Interest Rate Models 13

13.1 Summary

The results from the previous chapter will be applied to stochastic differential equations that were suggested in the literature for modeling interest rate dynamics. However, we do not model yield curves with various maturities, but consider the model for one interest rate only driven by one Wiener process (one-factor model). The next section starts with the general Ornstein-Uhlenbeck process which has the drawback of allowing for negative values. Subsequently, we discuss linear models for which negativity is ruled out. Finally, a class of nonlinear models will be considered.

13.2 Ornstein-Uhlenbeck Process (OUP)

We have already encountered the standard OUP in the chapter on Stieltjes integrals. Now, we discuss the general case, which has served as an interest rate model in the literature on finance.

Vasicek

We now assume constant coefficients for the inhomogeneous linear SDE with additive noise in (12.12):

$$c_1(t) = c_1 = const,\ c_2(t) = c_2 = const,\ \sigma_2(t) = \sigma_2 = const,\ \sigma_1(t) = 0\,.$$

This defines the general **Ornstein-Uhlenbeck process**,

$$dX(t) = (c_1\, X(t) + c_2)\ dt + \sigma_2\, dW(t)\,. \tag{13.1}$$

U. Hassler, *Stochastic Processes and Calculus*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-23428-1_13

According to Corollary 12.1 the solution is

$$X(t) = e^{c_1 t} \left[X(0) - \frac{c_2}{c_1} \left(e^{-c_1 t} - 1 \right) + \int_0^t \sigma_2 \, e^{-c_1 s} \, dW(s) \right] . \tag{13.2}$$

In particular for $c_2 = 0$ and $\sigma_2 = 1$ we obtain the standard OUP with $X(0) = 0$ from (9.3) in Sect. 9.4. The following equation sheds additional light on the OUP:

$$dX(t) = c_1 \, (X(t) - \mu) \, dt + \sigma_2 \, dW(t) \, , \quad \text{with } \mu := -\frac{c_2}{c_1} . \tag{13.3}$$

In this manner, Vasicek (1977) modeled the interest rate dynamics, cf. (1.7). Due to (13.2), the solution of this interest rate equation reads:

$$X(t) = e^{c_1 t} \left[X(0) + \mu \left(e^{-c_1 t} - 1 \right) + \int_0^t \sigma_2 \, e^{-c_1 s} \, dW(s) \right] .$$

Setting the starting value to μ, $X(0) = \mu$, then one obtains the form immediately corresponding to the standard OUP (9.3),

$$X(t) = \mu + e^{c_1 t} \int_0^t \sigma_2 \, e^{-c_1 s} \, dW(s) \, ,$$

with expectation μ. From (12.14) and (12.15) we obtain for an arbitrary fixed starting value $X(0)$:

$$\mu_1(t) = \mathrm{E}\left(X(t)\right) = e^{c_1 t} X(0) + \mu \left(1 - e^{c_1 t} \right) \, , \tag{13.4}$$

$$\mathrm{Var}(X(t)) = \frac{\sigma_2^2}{-2\, c_1} \left(1 - e^{2\, c_1 t} \right) . \tag{13.5}$$

Particularly the mean value function $\mu_1(t)$ results as a convex combination of the long-term mean value μ and the starting value $X(0)$. For $c_1 < 0$, these moments tend to a fixed value and the process can be understood as asymptotically stationary:

$$\mu_1(t) \; \to \; \mu \;\; \text{for } c_1 < 0 \, ,$$

$$\mathrm{Var}(X(t)) \; \to \; \frac{\sigma_2^2}{-2\, c_1} \;\; \text{for } c_1 < 0 \, ,$$

where the limits are taken as $t \to \infty$. Processes with this property are also called “mean-reverting”. The adjustment parameter $c_1 < 0$ measures the “speed of mean-reversion”, i.e. the strength of adjustment: The smaller (more negative) c_1, the more strongly $dX(t)$ reacts as a function of the deviation of $X(t)$ from μ. If $X(t) > \mu$, then $c_1 < 0$ causes $c_1(X(t) - \mu)$ to have a negative impact on $X(t)$ such that the process tends to decrease and therefore approaches μ; conversely, it holds that for $X(t) < \mu$ the process experiences a positive impulse. Furthermore, one observes the distinct influence of the parameter c_1 on the (asymptotic) variance: The smaller the negative c_1, the smaller is the asymptotic expression; however, for a negative c_1 near zero the variance becomes large and the OUP loses the property of "mean reversion" for $c_1 = 0$.

Determining the autocovariance function for $c_1 < 0$ is also useful. We denote it by $\gamma(t, t+h)$ at lag h:

$$\begin{aligned}\gamma(t, t+h) &= \mathrm{E}\left[(X(t) - \mu_1(t))(X(t+h) - \mu_1(t+h))\right], \quad h \geq 0, \\ &= \mathrm{E}\left[e^{c_1 t}\sigma_2 \int_0^t e^{-c_1 s}\, dW(s)\, e^{c_1(t+h)}\sigma_2 \int_0^{t+h} e^{-c_1 s}\, dW(s)\right] \\ &= \sigma_2^2\, \mathrm{E}\left[X_{c_1}(t)\, X_{c_1}(t+h)\right],\end{aligned}$$

where $X_{c_1}(t)$ is the standard OUP with zero expectation. From Proposition 9.4 we hence adopt

$$\begin{aligned}\gamma(t, t+h) &= \sigma_2^2\, e^{c_1 h}\, \frac{e^{2c_1 t} - 1}{2c_1} \\ &\to -\frac{\sigma_2^2 e^{c_1 h}}{2c_1}, \quad t \to \infty.\end{aligned}$$

Thus, for a large t there results an autocovariance function only depending on the temporal distance h. All in all, this is why the OUP with $c_1 < 0$ can be labeled as asymptotically ($t \to \infty$) weakly stationary.

