9 Stieltjes Integrals

9.1 Summary

Below, we will encounter Riemann-Stieltjes integrals (or more briefly: Stieltjes integrals) as solutions of certain stochastic differential equations. They can be reduced to the sum of a Riemann integral and a multiple of the Wiener process. Stieltjes integrals are again Gaussian. As an example we consider the Ornstein-Uhlenbeck process which is defined by a Stieltjes integral and which will be dealt with in detail in the chapter on interest rate models.

9.2 Definition and Partial Integration

As a first step towards the Ito integral, we define Stieltjes[1] integrals which can be reduced to Riemann integrals by integration by parts.

Definition

The Riemann-Stieltjes integral (or Stieltjes integral), as it is considered here, integrates over a deterministic function $f(s)$. Nevertheless, the Stieltjes integral is random as it is integrated with respect to the stochastic Wiener process $W(s)$. In order to understand what is meant by this, we recall the partition (8.1):

$$P_n([0, t]): \quad 0 = s_0 < s_1 < \ldots < s_n = t,$$

[1] Thomas J. Stieltjes lived from 1856 to 1894. The Dutch mathematician generalized the concept of integration by Riemann.

U. Hassler, *Stochastic Processes and Calculus*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-23428-1_9

with $s_i^* \in [s_{i-1}, s_i]$. Hence, the **Riemann-Stieltjes sum** is defined as

$$RS_n = \sum_{i=1}^{n} f(s_i^*)\ (W(s_i) - W(s_{i-1}))\,. \tag{9.1}$$

If an expression well-defined in mean square follows from this for $n \to \infty$ under (8.2), then we define it as a **Stieltjes integral** with the obvious notation

$$RS_n \overset{2}{\to} \int_0^t f(s)\, dW(s)\,.$$

Note that $dW(s)$ does not stand for the derivative of the Wiener process as it does not exist. It is just a common symbolic notation.
If f is **continuously differentiable**,[2] then the existence of the Stieltjes integral is guaranteed, see Soong (1973, Theorem 4.5.2).

Integration by Parts

If f is continuously differentiable, then the Stieltjes integral can be expressed by a Riemann integral and the WP. This relation is also known as integration by parts. In Chap. 11 we will understand that it is a special case of Ito's lemma, which is why we do not have to concern ourselves with a proof of Proposition 9.1 at this point.

Proposition 9.1 (Stieltjes Integral; Integration by Parts) *For a continuously differentiable, deterministic function f we have that*

(a) the Stieltjes sum from (9.1) converges in mean square if it holds that $\max(s_i - s_{i-1}) \to 0$*,*
(b) and

$$\int_0^t f(s)\, dW(s) = [f(s)\, W(s)]_0^t - \int_0^t W(s)\, df(s)$$
$$= f(t)\, W(t) - \int_0^t W(s) f'(s)\, ds\,.$$

where the last equality holds with probability one.[3]

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[2]We call a function continuously differentiable if it has a continuous first order derivative.

[3]Remember that we assumed $P(W(0) = 0) = 1$, which justifies the last statement. Whenever we have equalities in a stochastic setting, they are typically understood to hold with probability one for the rest of the book.

The result from (b) corresponds to the familiar rule of partial integration. As a refresher, we write this rule for two deterministic functions f and g:

$$\int_0^t f(s)\, g'(s)\, ds = [f(s)\, g(s)]_0^t - \int_0^t g(s) f'(s)\, ds\,. \tag{9.2}$$

Hence, this is the integral form of the product rule of differentiation:

$$\frac{d[f(s)\, g(s)]}{ds} = f'(s)\, g(s) + g'(s) f(s)\,,$$

or

$$d\,[f(s)\, g(s)] = g(s)\, df(s) + f(s)\, dg(s)\,,$$

or

$$[f(s)\, g(s)]_0^t = \int_0^t g(s)\, df(s) + \int_0^t f(s)\, dg(s)\,.$$

Therefore, one can make a mental note of Proposition 9.1 (b) by the well-known partial integration from (9.2).

