Spectra of Stationary Processes

4

4.1 Summary

Spectral analysis (or analysis in the frequency domain) aims at detecting cyclical movements in a time series. These may originate from seasonality, a trend component or from a business cycle. The theoretical spectrum of a stationary process is the quantity measuring how strongly cycles with a certain period, or frequency, account for total variance. Typically, elaborations on spectral analysis are formally demanding requiring e.g. knowledge of complex numbers and Fourier transformations. In this textbook we have tried for a way of presenting and deriving the relevant results being less elegant but in return managing with less mathematical burden. The next section provides the definitions and intuition behind spectral analysis. Section 4.3 is analytically more demanding containing some general theory. This theory is exemplified with the discussion of spectra from particular ARMA processes, hence building on the previous chapter.

4.2 Definition and Interpretation

In this chapter we assume the most general case considered previously, i.e. the infinite MA process that is only square summable, $\{x_t\}_{t\in\mathbb{T}}$, $\mathbb{T} \subseteq \mathbb{Z}$,

$$x_t = \mu + \sum_{j=0}^{\infty} c_j \, \varepsilon_{t-j} \,, \quad \sum_{j=0}^{\infty} c_j^2 < \infty \,, \; c_0 = 1 \,, \tag{4.1}$$

with $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$. The autocovariances,

$$\gamma(h) = \mathrm{Cov}(x_t, x_{t+h}) = \gamma(-h) \,, \quad h \in \mathbb{Z} \,,$$

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are given in Proposition 3.2 (a). We do not assume that $\{c_j\}$ and hence $\{\gamma(h)\}$ are absolutely summable,[1] simply because this will not hold under long memory treated in the next chapter. We wish to construct a function f that allows to express the autocovariances as weighted cosine waves of different periodicity,[2]

$$\gamma(h) = \int_{-\pi}^{\pi} \cos(\lambda h) f(\lambda)\, d\lambda\,.$$

The basic ingredient of an analysis of periodicity is the **cosine cycle** whose properties we want to recall as an introduction.

Periodic Cycles

By $c_\lambda(t)$ we denote the cycle based on the cosine,[3]

$$c_\lambda(t) = \cos(\lambda t)\ , \quad t \in \mathbb{R},$$

where λ with $\lambda > 0$ is called **frequency**. The frequency is inversely related to the **period** P,

$$P = \frac{2\pi}{\lambda}.$$

For $\lambda = 1$ one obtains the cosine function which is $2\pi-$periodic and even (symmetric about the ordinate):

$$c_1(t) = \cos(t) = \cos(t + 2\pi) = c_1(t + 2\pi)\,,$$

$$c_1(-t) = \cos(-t) = \cos(t) = c_1(t).$$

More generally, it holds with $P = 2\pi/\lambda$ that:

$$c_\lambda(t) = \cos(\lambda t) = \cos(\lambda t + 2\pi) = \cos(\lambda(t + P)) = c_\lambda(t + P)\,.$$

Therefore the cosine cycle $c_\lambda(t)$ with frequency λ has the period $P = 2\pi/\lambda$. Of course, the symmetry of $c_1(t)$ carries over:

$$c_\lambda(t) = c_\lambda(-t)\,.$$

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[1]The assumption of absolute summability underlies most textbooks when it comes to spectral analysis, see e.g. Hamilton (1994) or Fuller (1996).

[2]From Brockwell and Davis (1991, Coro. 4.3.1) in connection with Brockwell and Davis (Thm. 5.7.2) one knows that such an expression exists.

[3]Here, the so-called amplitude is equal to one ($|c_\lambda(t)| \leq 1$), and the phase shift is zero ($c_\lambda(0) = 1$).

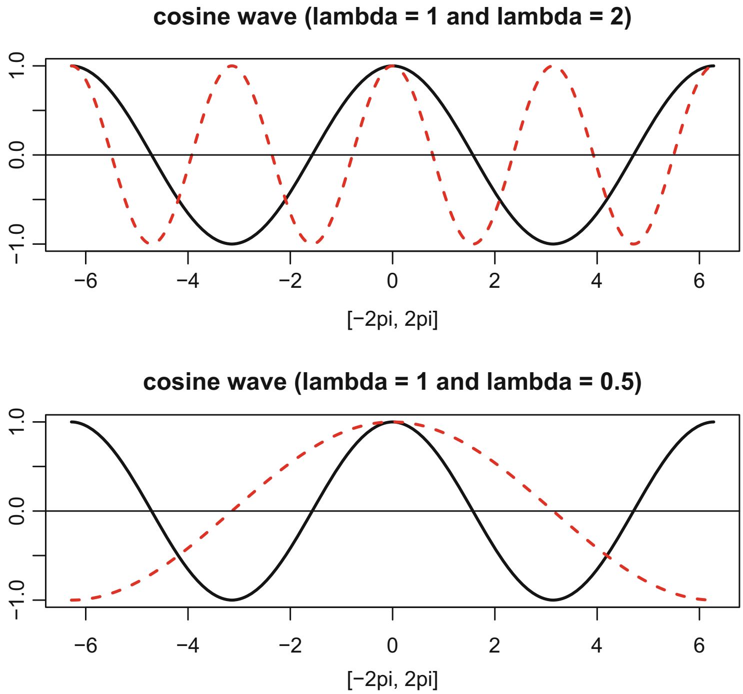

Fig. 4.1 Cosine cycle with different frequencies

For $\lambda = 1$, $\lambda = 2$ and $\lambda = 0.5$ these properties are graphically illustrated in Fig. 4.1.

Finally, remember the derivative of the cosine,

$$\frac{d\,c_\lambda(t)}{d\,t} = c'_\lambda(t) = -\lambda\ \sin(\lambda t)\,,$$

which we will use repeatedly.

Definition

For convenience, we now rephrase the MA(∞) process in terms of the lag polynomial $C(L)$ of infinite order,

$$x_t = \mu + C(L)\,\varepsilon_t \quad \text{with} \quad C(L) = \sum_{j=0}^{\infty} c_j L^j\,.$$

Next, we define the so-called **power transfer function** $T_C(\lambda)$ of this polynomial:[4]

$$T_C(\lambda) = \sum_{j=0}^{\infty} c_j^2 + 2\sum_{h=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+h}\cos(\lambda h)\,, \quad \lambda \in \mathbb{R}\setminus\{\lambda^*\}\,. \tag{4.2}$$

Note that T_C may not exist everywhere, there may be singularities at some frequency λ^* such that $T_C(\lambda)$ goes off to infinity as $\lambda \to \lambda^*$; but at least the power transfer function is integrable. The key result in Proposition 4.1 (e) is from from Brockwell and Davis (1991, Coro. 4.3.1, Thm. 5.7.2); it will be proved explicitly in Problem 4.1 under the simplifying assumption of absolute summability. The first four statements in the following proposition are rather straightforward and will be justified below.