Simulations

In the following, processes with $T = 20$ and $n = 1000$ are simulated. For reasons of graphical comparability, the same WP is always assumed, i.e. the 1000 random variables filtered by the recursion (12.16) are always the same ones.

First, we examine the impact of the adjustment parameter c_1 on the behavior of the OUP. In Fig. 13.1, $\sigma_2 = 0.01$. As we want to think about interest rates when looking at this figure, expectation and starting value are chosen to be $\mu = 5\,(\%)$. Here, it is obvious that the solid line deviates less strongly from the expected value for $c_1 = -0.9$ and is thus "more stationary" than the dashed graph for $c_1 = -0.1$. This is evident as the smaller (i.e. the more negative) c_1, the smaller is the variance $\sigma_2^2/(-2c_1)$ for t growing.

In the second figure, we have an OUP with the same parameter set-up for $c_1 = -0.9$, however, with a starting value different from $\mu = 5$, $X(0) = 5.1$. Furthermore, the expected value function $\mu_1(t)$ is given and one can observe how rather rapidly it approaches the value $\mu = 5$ (Fig. 13.2).

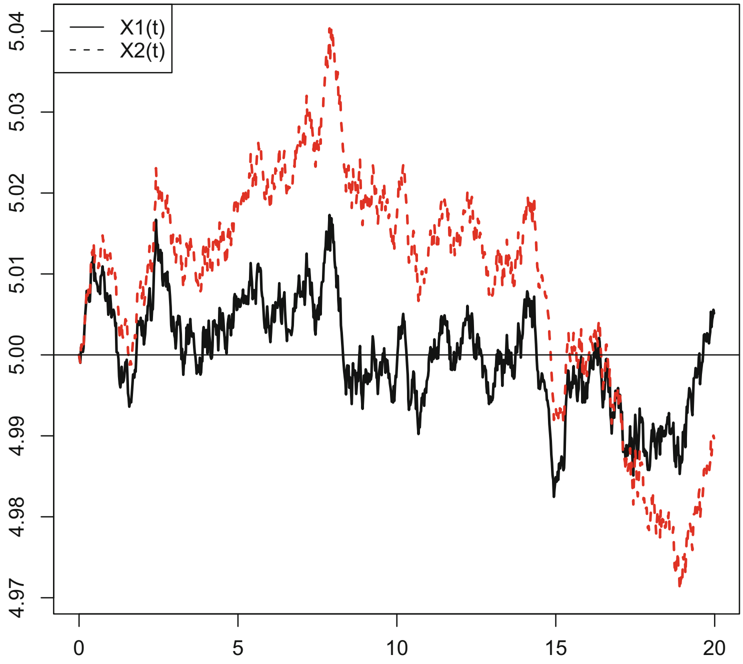

Fig. 13.1 OUP for $c_1 = -0.9$ (X_1) and $c_1 = -0.1$ (X_2) ($X(0) = \mu = 5$, $\sigma_2 = 0.01$)

Despite the convenient property of mean reversion, the OUP is only partly suitable for interest rate modeling: Note that the process takes on negative values with a positive probability. This is due to the fact that the OUP, as a Stieltjes integral, is Gaussian:

$$X(t) \sim \mathcal{N}\left(\mu_1(t), \mathrm{Var}(X(t))\right) .$$

Subsequent to Vasicek (1977), interest rate models without this drawback have been discussed.

13.3 Positive Linear Interest Rate Models

For now we stay in the class of linear SDEs, however, we restrict the discussion to the case in which positivity (more precisely: nonnegativity) is guaranteed.

Sufficient Condition

A sufficient condition for a positive evolution of the solution of a linear SDE is easy to be specified. For this purpose we naturally consider the general solution from

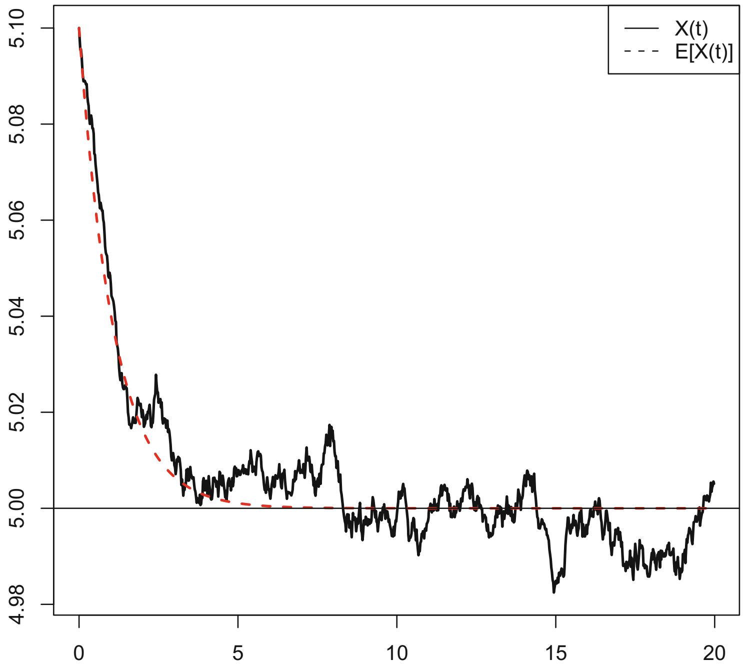

Fig. 13.2 OUP for $c_1 = -0.9$ and Starting Value $X(0) = 5.1$ including Expected Value Function ($\mu = 5, \sigma_2 = 0.01$)

