Stochastic Differential Equations (SDE) 12

12.1 Summary

In the following section we discuss the most general stochastic differential equation considered here, whose solution is a diffusion. Then, linear differential equations (with variable coefficients) will be studied extensively. Here we obtain analytical solutions by Ito's lemma. We discuss special cases that are widespread in the literature on finance. In the fourth section we turn to numerical solutions allowing to simulate processes. The sample paths displayed in the figures of Chap. 13 are constructed that way.

12.2 Definition and Existence

After a definition and a discussion of conditions for existence, we will consider the deterministic case as well. Deterministic differential equations are embedded into the stochastic ones as special cases.

Diffusions

We defined the solution of

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$$

as a diffusion process, where $\mu(t)$ and $\sigma(t)$ are allowed to depend on t and on $X(t)$ itself. As the most general case of this chapter we consider diffusions as in (10.6):

$$dX(t) = \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW(t)\,, \quad t \in [0, T]\,. \tag{12.1}$$

The solutions[1] of such differential equations can also be written in integral form:

$$X(t) = X(0) + \int_0^t \mu(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s) , \quad t \in [0, T] . \qquad (12.2)$$

Under what conditions is such a definition possible? In other words: Which requirements have to be met by the functions $\mu(t, x)$ and $\sigma(t, x)$, such that a solution of (12.1) exists at all – and uniquely so? This mathematical aspect is not to be overly deepened at this point, however, neither is it to be completely neglected. We consider stronger but simpler sufficient conditions than necessary. For a profound discussion see e.g. Øksendal (2003). The first assumption requires that μ and σ are smooth enough in the argument x:[2]

(E1)   *The partial derivatives of $\mu(t, x)$ and $\sigma(t, x)$ with respect to x exist and are continuous in x.*

Secondly, we maintain a linear restriction of the growth of the diffusion process:

(E2)   *There exist constants K_1 and K_2 with*

$$|\mu(t, x)| + |\sigma(t, x)| \le K_1 + K_2 |x| .$$

And finally we need a well defined starting value, which may be stochastic:

(E3)   *$X(0)$ is independent of $W(t)$ with $E(X^2(0)) < \infty$.*

Under these assumptions Øksendal (2003, Theorem 5.2.1) proves the following proposition.

Proposition 12.1 (Existence of a Unique Solution)   *Under the assumptions (E1) to (E3), Eq. (12.1) has a unique solution $X(t)$ of the form (12.2) with continuous paths and $E(X^2(t)) < \infty$.*

The assumption **(E3)** can always be met by assuming a fixed starting value. The second assumption is necessary for the existence of a (finite) solution while **(E1)**

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[1]Strictly speaking, this is a so-called "strong solution" in contrast to a "weak solution". For a weak solution the behavior of $X(t)$ is only characterized in distribution. We will not concern ourselves with weak solutions.

[2]Normally, one demands that they satisfy a Lipschitz condition. A function f is called Lipschitz continuous if it holds for all x and y that there exists a constant K with

$$|f(x) - f(y)| \le K |x - y| .$$

We can conceal this condition by requiring the stronger sufficient continuous differentiability.

guarantees the uniqueness of this solution. This is to be illustrated by means of two deterministic examples.

Example 12.1 (Violation of the Assumptions) We examine two examples known from the literature on deterministic differential equations where $\sigma(t, x) = 0$. Similar cases can be found e.g. in Øksendal (2003). In the first example we set $\mu(t, X(t)) = X^{2/3}(t)$:

$$dX(t) = X^{2/3}(t)\, dt\,, \quad X(0) = 0\,,\ t \geq 0\,.$$

We define for an arbitrary $a > 0$ infinitely many solutions:

$$X_a(t) = \begin{cases} 0\,, & t \leq a \\ \frac{(t-a)^3}{27}\,, & t > a\,. \end{cases}$$

By differentiating one can observe that any $X_a(t)$ indeed satisfies the given equation. The reason for the ambiguity of the solutions lies in the violation of **(E1)** as the partial derivative,

$$\frac{\partial \mu(t, x)}{\partial x} = \frac{2}{3} x^{-1/3}\,,$$

does not exist at $x = 0$.

The second example reads for $\mu(t, X(t)) = X^2(t)$:

$$dX(t) = X^2(t)\, dt\,, \quad X(0) = 1\,,\ t \in [0, 1)\,.$$

Again, by elementary means one proves that the solution reads

$$X(t) = (1 - t)^{-1}\,, \quad 0 \leq t < 1\,,$$

and hence tends to ∞ for $t \to 1$. The reason for this lies in a violation of **(E2)**: The quadratic function $\mu(t, x) = x^2$ cannot be linearly bounded. ■

Linear Coefficients

In order to be able to state analytical solutions, we frequently restrict generality and consider **linear differential equations**:

$$dX(t) = (c_1(t)\, X(t) + c_2(t))\ dt + (\sigma_1(t)\, X(t) + \sigma_2(t))\ dW(t)\,, \quad t \geq 0\,, \qquad (12.3)$$

where the variable coefficients $c_i(t)$ and $\sigma_i(t)$, $i = 1, 2$, are continuous deterministic functions of time. Here, $X(t)$ enters μ and σ just linearly. Obviously, the partial

derivatives from **(E1)** are constant (in x) and thus continuous. In addition, one obtains a linear bound:

$$\begin{aligned}
|\mu(t,x)| + |\sigma(t,x)| &\leq |c_1(t)| \, |x| + |c_2(t)| + |\sigma_1(t)| \, |x| + |\sigma_2(t)| \\
&= (|c_1(t)| + |\sigma_1(t)|) \, |x| + (|c_2(t)| + |\sigma_2(t)|) \\
&\leq K_2 \, |x| + K_1 \, .
\end{aligned}$$

As $c_i(t)$ and $\sigma_i(t)$ are continuous in t and hence bounded for finite t, positive constants K_1 and K_2 can be specified such that the inequality above holds true. Therefore, **(E2)** is satisfied. Therefore, a unique solution exists for linear stochastic differential equations. What is more: Ito's lemma will allow as well for the specification of an explicit form of this analytical solution from which one can determine first and second moments as functions of time. The next section is reserved for studying equation (12.3). Before, we consider the borderline case of a deterministic linear differential equation.