Proposition 4.1 (Spectrum) *Define for $\{x_t\}$ from (4.1) the spectrum*

$$f(\lambda) = T_C(\lambda)\,\frac{\sigma^2}{2\pi}\,.$$

It has the following properties:

(a) $f(-\lambda) = f(\lambda)$,
(b) $f(\lambda) = f(\lambda + 2\pi)$,
(c) $f(\lambda) \geq 0$,
(d) $f(\lambda)$ is continuous in λ under absolute summability, $\sum_j |c_j| < \infty$.
(e) For all $h \in \mathbb{Z}$:

$$\gamma(h) = \int_{-\pi}^{\pi} f(\lambda)\,\cos(\lambda h)\,d\lambda = 2\int_0^{\pi} f(\lambda)\,\cos(\lambda h)\,d\lambda\,.$$

Substituting the autocovariance expression from Proposition 3.2 into (4.2), the following representation of the spectrum exists:

$$f(\lambda) = \frac{\gamma(0)}{2\pi} + \frac{2}{2\pi}\sum_{h=1}^{\infty}\gamma(h)\,\cos(\lambda h) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma(h)\,\cos(\lambda h)\,. \tag{4.3}$$

The symmetry of the spectrum in Proposition 4.1 (a) immediately follows from the symmetry of the cosine function. From the periodicity of the cosine, (b) follows as well. Both results jointly explain why the spectrum is normally considered on the restricted domain $[0, \pi]$ only. Property (c) follows from the definition of the power transfer function, see Footnote 6 below. Finally, the continuity of $f(\lambda)$ claimed in

[4] A more detailed and technical exposition is reserved for the next section. Our expression in (4.2) can be derived from the expression in Brockwell and Davis (1991, eq. 5.7.9), which is given in terms of complex numbers.

(d) under absolute summability results from uniform convergence, see Fuller (1996, Thm. 3.1.9).

We call the function f (or f_x, if we want to emphasize that $\{x_t\}$ is the underlying process) the **spectrum** of $\{x_t\}$. Frequently, one also talks about spectral density or spectral density function as f is a non-negative function which could be standardized in such a way that the area beneath it would be equal to one.

Interpretation

The usual interpretation of the spectrum is based on Proposition 4.1. Result (e) and (4.3) jointly show the spectrum and the autocovariance series to result from each other. In a sense, spectrum and autocovariances are two sides of the same coin. The spectrum can be determined from the autocovariances by definition and having the spectrum, Proposition 4.1 provides the autocovariances. The case $h = 0$ with

$$\mathrm{Var}(x_t) = \gamma(0) = \int_{-\pi}^{\pi} f(\lambda)\, d\lambda = 2 \int_{0}^{\pi} f(\lambda)\, d\lambda$$

is particularly interesting. This equation implies: The spectrum at λ_0 measures how strongly the cycle with frequency λ_0 and therefore of period $P_0 = 2\pi/\lambda_0$ adds to the variance of the process. If f has a maximum at λ_0, then the dynamics of $\{x_t\}$ is dominated by the corresponding cycle or period; inversely, if the spectrum has a minimum at λ_0, then the corresponding cycle is of less relevance for the behavior of $\{x_t\}$ than all other cycles. For $\lambda \to 0$, period P converges to infinity. A cycle with an infinitely long period is interpreted as a trend or a long-run component. Hence, $f(0)$ indicates how strongly the process is dominated by a **trend component**.

Frequently, the analysis of the autocovariance structure or the autocorrelation structure of a process is called "analysis in the time domain" as $\gamma(h)$ measures the direct temporary dependence between x_t and x_{t+h}. Correspondingly, the spectral analysis is often referred to as "analysis in the frequency domain". Proposition 4.1 and the definition in (4.3) show how to move back and forth between time and frequency domain.

Examples

Example 4.1 (White Noise) Let us consider the white noise process $x_t = \varepsilon_t$ being free from serial correlation. By definition it immediately follows that the spectrum is constant:

$$f_\varepsilon(\lambda) = \sigma^2/2\pi\,, \quad \lambda \in [0, \pi]\,.$$

According to Proposition 4.1 all frequencies account equally strongly for the variance of the process. Analogously to the perspective in optics that the "color" white results if all frequencies are present equally strongly, serially uncorrelated processes are also often called "white noise". ■

Example 4.2 (Season) Let us consider the ordinary seasonal MA process from Example 3.1,

$$x_t = \varepsilon_t + b\varepsilon_{t-S}$$

with

$$\gamma(0) = \sigma^2\left(1 + b^2\right), \ \gamma(S) = \sigma^2 b$$

and $\gamma(h) = 0$ else. By definition we obtain for the spectrum from (4.3)

$$2\pi f(\lambda) = \gamma(0) + 2\gamma(S)\cos(\lambda S)$$

or

$$f(\lambda) = \left(1 + b^2 + 2b\cos(\lambda S)\right)\sigma^2/2\pi.$$

In Problem 4.2 we determine that there are extrema at

$$0, \ \frac{\pi}{S}, \ \frac{2\pi}{S}, \ \ldots, \ \frac{(S-1)\pi}{S}, \ \pi .$$

The corresponding values are

$$f(0) = f\left(\frac{2\pi}{S}\right) = \ldots = (1+b)^2\sigma^2/2\pi\,,$$
$$f\left(\frac{\pi}{S}\right) = f\left(\frac{3\pi}{S}\right) = \ldots = (1-b)^2\sigma^2/2\pi.$$

Depending on the sign of b, maxima and minima are followed by each other, respectively. In Fig. 4.2 we find two typical shapes of the spectrum of the seasonal MA process for[5] $S = 4$ (quarterly data) with $b = 0.7$ and $b = -0.5$. First, let us interpret the case $b > 0$. There are maxima at the frequencies $0, \pi/2$ and π. Corresponding cycles are of the period

$$P_0 = \frac{2\pi}{0} = \infty, \quad P_1 = \frac{2\pi}{\pi/2} = 4, \quad P_2 = \frac{2\pi}{\pi} = 2.$$

[5]The variance of the white noise is set to one , $\sigma^2 = 1$. This is also true for all spectra of this chapter depicted in the following.

