + (3!)x^4 + \cdots + (n-1)!x^n + \cdots .$$

This series is not convergent for any nonzero x.

The SOLDE solved in the preceding example is not normal. However, for *normal* SOLDEs, the power series of y in Eq. (14.27) converges to an analytic function, as the following theorem shows (for a proof, see [Birk 78, p. 95]):

Theorem 14.6.4 *For any choice of c_0 and c_1, the radius of convergence of any power series solution $y = \sum_{k=0}^{\infty} c_k x^k$ for the* normal *HSOLDE*

$$y'' + p(x)y' + q(x)y = 0$$

whose coefficients satisfy the recursion relation of (14.28) *is at least as large as the smaller of the two radii of convergence of the two series for $p(x)$ and $q(x)$.*

In particular, if $p(x)$ and $q(x)$ are analytic in an interval around $x = 0$, then the solution of the normal HSOLDE is also analytic in a neighborhood of $x = 0$.

Example 14.6.5 As an application of Theorem 14.6.2, let us consider the Legendre equation in its normal form

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0.$$

For $|x| < 1$ both p and q are analytic, and

$$p(x) = -2x \sum_{m=0}^{\infty} (x^2)^m = \sum_{m=0}^{\infty} (-2)x^{2m+1},$$

$$q(x) = \lambda \sum_{m=0}^{\infty} (x^2)^m = \sum_{m=0}^{\infty} \lambda x^{2m}.$$

Thus, the coefficients of Eq. (14.27) are

$$a_m = \begin{cases} 0 & \text{if } m \text{ is even,} \\ -2 & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad b_m = \begin{cases} \lambda & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

We want to substitute for a_m and b_m in Eq. (14.28) to find c_{n+1}. It is convenient to consider two cases: when n is odd and when n is even. For $n = 2r+1$, Eq. (14.28)—after some algebra—yields

$$(2r+1)(2r+2)c_{2r+2} = \sum_{m=0}^{r} (4r - 4m - \lambda)c_{2(r-m)}. \tag{14.29}$$

With $r \to r+1$, this becomes

$$\begin{aligned}
&(2r+3)(2r+4)c_{2r+4}\\
&\quad = \sum_{m=0}^{r+1}(4r+4-4m-\lambda)c_{2(r+1-m)}\\
&\quad = (4r+4-\lambda)c_{2(r+1)} + \sum_{m=1}^{r+1}(4r+4-4m-\lambda)c_{2(r+1-m)}\\
&\quad = (4r+4-\lambda)c_{2r+2} + \sum_{m=0}^{r}(4r-4m-\lambda)c_{2(r-m)}\\
&\quad = (4r+4-\lambda)c_{2r+2} + (2r+1)(2r+2)c_{2r+2}\\
&\quad = \bigl[-\lambda + (2r+3)(2r+2)\bigr]c_{2r+2},
\end{aligned}$$

where in going from the second equality to the third we changed the dummy index, and in going from the third equality to the fourth we used Eq. (14.29). Now we let $2r+2 \equiv k$ to obtain $(k+1)(k+2)c_{k+2} = [k(k+1)-\lambda]c_k$, or

$$c_{k+2} = \frac{k(k+1)-\lambda}{(k+1)(k+2)}c_k \quad \text{for even } k.$$

It is not difficult to show that starting with $n = 2r$, the case of even n, we obtain this same equation for odd k. Thus, we can write

$$c_{n+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)}c_n. \tag{14.30}$$

For arbitrary c_0 and c_1, we obtain two independent solutions, one of which has only even powers of x and the other only odd powers. The generalized ratio test (see [Hass 08, Chap. 5]) shows that the series is divergent for $x = \pm 1$ unless $\lambda = l(l+1)$ for some positive integer l. In that case the infinite series becomes a polynomial, the Legendre polynomial encountered in Chap. 8.

Equation (14.30) could have been obtained by substituting Eq. (14.27) directly into the Legendre equation. The roundabout way to (14.30) taken here shows the generality of Eq. (14.28). With specific differential equations it is generally better to substitute (14.27) directly.

quantum harmonic oscillator: power series method

Example 14.6.6 We studied Hermite polynomials in Chap. 8 in the context of classical orthogonal polynomials. Let us see how they arise in physics.

The one-dimensional time-independent Schroedinger equation for a particle of mass m in a potential $V(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi,$$

where E is the total energy of the particle.

For a harmonic oscillator, $V(x) = \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$ and

$$\psi'' - \frac{m^2\omega^2}{\hbar^2}x^2\psi + \frac{2m}{\hbar^2}E\psi = 0.$$

Substituting $\psi(x) = H(x)\exp(-m\omega x^2/2\hbar)$ and then making the change of variables $x = (1/\sqrt{m\omega/\hbar})y$ yields

$$H'' - 2yH' + \lambda H = 0 \quad \text{where } \lambda = \frac{2E}{\hbar\omega} - 1. \tag{14.31}$$

This is the Hermite differential equation in normal form. We assume the expansion $H(y) = \sum_{n=0}^{\infty} c_n y^n$ which yields

$$H'(y) = \sum_{n=1}^{\infty} nc_n y^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}y^n,$$

$$H''(y) = \sum_{n=1}^{\infty} n(n+1)c_{n+1}y^{n-1} = \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}y^n.$$