Proposition 12.2. Note that $Z(t)$, as an exponential function, is always positive. With the restriction $\sigma_2(t) = 0$ the following diffusion is obtained:

$$X(t) = Z(t)\left[X(0) + \int_0^t \frac{c_2(s)}{Z(s)} ds\right]. \tag{13.6}$$

With a positive starting value and $c_2(t) \geq 0$, a positive evolution of $X(t)$ is ensured. The models in this section are of the form (13.6).

Dothan

Let us consider a special case more extensively. Dothan (1978) suggested for the interest rate dynamics a special case of the geometric Brownian motion:

$$dX(t) = \sigma_1 X(t)\, dW(t), \quad X(0) > 0.$$

With $c_2(t) = \sigma_2(t) = 0$ it holds in this case that

$$X(t) = X(0)\,\exp\left\{\left(-\frac{1}{2}\,\sigma_1^2\right)\,t + \sigma_1\,W(t)\right\}\ ,$$

and hence, the interest rate $X(t)$ can in fact not get negative. Not $X(t)$ follows a Gaussian distribution but $\log(X(t))$ does. Furthermore, in Example 12.3 we have determined the moments (for a fixed starting value):

$$\mu_1(t) = X(0)\ \text{ and }\ \mathrm{Var}(X(t)) = X^2(0)\,\left(\exp(\sigma_1^2\,t) - 1\right)\ .$$

Thus, the variance of the process increases exponentially which is why the model may not be satisfactory for interest rates.

Brennan-Schwartz

Brennan and Schwartz (1980) suggested another attractive variant. It consists of a combination of Vasicek (1977) and Dothan (1978); we choose the drift component just as for the Ornstein-Uhlenbeck process and the volatility just as for the geometric Brownian motion:

$$dX(t) = c_1\,(X(t) - \mu)\,dt + \sigma_1\,X(t)\,dW(t)\,,\quad X(0) = \mu > 0\,, \tag{13.7}$$

where, for simplicity, the starting value is set equal to μ. For $c_1 < 0$ it holds that $c_2 = -c_1\mu > 0$ such that we have indeed a positive interest rate dynamics. For this model one can show (see Problem 13.4) that the expected value results just as for Dothan (1978),

$$\mu_1(t) = \mu = X(0)\,,$$

while it holds for the variance:

$$\mathrm{Var}(X(t)) = \frac{\mu^2\,\sigma_1^2}{2\,c_1 + \sigma_1^2}\,\left(\exp\left((2\,c_1 + \sigma_1^2)\,t\right) - 1\right)\ .$$

If $c_1 < -\sigma_1^2/2$ (i.e. $2\,c_1 + \sigma_1^2 < 0$), then it holds that the variance tends to a fixed positive value ($t \to \infty$):

$$\mathrm{Var}(X(t))\ \to\ -\frac{\mu^2\,\sigma_1^2}{2\,c_1 + \sigma_1^2}\quad \text{for } c_1 < -\frac{\sigma_1^2}{2}\,.$$

If the volatility parameter σ_1 is relatively small compared to the absolute value of the negative adjustment parameter c_1, then the model (13.7) provides a process with a fixed expected value and an asymptotically constant variance. Again, one speaks of "mean reversion". Interestingly, the variance is not only influenced by σ_1 and c_1 in an obvious manner: The greater σ_1, the greater $\text{Var}(X(t))$, and the greater c_1 in absolute value, the more strongly or the faster the adjustment happens (and the smaller is the variance). The parameter $\mu > 0$ from the drift function as well has a positive effect on the variance. Intuitively, this is obvious: The smaller μ (i.e. the closer to zero), the lesser $X(t)$ can spread as the process does not get negative; conversely, it holds that the scope for the variance between the zero line and μ increases with μ growing.

Simulations

Again, processes with $T = 20$ and $T = 1000$ were simulated. For reasons of graphical comparability, the same WP as in the previous section is assumed, i.e. the 1000 random variables being filtered by the recursion (12.16) are identical.

In Fig. 13.3 it is obvious how the variance of the geometric Brownian motion suggested by Dothan (1978) increases with the parameter σ_1. Although the expected value is constant and equal to the starting value, long periods are possible and probable in which the process does not cross the expected value. A more plausible interest rate dynamics can be observed in Fig. 13.4 for two values of the adjustment