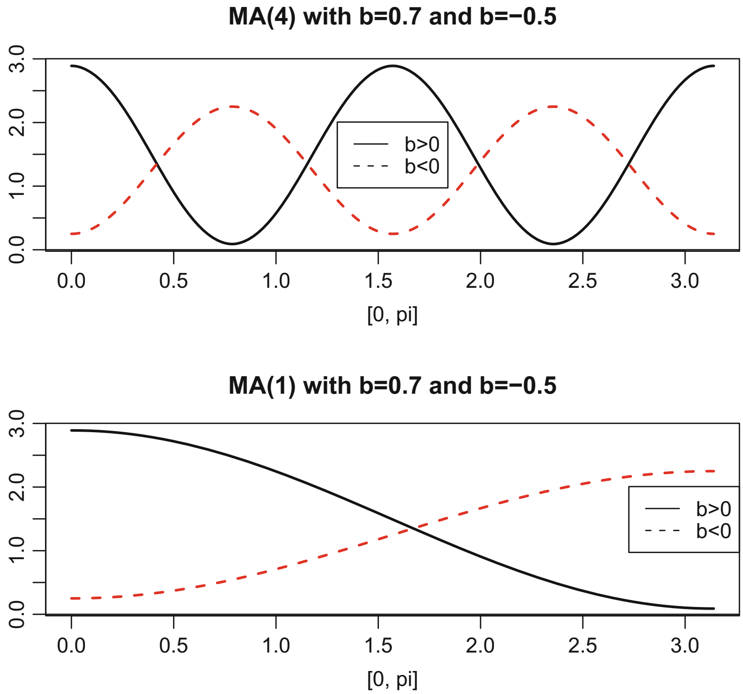

Fig. 4.2 Spectra ($2\pi f(\lambda)$) of the MA(S) process from Example 4.2

The trend is the first infinitely long "period". The second cycle has the period $P_1 = 4$, i.e. four quarters which is why this is the annual cycle. The third cycle with $P_2 = 2$ is the semi-annual cycle with only two quarters. These three cycles dominate the process for $b > 0$. Inversely, for $b < 0$ it holds that these very cycles add particularly little to the variance of the process. ■

Example 4.3 (MA(1)) Specifically for $S = 1$ the seasonal MA process passes into the MA(1) process. Accordingly, one obtains two extrema at zero and π:

$$f(0) = (1+b)^2 \sigma^2/2\pi, \quad f(\pi) = (1-b)^2 \sigma^2/2\pi.$$

In between the spectrum reads

$$f(\lambda) = \left(1 + b^2 + 2b\cos(\lambda)\right) \sigma^2/2\pi.$$

For $b = 0.7$ and $b = -0.5$, respectively, the spectra were calculated, see Fig. 4.2. For $b < 0$ one spots the relative absence of a trend (frequency zero matters least) while for $b > 0$ precisely the long-run component as a trend dominates the process. ■

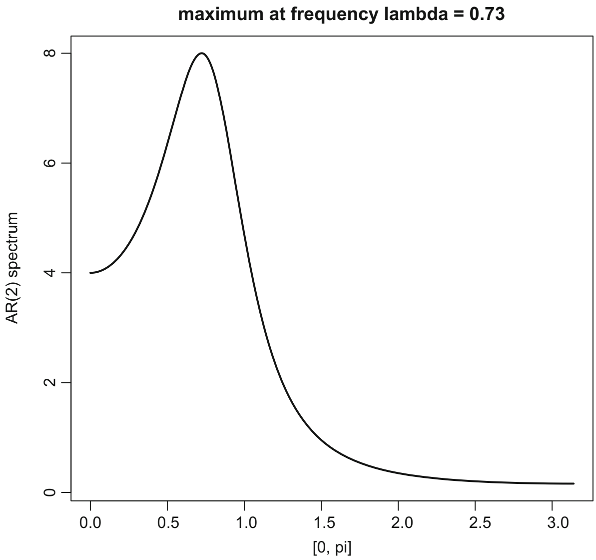

Fig. 4.3 Spectrum ($2\pi f(\lambda)$) of business cycle with a period of 8.6 years

Example 4.4 (Business Cycle) The spectrum is not only used for modelling seasonal patterns but as well for determining the length of a typical business cycle. Let us assume a process with annual observations having the spectrum depicted in Fig. 4.3. The maximum is at $\lambda = 0.73$. How do we interpret this fact with regard to contents? The dominating frequency $\lambda = 0.73$ corresponds to a period of about 8.6 (years). A frequency of this magnitude is often called "business cycle frequency" being interpreted as the frequency which corresponds to the business cycle. In fact, Fig. 4.3 does not comprise an empirical spectrum. Rather, one detects the theoretical spectrum of the AR(2) model whose autocorrelogram is depicted in Fig. 3.4 down to the right. The cycle, which can be seen in the autocorrelogram there, translates into the spectral maximum from Fig. 4.3. ■

4.3 Filtered Processes

The ARMA process or more generally the infinite MA process have been defined as filtered white noise. In order to systematically derive a formal expression for their spectra, we start quite generally with the relation between input and output of a filter in the frequency domain.

Filtered Processes

As in the previous chapter, we consider the causal, time-invariant, linear filter $F(L)$,

$$F(L) = \sum_{j=0}^{p} w_j L^j,$$

where L again denotes the lag operator and $p = \infty$ is allowed for. The filter is assumed to be absolutely summable, which trivially holds true for finite order p. Let the process $\{x_t\}$ be generated by filtering of the stationary process $\{e_t\}$,

$$x_t = F(L)\, e_t.$$

Then, how does the corresponding spectrum of $\{x_t\}$ for a given spectrum f_e of $\{e_t\}$ read? The answer is based on the **power transfer function** $T_F(\lambda)$ that we briefly touched upon in the previous section[6]:

$$T_F(\lambda) = \sum_{j=0}^{\infty} w_j^2 + 2 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} w_j w_{j+h} \cos(\lambda h) \ . \tag{4.4}$$

At a first glance, this expression appears cumbersome. However, in the next section we will see that for concrete ARMA processes it simplifies radically. If $F(L)$ is a finite filter (i.e. with finite p), then the sums of $T_F(\lambda)$ are truncated accordingly, see (4.8) in the next section. With $T_F(\lambda)$ the following proposition considerably simplifies the calculation of theoretical spectra (for a proof of an even more general result see Brockwell and Davis (1991, Thm. 4.4.1), while Fuller (1996, Thm. 4.3.1) covers our case where $\{e_t\}$ has absolutely summable autocovariances).

[6] The mathematically experienced reader will find the expression in (4.4) to be unnecessarily complicated as the transformation $T_F(\lambda)$ can be written considerably more compactly by using the exponential function in the complex space. It holds that

$$T_F(\lambda) = \left|F(e^{-i\lambda})\right|^2 = F(e^{i\lambda})F(e^{-i\lambda}),$$

where Euler's formula allows for expressing the complex-valued exponential function by sine and cosine,

$$e^{i\lambda} = \cos\lambda + i \sin\lambda, \quad i^2 = -1,$$

with the conjugate complex number $e^{-i\lambda} = \cos\lambda - i\sin\lambda$, where i denotes the imaginary unit. Instead of burdening the reader with complex numbers and functions, we rather expect him or her to handle the more cumbersome definition from (4.4). By the way, the term "power transfer function" stems from calling $F(e^{-i\lambda})$ alone transfer function of the filter $F(L)$, and $T_F(\lambda) = \left|F(e^{-i\lambda})\right|^2$ being the power thereof.