Example 9.1 (Corollary) As an application of Proposition 9.1 we consider Riemann-Stieltjes integrals for three particularly simple functions. We will encounter these relations repeatedly. The proof amounts to a simple exercise in substitution. It holds

(a) for the identity function $f(s) = s$:

$$\int_0^t s\, dW(s) = t\, W(t) - \int_0^t W(s)\, ds\,;$$

(b) for $f(s) = 1 - s$:

$$\int_0^t (1-s)\, dW(s) = (1-t)\, W(t) + \int_0^t W(s)\, ds\,;$$

(c) for the constant function $f(s) = 1$:

$$\int_0^t dW(s) = W(t)\,.$$

In (c) we again observe a formal analogy of the WP with the random walk. Just like the latter is defined as the sum over the past of a pure random process, see (1.8), the WP is the integral of its past independent increments. ■

9.3 Gaussian Distribution and Autocovariances

The reduction of Stieltjes integrals to Riemann integrals suggests that there are Gaussian processes hiding behind them. In fact, it holds that all Stieltjes integrals follow Gaussian distributions with expectation zero.

Gaussian Distribution

The Gaussian distribution itself is obvious: The Riemann-Stieltjes sum from (9.1) is, as the sum of multivariate Gaussian random variables, Gaussian as well. Then, this also holds for the limit of the sum due to Lemma 8.1. The expected value is zero due to Propositions 9.1(b) and 8.3. The variance results as a special case of the autocovariance given in Proposition 9.3. Hence, we obtain the following proposition.

Proposition 9.2 (Normality of Stieltjes integrals) *For a continuously differentiable, deterministic function f, it holds that*

$$\int_0^t f(s)\, dW(s) \sim \mathcal{N}\left(0\,,\int_0^t f^2(s)\, ds\right) .$$

The variance of the Stieltjes integral is well motivated as follows. For the variance of the Riemann-Stieltjes sum,

$$\mathrm{Var}\left(\sum_{i=1}^{n} f(s_i^*)\ (W(s_i) - W(s_{i-1}))\right) ,$$

it follows for $n \to \infty$, due to the independence of the increments of the WP, that:

$$\begin{aligned}\sum_{i=1}^{n} f^2(s_i^*)\,\mathrm{Var}\,(W\,(s_i) - W\,(s_{i-1})) &= \sum_{i=1}^{n} f^2(s_i^*)\,(s_i - s_{i-1}) \\ &\to \int_0^t f^2(s)\, ds\,.\end{aligned}$$

The convergence takes place as f^2 is continuous and thus Riemann-integrable. Hence, for $n \to \infty$ the expression from Proposition 9.2 is obtained.

Let us consider the integrals from Example 9.1 and calculate the variances for $t = 1$ (see Problem 9.1).

Example 9.2 (Corollary) For the functions from Example 9.1 it holds:

(a) for the identity function $f(s) = s$:

$$\int_0^1 s\, dW(s) \sim \mathcal{N}(0, 1/3)\,;$$

(b) for $f(s) = 1 - s$:

$$\int_0^1 (1-s)\, dW(s) \sim \mathcal{N}(0, 1/3)\,;$$

(c) for the constant function $f(s) = 1$:

$$W(t) = \int_0^t dW(s) \sim \mathcal{N}(0, t)\,. \quad \blacksquare$$

Autocovariance Function

As a generalization of the variance, an expression for the covariance is to be found. Hence, let us define the process $Y(t) = \int_0^t f(s)dW(s)$. The autocovariance of $Y(t)$ and $Y(t+h)$ with $h \geq 0$ can be well justified if one takes into account that the increments $dW(t)$ of the WP are stochastically independent provided they do not overlap. Therefore, one should expect $\int_0^t f(s)dW(s)$ and $\int_t^{t+h} f(r)dW(r)$ to be uncorrelated:

$$\mathrm{E}\left[\int_0^t f(s)dW(s) \int_t^{t+h} f(r)dW(r)\right] = 0.$$

If this is true, then, due to

$$\int_0^{t+h} f(r)dW(r) = \int_0^t f(r)dW(r) + \int_t^{t+h} f(r)dW(r)$$

the following result is obtained:

$$\begin{aligned}\mathrm{E}\left[\int_0^t f(s)dW(s) \int_0^{t+h} f(r)dW(r)\right] &= \mathrm{E}\left[\int_0^t f(s)dW(s) \int_0^t f(r)dW(r)\right] \\ &= \mathrm{Var}\left(\int_0^t f(s)dW(s)\right).\end{aligned}$$