Substituting in Eq. (14.31) gives

$$\sum_{n=0}^{\infty}\big[(n+1)(n+2)c_{n+2} + \lambda c_n\big]y^n - 2\sum_{n=0}^{\infty}(n+1)c_{n+1}y^{n+1} = 0,$$

or

$$2c_2 + \lambda c_0 + \sum_{n=0}^{\infty}\big[(n+2)(n+3)c_{n+3} + \lambda c_{n+1} - 2(n+1)c_{n+1}\big]y^{n+1} = 0.$$

Setting the coefficients of powers of y equal to zero, we obtain

$$c_2 = -\frac{\lambda}{2}c_0,$$

$$c_{n+3} = \frac{2(n+1) - \lambda}{(n+2)(n+3)}c_{n+1} \quad \text{for } n \geq 0,$$

or, replacing n with $n-1$,

$$c_{n+2} = \frac{2n - \lambda}{(n+1)(n+2)}c_n, \quad n \geq 1. \tag{14.32}$$

The ratio test yields easily that the series is convergent for all values of y.

Thus, the infinite series whose coefficients obey the recursive relation in Eq. (14.32) converges for all y. However, if we demand that $\lim_{x\to\infty}\psi(x) = 0$, which is necessary on physical grounds, the series must be truncated. This happens only if $\lambda = 2l$ for some integer l (see Problem 14.22 and [Hass 08, Chap. 13]), and in that case we obtain a polynomial, the Hermite polynomial of order l. A consequence of such a truncation is the quantization of harmonic oscillator energy:

$$2l = \lambda = \frac{2E}{\hbar\omega} - 1 \quad \Rightarrow \quad E = \left(l + \frac{1}{2}\right)\hbar\omega.$$

Two solutions are generated from Eq. (14.32), one including only even powers and the other only odd powers. These are clearly linearly independent. Thus, knowledge of c_0 and c_1 determines the general solution of the HSOLDE of (14.31).

14.6.2 Quantum Harmonic Oscillator

The preceding two examples show how certain special functions used in mathematical physics are obtained in an analytic way, by solving a differential equation. We saw in Chap. 13 how to obtain spherical harmonics and Legendre polynomials by algebraic methods. It is instructive to solve the harmonic oscillator problem using algebraic methods.

The Hamiltonian of a one-dimensional harmonic oscillator is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where $\mathbf{p} = -i\hbar d/dx$ is the momentum operator. Let us find the eigenvectors and eigenvalues of $\mathbf{H}$.

We define the operators

$$\mathbf{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{\mathbf{p}}{\sqrt{2m\hbar\omega}} \quad \text{and} \quad \mathbf{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{\mathbf{p}}{\sqrt{2m\hbar\omega}}.$$

Using the commutation relation $[x, \mathbf{p}] = i\hbar\mathbf{1}$, we can show that

$$\left[\mathbf{a}, \mathbf{a}^\dagger\right] = \mathbf{1} \quad \text{and} \quad \mathbf{H} = \hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\hbar\omega\mathbf{1}. \tag{14.33}$$

creation and annihilation operators

Furthermore, one can readily show that

$$[\mathbf{H}, \mathbf{a}] = -\hbar\omega\mathbf{a}, \qquad \left[\mathbf{H}, \mathbf{a}^\dagger\right] = \hbar\omega\mathbf{a}^\dagger. \tag{14.34}$$

Let $|\psi_E\rangle$ be the eigenvector corresponding to the eigenvalue E: $\mathbf{H}|\psi_E\rangle = E|\psi_E\rangle$, and note that Eq. (14.34) gives

$$\mathbf{H}\mathbf{a}|\psi_E\rangle = (\mathbf{a}\mathbf{H} - \hbar\omega\mathbf{a})|\psi_E\rangle = (E - \hbar\omega)\mathbf{a}|\psi_E\rangle$$

and

$$\mathbf{H}\mathbf{a}^\dagger|\psi_E\rangle = (E + \hbar\omega)\mathbf{a}^\dagger|\psi_E\rangle.$$

Thus, $\mathbf{a}|\psi_E\rangle$ is an eigenvector of $\mathbf{H}$, with eigenvalue $E - \hbar\omega$, and $\mathbf{a}^\dagger|\psi_E\rangle$ is an eigenvector with eigenvalue $E + \hbar\omega$. That is why $\mathbf{a}^\dagger$ and $\mathbf{a}$ are called the **raising** and **lowering** (or creation and annihilation) **operators**, respectively. We can write

$$\mathbf{a}|\psi_E\rangle = c_E|\psi_{E-\hbar\omega}\rangle.$$

By applying $\mathbf{a}$ repeatedly, we obtain states of lower and lower energies. But there is a limit to this because $\mathbf{H}$ is a positive operator: It cannot have a negative eigenvalue. Thus, there must exist a **ground state**, $|\psi_0\rangle$, such that

$\mathbf{a}|\psi_0\rangle = 0$. The energy of this ground state (or the eigenvalue corresponding to $|\psi_0\rangle$) can be obtained:[7]

$$\mathbf{H}|\psi_0\rangle = \left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\hbar\omega\right)|\psi_0\rangle = \frac{1}{2}\hbar\omega|\psi_0\rangle.$$

Repeated application of the raising operator yields both higher-level states and eigenvalues. We thus define $|\psi_n\rangle$ by

$$\left(\mathbf{a}^\dagger\right)^n|\psi_0\rangle = c_n|\psi_n\rangle, \tag{14.35}$$

where c_n is a normalizing constant. The energy of $|\psi_n\rangle$ is n units higher than the ground state's, or

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,$$

which is what we obtained in the preceding example.