Proposition 4.2 (Spectra of Filtered Processes) *Let* $\{e_t\}$ *be a stationary process with spectrum* $f_e(\lambda)$. *The filter*

$$F(L) = \sum_{j=0}^{\infty} w_j L^j$$

be absolutely summable, $\sum_{j=0}^{\infty} | w_j | < \infty$, *and* $\{x_t\}$ *be*

$$x_t = F(L)\, e_t .$$

Then, $\{x_t\}$ *is stationary with spectrum*

$$f_x(\lambda) = T_F(\lambda) f_e(\lambda) \ , \quad \lambda \in [0, \pi] \, ,$$

where $T_F(\lambda)$ *is defined in (4.4).*

Example 4.5 (Infinite MA) Let $e_t = \varepsilon_t$ from Proposition 4.2 be white noise with

$$f_\varepsilon(\lambda) = \sigma^2/2\pi \ ,$$

and consider an absolutely summable MA(∞) process,

$$x_t = C(L)\, \varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} \, .$$

Then Proposition 4.2 kicks in:

$$f_x(\lambda) = \left(\sum_{j=0}^{\infty} c_j^2 + 2 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} c_j c_{j+h} \cos(\lambda h) \right) \frac{\sigma^2}{2\pi} \, , \quad \lambda \in [0, \pi] \, . \tag{4.5}$$

This special case of Proposition 4.2 will be verified in Problem 4.3. Note that the spectrum given in (4.5) equals of course the result from Proposition 4.1 with (4.2), which continues to hold without absolute summability. ■

Persistence

We now return more systematically to the issue of **persistence** that we have touched upon in the example of the AR(1) process in the previous chapter. Loosely speaking, we understand by persistence the degree of (positive) autocorrelation such that subsequent observations form clusters: positive observations tend to be followed by positive ones, while negative observations tend to induce negative ones. With persistence we try to capture the strength of such a tendency, which depends not only on the autocorrelation coefficient at lag one but also on higher order lags. In the previous chapter we mentioned that it has been suggested to measure persistence by means of the cumulated impulse responses *CIR* defined in (3.3). This quantity shows up in the spectrum at frequency zero by Proposition 4.2. Assume that $\{x_t\}$ is an MA(∞) process with absolutely summable impulse response sequence $\{c_j\}$. We then have:

$$f_x(0) = T_C(0)\frac{\sigma^2}{2\pi} = \left(\sum_{j=0}^{\infty} c_j^2 + 2\sum_{h=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+h}\right)\frac{\sigma^2}{2\pi} = \left(\sum_{j=0}^{\infty} c_j\right)^2 \frac{\sigma^2}{2\pi},$$

or

$$f_x(0) = (CIR)^2\,\frac{\sigma^2}{2\pi}.$$

Hence, the larger *CIR*, the stronger is the contribution of the trend component at frequency zero to the variance of the process, which formalizes our concept of persistence. Cogley and Sargent (2005) applied as relative spectral measure for persistence the ratio of $2\pi f_x(0)/\gamma_x(0)$ pioneered previously by Cochrane (1988); it can be interpreted as a variance ratio and is hence abbreviated as *VR*:

$$VR := \frac{2\pi f_x(0)}{\gamma_x(0)} = \frac{\left(\sum_{j=0}^{\infty} c_j\right)^2}{\sum_{j=0}^{\infty} c_j^2}. \tag{4.6}$$

In the case of a stationary AR(1) process, $x_t = a_1 x_{t-1} + \varepsilon_t$, it holds that (see Problem 4.6)

$$VR = \frac{1-a_1^2}{(1-a_1)^2} = \frac{1+a_1}{1-a_1} \begin{cases} > 1 \text{ if } a_1 > 0 \\ = 1 \text{ if } a_1 = 0 \\ < 1 \text{ if } a_1 < 0 \end{cases}. \tag{4.7}$$

In the case of $a_1 = 0$ (white noise) we have no persistence, and $VR = 1$. For $a_1 > 0$ the process is all the more persistent the larger a_1 is. Following Hassler (2014), one may say that a process has negative persistence if $VR < 1$. The plot of a series under negative persistence will typically display a zigzag pattern as observed in the last plot in Fig. 3.2. The limiting cases of $VR = 0$ (also called antipersistent) and $VR = \infty$ (also called strongly persistent) will be dealt with in Chap. 5.

ARMA Spectra

As a consequence of the previous proposition, we can derive what the spectrum of a stationary ARMA process $\{x_t\}$ looks like. Remember the definition from (3.13),

$$A\,(L)\,x_t = B\,(L)\,\varepsilon_t\,.$$

Now, define

$$y_t = A\,(L)\,x_t = B\,(L)\,\varepsilon_t\,.$$

By Proposition 4.2 one obtains for the spectra

$$f_y(\lambda) = T_A\,(\lambda)\,f_x(\lambda) = T_B\,(\lambda)\;\sigma^2/2\pi\,.$$

The assumption of a stationary MA(∞) representation[7] implies $T_A\,(\lambda) > 0$. Consequently, one may solve for f_x rendering the following corollary.

Corollary 4.1 (ARMA Spectra) *Let $\{x_t\}$ be a stationary ARMA(p, q) process*

$$A\,(L)\,x_t = \nu + B\,(L)\,\varepsilon_t.$$

Its spectrum is given by

$$f_x\,(\lambda) = \frac{T_B\,(\lambda)}{T_A\,(\lambda)}\,\frac{\sigma^2}{2\pi}\,, \quad \lambda \in [0, \pi]\,,$$

where $T_B\,(\lambda)$ and $T_A\,(\lambda)$ are the power transfer functions of $B\,(L)$ and $A\,(L)$.

Often, we restrict the class of stationary ARMA processes to the **invertible** ones, meaning we assume that the moving average polynomial $B(L)$ satisfies the invertibility condition of Proposition 3.3: All solutions of $B(z) = 0$ are larger than 1 in absolute value. This implies as in Footnote 7 that $T_B(\lambda) > 0$, such that the invertible ARMA spectrum is strictly positive for all λ.

[7] According to Proposition 3.5 we rule out autoregressive roots on the unit circle, such that $A(e^{-i\lambda}) \neq 0$, and $\left|A(e^{-i\lambda})\right|^2 > 0$. By assumption, $|z| = 1$ implies $A(z) \neq 0$, and here, $z = e^{-i\lambda}$ with

$$|e^{-i\lambda}|^2 = (\cos\lambda)^2 + (\sin\lambda)^2 = 1\,.$$

In the next section we will learn that the calculation of the functions $T_A(\lambda)$ and $T_B(\lambda)$ and thereby the calculation of the spectra do not pose any problem, cf. Eq. (4.8).

4.4 Examples of ARMA Spectra

The ARMA filters $A(L)$ and $B(L)$ are assumed to be of finite order. In order to calculate the spectrum, the power transfer function is needed due to Corollary 4.1. Thus, next we will get to know a simple trick allowing for quickly calculating the power transfer function of a finite filter.