Therefore, for an arbitrary $h \geq 0$ the autocovariance coincides with the variance in t. In fact, this result can be verified more rigorously (see Problem 9.5).

Proposition 9.3 (Autocovariance of Stieltjes Integrals) *For a continuously differentiable, deterministic function f it holds that*

$$E\left[\int_0^t f(s)dW(s) \int_0^{t+h} f(s)dW(s)\right] = \int_0^t f^2(s)ds$$

with $h \geq 0$.

Of course, for $h = 0$ the variance from Proposition 9.2 is obtained.

Example 9.3 (Autocovariance of the WP) As an example, let us consider $f(s) = 1$ with

$$W(t) = \int_0^t dW(s).$$

Then, it follows for $h \geq 0$:

$$\mathrm{E}(W(t)W(t+h)) = \int_0^t ds = t = \min(t, t+h).$$

Trivially, this just reproduces the autocovariance structure of the Wiener process already known from (7.4). ■

9.4 Standard Ornstein-Uhlenbeck Process

The so-called Ornstein-Uhlenbeck process has been introduced in a publication by the physicists Ornstein and Uhlenbeck in 1930.

Definition

We define the **Ornstein-Uhlenbeck process** (OUP) with starting value $X_c(0) = 0$ for an arbitrary real c as a Stieltjes integral,

$$X_c(t) := e^{ct} \int_0^t e^{-cs}\, dW(s), \quad t \geq 0,\ X_c(0) = 0\,. \tag{9.3}$$

For $c = 0$ in (9.3) the Wiener process, $X_0(t) = W(t)$, is obtained. More precisely, $X_c(t)$ from (9.3) is a standard OUP; a generalization will be offered in the chapter on interest rate dynamics. By definition, it holds that:

$$\begin{aligned} X_c(t+1) &= e^{ct}\, e^{c} \left[\int_0^t e^{-cs} dW(s) + \int_t^{t+1} e^{-cs} dW(s) \right] \\ &= e^{c} X_c(t) + e^{c(t+1)} \int_t^{t+1} e^{-cs} dW(s) \\ &= e^{c} X_c(t) + \varepsilon(t+1), \end{aligned}$$

where $\varepsilon(t+1)$ was defined implicitly. Note that the increments $dW(s)$ from t on in $\varepsilon(t+1)$ are independent of the increments up to t as they appear in $X_c(t)$. Hence, the OUP is a continuous counterpart of the AR(1) process from Chap. 3 where the autoregressive parameter is denoted by e^c. For $c < 0$ this parameter is less than one, such that in this case we expect a stable adjustment or, in a way, a quasi-stationary behavior. This will be reflected by the behavior of the variance and the covariance function which are given, among others, in the following proposition.

Properties

The proof of Proposition 9.4 will be given in an exercise problem. It comprises an application of Propositions 9.1, 9.2 and 9.3.

Proposition 9.4 (Ornstein-Uhlenbeck Process) *It holds for the Ornstein-Uhlenbeck process from (9.3) that:*

$$\begin{aligned} &(a)\ X_c(t) = W(t) + c\, e^{ct} \int_0^t e^{-cs}\, W(s)\, ds\,, \\ &(b)\ X_c(t) \sim \mathcal{N}(0,\ (e^{2ct} - 1)/2c)\,, \\ &(c)\ E\left(X_c(t)\, X_c(t+h)\right) = e^{ch}\, Var\left(X_c(t)\right)\,, \end{aligned}$$

where $h \geq 0$.