To find c_n, we demand orthonormality for the $|\psi_n\rangle$. Taking the inner product of (14.35) with itself, we can show (see Problem 14.23) that

$$|c_n|^2 = n|c_{n-1}|^2 \quad \Rightarrow \quad |c_n|^2 = n!|c_0|^2,$$

which for $|c_0| = 1$ and real c_n yields $c_n = \sqrt{n!}$. It follows, then, that

$$|\psi_n\rangle = \frac{1}{\sqrt{n!}}\left(\mathbf{a}^\dagger\right)^n|\psi_0\rangle. \tag{14.36}$$

quantum harmonic oscillator: connection between algebraic and analytic methods

In terms of functions and derivative operators, $\mathbf{a}|\psi_0\rangle = 0$ gives

$$\left(\sqrt{\frac{m\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)\psi_0(x) = 0$$

with the solution $\psi_0(x) = c\exp(-m\omega x^2/2\hbar)$. Normalizing $\psi_0(x)$ gives

$$1 = \langle\psi_0|\psi_0\rangle = c^2\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega x^2}{\hbar}\right)dx = c^2\left(\frac{\hbar\pi}{m\omega}\right)^{1/2}.$$

Thus,

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}e^{-m\omega x^2/(2\hbar)}.$$

We can now write Eq. (14.36) in terms of differential operators:

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\left(\sqrt{\frac{m\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)^n e^{-m\omega x^2/(2\hbar)}.$$

Defining a new variable $y = \sqrt{m\omega/\hbar}x$ transforms this equation into

$$\psi_n = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}.$$

[7]From here on, the unit operator $\mathbf{1}$ will not be shown explicitly.

From this, the relation between Hermite polynomials, and the solutions of the one-dimensional harmonic oscillator as given in the previous example, we can obtain a general formula for $H_n(x)$. In particular, if we note that (see Problem 14.23)

$$e^{y^2/2}\left(y-\frac{d}{dy}\right)e^{-y^2/2}=-e^{y^2}\frac{d}{dy}e^{-y^2}$$

and, in general,

$$e^{y^2/2}\left(y-\frac{d}{dy}\right)^n e^{-y^2/2}=(-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2},$$

we recover the generalized Rodriguez formula of Chap. 8 for Hermite polynomials.

14.7 SOLDEs with Constant Coefficients

The solution to a SOLDE with constant coefficients can always be found in closed form. In fact, we can treat an nth-order linear differential equation (NOLDE) with constant coefficients with no extra effort. This section outlines the procedure for solving such an equation. For details, the reader is referred to any elementary book on differential equations (see also [Hass 08]). The most general nth-order linear differential equation (NOLDE) with constant coefficients can be written as

$$\mathbf{L}[y]\equiv y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y=r(x). \tag{14.37}$$

The corresponding homogeneous NOLDE (HNOLDE) has $r(x)=0$. The solution to the HNOLDE

$$\mathbf{L}[y]\equiv y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y=0 \tag{14.38}$$

can be found by making the exponential substitution $y=e^{\lambda x}$, which results in the equation

$$\mathbf{L}\big[e^{\lambda x}\big]=\big(\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0\big)e^{\lambda x}=0.$$

characteristic polynomial of an HNOLDE

This equation will hold only if λ is a root of the **characteristic polynomial**

$$p(\lambda)\equiv\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0,$$

which, by the fundamental theorem of algebra (Theorem 10.5.6), can be written as

$$p(\lambda)=(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m}, \tag{14.39}$$

where λ_j is the distinct (complex) root of $p(\lambda)$ with multiplicity k_j.

Theorem 14.7.1 *Let $\{\lambda_j\}_{j=1}^m$ be the distinct roots of the characteristic polynomial of the real HNOLDE of Eq.* (14.38) *with multiplicities $\{k_j\}_{j=1}^m$. Then the functions*

$$\left\{\left\{x^{r_j} e^{\lambda_j x}\right\}_{r_j=0}^{k_j-1}\right\}_{j=1}^m \equiv \left\{e^{\lambda_j x}, x e^{\lambda_j x}, \dots, x^{k_j-1} e^{\lambda_j x}\right\}_{j=1}^m$$

are a basis of solutions of Eq. (14.38).

When λ_i is complex, one can write its corresponding solution in terms of trigonometric functions.

Example 14.7.2 An equation that is used in both mechanics and circuit theory is

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0 \quad \text{for } a, b > 0. \tag{14.40}$$

Its characteristic polynomial is $p(\lambda) = \lambda^2 + a\lambda + b$, which has the roots

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \text{and} \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

We can distinguish three different possible motions depending on the relative sizes of a and b.