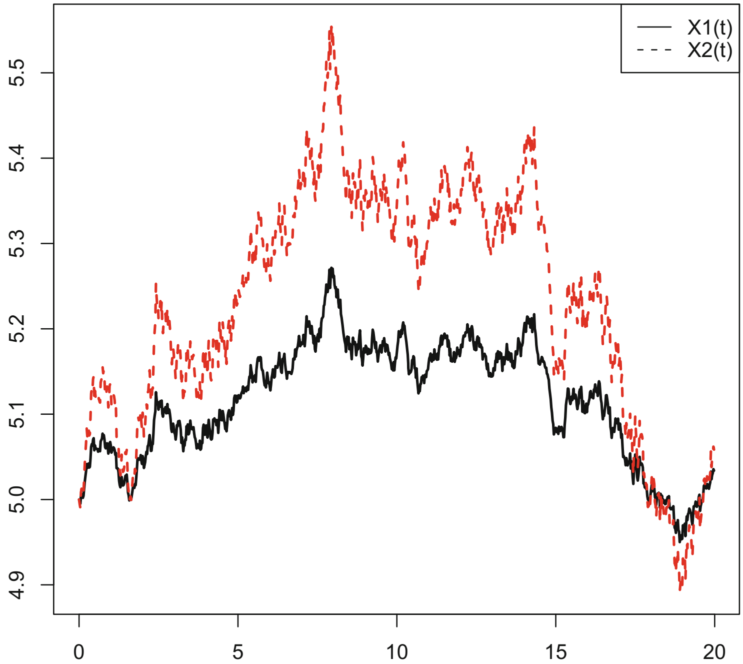

Fig. 13.3 Dothan for $\sigma_1 = 0.01$ (X_1) and $\sigma_1 = 0.02$ (X_2) ($X(0) = \mu = 5$)

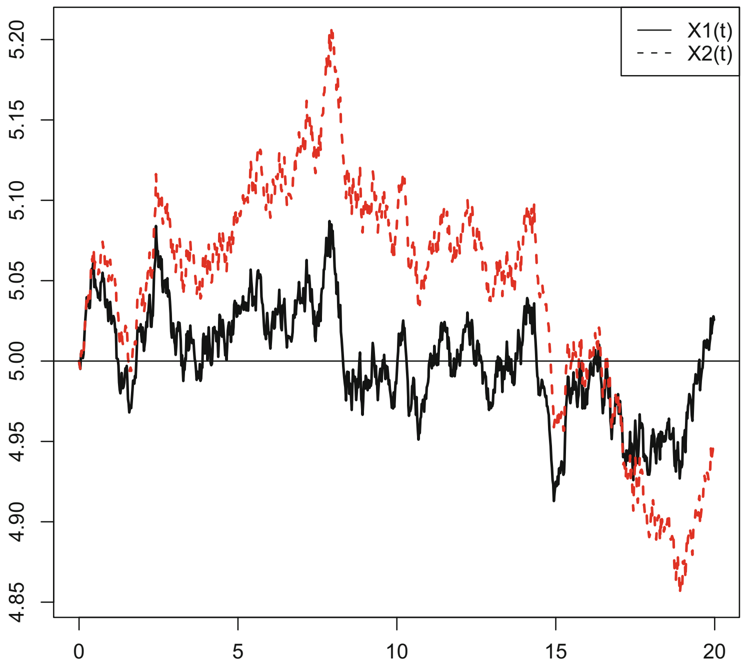

Fig. 13.4 Brennan-Schwartz for $c_1 = -0.9$ (X_1) and $c_1 = -0.1$ (X_2) ($X(0) = \mu = 5$, $\sigma_1 = 0.01$)

parameter c_1. The values are chosen small enough (relative to σ_1), such that the variance remains bounded and converges to a fixed value. It is obvious: The greater c_1 in absolute value, the smaller the variance.

13.4 Nonlinear Models

Chan, Karolyi, Longstaff, and Sanders (1992) [in short: CKLS] considered the following class of nonlinear equations for modeling short-term interest rates which is covered in this section:

$$dX(t) = c_1\,(X(t) - \mu)\,dt + \sigma\,X^{\gamma}(t)\,dW(t)\,, \quad \mu > 0\,,\ 0 \leq \gamma \leq 1\,. \tag{13.8}$$

Thus, the modeling of the drift component always corresponds to the one by Vasicek (1977). The OUP from (13.1) just results for $\gamma = 0$ while $\gamma = 1$ leads to the just discussed process from (13.7). Noninteger values of γ in between provide a nonlinear interest rate dynamics. The process from (13.8) is sometimes also called model with constant elasticity as it holds for the elasticity with the derivative of the volatility $\sigma X^{\gamma}(t)$ with respect to X that:

$$\frac{d(\sigma X^{\gamma})}{dX}\,\frac{X}{\sigma X^{\gamma}} = \gamma\,.$$

In order to show the interpretation of γ as an elasticity, we consider a discretization of the CKLS process as for the computer simulation. For this purpose, we define for discrete steps of the length 1, $t = 1, 2, \ldots, T$:

$$x_t := X(t)\,, \quad \varepsilon_t := \Delta W(t) = W(t) - W(t-1)\,.$$

The discrete-time version of (13.8) hence reads

$$\Delta x_t = c_1\ (x_{t-1} - \mu) + \sigma x_{t-1}^{\gamma}\varepsilon_t\,, \quad \varepsilon_t \sim \mathrm{ii}\mathcal{N}(0,1)\,,$$

or

$$x_t = x_{t-1} + c_1\ (x_{t-1} - \mu) + \sigma x_{t-1}^{\gamma}\varepsilon_t\,.$$

For the conditional variance it holds:

$$\mathrm{Var}(x_t|x_{t-1}) = \sigma^2 x_{t-1}^{2\gamma}\,.$$

Correspondingly, it holds for the conditional standard deviation that e.g. a doubling of x_{t-1} leads to a multiplication by the factor 2^{γ}:

$$\begin{aligned}\sqrt{\mathrm{Var}\,(x_t|\tilde{x}_{t-1})} &= \sigma\tilde{x}_{t-1}^{\gamma} \quad \text{for} \quad \tilde{x}_{t-1} = 2\,x_{t-1}\,,\\ &= \sigma\,2^{\gamma}x_{t-1}^{\gamma}\\ &= 2^{\gamma}\sqrt{\mathrm{Var}\,(x_t|x_{t-1})}\,.\end{aligned}$$

Two simulated paths of the CKLS model are depicted in Fig. 13.5. They only differ in the elasticity γ. It is not surprising that the deviations from μ get greater with a greater γ.