Deterministic Case

By setting $\sigma_1(t) = \sigma_2(t) = 0$ in (12.3), we obtain a deterministic linear differential equation (in small letters to distinguish from the stochastic case),

$$dx(t) = (c_1(t) \, x(t) + c_2(t)) \, dt \, , \quad t \geq 0 \, , \tag{12.4}$$

or as well

$$x'(t) = c_1(t) \, x(t) + c_2(t) \, .$$

Frequently, one speaks of first order differential equations, as only the first derivative is involved. As is well known, the solution reads (see Problem 12.1)

$$x(t) = z(t) \left[x(0) + \int_0^t \frac{c_2(s)}{z(s)} \, ds \right] \tag{12.5}$$

with

$$z(t) = \exp \left\{ \int_0^t c_1(s) \, ds \right\} \, . \tag{12.6}$$

For $c_2(t) = 0$ one obtains from (12.4) the related **homogeneous differential equation** (with starting value 1),

$$dz(t) = c_1(t) \, z(t) \, dt \, , \quad z(0) = 1 \, ,$$

which just has $z(t)$ from (12.6) as a solution. The following example presents the special case of constant coefficients.

Example 12.2 (Constant Coefficients) In the case of constant coefficients,

$$c_1(t) = c_1 = const\,, \quad c_2(t) = c_2 = const\,,$$

the solution from (12.5) simplifies, see Problem 12.1:

$$x(t) = e^{c_1 t} \left[x(0) + \frac{c_2}{c_1} \left(1 - e^{-c_1 t} \right) \right]$$

$$= e^{c_1 t} \left[x(0) + \frac{c_2}{c_1} \right] - \frac{c_2}{c_1}\,.$$

Hence, for negative values of c_1 it holds that the equation is stable in the sense that the solution tends towards a fixed value:

$$x(t) \overset{t \to \infty}{\longrightarrow} -\frac{c_2}{c_1} =: \mu\,, \quad c_1 < 0\,.$$

Basically, one can already observe this from the equation itself:

$$dx(t) = (c_1 x(t) + c_2)\, dt$$

$$= c_1 (x(t) - \mu)\, dt\,.$$

Namely, if $x(t)$ lies above the limit μ, then the expression in brackets is positive and hence the change is negative such that $x(t)$ adjusts towards the limit μ. Conversely, $x(t) < \mu$ causes a positive derivative such that $x(t)$ grows and moves towards the limit. All in all, for $c_1 < 0$ a convergence to μ is modeled. ■

In the following, we will see that the solution of the deterministic linear equation is embedded into the stochastic one for $\sigma_1(t) = \sigma_2(t) = 0$.

12.3   Linear Stochastic Differential Equations

For the solution of the equation (12.3) we expect a similar structure as in the deterministic case, (12.5), i.e. a homogeneous solution as a multiplicative factor has to be expected. Hence, we start with the solution of a homogeneous stochastic equation.

Homogeneous Solution

For $c_2(t) = \sigma_2(t) = 0$ one obtains from (12.3) the corresponding homogeneous linear equation. In doing so, we rename X and choose 1 as the starting value[3]:

$$dZ(t) = c_1(t)\,Z(t)\,dt + \sigma_1(t)Z(t)\,dW(t)\,, \quad Z(0) = 1. \tag{12.7}$$

Now, Ito's lemma (Proposition 11.1) is applied to $g(Z(t)) = \log(Z(t))$. Thus, we obtain as the solution of (12.7),

$$Z(t) = \exp\left\{\int_0^t \left(c_1(s) - \frac{1}{2}\sigma_1^2(s)\right) ds + \int_0^t \sigma_1(s)\,dW(s)\right\}, \tag{12.8}$$

see Problem 12.2. Hence, for $\sigma_1(t) = 0$ the deterministic solution from (12.6) is reproduced. The solution with an arbitrary starting value different from zero therefore reads

$$X(t) = X(0)\exp\left\{\int_0^t \left(c_1(s) - \frac{1}{2}\sigma_1^2(s)\right) ds + \int_0^t \sigma_1(s)\,dW(s)\right\}.$$

General Solution

Let us return to the solution of equation (12.3). Now, analogously to the deterministic case (12.5), let us define $Z(t)$ from (12.8) as a homogeneous solution. At the end of the section we will establish the following proposition whilst applying two versions of Ito's lemma. Two interesting, alternative proofs will be given in exercise problems.