(a) $a^2 > 4b$ (overdamped): Here we have two distinct simple roots. The multiplicities are both one ($k_1 = k_2 = 1$); therefore, the power of x for both solutions is zero ($r_1 = r_2 = 0$). Let $\gamma \equiv \frac{1}{2}\sqrt{a^2 - 4b}$. Then the most general solution is

overdamped oscillation

$$y(t) = e^{-at/2}\left(c_1 e^{\gamma t} + c_2 e^{-\gamma t}\right).$$

Since $a > 2\gamma$, this solution starts at $y = c_1 + c_2$ at $t = 0$ and continuously decreases; so, as $t \to \infty$, $y(t) \to 0$.

(b) $a^2 = 4b$ (critically damped): In this case we have one multiple root of order 2 ($k_1 = 2$); therefore, the power of x can be zero or 1 ($r_1 = 0, 1$). Thus, the general solution is

critically damped oscillation

$$y(t) = c_1 t e^{-at/2} + c_0 e^{-at/2}.$$

This solution starts at $y(0) = c_0$ at $t = 0$, reaches a maximum (or minimum) at $t = 2/a - c_0/c_1$, and subsequently decays (grows) exponentially to zero.

(c) $a^2 < 4b$ (underdamped): Once more, we have two distinct simple roots. The multiplicities are both one ($k_1 = k_2 = 1$); therefore, the power of x for both solutions is zero ($r_1 = r_2 = 0$). Let $\omega \equiv \frac{1}{2}\sqrt{4b - a^2}$. Then $\lambda_1 = -a/2 + i\omega$ and $\lambda_2 = \lambda_1^*$. The roots are complex, and the most general solution is thus of the form

underdamped oscillation

$$y(t) = e^{-at/2}(c_1 \cos\omega t + c_2 \sin\omega t) = Ae^{-at/2}\cos(\omega t + \alpha).$$

The solution is a harmonic variation with a decaying amplitude $A\exp(-at/2)$. Note that if $a = 0$, the amplitude does not decay. That is why a is called the **damping factor** (or the damping constant).

These equations describe either a mechanical system oscillating (with no external force) in a viscous (dissipative) fluid, or an electrical circuit consisting of a resistance R, an inductance L, and a capacitance C. For RLC circuits, $a = R/L$ and $b = 1/(LC)$. Thus, the damping factor depends on the relative magnitudes of R and L. On the other hand, the frequency

$$\omega \equiv \sqrt{b - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

depends on all three elements. In particular, for $R \geq 2\sqrt{L/C}$ the circuit does not oscillate.

A physical system whose behavior in the absence of a driving force is described by a HNOLDE will obey an inhomogeneous NOLDE in the presence of the driving force. This driving force is simply the inhomogeneous term of the NOLDE. The best way to solve such an inhomogeneous NOLDE in its most general form is by using Fourier transforms and Green's functions, as we will do in Chap. 20. For the particular, but important, case in which the inhomogeneous term is a product of polynomials and exponentials, the solution can be found in closed form.

Theorem 14.7.3 *The INOLDE* $\mathbf{L}[y] = e^{\lambda x} S(x)$, *where* $S(x)$ *is a polynomial, has the particular solution* $e^{\lambda x} q(x)$, *where* $q(x)$ *is also a polynomial. The degree of* $q(x)$ *equals that of* $S(x)$ *unless* $\lambda = \lambda_j$, *a root of the characteristic polynomial of* $\mathbf{L}$, *in which case the degree of* $q(x)$ *exceeds that of* $S(x)$ *by* k_j, *the multiplicity of* λ_j.

Once we know the form of the particular solution of the NOLDE, we can find the coefficients in the polynomial of the solution by substituting in the NOLDE and matching the powers on both sides.

Example 14.7.4 Let us find the most general solutions for the following two differential equations subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$.

(a) The first DE we want to consider is

$$y'' + y = xe^x. \tag{14.41}$$

The characteristic polynomial is $\lambda^2 + 1$, whose roots are $\lambda_1 = i$ and $\lambda_2 = -i$. Thus, a basis of solutions is $\{\cos x, \sin x\}$. To find the particular solution we note that λ (the coefficient of x in the exponential part of the inhomogeneous term) is 1, which is neither of the roots λ_1 and λ_2. Thus, the particular solution is of the form $q(x)e^x$, where $q(x) = Ax + B$ is of degree 1 [same degree as that of $S(x) = x$]. We now substitute $y = (Ax + B)e^x$ in Eq. (14.41) to obtain the relation

$$2Axe^x + (2A + 2B)e^x = xe^x.$$

Matching the coefficients, we have

$$2A = 1 \quad \text{and} \quad 2A + 2B = 0 \quad \Rightarrow \quad A = \frac{1}{2} = -B.$$

Thus, the most general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}(x-1)e^x.$$

Imposing the given boundary conditions yields $0 = y(0) = c_1 - \frac{1}{2}$ and $1 = y'(0) = c_2$. Thus,

$$y = \frac{1}{2}\cos x + \sin x + \frac{1}{2}(x-1)e^x$$

is the unique solution.