Cox, Ingersoll & Ross [CIR]

A particularly prominent representative of (13.8) is obtained for $\gamma = 0.5$. This model is often used following Cox, Ingersoll, and Ross (1985):

$$dX(t) = c_1\ (X(t) - \mu)\ dt + \sigma\,\sqrt{X(t)}\,dW(t)\,, \quad \mu > 0\,,\ c_1 < 0\,. \tag{13.9}$$

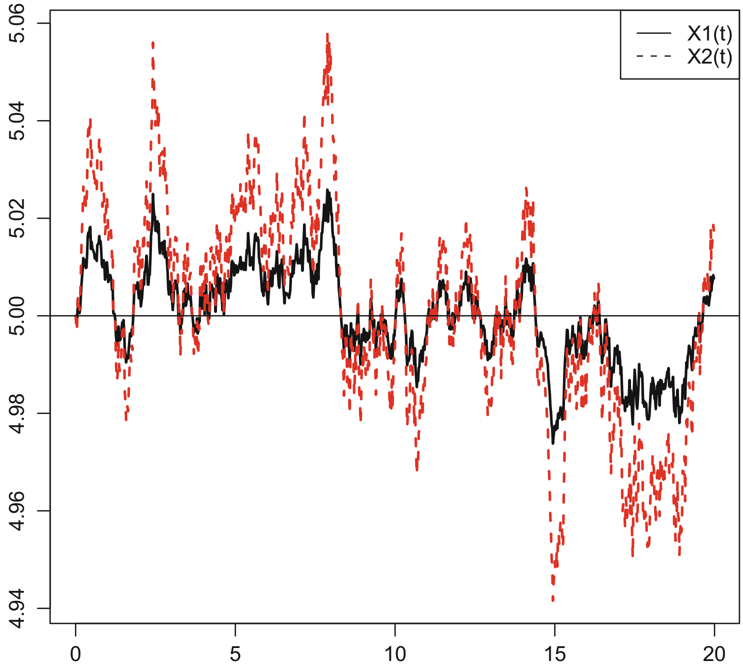

Fig. 13.5 CKLS with $\gamma = 0.25$ (X_1) and $\gamma = 0.75$ (X_2) for $c_1 = -0.9$ ($X(0) = \mu = 5$, $\sigma = 0.01$)

The conditional standard deviation is modeled as a square root which is why one also speaks of (13.9) as a "square root process". Consequently, the conditional variance of the increments is proportional to the level of the process.

For this nonlinear SDE it can be formally shown, which is also intuitive: If $X(t)$ (starting from a positive starting value $X(0) > 0$) takes on the value zero, then the variance is zero as well, but the change $dX(t)$ gets a positive impulse of the strength $-c_1\,\mu$ such that the process is reflected on the zero line for $\mu > 0$. Insofar the square root process overcomes the deficiency of the OUP as an interest rate model. However, an analytical representation of the solution of (13.9) is not known.

Already for the ordinary square root process from (13.9) with $\sigma(t, x) = \sigma\sqrt{x}$ the condition of existence **(E1)** from Proposition 12.1 is not fulfilled anymore as the derivative at zero does not exist. Fortunately, there are weaker conditions ensuring the existence of a solution of (13.9) – however, they do not guarantee the finiteness of the first two moments anymore. In order to show that finite moments exist up to the second order, we would need more fundamental arguments. Instead, we start with calculating the moments (finiteness assumed).

For reasons of simplicity, assume a fixed starting value equal to μ in the following: $X(0) = \mu$. Then we obtain (see Problem 13.5), as for the OUP, on average

$$\mu_1(t) = \mathrm{E}(X(t)) = e^{c_1 t}X(0) + \mu\left(1 - e^{c_1 t}\right) = \mu\,.$$

Under the assumption on the starting value $X(0) = \mu$, for the second moment we obtain (cf. Problem 13.6)

$$\mu_2(t) = \mu^2 - \frac{\sigma^2\,\mu}{2\,c_1}\,(1 - e^{2\,c_1\,t})\,,$$

from which it immediately follows for the variance

$$\mathrm{Var}(X(t)) = \frac{\sigma^2\,\mu}{-2\,c_1}\,(1 - e^{2\,c_1\,t}) \;\to\; \frac{\sigma^2\,\mu}{-2\,c_1}\,, \qquad t \to \infty\,.$$

The asymptotic variance for $t \to \infty$ hence coincides with the one of the OUP if $\mu = 1$; for $\mu < 1$ it turns out to be smaller (as the process is reflected on the zero line and therefore varies in a narrow band) while it is obviously greater for $\mu > 1$. The border case $\mu = 0$ makes sense as well: Here, the asymptotic variance is zero as, sooner or later, the process is absorbed by the zero line.