Summation over the Diagonal

We consider for finite p the filter $F(L)$ with the coefficients $w_0, w_1, \ldots, w_p$ being collected in a vector:

$$w = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{p-1} \\ w_p \end{pmatrix}.$$

The outer product yields a matrix where w' stands for the transposition of the column w:

$$ww' = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{p-1} \\ w_p \end{pmatrix} \left(w_0, w_1, \ldots, w_{p-1}, w_p\right)$$

$$= \begin{pmatrix} w_0^2 & w_0 w_1 & \ldots & w_0 w_{p-1} & w_0 w_p \\ w_1 w_0 & w_1^2 & \ldots & w_1 w_{p-1} & w_1 w_p \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_{p-1} w_0 & w_{p-1} w_1 & \cdots & w_{p-1}^2 & w_{p-1} w_p \\ w_p w_0 & w_p w_1 & \cdots & w_p w_{p-1} & w_p^2 \end{pmatrix}.$$

Obviously, the matrix is symmetric. Now, we add the cosine as function of $|j-i|$, $\cos(\lambda\ |j-i\ |)$, to the entries $w_i w_j$. Let the resulting matrix be called $M_F(\lambda)$. It becomes:

$$\begin{pmatrix} w_0^2\cos(0) & w_0w_1\cos(\lambda) & \dots & w_0w_p\cos(\lambda p) \\ w_0w_1\cos(\lambda) & w_1^2\cos(0) & \dots & w_1w_p\cos(\lambda(p-1)) \\ \vdots & \vdots & \cdots & \vdots \\ w_0w_{p-1}\cos(\lambda(p-1)) & w_1w_{p-1}\cos(\lambda(p-2)) & \cdots & w_{p-1}w_p\cos(\lambda) \\ w_0w_p\cos(\lambda p) & w_1w_p\cos(\lambda(p-1)) & \cdots & w_p^2\cos(0) \end{pmatrix}$$

The rule for calculating $T_F(\lambda)$ reads in words: "Add up the sums over all diagonals of $M_F(\lambda)$":

$$\left[w_0^2 + \dots w_p^2\right] + 2\left[w_0w_1 + \cdots w_{p-1}w_p\right]\cos(\lambda) + \cdots + 2\left[w_0w_p\right]\cos(\lambda p).$$

This corresponds exactly to (4.4) for finite p:

$$T_F(\lambda) = \sum_{j=0}^{p} w_j^2 + 2\sum_{h=1}^{p}\left[\sum_{j=0}^{p-h} w_j w_{j+h}\right]\cos(\lambda h). \tag{4.8}$$

AR(1) Spectra

The autoregressive polynomial of order one reads

$$A(L) = 1 - a_1 L,$$

i.e. the filter coefficients are

$$w_0 = 1 \text{ and } w_1 = -a_1.$$

Hence, for the power transfer function, (4.8) provides us with

$$T_A(\lambda) = 1 + a_1^2 - 2a_1\cos(\lambda),$$

and Corollary 4.1 yields for the spectrum

$$2\pi f(\lambda) = \frac{\sigma^2}{1 + a_1^2 - 2a_1\cos(\lambda)}.$$

In Problem 4.4 we will show that there are extrema at $\lambda = 0$ and $\lambda = \pi$, where the slope of the spectrum is zero. For $a_1 > 0$ the spectrum decreases on $[0, \pi]$, i.e. the most significant frequency is $\lambda = 0$: The process is dominated by trending behavior. Figure 4.4 shows that this is the more true the greater a_1 is: The greater a_1, the steeper and higher grows the spectrum in the area around zero. Mirror-inversely,

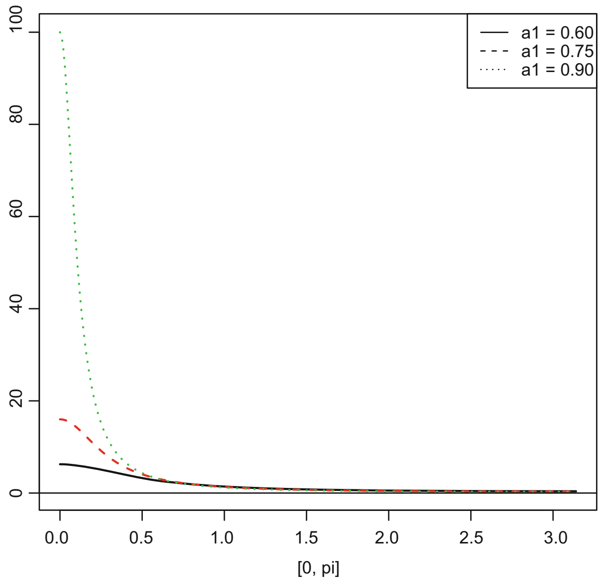

Fig. 4.4 AR(1) spectra ($2\pi f(\lambda)$) with positive autocorrelation

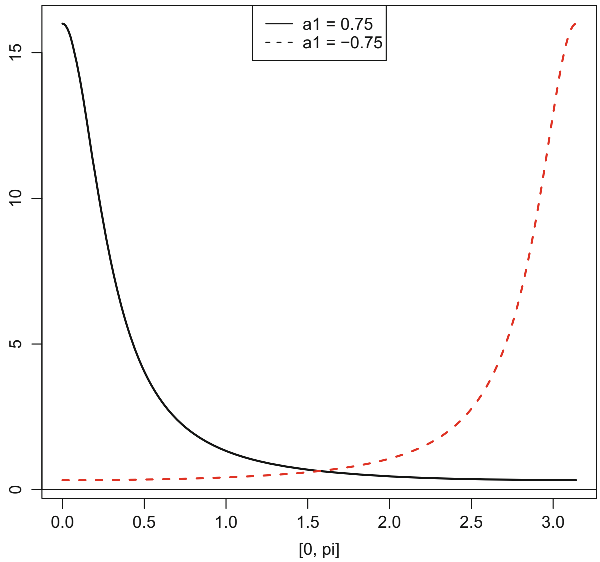

Fig. 4.5 AR(1) spectra ($2\pi f(\lambda)$), cf. Fig. 3.2

for $a_1 < 0$ it holds that the trend component matters least, see Fig. 4.5. The direct comparison to the time domain in Fig. 3.2 is also interesting. The case in which $a_1 > 0$ with the spectral maximum at $\lambda = 0$ translates in persistence of the process:

Observations temporarily lying close together have similar numerical values, i.e. the autocorrelation function is positive. For $a_1 < 0$, however, observations following each other have the tendency to change their sign as, in this case, there is just no trending behavior.