(b) The next DE we want to consider is

$$y'' - y = xe^x. \tag{14.42}$$

Here $p(\lambda) = \lambda^2 - 1$, and the roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. A basis of solutions is $\{e^x, e^{-x}\}$. To find a particular solution, we note that $S(x) = x$ and $\lambda = 1 = \lambda_1$. Theorem 14.7.3 then implies that $q(x)$ must be of degree 2, because λ_1 is a simple root, i.e., $k_1 = 1$. We therefore try

$$q(x) = Ax^2 + Bx + C \quad \Rightarrow \quad y = \big(Ax^2 + Bx + C\big)e^x.$$

Taking the derivatives and substituting in Eq. (14.42) yields two equations,

$$4A = 1 \quad \text{and} \quad A + B = 0,$$

whose solution is $A = -B = \frac{1}{4}$. Note that C is not determined, because Ce^x is a solution of the homogeneous DE corresponding to Eq. (14.42), so when $\mathbf{L}$ is applied to y, it eliminates the term Ce^x. Another way of looking at the situation is to note that the most general solution to (14.42) is of the form

$$y = c_1 e^x + c_2 e^{-x} + \left(\frac{1}{4}x^2 - \frac{1}{4}x + C\right)e^x.$$

The term Ce^x could be absorbed in $c_1 e^x$. We therefore set $C = 0$, apply the boundary conditions, and find the unique solution

$$y = \frac{5}{4}\sinh x + \frac{1}{4}\big(x^2 - x\big)e^x.$$

14.8 The WKB Method

In this section, we treat the somewhat specialized method of obtaining an approximate solution to a particular type of second-order DE arising from the Schrödinger equation in one dimension. The method's name comes from Wentzel, Kramers, and Brillouin, who invented it and applied it for the first time.

Suppose we are interested in finding approximate solutions of the DE

$$\frac{d^2y}{dx^2} + q(x)y = 0 \tag{14.43}$$

in which q varies "slowly" with respect to x in the sense discussed below. If q varies infinitely slowly, i.e., if it is a constant, the solution to Eq. (14.43) is simply an imaginary exponential (or trigonometric). So, let us define $\phi(x)$ by $y = e^{i\phi(x)}$ and rewrite the DE as

$$(\phi')^2 + i\phi'' - q = 0. \tag{14.44}$$

Assuming that ϕ'' is small (compared to q), so that y does not oscillate too rapidly, we can find an approximate solution to the DE:

$$\phi' = \pm\sqrt{q} \quad \Rightarrow \quad \phi = \pm\int \sqrt{q(x)}\,dx. \tag{14.45}$$

The condition of validity of our assumption is obtained by differentiating (14.45):

$$|\phi''| \approx \frac{1}{2}\left|\frac{q'}{\sqrt{q}}\right| \ll |q|.$$

It follows from Eq. (14.45) and the definition of ϕ that $1/\sqrt{q}$ is approximately $1/(2\pi)$ times one "wavelength" of the solution y. Therefore, the approximation is valid if the change in q in one wavelength is small compared to $|q|$.

The approximation can be improved by inserting the derivative of (14.45) in the DE and solving for a new ϕ:

$$(\phi')^2 \approx q \pm \frac{i}{2}\frac{q'}{\sqrt{q}} \quad \Rightarrow \quad \phi' \approx \pm\left(q \pm \frac{i}{2}\frac{q'}{\sqrt{q}}\right)^{1/2},$$

or

$$\begin{aligned}\phi' &\approx \pm\sqrt{q}\left(1 \pm \frac{i}{2}\frac{q'}{q^{3/2}}\right)^{1/2} = \pm\sqrt{q}\left(1 \pm \frac{i}{4}\frac{q'}{q^{3/2}}\right)\\ &= \pm\sqrt{q} + \frac{i}{4}\frac{q'}{q} \quad \Rightarrow \quad \phi(x) \approx \pm\int \sqrt{q}\,dx + \frac{i}{4}\ln q.\end{aligned}$$

The two choices give rise to two different solutions, a linear combination of which gives the most general solution. Thus,

$$y \approx \frac{1}{\sqrt[4]{q(x)}}\left\{c_1 \exp\left[i\int \sqrt{q}\,dx\right] + c_2 \exp\left[-i\int \sqrt{q}\,dx\right]\right\}. \tag{14.46}$$

Equation (14.46) gives an approximate solution to (14.43) in any region in which the condition of validity holds. The method fails if q changes too rapidly or if it is zero at some point of the region. The latter is a serious difficulty, since we often wish to join a solution in a region in which $q(x) > 0$ to one in a region in which $q(x) < 0$. There is a general procedure for deriving the so-called *connection formulas* relating the constants c_1 and c_2 of the two solutions on either side of the point where $q(x) = 0$. We shall not go into the details of such a derivation, as it is not particularly illuminating.[8] We simply quote a particular result that is useful in applications.

connection formulas

Suppose that q passes through zero at x_0, is positive to the right of x_0, and satisfies the condition of validity in regions both to the right and to the left of x_0. Furthermore, assume that the solution of the DE decreases exponentially to the left of x_0. Under such conditions, the solution to the left will be of the form

$$\frac{1}{\sqrt[4]{-q(x)}} \exp\left[-\int_x^{x_0} \sqrt{-q(x)}\, dx\right], \tag{14.47}$$

while to the right, we have

$$2\frac{1}{\sqrt[4]{q(x)}} \cos\left[\int_{x_0}^{x} \sqrt{q(x)}\, dx - \frac{\pi}{4}\right]. \tag{14.48}$$

A similar procedure gives connection formulas for the case where q is positive on the left and negative on the right of x_0.