Proposition 12.2 (Solution of Linear SDE with Variable Coefficients) *The solution of (12.3) with in t continuous deterministic coefficients is*

$$X(t) = Z(t)\left[X(0) + \int_0^t \frac{c_2(s) - \sigma_1(s)\sigma_2(s)}{Z(s)}\,ds + \int_0^t \frac{\sigma_2(s)}{Z(s)}\,dW(s)\right] \tag{12.9}$$

with the homogeneous solution

$$Z(t) = \exp\left\{\int_0^t \left(c_1(s) - \frac{1}{2}\sigma_1^2(s)\right) ds + \int_0^t \sigma_1(s)\,dW(s)\right\}.$$

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[3]The renaming justifies the assumption regarding the starting value. Consider

$$dX(t) = c_1(t)\,X(t)\,dt + \sigma_1(t)X(t)\,dW(t)\,, \quad X(0) \neq 0\,,$$

with a starting value different from zero, then by division one can normalize $Z(t) = X(t)/X(0)$.

For $\sigma_1(t) = \sigma_2(t) = 0$ we again obtain the known result of a deterministic differential equation, cf. (12.5).

Expected Value and Variance

The process defined by (12.3) reads in integral notation

$$X(t) = X(0) + \int_0^t (c_1(s)\,X(s) + c_2(s))\;ds + \int_0^t (\sigma_1(s)\,X(s) + \sigma_2(s))\;dW(s)\,.$$

Let us define the expectation function as

$$\mu_1(t) := \mathrm{E}(X(t))\,,$$

then it holds due to Propositions 8.2 (Fubini) and 10.3 that:

$$\mu_1(t) = \mathrm{E}\,(X(0)) + \int_0^t (c_1(s)\,\mathrm{E}\,(X(s)) + c_2(s))\;ds + 0$$

$$= \mu_1(0) + \int_0^t (c_1(s)\,\mu_1(s) + c_2(s))\;ds\,.$$

This corresponds exactly with the deterministic equation (12.4). Hence, the solution is known from (12.5) and one obtains the form given in Proposition 12.3. The derivation of an expression for the second moment is somewhat more complex,

$$\mu_2(t) := \mathrm{E}(X^2(t))\,,$$

see Problem 12.3.

Proposition 12.3 (Moments of the Solution of a Linear SDE)   *Under the assumptions of Proposition 12.2 it holds that*

$$\mu_1(t) = z(t)\left[\mu_1(0) + \int_0^t \frac{c_2(s)}{z(s)}\,ds\right],\quad z(t) = \exp\left\{\int_0^t c_1(s)ds\right\}\qquad (12.10)$$

and

$$\mu_2(t) = \zeta(t)\left[\mu_2(0) + \int_0^t \frac{\gamma_2(s)}{\zeta(s)}\,ds\right],\quad \zeta(t) = \exp\left\{\int_0^t \gamma_1(s)ds\right\},\qquad (12.11)$$

where

$$\gamma_1(t) = 2\,c_1(t) + \sigma_1^2(t)\,,\quad \gamma_2(t) = 2\,[c_2(t) + \sigma_1(t)\,\sigma_2(t)]\,\mu_1(t) + \sigma_2^2(t)\,.$$

Example 12.3 (Homogeneous Linear SDE (Constant Coefficients))   Since the works by Black and Scholes (1973) and Merton (1973) one assumes for the stock price $X(t)$ the model of a homogeneous linear SDE with constant coefficients (and starting value $X(0)$, cf. (1.3)):

$$dX(t) = c_1 X(t) \, dt + \sigma_1 X(t) \, dW(t) \, .$$

The solution resulting from (12.9) or rather from Proposition 12.2 is a geometric Brownian motion,

$$X(t) = X(0) \, \exp \left\{ \left(c_1 - \frac{1}{2} \sigma_1^2 \right) t + \sigma_1 \, W(t) \right\} \, .$$

This process has already been discussed in Chap. 7. With the generally derived formulas we can now recheck the moment functions from (7.9). Proposition 12.3 yields (see Problem 12.4)

$$\mu_1(t) = \mu_1(0) \, \exp \left(c_1 \, t \right) \, ,$$

$$\mu_2(t) = \mu_2(0) \, \exp \left\{ \left(2 \, c_1 + \sigma_1^2 \right) t \right\} \, .$$

Now, assume a fixed starting value $X(0)$. Then, it holds that

$$\mu_1(0) = X(0) \text{ and } \mu_2(0) = X^2(0) \, ,$$

and hence

$$\begin{aligned} \text{Var}(X(t)) &= \mu_2(t) - \mu_1^2(t) \\ &= X^2(0) \, \exp \left(2 \, c_1 \, t \right) \left(\exp \left(\sigma_1^2 \, t \right) - 1 \right) \, . \end{aligned}$$

With $X(0) = 1$, $\mu = c_1 - \frac{1}{2} \sigma_1^2$ and $\sigma = \sigma_1$ this corresponds to the notation from Chap. 7. The moments from (7.9) are indeed reproduced.   ∎

Inhomogeneous Linear SDE with Additive Noise

For $c_2(t) \neq 0$ the linear SDE is inhomogeneous. However, at the same time the increments of the Wiener process ("noise") enter into (12.3) additively, i.e. $\sigma_1(t) = 0$:

$$dX(t) = (c_1(t) \, X(t) + c_2(t)) \, dt + \sigma_2(t) \, dW(t) \, . \tag{12.12}$$

The solution results from (12.9) in Proposition 12.2 as

$$X(t) = z(t) \left[X(0) + \int_0^t \frac{c_2(s)}{z(s)} \, ds + \int_0^t \frac{\sigma_2(s)}{z(s)} \, dW(s) \right],$$ (12.13)

where $z(t)$ is a deterministic function:

$$z(t) = \exp \left\{ \int_0^t c_1(s) \, ds \right\}.$$

Note that $X(t)$, as a Stieltjes integral, is a Gaussian process due to Proposition 9.2. Its moments result correspondingly (for a fixed starting value $X(0)$). We collect these results in a corollary.