Statement (a) establishes the usual relation between Stieltjes and Riemann integrals and, seen individually, it is not that thrilling. As for $c = 0$ the OUP coincides with the WP, it is interesting to examine the variance from (b) for $c \to 0$. **L'Hospital's rule** yields:

$$\lim_{c\to 0} \frac{e^{2ct} - 1}{2c} = \lim_{c\to 0} \frac{2te^{2ct}}{2} = t.$$

Hence, for $c \to 0$ the variance of the WP is embedded in (b). The covariance from (c) allows for determining the autocorrelation:

$$
\begin{aligned}
corr(X_c(t), X_c(t+h)) &= \frac{e^{ch}\mathrm{Var}(X_c(t))}{\sqrt{\mathrm{Var}(X_c(t))}\sqrt{\mathrm{Var}(X_c(t+h))}} \\
&= e^{ch}\frac{\sqrt{\mathrm{Var}(X_c(t))}}{\sqrt{\mathrm{Var}(X_c(t+h))}}.
\end{aligned}
$$

Now, let us assume that $c < 0$. Then it holds for t growing that:

$$
\lim_{t\to\infty} \mathrm{Var}(X_c(t)) = -\frac{1}{2c} > 0.
$$

Accordingly, it holds for the autocorrelation that:

$$
\lim_{t\to\infty} corr(X_c(t), X_c(t+h)) = e^{ch}, \quad c < 0.
$$

Thus, for $c < 0$ we obtain the "asymptotically stationary" case with asymptotically constant variance and an autocorrelation being asymptotically dependent on the lag h only. Thereby, the autocorrelation results as the h-th power of the "autoregressive parameter" $a = e^c$. With h growing, the autocovariance decays gradually. This finds its counterpart in the discrete-time AR(1) process. Just as the random walk arises from the AR(1) process with the parameter value one, the WP with $c = 0$, i.e. $a = e^0 = 1$, is the corresponding special case of the OUP. Hence, we can definitely consider the OUP as a continuous-time analog to the AR(1) process.

Simulation

The theoretical properties of the process for $c < 0$ can be illustrated graphically. In Fig. 9.1 the simulated paths of two parameter constellations are shown. It can be observed that the process oscillates about the zero line where the variance or the deviation from zero for $c = -0.1$ is much larger[4] than in the case $c = -0.9$. This is clear against the background of (b) from Proposition 9.4 in which the first moment and the variance are given: The expected value is zero and the variance decreases with the absolute value of c increasing. The positive autocorrelation (cf. Proposition 9.4(c)) is obvious as well: Positive values tend to be followed by positive values and the inverse holds for negative observations. The closer to zero c is, the stronger the autocorrelation gets. That is why the graph for $c = -0.1$ is strongly

[4]If the arithmetic mean of the 1000 observations of this time series is calculated, then by -0.72344 a notably negative number is obtained although the theoretical expected value is zero. Details on the simulation of OUP paths are to follow in Sect. 13.2.