For Fig. 13.6 an OUP with $c_1 = -0.9$ and $\sigma_2 = 0.01$ was simulated but the expected value of 5 % is now written as 0.05. In the example it becomes clear that the OUP can definitely become negative. In comparison, we observe a numerical solution of the corresponding square root process from (13.9) with the

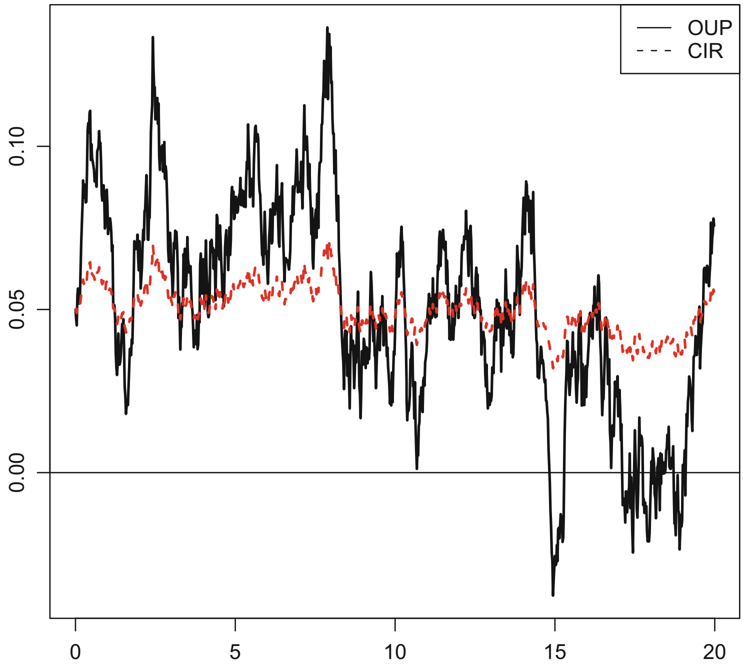

Fig. 13.6 OUP and CIR for $c_1 = -0.9$ ($X(0) = \mu = 0.05$, $\sigma = \sigma_2 = 0.01$)

same volatility parameter and the same drift component. The picture confirms the theoretical considerations: The process exhibits a smaller variance and does not get negative.

Further Models and Parameter Estimation

Marsh and Rosenfeld (1983) mention the variant with $\mu = 0$ as a borderline case of (13.8). Cox, Ingersoll, and Ross (1980) consider a version with $\gamma > 1$ for a special investigation:

$$dX(t) = \sigma\, X^{3/2}(t)\, dW(t)\,.$$

Finally, some models are applied which leave the framework of CKLS from (13.8) entirely, e.g. Constantinides and Ingersoll (1984) with

$$dX(t) = c\, X^2(t)\, dt + \sigma\, X^{3/2}(t)\, dW(t)\,,$$

where both drift and volatility are nonlinear.

Given the copious possibilities for specifying a diffusion process, it is not surprising that it has been tried to, first, estimate unknown parameters and second to statistically discriminate between the different model classes. Beside the work by CKLS, the papers by Broze, Scaillet, and Zakoïan (1995) and Tse (1995) should be mentioned. As a first introduction to the topic of estimation of diffusion parameters, the corresponding chapter by Gourieroux and Jasiak (2001) is recommended.

13.5 Problems and Solutions

Problems

13.1 Derive the solution (13.2) of Eq. (13.1).

13.2 Derive the moments, (13.4) and (13.5), of the OUP (a fixed starting value $X(0)$ assumed).

13.3 Discuss Eq. (13.1) for $c_1 = 0$ as a special case of the OUP (a proposal by Merton, 1973). For this purpose, consider the solution, the expected value and the variance for $c_1 \to 0$, if necessary with L'Hospital's rule. (You should be familiar with the results. By which name do you know the process as well?)

13.4 Consider now, as a combination of the interest models by Vasicek (1977) and Dothan (1978), the process from (13.7) by Brennan and Schwartz (1980),

$$dX(t) = c_1\,(X(t) - \mu)\, dt + \sigma_1\, X(t)\, dW(t)\,, \quad \mu = X(0) > 0\,,$$

particularly with the starting value $X(0) = \mu$. Determine expectation and variance. How do these behave for $t \to \infty$ if it holds that $2\,c_1 < -\sigma_1^2$?

13.5 Consider the square root process (13.9) by Cox et al. (1985). Under the assumption $X(0) = \mu$ for the starting value, derive an expression for the expected value.

13.6 Again, consider the square root process (13.9) by Cox et al. (1985). Under the assumption $X(0) = \mu$ for the starting value, derive an expression for the variance.

Solutions

13.1 Equation (13.1) is a special case of (12.12) with constant coefficients. Hence, the solution results from (12.13) with

$$z(t) = \exp\left\{\int_0^t c_1 ds\right\} = e^{c_1 t}.$$

From this it follows

$$\begin{aligned}\int_0^t \frac{c_2}{z(s)}\,ds &= c_2 \int_0^t e^{-c_1 s}\,ds \\ &= -\frac{c_2}{c_1}\,e^{-c_1 s}\Big|_0^t \\ &= -\frac{c_2}{c_1}\,(e^{-c_1 t} - 1).\end{aligned}$$

Thus, from (12.13) we obtain the desired result:

$$X(t) = e^{c_1 t}\left[X(0) - \frac{c_2}{c_1}\,\left(e^{-c_1 t} - 1\right) + \int_0^t \sigma_2\,e^{-c_1 s}\,dW(s)\right].$$

13.2 The expected value function is determined from (12.14):

$$\begin{aligned}\mu_1(t) &= e^{c_1 t}\left[X(0) + \int_0^t c_2\,e^{-c_1 s}\,ds\right] \\ &= e^{c_1 t}\left[X(0) - \frac{c_2}{c_1}\,(e^{-c_1 t} - 1)\right].\end{aligned}$$

With $\mu = -\frac{c_2}{c_1}$ the formula from (13.4) results.