AR(2) Spectra

The AR(2) process is given by

$$x_t = \frac{\varepsilon_t}{A(L)} \quad \text{with} \quad A(L) = 1 - a_1 L - a_2 L^2.$$

In Problem 4.5 we recapitulate the principle of the "summation over the diagonal" and thus we show

$$T_A(\lambda) = 1 + a_1^2 + a_2^2 + 2\left[a_1\,(a_2 - 1)\cos(\lambda) - a_2 \cos\,(2\lambda)\right].$$

Therefore, due to Corollary 4.1, the corresponding spectrum reads

$$2\pi f(\lambda) = \frac{\sigma^2}{T_A(\lambda)}.$$

For $a_2 = 0$ one obtains the AR(1) case.

In Fig. 4.6 spectra for four parameter constellations are depicted; these are exactly the four cases for which autocorrelograms are given in Fig. 3.4. The top left case could be well approximated by an AR(1) process. This is also roughly true for the top right case; however, closer inspection reveals that the AR(2) spectrum is not minimal at frequency zero. Both the lower spectra entirely burst the AR(1) scheme. On the bottom right we have the example of the business cycle, see Fig. 4.3. The spectrum on the bottom left is even more extreme: Except for a rather small area around $\lambda = 2$, it is zero almost everywhere which is why there is no trend component. The process is determined by almost only one cycle which can be seen in the autocorrelogram as well.

ARMA(1,1) Spectra

Now, let us consider the two filters

$$A(L) = 1 - a_1 L \quad \text{and} \quad B(L) = 1 + b_1 L.$$

We know the filter transfer function of $B(L)$ from Example 4.3:

$$T_B(\lambda) = 1 + b_1^2 + 2\,b_1\,\cos(\lambda).$$

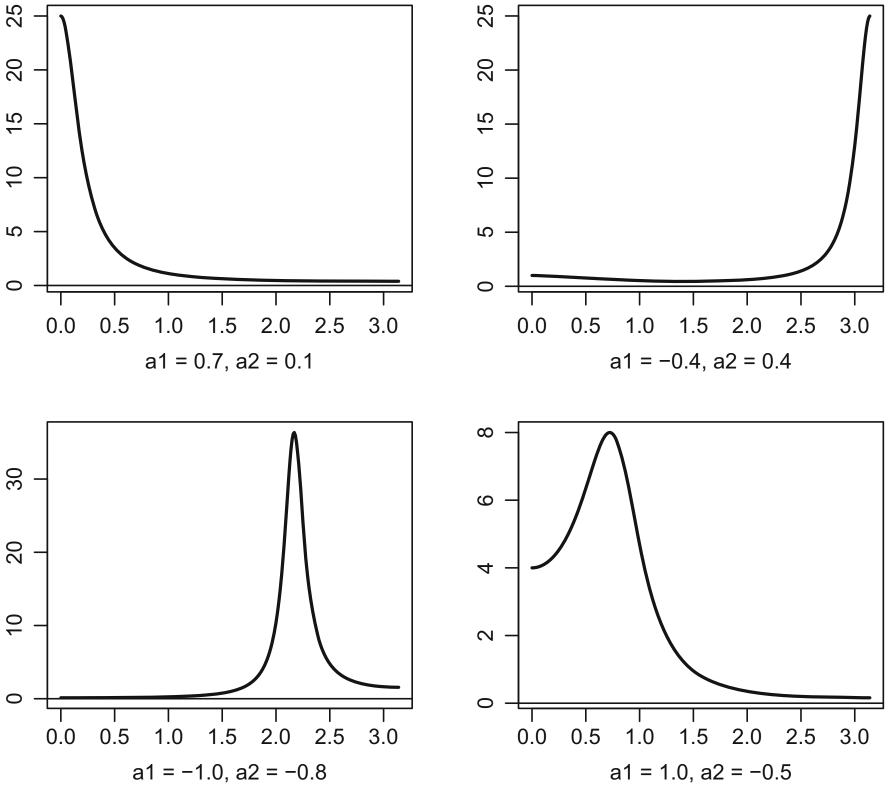

Fig. 4.6 AR(2) spectra ($2\pi f(\lambda)$), cf. Fig. 3.4

The transformation of $A(L)$ was determined at the beginning of this section. Due to Corollary 4.1, we put the spectrum together as follows:

$$2\pi f(\lambda) = \frac{1 + b_1^2 + 2\, b_1 \cos(\lambda)}{1 + a_1^2 - 2\, a_1 \cos(\lambda)}\, \sigma^2\,, \quad \lambda \in [0, \pi]\,.$$

In order to have this illustrated, consider the examples from Fig. 4.7. The cases correspond in their graphical arrangement to the autocorrelograms from Fig. 3.5. The cases top right and bottom left are interesting. At the top on the right, the entire absence of a trend is reflected in a negative autocorrelogram close to zero. At the bottom on the left, beside the trend, cycles of higher frequencies add to the process as well, the process consequently being positively autocorrelated of the first order and then exhibiting an alternating pattern of autocorrelation.

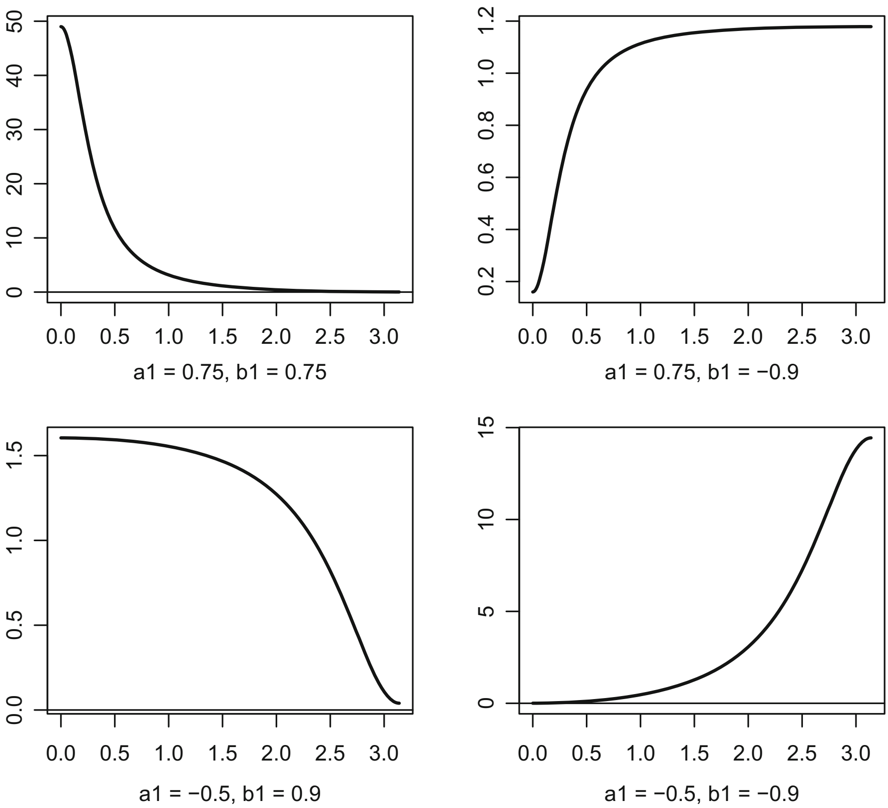

Fig. 4.7 ARMA(1,1) spectra ($2\pi f(\lambda)$), cf. Fig. 3.5

Multiplicative Seasonal AR Process

If one wants to have a decaying autocorrelation function not dropping to zero, then one does not choose a pure MA model as in Example 4.2. The most basic seasonal autoregressive model is based on the filter $(1-a_S L^S)$. Frequently, the trend component is to have an additional weight which is why one adds the AR(1) factor $(1 - a_1 L)$:

$$\begin{aligned} A(L) &= (1 - a_1 L)\,(1 - a_S L^S) \\ &= 1 - a_1 L - a_S L^S + a_1\, a_S L^{S+1}\,. \end{aligned}$$