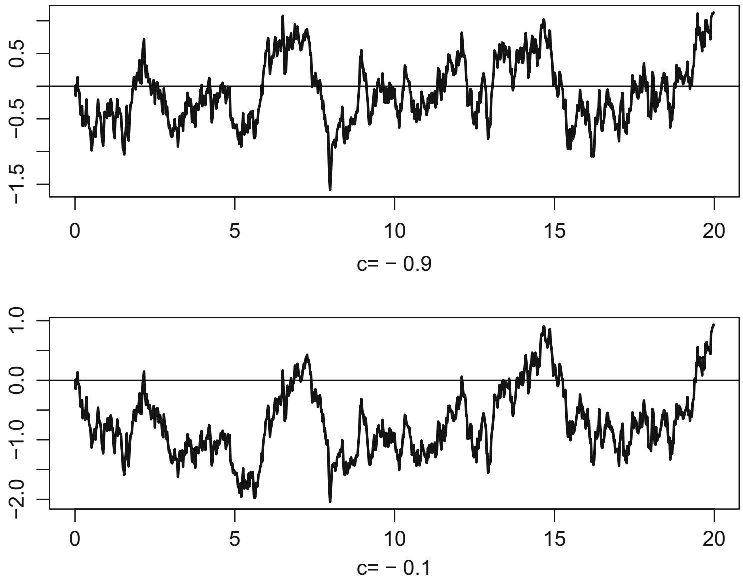

Fig. 9.1 Standard Ornstein-Uhlenbeck processes

determined by the "local" trend and does not cross the zero line for longer time spans while for $c = -0.9$ the force which pulls the observations back to the zero line is more effective such that the graph looks "more stationary" for $c = -0.9$.

9.5 Problems and Solutions

Problems

9.1 Calculate the variances from Example 9.2.

9.2 Verify the Gaussian distribution from Corollary 8.1(b).

9.3 Verify the following equality (with probability 1):

$$\int_0^t s^2 \, dW(s) = t^2 \, W(t) - 2 \int_0^t s \, W(s) \, ds \, .$$

9.4 Determine the variance of the process $X(t)$ with

$$X(t) = \int_0^t s^2 \, dW(s) \, .$$

9.5 Prove Proposition 9.3.

9.6 Prove (a) from Proposition 9.4.

9.7 Show (b) from Proposition 9.4.

9.8 Prove (c) from Proposition 9.4.

Solutions

9.1 From Proposition 9.2 it obviously follows for (a) that:

$$\int_0^1 s^2 ds = \left[\frac{s^3}{3}\right]_0^1 = \frac{1}{3}.$$

Equally, one shows (b):

$$\int_0^1 (1-s)^2 ds = \left[-\frac{(1-s)^3}{3}\right]_0^1 = \frac{1}{3}.$$

Finally, the result from (c) is known anyway.

9.2 The result follows from the examples of this chapter. From Example 9.1(a) we obtain for $t = 1$:

$$W(1) - \int_0^1 W(s)\,ds = \int_0^1 s\,dW(s)\,.$$

Due to Example 9.2(a) the claim is verified.

9.3 This is a straightforward application of Proposition 9.1. With $f(s) = s^2$ and $f'(s) = 2s$ the claim is established.

9.4 From Proposition 9.2 with $f(s) = s^2$ it follows for the variance

$$\text{Var}\left(\int_0^t s^2 dW(s)\right) = \int_0^t s^4 ds = \left[\frac{1}{5}s^5\right]_0^t = \frac{t^5}{5}.$$

9.5 With $Y(t) = \int_0^t f(s)dW(s)$ we know from Proposition 9.1 that:

$$Y(t) = f(t)\,W(t) - \int_0^t f'(s)W(s)ds.$$

Hence, the covariance results as

$$\mathrm{E}(Y(t)Y(t+h)) = A - B - C + D$$

where the expressions on the right-hand side are defined by multiplying $Y(t)$ and $Y(t+h)$. Now, we consider them one by one.

For A we obtain immediately:

$$\begin{aligned} A &= \mathrm{E}\left[f(t)f(t+h)W(t)\,W(t+h)\right] \\ &= f(t)f(t+h)\min(t, t+h) \\ &= f(t)f(t+h)t. \end{aligned}$$

By Fubini's theorem it holds for B that:

$$\begin{aligned} B &= \mathrm{E}\left[f(t+h)\int_0^t f'(s)W(s)W(t+h)ds\right] \\ &= f(t+h)\int_0^t f'(s)\min(s, t+h)ds \\ &= f(t+h)\int_0^t f'(s)sds. \end{aligned}$$