The variance expression follows from (12.15):

$$\begin{aligned}\operatorname{Var}(X(t)) &= \left(e^{c_1 t}\right)^2 \int_0^t \left(\frac{\sigma_2}{e^{c_1 s}}\right)^2 ds \\ &= e^{2c_1 t}\,\sigma_2^2 \int_0^t e^{-2c_1 s}\, ds \\ &= e^{2c_1 t}\,\sigma_2^2 \left(\frac{-1}{2c_1}\right) e^{-2c_1 s}\Big|_0^t \\ &= -e^{2c_1 t}\,\frac{\sigma_2^2}{2c_1}\left(e^{-2c_1 t} - 1\right).\end{aligned}$$

This expression coincides with (13.5).

13.3 L'Hospital's rule provides for $c_1 \to 0$:

$$\lim_{c_1 \to 0} \frac{e^{-c_1 t} - 1}{c_1} = \lim_{c_1 \to 0} \frac{-t\, e^{-c_1 t}}{1} = -t.$$

Hence, $X(t)$ from (13.2) merges for $c_1 \to 0$ into

$$\begin{aligned}X(t) &= \left[X(0) + c_2\, t + \int_0^t \sigma_2\, dW(s)\right] \\ &= X(0) + c_2\, t + \sigma_2\, W(t).\end{aligned}$$

Equation (13.4) is not suitable for determining the expected value as $\mu = \frac{-c_2}{c_1}$ is not defined for $c_1 \to 0$. Instead, one directly obtains (for X(0) fixed):

$$\mu_1(t) = \mathrm{E}(X(t)) = X(0) + c_2\, t + 0.$$

This linear growth is distinctive for the Brownian motion with drift, cf. Chap. 7.

In order to determine the variance from (13.5), it is again argued with L'Hospital's rule:

$$\lim_{c_1 \to 0} \frac{1 - e^{2c_1 t}}{2\, c_1} = \lim_{c_1 \to 0} \frac{-2t\, e^{2c_1 t}}{2} = -t.$$

Therefore it holds that

$$\lim_{c_1 \to 0} \operatorname{Var}(X(t)) = \sigma_2^2\, t = \operatorname{Var}(\sigma_2\, W(t)).$$

This is the familiar variance of a Brownian motion with drift. Indeed, a Brownian motion with drift is the same as the process resulting from the OUP for $c_1 = 0$ or $c_1 \to 0$.

13.4 This equation is linear and does not belong to the category "additive noise". It is rather a special case of (12.3) with

$$c_1(t) = c_1, \quad c_2(t) = -\mu c_1, \quad \sigma_1(t) = \sigma_1, \quad \sigma_2(t) = 0.$$

With

$$z(t) = e^{c_1 t}$$

due to Proposition 12.3 the expected value is given by

$$\begin{aligned} \mathrm{E}(X(t)) &= e^{c_1 t}\left[\mathrm{E}(X(0)) - \mu c_1 \int_0^t e^{-c_1 s} ds\right] \\ &= e^{c_1 t}[\mathrm{E}(X(0)) + \mu e^{-c_1 t} - \mu] \\ &= \mu + e^{c_1 t}[\mathrm{E}(X(0)) - \mu]. \end{aligned}$$

Hence, for $c_1 < 0$ it holds:

$$\mathrm{E}(X(t)) \to \mu, \quad t \to \infty.$$

For the second moment, we determine from (12.11):

$$\mu_2(t) = \exp\{(2c_1 + \sigma_1^2)t\}\left[\mu_2(0) - \int_0^t \frac{2c_1\mu\mu_1(s)}{\exp\{(2c_1+\sigma_1^2)s\}} ds\right].$$

In particular for $X(0) = \mu$, this simplifies, due to $\mathrm{E}(X(t)) = \mu$, to (with $\mu_2(0) = \mu^2$):

$$\begin{aligned} \mu_2(t) &= \exp\{(2c_1 + \sigma_1^2)t\}\left[\mu^2 - 2c_1\mu^2 \int_0^t \exp\{-(2c_1+\sigma_1^2)s\} ds\right] \\ &= \exp\{(2c_1 + \sigma_1^2)t\}\left[\mu^2 + 2c_1\mu^2 \left[\frac{\exp\{-(2c_1+\sigma_1^2)s\}}{2c_1+\sigma_1^2}\right]_0^t\right] \\ &= \exp\{(2c_1 + \sigma_1^2)t\}\left[\mu^2 + \frac{2c_1\mu^2}{2c_1+\sigma_1^2}(\exp\{-(2c_1+\sigma_1^2)t\} - 1)\right] \\ &= \frac{2c_1\mu^2}{2c_1+\sigma_1^2} + \frac{\sigma_1^2\mu^2\exp\{(2c_1+\sigma_1^2)t\}}{2c_1+\sigma_1^2}. \end{aligned}$$