Corollary 12.1 (Additive Noise) *The solution of (12.12) with in t continuous deterministic coefficients is given by (12.13). The starting value $X(0)$ be deterministic. Then, the process is Gaussian with:*

$$\mu_1(t) = z(t) \left[X(0) + \int_0^t \frac{c_2(s)}{z(s)} \, ds \right], \quad z(t) = \exp \left\{ \int_0^t c_1(s) \, ds \right\},$$ (12.14)

$$Var(X(t)) = z^2(t) \int_0^t \left(\frac{\sigma_2(s)}{z(s)} \right)^2 \, ds.$$ (12.15)

We illustrate the corollary with the following example.

Example 12.4 (Convergence to Zero) As a concrete example, let us consider the process given by the following equation with starting value 0:

$$dX(t) = -X(t) \, dt + \frac{dW(t)}{\sqrt{1+t}}, \quad t \geq 0, \, X(0) = 0.$$

This equation is a special case of additive noise as it holds that $\sigma_1(t) = 0$. The remaining coefficient restrictions read:

$$c_1(t) = -1, \quad c_2(t) = 0, \quad \sigma_2(t) = \frac{1}{\sqrt{1+t}}.$$

What behavior is to be expected intuitively for $X(t)$? The volatility term, $\sigma_2(t)$, tends to zero with t growing; does this also hold true for the variance of the process? And $c_1(t) = -1$ implies that positive values influence the change negatively and vice versa; does the process hence fluctuate around the expectation of zero? In fact, we can show that the process with vanishing variance varies around zero and therefore

converges to zero.[4] For this we need the first two moments. These can be obtained from (12.14) and (12.15):

$$E(X(t)) = 0,$$

$$\text{Var}(X(t)) = e^{-2t} \int_0^t \frac{e^{2s}}{1+s} ds.$$

What can be learned from this about the variance for t increasing? In Problem 12.7 we show

$$\int_0^t \frac{e^{2s}}{1+s} ds \leq \frac{e^{2t}}{1+t} - 1.$$

Then, this proves $\text{Var}(X(t)) \to 0$ for $t \to \infty$. Hence, it is obvious that $X(t)$ indeed tends to zero in mean square. $\blacksquare$

Proof of Proposition 12.2

With the homogeneous solution

$$Z(t) := \exp\left\{ \int_0^t \left(c_1(s) - \frac{1}{2} \sigma_1^2(s) \right) ds + \int_0^t \sigma_1(s) \, dW(s) \right\},$$

of

$$dZ(t) = c_1(t) \, Z(t) \, dt + \sigma_1(t) Z(t) \, dW(t)$$

we define the two auxiliary quantities

$$X_1(t) := Z^{-1}(t), \quad X_2(t) := X(t).$$

Note that $X(t)$ is the process defined by (12.3) such that the differential of $X_2(t)$ is shown in (12.3). The proof suggested here uses the product rule for $d(X_1(t) X_2(t))$. However, for a valid application the derivation of $dX_1(t)$ is necessary as well.

As a first step we use Ito's lemma in the form of Proposition 11.1 in order to determine the differential for $X_1(t)$ with

$$g(Z) = Z^{-1}, \quad g'(Z) = -Z^{-2}, \quad g''(Z) = 2Z^{-3}.$$

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[4]For this purpose we do not need an explicit expression for the process which, however, can be easily obtained from (12.13) with $X(0) = 0$:

$$X(t) = e^{-t} \int_0^t \frac{e^s}{\sqrt{1+s}} dW(s).$$

The differential becomes

$$dX_1(t) = g'(Z(t))\, dZ(t) + \frac{1}{2}\, g''(Z(t))\, \sigma_1^2(t)\, Z^2(t)\, dt$$

$$= -\frac{c_1(t)\, Z(t)\, dt + \sigma_1(t) Z(t)\, dW(t)}{Z^2(t)} + \frac{2}{2} \frac{\sigma_1^2(t)\, Z^2(t)}{Z^3(t)}\, dt$$

$$= \frac{\sigma_1^2(t) - c_1(t)}{Z(t)}\, dt - \frac{\sigma_1(t)}{Z(t)}\, dW(t)$$

$$= \left(\sigma_1^2(t) - c_1(t)\right) X_1(t)\, dt - \sigma_1(t)\, X_1(t)\, dW(t)\,.$$

In a second step, we can now apply the stochastic product rule (see Eq. (11.2)) as an implication of Proposition 11.2 to the auxiliary quantities[5]:

$$d\,(X_1(t)\, X_2(t)) = X_1(t)\, dX_2(t) + X_2(t)\, dX_1(t)$$

$$- \left(\sigma_1(t)\, X_2(t) + \sigma_2(t)\right)\, \sigma_1(t)\, X_1(t)\, dt\,.$$

If the differentials $dX_1(t)$ and $dX_2(t)$ are plugged in, then some terms cancel each other such that it just remains:

$$d\,(X_1(t)\, X_2(t)) = X_1(t)\,(c_2(t)\, dt + \sigma_2(t)\, dW(t)) - \sigma_1(t)\, \sigma_2(t) X_1(t)\, dt$$

$$= \frac{c_2(t) - \sigma_1(t)\sigma_2(t)}{Z(t)}\, dt + \frac{\sigma_2(t)}{Z(t)}\, dW(t)\,.$$

Due to

$$X_1(t)\, X_2(t) = \frac{X(t)}{Z(t)}\,,$$

it follows by integrating in a third step:

$$\frac{X(t)}{Z(t)} = \frac{X(0)}{Z(0)} + \int_0^t \frac{c_2(s) - \sigma_1(s)\sigma_2(s)}{Z(s)}\, ds + \int_0^t \frac{\sigma_2(s)}{Z(s)}\, dW(s)\,.$$

As $Z(0) = 1$, we have established (12.9) and hence completed the proof. Two alternative proofs, which are again based on Ito's lemma (or implications thereof), are covered as exercise problems.