Integration by parts in the following form,

$$\int_0^t f(r)rdr = F(t)t - \int_0^t F(r)dr \quad \text{with } F' = f, \tag{9.4}$$

applied to f' yields:

$$B = f(t+h)[f(t)t - F(t) + F(0)],$$

where $F(s)$ denotes the antiderivative of $f(s)$. In the same way, we obtain

$$\begin{aligned} C &= \mathrm{E}\left[f(t)\int_0^{t+h} f'(s)W(s)W(t)ds\right] \\ &= f(t)\int_0^{t+h} f'(s)\min(s, t)ds \\ &= f(t)\left[\int_0^t f'(s)sds + \int_t^{t+h} f'(s)tds\right] \\ &= f(t)\left[f(t)t - F(t) + F(0) + t(f(t+h) - f(t))\right] \\ &= f(t)\left[F(0) - F(t) + tf(t+h)\right]\,. \end{aligned}$$

For the fourth expression Proposition 8.4 provides us with f' instead of f:

$$D = \mathrm{E}\left[\int_0^t f'(s)W(s)ds \int_0^{t+h} f'(r)W(r)dr\right]$$
$$= \int_0^t f'(s)\left[f(s)s - (F(s) - F(0)) + s(f(t+h) - f(s))\right] ds$$
$$= \int_0^t f'(s)dsF(0) - \int_0^t f'(s)F(s)ds + f(t+h)\int_0^t sf'(s)ds.$$

In addition to (9.4), we apply integration by parts in the form of

$$\int_0^t f'(s)F(s)ds = f(t)F(t) - f(0)F(0) - \int_0^t f^2(s)ds\,.$$

Then it holds that:

$$D = (f(t) - f(0))F(0) - f(t)F(t) + f(0)F(0) + \int_0^t f^2(s)ds$$
$$+ f(t+h)\,(f(t)t - F(t) + F(0))$$
$$= \int_0^t f^2(s)ds + (F(0) - F(t))(f(t) + f(t+h)) + f(t)f(t+h)t.$$

If we assemble the terms, then we obtain the autocovariance function in the desired form:

$$\mathrm{E}\left[\int_0^t f(s)dW(s)\int_0^{t+h} f(r)dW(r)\right] = A - B - C + D = \int_0^t f^2(s)ds.$$

9.6 We use Proposition 9.1 with $f(s) = e^{-cs}$:

$$\int_0^t e^{-cs}dW(s) = e^{-ct}W(t) + c\int_0^t e^{-cs}W(s)\,ds.$$

Multiplying by e^{ct} yields

$$e^{ct}\int_0^t e^{-cs}dW(s) = W(t) + c\,e^{ct}\int_0^t e^{-cs}W(s)\,ds.$$

On the left-hand side, we have the OUP $X_c(t)$ by definition which was to be verified.

9.7 Due to Proposition 9.2 the OUP is Gaussian with expectation zero and variance

$$\begin{aligned}\mathrm{Var}(X_c(t)) &= e^{2ct} \int_0^t e^{-2cs} ds \\ &= e^{2ct} \left[\frac{e^{-2cs}}{-2c} \right]_0^t \\ &= e^{2ct} \frac{e^{-2ct} - 1}{-2c} \\ &= \frac{1 - e^{2ct}}{-2c}.\end{aligned}$$

This is equal to the claimed variance.

9.8 As for the derivation of the variance, we use

$$\int_0^t e^{-2cs} ds = \frac{1 - e^{-2ct}}{2c}.$$

From Proposition 9.3 we know that this is also the expression for the autocovariance of the Stieltjes integrals ($h \geq 0$):

$$\mathrm{E}\left[\int_0^t e^{-cs} dW(s) \int_0^{t+h} e^{-cs} dW(s) \right] = \frac{1 - e^{-2ct}}{2c}.$$

Hence, we obtain for the OUP:

$$\begin{aligned}\mathrm{E}(X_c(t)\, X_c(t+h)) &= e^{ct} e^{c(t+h)} \frac{1 - e^{-2ct}}{2c} \\ &= e^{ch} \frac{e^{2ct} - 1}{2c} \\ &= e^{ch}\, \mathrm{Var}(X_c(t)).\end{aligned}$$

Reference

Soong, T. T. (1973). *Random differential equations in science and engineering*. New York: Academic Press.