Therefore, we have an AR($S + 1$) model with parameter restrictions. The spectrum is adopted from Problem 4.6 in which $T_A(\lambda)$ is given:

$$2\pi f(\lambda) = \frac{\sigma^2}{T_A(\lambda)}\,.$$

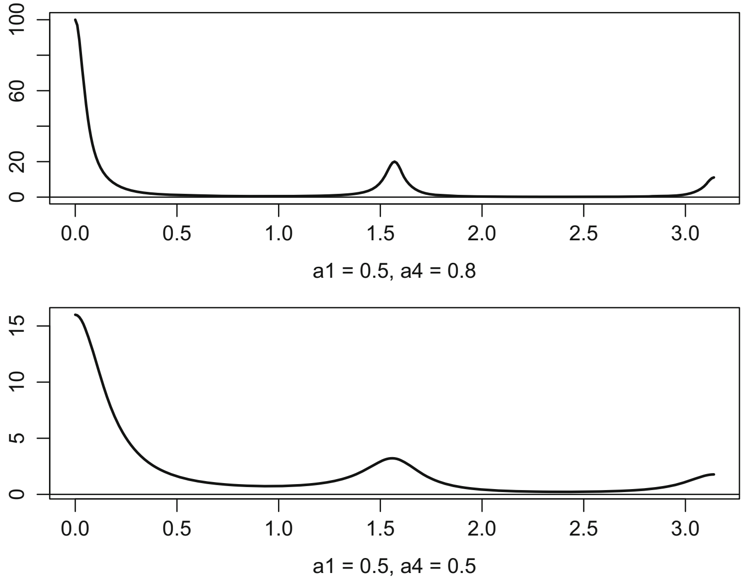

Fig. 4.8 Spectra ($2\pi f(\lambda)$) of multiplicative seasonal AR processes ($S = 4$)

In Fig. 4.8, we show two examples for the quarterly case ($S = 4$). With the frequencies $\lambda = \pi$ and $\lambda = \pi/2$ the semi-annual cycles with $P = 2$ quarters period and the annual cycles with $P = 4$ quarters length are modelled (one also talks about seasonal cycles). As $a_1 = 0.5$ is positive in both the spectra, the trend (at frequency zero) dominates the seasonal cycles. The annual and semi-annual cycles add both equally strongly to the variance of the process. However, in the case $a_4 = 0.8$, the seasonal component is more pronounced than in the case $a_4 = 0.5$ as in the upper spectrum both the seasonal peaks are not only higher than in the lower one (note the scale on the ordinate) but most of all steeper: In the upper graph, the area beneath the spectrum substantially concentrates on the three frequencies $0, \pi/2$ and π, whereas it is more spread over all frequencies in the lower one.

4.5 Problems and Solutions

Problems

4.1 Prove Proposition 4.1 (e) under the additional assumption of absolute summability.

4.2 Determine the extrema in the spectrum of the seasonal MA process from Example 4.2.

4.3 Prove the structure of the spectrum (4.5) for absolutely summable MA(∞) processes.

4.4 Determine the extrema of the AR(1) spectrum.

4.5 Determine the power transfer function $T_A(\lambda)$ of the filter $A(L) = 1 - a_1 L - a_2 L^2$.

4.6 Determine the power transfer function $T_A(\lambda)$ of the multiplicative quarterly AR filter $A(L) = (1 - a_1 L)(1 - a_4 L^4) = 1 - a_1 L - a_4 L^4 + a_1 a_4 L^5$.

4.7 Determine the persistence measure *VR* from (4.6) for a stationary and invertible ARMA(1,1) process. Discuss its behavior in particular for the MA(1) model (in comparison with the AR(1) case given in (4.7)).

Solutions

4.1 We define the entity A_h and will show that it equals $\gamma(h)$. Due to the symmetry of the cosine function and of the even spectrum it holds by definition that:

$$
\begin{aligned}
A_h &:= 2 \int_0^{\pi} f(\lambda) \cos(\lambda h)\, d\lambda \\
&= \int_{-\pi}^{\pi} f(\lambda) \cos(\lambda h)\, d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{l=-\infty}^{\infty} \gamma(l) \cos(\lambda l) \cos(\lambda h)\, d\lambda .
\end{aligned}
$$

Because of the absolute summability, the order of summation and integration is interchangeable:

$$
\begin{aligned}
2\pi A_h &= \sum_{l=-\infty}^{\infty} \gamma(l) \int_{-\pi}^{\pi} \cos(\lambda l) \cos(\lambda h)\, d\lambda \\
&= \gamma(0) \int_{-\pi}^{\pi} \cos(\lambda h)\, d\lambda + 2 \sum_{l=1}^{\infty} \gamma(l) \int_{-\pi}^{\pi} \cos(\lambda l) \cos(\lambda h)\, d\lambda .
\end{aligned}
$$

For $h = 0$ it holds that[8]

$$2\pi A_0 = \gamma(0)\, 2\pi + 2\sum_{l=1}^{\infty} \gamma(l)\, \frac{\sin(\pi l) - \sin(-\pi l)}{l}$$
$$= 2\pi\,\gamma(0) + 0.$$

Accordingly, for $h \neq 0$ it holds that

$$2\pi A_h = 0 + 2\sum_{l=1}^{\infty} \gamma(l) \int_{-\pi}^{\pi} \frac{\cos(\lambda(l-h)) + \cos(\lambda(l+h))}{2}\, d\lambda,$$

where the trigonometric formula

$$2\cos x \cos y = \cos(x-y) + \cos(x+y)$$

was used. By this we obtain

$$2\pi A_h = \gamma(h) \int_{-\pi}^{\pi} (1 + \cos(2\lambda h))\, d\lambda$$

as one can see that for $k \in \mathbb{Z}\backslash\{0\}$ the integral is

$$\int_{-\pi}^{\pi} \cos(\lambda k)\, d\lambda = \frac{\sin(\pi k) - \sin(-\pi k)}{k} = 0\,.$$

So, we finally obtain

$$2\pi A_h = \gamma(h)\,(2\pi + 0) = 2\pi\,\gamma(h)$$

for $h \neq 0$ as well. Hence, $A_h = \gamma(h)$ for all h, and the proof is complete.