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[5]There is the risk of confusing the symbols σ_i, $i = 1, 2$, from Eq. (12.3) with the ones from Eq. (11.1). Note that the volatility of X_1 (i.e. "σ_1") is given by $-\sigma_1 X_1$ while the volatility term "σ_2" of X_2 just reads $\sigma_1 X_2 + \sigma_2$!

12.4   Numerical Solutions

Even if an analytical expression for the solution of a SDE is known, numerical solutions in the sense of simulated approximations to paths of a process are of interest. Such a simulation of a solution is, on the one hand, desired for reasons of a graphic illustration; on the other hand, in practice a whole family of numerical solutions is simulated in order to obtain a whole scenario of possible trajectories.

Euler Approximation

The interval $[0, T]$ from (12.1) is divided w.l.o.g. in n equidistant intervals of the length $\frac{T}{n}$. The corresponding partition reads:

$$0 = t_0 < t_1 = \frac{T}{n} < \ldots < t_i = \frac{iT}{n} < \ldots < t_n = T.$$

The theoretical solution from (12.2) of an arbitrary diffusion is now considered on the subinterval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$:

$$X(t_i) = X(t_{i-1}) + \int_{t_{i-1}}^{t_i} \mu(s, X(s)) \, ds + \int_{t_{i-1}}^{t_i} \sigma(s, X(s)) \, dW(s).$$

This allows for the following approximation[6] as it is discussed e.g. in Mikosch (1998):

$$X(t_i) \approx X(t_{i-1})$$
$$+ \int_{t_{i-1}}^{t_i} \mu(t_{i-1}, X(t_{i-1})) \, ds + \int_{t_{i-1}}^{t_i} \sigma(t_{i-1}, X(t_{i-1})) \, dW(s),$$

which can also be written as:

$$X(t_i) \approx X(t_{i-1})$$
$$+ \mu(t_{i-1}, X(t_{i-1})) \frac{T}{n} + \sigma(t_{i-1}, X(t_{i-1})) (W(t_i) - W(t_{i-1})).$$

For this purpose

$$\int_{t_{i-1}}^{t_i} ds = t_i - t_{i-1} = \frac{T}{n} \quad \text{and} \quad \int_{t_{i-1}}^{t_i} dW(s) = W(t_i) - W(t_{i-1})$$

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[6] In the literature, one speaks of an Euler approximation. An improvement is known under the keyword Milstein approximation. In order to explain what is meant by "improve" in this case, one would have to become more involved in numerics.

was used. Hence, we have a recursive scheme. Given $X_0 = X(0)$ one calculates for $i = 1$:

$$X_1 = X_0 + \mu(0, X_0) \frac{T}{n} + \sigma(0, X_0) \left(W \left(\frac{T}{n} \right) - W(0) \right),$$

and in general, for $i = 1, \ldots, n$:

$$X_i = X_{i-1} + \mu(t_{i-1}, X_{i-1}) \frac{T}{n} + \sigma(t_{i-1}, X_{i-1}) (W(t_i) - W(t_{i-1})). \qquad (12.16)$$

Thus we obtain n observations X_i (i.e. $n + 1$ observations including the starting value), with which a path of the continuous-time process $X(t)$ on $[0, T]$ is simulated. However, this simulation requires Gaussian pseudo random numbers in (12.16),

$$W(t_i) - W(t_{i-1}) = W \left(\frac{iT}{n} \right) - W \left(\frac{(i-1)T}{n} \right) \sim \text{ii} \mathcal{N} \left(0, \frac{T}{n} \right).$$

For this purpose, a series of stochastically independent $\mathcal{N} \left(0, \frac{T}{n} \right)$-distributed random variables ε_i need to be simulated instead of $W(t_i) - W(t_{i-1})$, in order to obtain a numerical solution X_i, $i = 1, \ldots, n$ for the diffusion $X(t)$ from (12.2) according to (12.16). Naturally, with n growing the approximation of a numerical solution improves.

12.5 Problems and Solutions

Problems

12.1 Show that the function given in (12.5) solves the deterministic differential equation (12.4). How does it look like in the case of constant coefficients?

12.2 Show that $Z(t)$ from (12.8) solves the homogeneous SDE (12.7) with $Z(0) = 1$.

Hint: See the text.

12.3 Prove (12.11) from Proposition 12.3.

Hint: Determine for $g(X(t)) = X^2(t)$ an expression with Ito's lemma.

12.4 Derive the expectation and the variance of the geometric Brownian motion with Proposition 12.3,

$$X(t) = X(0) \exp \left\{ \left(c_1 - \frac{1}{2} \sigma_1^2 \right) t + \sigma_1 W(t) \right\}.$$

12.5 Determine the process $X(t)$ for which it holds that:

$$dX(t) = X(t)\,dW(t)\,, \quad X(0) = 1\,.$$

Hint: Proposition 12.2.

12.6 Find the solution of

$$dX(t) = \frac{-X(t)}{1+t}\,dt + \frac{dW(t)}{1+t}\,, \quad t \geq 0\,,$$

for $X(0) = 0$. Show that it tends to zero in mean square.
Hint: Proposition 12.2.