Example 14.8.1 Consider the Schrödinger equation in one dimension

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi = 0$$

where $V(x)$ is a potential well meeting the horizontal line of constant E at $x = a$ and $x = b$, so that

$$q(x) = \frac{2m}{\hbar^2}\left[E - V(x)\right]\begin{cases} > 0 & \text{if } a < x < b, \\ < 0 & \text{if } x < a \text{ or } x > b. \end{cases}$$

The solution that is bounded to the left of a must be exponentially decaying. Therefore, in the interval (a, b) the approximate solution, as given by Eq. (14.48), is

$$\psi(x) \approx \frac{A}{(E - V)^{1/4}} \cos\left(\int_a^x \sqrt{\frac{2m}{\hbar^2}\left[E - V(x)\right]}\, dx - \frac{\pi}{4}\right),$$

where A is some arbitrary constant. The solution that is bounded to the right of b must also be exponentially decaying. Hence, the solution for $a < x < b$ is

$$\psi(x) \approx \frac{B}{(E - V)^{1/4}} \cos\left(\int_x^b \sqrt{\frac{2m}{\hbar^2}\left[E - V(x)\right]}\, dx - \frac{\pi}{4}\right).$$

[8] The interested reader is referred to the book by Mathews and Walker, pp. 27–37.

Since these two expressions give the same function in the same region, they must be equal. Thus, $A = B$, and, more importantly,

$$\cos\left(\int_a^x \sqrt{\frac{2m}{\hbar^2}\big[E - V(x)\big]}\,dx - \frac{\pi}{4}\right) = \cos\left(\int_x^b \sqrt{\frac{2m}{\hbar^2}\big[E - V(x)\big]}\,dx - \frac{\pi}{4}\right),$$

or

$$\int_a^b \sqrt{2m\big[E - V(x)\big]}\,dx = \left(n + \frac{1}{2}\right)\pi\hbar.$$

Bohr-Sommerfeld quantization condition

This is essentially the Bohr-Sommerfeld quantization condition of pre-1925 quantum mechanics.

14.8.1 Classical Limit of the Schrödinger Equation

As long as we are approximating solutions of second-order DEs that arise naturally from the Schrödinger equation, it is instructive to look at another approximation to the Schrödinger equation, its classical limit in which the Planck constant goes to zero.

The idea is to note that since $\psi(\mathbf{r}, t)$ is a complex function, one can write it as

$$\psi(\mathbf{r}, t) = A(\mathbf{r}, t)\exp\left[\frac{i}{\hbar}S(\mathbf{r}, t)\right], \tag{14.49}$$

where $A(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ are real-valued functions. Substituting (14.49) in the Schrödinger equation and separating the real and the imaginary parts yields

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{\boldsymbol{\nabla} S \cdot \boldsymbol{\nabla} S}{2m} + V &= \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}, \\ m\frac{\partial A}{\partial t} + \boldsymbol{\nabla} S \cdot \boldsymbol{\nabla} A + \frac{A}{2}\nabla^2 S &= 0. \end{aligned} \tag{14.50}$$

These two equations are completely equivalent to the Schrödinger equation. The second equation has a direct physical interpretation. Define

$$\rho(\mathbf{r}, t) \equiv A^2(\mathbf{r}, t) = \big|\psi(\mathbf{r}, t)\big|^2 \quad \text{and} \quad \mathbf{J}(\mathbf{r}, t) \equiv A^2(\mathbf{r}, t)\underbrace{\frac{\nabla S}{m}}_{\equiv \mathbf{v}} = \rho\mathbf{v}, \tag{14.51}$$

multiply the second equation in (14.50) by $2A/m$, and note that it then can be written as

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{14.52}$$

which is the continuity equation for probability. The fact that $\mathbf{J}$ is indeed the probability current density is left for Problem 14.32.

The first equation of (14.50) gives an interesting result when $\hbar \to 0$ because in this limit, the RHS of the equation will be zero, and we get

$$\frac{\partial S}{\partial t} + \frac{1}{2}mv^2 + V = 0.$$

Taking the gradient of this equation, we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)m\mathbf{v} + \nabla V = 0,$$

which is the equation of motion of a classical fluid with velocity field $\mathbf{v} = \nabla S/m$. We thus have the following:

Proposition 14.8.2 *In the classical limit, the solution of the Schrödinger equation describes a fluid (statistical mixture) of noninteracting classical particles of mass m subject to the potential* $V(\mathbf{r})$. *The density and the current density of this fluid are, respectively, the probability density* $\rho = |\psi|^2$ *and the probability current density* $\mathbf{J}$ *of the quantum particle.*

Schrödinger equation describes a classical statistical mixture when $\hbar \to 0$.