Thus, the variance for $X(0) = \mu$ reads:

$$\begin{aligned}
\mathrm{Var}(X(t)) &= \mu_2(t) - \mu^2 \\
&= \frac{\sigma_1^2 \mu^2 \exp\{(2c_1 + \sigma_1^2)t\}}{2c_1 + \sigma_1^2} + \frac{2c_1 \mu^2}{2c_1 + \sigma_1^2} - \frac{(2c_1 + \sigma_1^2)\mu^2}{2c_1 + \sigma_1^2} \\
&= \frac{\sigma_1^2 \mu^2 \exp\{(2c_1 + \sigma_1^2)t\}}{2c_1 + \sigma_1^2} - \frac{\sigma_1^2}{2c_1 + \sigma_1^2}\,\mu^2 \\
&= \frac{\sigma_1^2\,\mu^2}{2c_1 + \sigma_1^2}\,\left(\exp\{(2c_1 + \sigma_1^2)\,t\} - 1\right)\,.
\end{aligned}$$

If $2c_1 < -\sigma_1^2$, then it hence holds

$$\mathrm{Var}(X(t)) \to \frac{-\sigma_1^2}{2c_1 + \sigma_1^2}\mu^2 > 0\,,$$

as $t \to \infty$.

13.5 In order to determine the expected value function, we write Eq. (13.9) in integral form:

$$X(t) = X(0) + \int_0^t c_1(X(s) - \mu)ds + \sigma \int_0^t \sqrt{X(s)}dW(s).$$

By assumption, $\mu_1(0) = \mathrm{E}(X(0)) = \mathrm{E}(\mu) = \mu$. Thus, Propositions 8.2 and 10.3(b) yield:

$$\mu_1(t) = \mu + \int_0^t c_1(\mu_1(s) - \mu)ds + 0$$

or rather

$$d\mu_1(t) = c_1(\mu_1(t) - \mu)dt.$$

Due to (12.5), the solution of this deterministic differential equation reads:

$$\begin{aligned}
\mu_1(t) &= z(t)\left[\mu_1(0) + \int_0^t \frac{(-c_1\mu)}{z(s)}ds\right] \\
&= e^{c_1 t}\left[\mu - \mu c_1 \int_0^t e^{-c_1 s}ds\right] \\
&= e^{c_1 t}\left[\mu + \mu e^{-c_1 s}|_0^t\right] \\
&= \mu e^{c_1 t}\left[1 + e^{-c_1 t} - 1\right] \\
&= \mu.
\end{aligned}$$

13.6 In order to determine the function of the second moment analogously to the expected value in Problem 13.5, we search for an integral equation for $X^2(t)$. This is provided by Ito's lemma for $g(X) = X^2$ with $g'(X) = 2X$ and $g''(X) = 2$:

$$\begin{aligned} dX^2(t) &= 2X(t)dX(t) + \frac{1}{2}2\sigma^2(t)dt \\ &= 2X(t)dX(t) + \sigma^2 X(t)dt, \end{aligned}$$

where $\sigma(t) = \sigma\sqrt{X(t)}$ from (13.9) was substituted. Plugging in the definition of $dX(t)$ further provides:

$$dX^2(t) = \left(2c_1X^2(t) - 2\mu c_1X(t) + \sigma^2 X(t)\right)dt + 2\sigma X(t)\sqrt{X(t)}dW(t),$$

or rather

$$X^2(t) = X^2(0) + \int_0^t \left(2c_1X^2(s) + (\sigma^2 - 2\mu c_1)X(s)\right)ds + 2\sigma\int_0^t X(s)\sqrt{X(s)}dW(s).$$

With $X(0) = \mu$ forming expectation yields from Propositions 8.2 and 10.3(b):

$$\mu_2(t) = \mu^2 + \int_0^t \left(2c_1\mu_2(s) + (\sigma^2 - 2\mu c_1)\mu\right)ds + 0,$$

as $\mu_1(t) = \mu$ is constant. Thus, for $\mu_2(t)$ a deterministic differential equation results:

$$d\mu_2(t) = (2c_1\mu_2(t) + \sigma^2\mu - 2\mu^2 c_1)dt.$$

With

$$z(t) = \exp\left\{\int_0^t 2c_1 ds\right\} = e^{2c_1 t}$$

the solution reads

$$\begin{aligned} \mu_2(t) &= e^{2c_1 t}\left[\mu^2 + \int_0^t \frac{\sigma^2\mu - 2\mu^2 c_1}{e^{2c_1 s}}ds\right] \\ &= e^{2c_1 t}\left[\mu^2 - \frac{(\sigma^2\mu - 2\mu^2 c_1)}{2c_1}e^{-2c_1 s}\Big|_0^t\right] \\ &= \frac{e^{2c_1 t}}{2c_1}\left[2c_1\mu^2 - (\sigma^2\mu - 2\mu^2 c_1)(e^{-2c_1 t} - 1)\right] \end{aligned}$$

$$= \frac{e^{2c_1 t}}{2c_1}\left[\sigma^2\mu - (\sigma^2\mu - 2\mu^2 c_1)e^{-2c_1 t}\right]$$
$$= \mu^2 - \frac{\sigma^2\mu}{2c_1} + \frac{\sigma^2\mu}{2c_1}e^{2c_1 t}.$$

Thus, in a last step the required variance is calculated as

$$\begin{aligned}\mathrm{Var}(X(t)) &= \mu_2(t) - \mu_1^2(t)\\ &= \frac{\sigma^2\mu}{2c_1}(e^{2c_1 t} - 1).\end{aligned}$$

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