12.7 Show for the Example 12.4:

$$\int_0^t \frac{e^{2s}}{1+s}\,ds \; \leq \; \frac{e^{2t}}{1+t} - 1\,.$$

12.8 Determine the solution of

$$dX(t) = -\frac{X(t)}{1-t}\,dt + dW(t)\,, \quad 0 \leq t < 1\,,$$

with $X(0) = 0$. Show that $\text{Var}(X(t)) = (1-t)\,t$ and hence that $X(t)$ tends to zero in mean square for $t \to 1$. (This reminds us of the Brownian bridge, see (7.6). In fact, the above SDE defines a Brownian bridge, cf. Grimmett & Stirzaker, 2001, p. 535.)

12.9 Prove Proposition 12.2 by directly applying Proposition 11.2.
Hint: Choose $g(X, Z) = X/Z$.

12.10 Prove Proposition 12.2 with the quotient rule from (11.5).
Hint: First derive the quotient rule for the one-factor case ($d = 1$) as a special case of (11.5).

Solutions

12.1 The solution from (12.5) reads

$$x(t) = z(t)\left[x(0) + \int_0^t \frac{c_2(s)}{z(s)}\,ds\right]$$

with

$$z(t) = \exp\left\{\int_0^t c_1(s)ds\right\}.$$

Let us define the square bracket as $b(t)$:

$$b(t) = \left[x(0) + \int_0^t \frac{c_2(s)}{z(s)}ds\right] = \frac{x(t)}{z(t)}$$

and

$$b'(t) = \frac{c_2(t)}{z(t)}.$$

The derivative of $z(t)$ is

$$z'(t) = c_1(t) \exp\left\{\int_0^t c_1(s)ds\right\} = c_1(t)z(t).$$

Hence, the product rule yields:

$$x'(t) = z'(t) b(t) + z(t) b'(t)$$
$$= c_1(t) z(t) \frac{x(t)}{z(t)} + z(t) \frac{c_2(t)}{z(t)}$$
$$= c_1(t) x(t) + c_2(t),$$

which just corresponds to the claim.

In the case of constant coefficients, $x(t)$ from (12.5) with $z(t) = e^{c_1 t}$ becomes:

$$x(t) = e^{c_1 t}[x(0) + c_2 \int_0^t e^{-c_1 s}ds]$$
$$= e^{c_1 t}[x(0) - \frac{c_2}{c_1}(e^{-c_1 t} - 1)]$$
$$= e^{c_1 t}\left(x(0) + \frac{c_2}{c_1}\right) - \frac{c_2}{c_1}.$$

12.2 For $Z(t)$ from (12.7) it holds that

$$dZ(t) = \mu(t, Z(t))dt + \sigma(t, Z(t))dW(t)$$

with

$$\mu(t, Z(t)) = c_1(t)Z(t), \quad \sigma(t, Z(t)) = \sigma_1(t)Z(t).$$

Therefore, Proposition 11.1 yields:

$$dg(Z(t)) = g'(Z(t))dZ(t) + \frac{1}{2}g''(Z(t))\sigma^2(t, Z(t))dt.$$

With

$$g(x) = \log(x), \quad g'(x) = \frac{1}{x}, \quad g''(x) = -\frac{1}{x^2},$$

we hence obtain

$$d\log(Z(t)) = \frac{\mu(t, Z(t))dt + \sigma(t, Z(t))dW(t)}{Z(t)} - \frac{1}{2}\frac{\sigma^2(t, Z(t))}{Z^2(t)}dt$$

$$= c_1(t)dt + \sigma_1(t)dW(t) - \frac{\sigma_1^2(t)}{2}dt.$$

Integration yields

$$\log(Z(t)) = \log(Z(0)) + \int_0^t \left(c_1(s) - \frac{\sigma_1^2(s)}{2}\right)ds + \int_0^t \sigma_1(s)dW(s).$$

Because of $Z(0) = 1$, the exponential function yields as desired:

$$Z(t) = \exp\left\{\int_0^t \left(c_1(s) - \frac{\sigma_1^2(s)}{2}\right)ds + \int_0^t \sigma_1(s)dW(s)\right\}.$$

12.3 Proposition 11.1 with

$$g(x) = x^2, \quad g'(x) = 2x, \quad g''(x) = 2$$

is applied to X^2 where the differential $dX(t)$ is given by Eq. (12.3). This leads to

$$dX^2(t) = 2X(t)dX(t) + (\sigma_1(t)X(t) + \sigma_2(t))^2 dt$$

$$= [2X(t)(c_1(t)X(t) + c_2(t)) + (\sigma_1(t)X(t) + \sigma_2(t))^2]dt$$

$$+ 2X(t)(\sigma_1(t)X(t) + \sigma_2(t))dW(t).$$

As an integral equation this reads as follows:

$$X^2(t) = X^2(0) + \int_0^t \left[2X(s)(c_1(s)X(s) + c_2(s)) + (\sigma_1(s)X(s) + \sigma_2(s))^2\right]ds$$

$$+ 2\int_0^t X(s)(\sigma_1(s)X(s) + \sigma_2(s))dW(s).$$

The expectation of the second integral is zero due to Proposition 10.3. The expectation of the first integral results due to Fubini's theorem as

$$\int_0^t \mathrm{E}\left[2c_1(s)X^2(s) + 2c_2(s)X(s) + \sigma_1^2(s)X^2(s) + 2\sigma_1(s)\sigma_2(s)X(s) + \sigma_2^2(s)\right] ds.$$