14.9 Problems

14.1 Let $u(x)$ be a differentiable function satisfying the *differential inequality* $u'(x) \le Ku(x)$ for $x \in [a,b]$, where K is a constant. Show that $u(x) \le u(a)e^{K(x-a)}$. Hint: Multiply both sides of the inequality by e^{-Kx}, and show that the result can be written as the derivative of a nonincreasing function. Then use the fact that $a \le x$ to get the final result.

14.2 Prove Proposition 14.4.2.

14.3 Let $f_1(x) = x$ and $f_2(x) = |x|$ for $x \in [-1, 1]$. Show that these two functions are linearly independent in the given interval, and that their Wronskian vanishes. Is this a violation of Theorem 14.4.3?

14.4 How would you generalize the Wronskian to n functions which have derivatives up to nth order? Prove that the Wronskian vanishes if the functions are linearly dependent.

14.5 Let f and g be two differentiable functions that are linearly dependent. Show that their Wronskian vanishes.

14.6 Show that if (f_1, f_1') and (f_2, f_2') are linearly dependent at one point, then f_1 and f_2 are linearly dependent at all $x \in [a,b]$. Here f_1 and f_2 are solutions of the DE of (14.12). Hint: Derive the identity

$$W(f_1, f_2; x_2) = W(f_1, f_2; x_1)\exp\left\{-\int_{x_1}^{x_2} p(t)\,dt\right\}.$$

14.7 Show that the solutions to the SOLDE $y'' + q(x)y = 0$ have a constant Wronskian.

14.8 Find (in terms of an integral) $G_n(x)$, the linearly independent "partner" of the Hermite polynomial $H_n(x)$. Specialize this to $n = 0, 1$. Is it possible to find $G_0(x)$ and $G_1(x)$ in terms of elementary functions?

14.9 Let f_1, f_2, and f_3 be any three solutions of $y'' + py' + qy = 0$. Show that the (generalized 3×3) Wronskian of these solutions is zero. Thus, any three solutions of the HSOLDE are linearly dependent.

14.10 For the HSOLDE $y'' + py' + qy = 0$, show that

$$p = -\frac{f_1 f_2'' - f_2 f_1''}{W(f_1, f_2)} \quad \text{and} \quad q = \frac{f_1' f_2'' - f_2' f_1''}{W(f_1, f_2)}.$$

Thus, knowing two solutions of an HSOLDE allows us to reconstruct the DE.

14.11 Let f_1, f_2, and f_3 be three solutions of the third-order linear differential equation $y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$. Derive a FODE satisfied by the (generalized 3×3) Wronskian of these solutions.

14.12 Prove Corollary 14.4.12. Hint: Consider the solution $u = 1$ of the DE $u'' = 0$ and apply Theorem 14.4.10.

14.13 Show that the adjoint of **M** given in Eq. (14.20) is the original **L**.

Abel's identity

14.14 Show that if $u(x)$ and $v(x)$ are solutions of the self-adjoint DE $(pu')' + qu = 0$, then **Abel's identity**, $p(uv' - vu') = \text{constant}$, holds.

14.15 Reduce each DE to self-adjoint form.

$$\text{(a)} \quad x^2 y'' + xy' + y = 0, \qquad \text{(b)} \quad y'' + y' \tan x = 0.$$

14.16 Reduce the self-adjoint DE $(py')' + qy = 0$ to $u'' + S(x)u = 0$ by an appropriate change of the dependent variable. What is $S(x)$? Apply this reduction to the Legendre DE for $P_n(x)$, and show that

$$S(x) = \frac{1 + n(n+1) - n(n+1)x^2}{(1 - x^2)^2}.$$

Now use this result to show that every solution of the Legendre equation has at least $(2n + 1)/\pi$ zeros on $(-1, +1)$.

14.17 Substitute $v = y'/y$ in the *homogeneous* SOLDE

$$y'' + p(x)y' + q(x)y = 0$$

and:

(a) Show that it turns into $v' + v^2 + p(x)v + q(x) = 0$, which is a first-order *nonlinear* equation called the **Riccati equation**. Would the same substitution work if the DE were inhomogeneous?

Riccati equation

(b) Show that by an appropriate transformation, the Riccati equation can be directly cast in the form $u' + u^2 + S(x) = 0$.

14.18 For the function $S(x)$ defined in Example 14.6.1, let $S^{-1}(x)$ be the inverse, i.e., $S^{-1}(S(x)) = x$. Show that

$$\frac{d}{dx}\left[S^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}},$$

and given that $S^{-1}(0) = 0$, conclude that

$$S^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

14.19 Define $\sinh x$ and $\cosh x$ as the solutions of $y'' = y$ satisfying the boundary conditions $y(0) = 0$, $y'(0) = 1$ and $y(0) = 1$, $y'(0) = 0$, respectively. Using Example 14.6.1 as a guide, show that

(a) $\cosh^2 x - \sinh^2 x = 1$,

(b) $\cosh(-x) = \cosh x$,

(c) $\sinh(-x) = -\sinh x$.

(d) $\sinh(a + x) = \sinh a \cosh x + \cosh a \sinh x$.

14.20 For Example 14.6.5, derive

(a) Equation (14.29), and
(b) Equation (14.30) by direct substitution.
(c) Let $\lambda = l(l+1)$ and calculate the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3$, subject to the condition $P_l(1) = 1$.