4.2 The spectrum

$$f(\lambda) = \left(1 + b^2 + 2b\cos(\lambda S)\right)\sigma^2/2\pi$$

[8] We use

$$\int \cos(\lambda\ell) d\lambda = \frac{\sin(\lambda\ell)}{\ell}\,,$$

and $\sin(\pi k) = 0$ for $k \in \mathbb{Z}$.

is given. In order to determine the extrema, we consider the derivative,

$$f'(\lambda) = -2bS\sin(\lambda S)\ \sigma^2/2\pi,$$

with $(S+1)$ zeros

$$0,\ \frac{\pi}{S},\ \frac{2\pi}{S},\ \ldots,\ \frac{(S-1)\,\pi}{S},\ \pi$$

on the interval $[0,\pi]$. The sign of the second derivative depends on b:

$$f''(\lambda) = -2bS^2\cos(\lambda S)\,\sigma^2/2\pi.$$

One obtains

$$f''(0) = f''\left(\frac{2\pi}{S}\right) = \ldots = -2bS^2\,\sigma^2/2\pi\ ,$$
$$f''\left(\frac{\pi}{S}\right) = f''\left(\frac{3\pi}{S}\right) = \ldots = +2bS^2\,\sigma^2/2\pi.$$

Accordingly, maxima and minima follow each other. For $b > 0$, the sequence of extrema begins with a maximum at zero; for $b < 0$, one obtains a minimum, inversely.

4.3 The autocovariances of

$$x_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$$

are known from Proposition 3.2:

$$\gamma_x(h) = \sigma^2 \sum_{j=0}^{\infty} c_j c_{j+h}.$$

For the spectrum $f_x(\lambda)$ it follows:

$$2\pi\frac{f_x(\lambda)}{\sigma^2} = \frac{\gamma_x(0)}{\sigma^2} + 2\sum_{h=1}^{\infty}\frac{\gamma_x(h)}{\sigma^2}\cos(\lambda h)$$
$$= \sum_{j=0}^{\infty} c_j^2 + 2\sum_{h=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+h}\cos(\lambda h)\,.$$

Hence, the claim is verified.

4.4 For

$$f(\lambda) = \left(1 + a_1^2 - 2\,a_1\,\cos(\lambda)\right)^{-1}\,\sigma^2/2\pi$$

one obtains by differentiation

$$f'(\lambda) = -\left(1 + a_1^2 - 2\,a_1\,\cos(\lambda)\right)^{-2}(-2\,a_1)\,(-\sin(\lambda))\,\sigma^2/2\pi\,.$$

Obviously, candidates for extrema are $\lambda = 0$ and $\lambda = \pi$:

$$f'(0) = f'(\pi) = 0\,.$$

The sign of the derivative depends on a_1 only:

$$f'(\lambda) < 0\,,\ \lambda \in [0, \pi] \quad \Longleftrightarrow \quad a_1 > 0\,.$$

Accordingly,

$$f(0) = (1 - a_1)^{-2}\,\sigma^2/2\pi \quad \text{and} \quad f(\pi) = (1 + a_1)^{-2}\,\sigma^2/2\pi$$

are maxima and minima, depending on the sign of a_1.

4.5 With the vector of coefficients

$$a = \begin{pmatrix} 1 \\ -a_1 \\ -a_2 \end{pmatrix}$$

we obtain as outer product

$$a\,a' = \begin{pmatrix} 1 & -a_1 & -a_2 \\ -a_1 & a_1^2 & a_1 a_2 \\ -a_2 & a_1 a_2 & a_2^2 \end{pmatrix}.$$

Adding the cosine, it follows that

$$M_A(\lambda) = \begin{pmatrix} 1 & -a_1\,\cos(\lambda) & -a_2\,\cos(2\,\lambda) \\ -a_1\,\cos(\lambda) & a_1^2 & a_1 a_2\,\cos(\lambda) \\ -a_2\,\cos(2\,\lambda) & a_1 a_2\,\cos(\lambda) & a_2^2 \end{pmatrix}.$$

By summation over the diagonal we obtain due to symmetry

$$T_A(\lambda) = 1 + a_1^2 + a_2^2 + 2\,[-a_1 + a_1\,a_2]\,\cos(\lambda) + 2\,[-a_2]\,\cos(2\lambda)\,,$$

which results from (4.8) as well. This is in accordance with the result in the text.

4.6 Using (4.8) with $p = 5$ yields the following expression:

$$T_A(\lambda) = 1 + a_1^2 + a_4^2 + a_1^2 a_4^2 \\ +2\left[-a_1 - a_1 a_4^2\right] \cos(\lambda) + 2\left[a_1 a_4\right] \cos(3\lambda) \\ +2\left[-a_4 - a_1^2 a_4\right] \cos(4\lambda) + 2\left[a_1 a_4\right] \cos(5\lambda) .$$

This is simply an exercise in concentration and is simplified by the following equalities:

$$w_0 = 1, \quad w_1 = -a_1, \quad w_2 = w_3 = 0, \quad w_4 = -a_4, \quad w_5 = a_1 a_4 .$$

4.7 In the previous section we discussed the ARMA(1,1) process with the polynomials

$$A(L) = 1 - a_1 L \quad \text{and} \quad B(L) = 1 + b_1 L .$$

Evaluating the spectrum given there we have:

$$2\pi f(0) = \frac{1 + b_1^2 + 2 b_1}{1 + a_1^2 - 2 a_1} \sigma^2 = \frac{(1 + b_1)^2}{(1 - a_1)^2} \sigma^2 .$$

The variance we copy from Chap. 3:

$$\gamma(0) = \frac{(1 + b_1^2 + 2 a_1 b_1)}{1 - a_1^2} \sigma^2 .$$

By (4.6) we obtain

$$VR = \frac{1 + a_1}{1 - a_1} \frac{(1 + b_1)^2}{1 + b_1^2 + 2 a_1 b_1} .$$

If $b_1 = 0$, the AR(1) case from (4.7) is of course reproduced. If $a_1 = 0$, the MA(1) case results as

$$VR = \frac{(1 + b_1)^2}{1 + b_1^2} \begin{cases} > 1 \text{ if } b_1 > 0 \\ = 1 \text{ if } b_1 = 0 \\ < 1 \text{ if } b_1 < 0 \end{cases} .$$

We hence have negative persistence for $b_1 < 0$, which reflects the negative autocorrelation. For $b_1 > 0$, it is straightforward to verify that VR is growing with b_1, reaching a maximum value of $VR = 2$ for $b_1 = 1$. This corresponds to the persistence of an AR(1) process with $a_1 = 1/3$. Hence, the invertible MA(1) process with $|b_1| < 1$ can only capture very moderate persistence in comparison with the AR(1) case where VR grows with a_1 beyond any limit.

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