With the definition of $\mu_1(s)$ and $\mu_2(s)$ it thus follows that

$$\mu_2(t) = \mu_2(0) + \int_0^t \left[\left(2\,c_1(s) + \sigma_1^2(s)\right)\mu_2(s)\right] ds$$

$$+ 2\int_0^t \left[\left(c_2(s) + \sigma_1(s)\,\sigma_2(s)\right)\mu_1(s) + \sigma_2^2(s)\right] ds.$$

In differential notation this equation reads

$$d\mu_2(t) = \left(\gamma_1(t)\,\mu_2(t) + \gamma_2(t)\right)\,dt,$$

where the functions $\gamma_1(t)$ and $\gamma_2(t)$ in the proposition following (12.11) were adequately defined. Therefore, the second moment results as the solution of a deterministic differential equation of the form (12.4). Its solution can be found in (12.5). Hence, the proposition is verified.

12.4 The geometric Brownian motion solves the homogeneous linear equation with constant coefficients:

$$c_2(t) = \sigma_2(t) = 0, \quad c_1(t) = c_1 = \text{const.}, \quad \sigma_1(t) = \sigma_1 = \text{const.}$$

For a stochastic starting value it holds that:

$$\mu_1(0) = \mathrm{E}(X(0)), \quad \mu_2(0) = \mathrm{E}(X^2(0)).$$

By plugging in, Proposition 12.3 yields

$$\mu_1(t) = e^{c_1 t}[\mu_1(0) + 0] = \mu_1(0)\,\exp(c_1 t).$$

With the definitions from Proposition 12.3 one determines

$$\gamma_1(t) = 2\,c_1 + \sigma_1^2 =: \gamma_1 \quad \text{and} \quad \gamma_2(t) = 0.$$

Hence, substitution yields

$$\mu_2(t) = e^{\gamma_1 t}[\mu_2(0) + 0] = \mu_2(0)\,\exp(\gamma_1 t).$$

Thus, the variance is calculated as

$$\text{Var}(X(t)) = \mu_2(t) - \mu_1^2(t)$$
$$= \mu_2(0) \exp\{(2c_1 + \sigma_1^2)t\} - \mu_1^2(0) \exp\{2c_1 t\}.$$

12.5 The equation at hand is linear with

$$c_1(t) = c_2(t) = 0, \quad \sigma_2(t) = 0.$$

Furthermore, it holds that

$$\sigma_1(t) = 1.$$

Therefore, the solution deduced from Proposition 12.2 reads:

$$X(t) = Z(t)[X(0) + 0]$$

with

$$Z(t) = \exp\left\{-\frac{t}{2} + W(t)\right\}.$$

In particular for $X(0) = 1$ (analogously to $e^0 = 1$) it hence holds that:

$$X(t) = \exp\left\{W(t) - \frac{t}{2}\right\}.$$

Due to the analogy to $de^t = e^t\, dt$ with $e^0 = 1$ this process $X(t)$ is sometimes called "Ito exponential". It is noteworthy that the Ito exponential is not given by $\exp\{W(t)\}$.

12.6 The equation is linear, see Eq. (12.3), and corresponds to the special case of additive noise, cf. (12.12), i.e. $\sigma_1(t) = 0$. The remaining coefficients read:

$$c_1(t) = -\frac{1}{1+t}, \quad c_2(t) = 0, \quad \sigma_2(t) = \frac{1}{1+t}.$$

Hence, the expression for the solution from (12.13) yields with $X(0) = 0$:

$$X(t) = z(t) \int_0^t \frac{1}{(1+s)z(s)} dW(s),$$

where

$$z(t) = \exp\left\{-\int_0^t (1+s)^{-1} ds\right\}$$
$$= \exp\{-[\log(1+s)]_0^t\}$$

$$= \exp\{-\log(1 + t) + 0\}$$

$$= \exp\left\{\log\left(\frac{1}{1+t}\right)\right\} = \frac{1}{1+t}.$$

Since $\sigma_2(t)/z(t) = 1$, the solution simplifies radically:

$$X(t) = \frac{1}{1+t} \int_0^t dW(s) = \frac{W(t)}{1+t}.$$

For this solution it obviously holds that:

$$E(X(t)) = 0$$

$$\text{Var}(X(t)) = \frac{t}{(1+t)^2} \to 0, \quad t \to \infty.$$

Thus, for $t \to \infty$ we have established

$$\text{MSE}(X(t), 0) = E[(X(t) - 0)^2] \to 0$$

which just corresponds to the required convergence in mean square.

12.7 In order to prove the inequality claimed, we define the function

$$g(s) = \frac{1}{2} \frac{e^{2s}}{1+s}$$

with the derivative (quotient rule)

$$g'(s) = \frac{e^{2s}}{1+s} - \frac{1}{2} \frac{e^{2s}}{(1+s)^2}.$$

Let us call the integral of interest I,

$$I = \int_0^t \frac{e^{2s}}{1+s} ds.$$

Then it follows

$$I = \int_0^t g'(s)ds + \frac{1}{2} \int_0^t \frac{e^{2s}}{(1+s)^2} ds$$

$$= g(t) - g(0) + \frac{1}{2} \int_0^t \frac{e^{2s}}{(1+s)^2} ds$$

$$\leq g(t) - g(0) + \frac{1}{2} I,$$

where the bound follows from $(1 + s) \leq (1 + s)^2$. By rearranging terms it results that

$$I \leq 2(g(t) - g(0)).$$

With the definition of g it follows

$$I \leq \frac{e^{2t}}{(1 + t)} - 1,$$

which was to be shown.