14.21 Use Eq. (14.32) of Example 14.6.6 to generate the first three Hermite polynomials. Use the normalization

$$\int_{-\infty}^{\infty} \left[H_n(x)\right]^2 e^{-x^2} dx = \sqrt{\pi}\, 2^n n!$$

to determine the arbitrary constant.

14.22 The function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{where} \quad c_{n+2} = \frac{2n - \lambda}{(n+1)(n+2)} c_n,$$

can be written as $f(x) = c_0 g(x) + c_1 h(x)$, where g is even and h is odd in x. Show that $f(x)$ goes to infinity at least as fast as e^{x^2} does, i.e., $\lim_{x\to\infty} f(x)e^{-x^2} \neq 0$. Hint: Consider $g(x)$ and $h(x)$ separately and show that

$$g(x) = \sum_{n=0}^{\infty} b_n x^n, \quad \text{where} \quad b_{n+1} = \frac{4n - \lambda}{(2n+1)(2n+2)} b_n.$$

Then concentrate on the ratio $g(x)/e^{x^2}$, where g and e^{x^2} are approximated by polynomials of very high degrees. Take the limit of this ratio as $x \to \infty$, and use recursion relations for g and e^{x^2}. The odd case follows similarly.

14.23 Refer to Sect. 14.6.2 for this problem.

(a) Derive the commutation relation $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$.
(b) Show that the Hamiltonian can be written as given in Eq. (14.33).
(c) Derive the commutation relation $[\mathbf{a}, (\mathbf{a}^\dagger)^n] = n(\mathbf{a}^\dagger)^{n-1}$.
(d) Take the inner product of Eq. (14.35) with itself and use (c) to show that $|c_n|^2 = n|c_{n-1}|^2$. From this, conclude that $|c_n|^2 = n!|c_0|^2$.
(e) For any function $f(y)$, show that

$$\left(y - \frac{d}{dy}\right)(e^{y^2/2} f) = -e^{y^2/2}\frac{df}{dy}.$$

Apply $(y - d/dy)$ repeatedly to both sides of the above equation to obtain

$$\left(y - \frac{d}{dy}\right)^n (e^{y^2/2} f) = (-1)^n e^{y^2/2}\frac{d^n f}{dy^n}.$$

(f) Choose an appropriate $f(y)$ in part (e) and show that

$$e^{y^2/2}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2} = (-1)^n e^{y^2}\frac{d^n}{dy^n}\left(e^{-y^2}\right).$$

Airy's DE

14.24 Solve **Airy's DE**, $y'' + xy = 0$, by the power-series method. Show that the radius of convergence for both independent solutions is infinite. Use the comparison theorem to show that for $x > 0$ these solutions have infinitely many zeros, but for $x < 0$ they can have at most one zero.

14.25 Show that the functions $x^r e^{\lambda x}$, where $r = 0, 1, 2, \ldots, k$, are linearly independent. Hint: Apply appropriate powers of $\mathbf{D} - \lambda$ to a linear combination of $x^r e^{\lambda x}$ for all possible r's.

14.26 Find a basis of real solutions for each DE.

(a) $y'' + 5y' + 6 = 0$, (b) $y''' + 6y'' + 12y' + 8y = 0$,

(c) $\dfrac{d^4 y}{dx^4} = y$, (d) $\dfrac{d^4 y}{dx^4} = -y$.

14.27 Solve the following initial value problems.

(a) $\dfrac{d^4 y}{dx^4} = y, \qquad y(0) = y'(0) = y'''(0) = 0, \qquad y''(0) = 1,$

(b) $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 0, \qquad y(0) = y''(0) = y'''(0) = 0, \qquad y'(0) = 1,$

(c) $\dfrac{d^4y}{dx^4} = 0, \qquad y(0) = y'(0) = y''(0) = 0, \qquad y'''(0) = 2.$

14.28 Solve $y'' - 2y' + y = xe^x$ subject to the initial conditions $y(0) = 0$, $y'(0) = 1$.

14.29 Find the general solution of each equation,

(a) $y'' = xe^x$, (b) $y'' - 4y' + 4y = x^2$,

(c) $y'' + y = \sin x \sin 2x$, (d) $y'' - y = \left(1 + e^{-x}\right)^2$,

(e) $y'' - y = e^x \sin 2x$, (f) $y^{(6)} - y^{(4)} = x^2$,

(g) $y'' - 4y' + 4 = e^x + xe^{2x}$, (h) $y'' + y = e^{2x}$.

14.30 Consider the **Euler equation**,

the Euler equation

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = r(x).$$

Substitute $x = e^t$ and show that such a substitution reduces this to a DE with constant coefficients. In particular, solve $x^2y'' - 4xy' + 6y = x$.

14.31 Show that

(a) the substitution (14.49) reduces the Schrödinger equation to (14.50), and
(b) derive the continuity equation for probability from the second equation of (14.50).

14.32 Show that the usual definition of probability current density,

$$\mathbf{J} = \mathrm{Re}\left[\psi^* \frac{\hbar}{im} \nabla \psi\right],$$

reduces to that in Eq. (14.51) if we use (14.49).