12.8 This is again an inhomogeneous linear equation with additive noise:

$$c_1(t) = -\frac{1}{1 - t}, \quad c_2(t) = 0, \quad \sigma_1(t) = 0, \quad \sigma_2(t) = 1.$$

With $X(0) = 0$, $X(t)$ from (12.13) turns out to be:

$$X(t) = z(t) \int_0^t (z(s))^{-1} \, dW(s)$$

with

$$\begin{aligned}
z(t) &= \exp\left\{ -\int_0^t \frac{1}{1 - s} \, ds \right\} \\
&= \exp\left\{ [\log(1 - s)]_0^t \right\} \\
&= 1 - t,
\end{aligned}$$

i.e.

$$X(t) = (1 - t) \int_0^t \frac{1}{1 - s} \, dW(s).$$

Due to $c_2(t) = X(0) = 0$, (12.14) yields:

$$E(X(t)) = 0.$$

Due to (12.15), the variance is:

$$\begin{aligned}
\mathrm{Var}(X(t)) &= (1 - t)^2 \int_0^t \frac{1}{(1 - s)^2} \, ds \\
&= (1 - t)^2 \left[\frac{1}{(1 - s)} \right]_0^t
\end{aligned}$$

$$= (1-t)^2 \left(\frac{1}{1-t} - 1 \right) =$$

$$= (1-t)^2 \frac{t}{1-t} = (1-t)t.$$

For $t \to 1$ the variance shrinks to zero such that it holds that $X(t)$ tends to 0 in mean square:

$$\text{MSE}(X(t), 0) = \text{Var}(X(t)) = \text{E}[(X(t) - 0)^2] \to 0.$$

12.9 The key problem with this exercise is not to confuse the different meanings of $\sigma_i(t)$, $i = 1, 2$, in Proposition 11.2 and Eq. (12.3). Hence, firstly we adapt Proposition 11.2 for the processes $X(t)$ from (12.3) and $Z(t)$ from (12.7):

$$dX(t) = \mu_x(t)\, dt + \sigma_x(t)\, dW(t)\,,$$

$$\mu_x(t) = c_1(t)\, X(t) + c_2(t)\,, \quad \sigma_x(t) = \sigma_1(t)\, X(t) + \sigma_2(t)\,,$$

$$dZ(t) = \mu_z(t)\, dt + \sigma_z(t)\, dW(t)\,,$$

$$\mu_z(t) = c_1(t)\, Z(t)\,, \quad \sigma_z(t) = \sigma_1(t)\, Z(t)\,.$$

Following the hint, we consider

$$g(X, Z) = \frac{X}{Z} = X Z^{-1}$$

with

$$\frac{\partial g}{\partial X} = Z^{-1}\,, \quad \frac{\partial^2 g}{\partial X^2} = 0$$

$$\frac{\partial g}{\partial Z} = -X Z^{-2}\,, \quad \frac{\partial^2 g}{\partial Z^2} = 2 X Z^{-3}\,, \quad \frac{\partial^2 g}{\partial X\, \partial Z} = -Z^{-2}\,.$$

Hence, Ito's lemma (Proposition 11.2) yields:

$$d\left(\frac{X}{Z} \right) = Z^{-1}\, dX - X Z^{-2}\, dZ + \frac{1}{2} \left[0 + 2 X Z^{-3}\, \sigma_z^2 \right] dt - Z^{-2} \sigma_x \sigma_z\, dt$$

$$= Z^{-1}\, (c_1 X + c_2)\, dt + Z^{-1}\, (\sigma_1 X + \sigma_2)\, dW - X Z^{-2} c_1 Z\, dt$$

$$- X Z^{-2} \sigma_1 Z\, dW + X Z^{-3} \sigma_1^2 Z^2\, dt - Z^{-2} (\sigma_1 X + \sigma_2)\, \sigma_1 Z\, dt$$

$$= \left(Z^{-1} c_2 - Z^{-1} \sigma_1 \sigma_2 \right) dt + Z^{-1} \sigma_2\, dW.$$

Integration yields:

$$\frac{X(t)}{Z(t)} = \frac{X(0)}{Z(0)} + \int_0^t \frac{c_2(s) - \sigma_1(s)\sigma_2(s)}{Z(s)} \, ds + \int_0^t \frac{\sigma_2(s)}{Z(s)} \, dW(s) \, .$$

If this equation is multiplied by $Z(t)$, then, due to $Z(0) = 1$, one obtains the desired result.

12.10 We apply (11.5) with

$$X_1 = X \quad \text{and} \quad X_2 = Z \, ,$$

where X and Z are driven by the same Wiener process, say $W_1 = W$. Then the one-factor quotient rule is obtained by the following restrictions:

$$\sigma_{11} = \sigma_x \quad \text{and} \quad \sigma_{12} = 0 \, ,$$

$$\sigma_{21} = \sigma_z \quad \text{and} \quad \sigma_{22} = 0 \, .$$

For this purpose, σ_x and σ_z were defined in the previous problem. Then, the one-factor quotient rule yields:

$$\begin{aligned}
d\left(\frac{X}{Z}\right) &= \frac{ZdX - XdZ}{Z^2} + \frac{XZ^{-1}\sigma_z^2 - \sigma_x\sigma_z}{Z^2} \, dt \\
&= \frac{Z(c_1X + c_2)\, dt + Z(\sigma_1 X + \sigma_2)\, dW - Xc_1 Z\, dt - X\sigma_1 Z\, dW}{Z^2} \\
&\quad + \frac{XZ^{-1}\sigma_1^2 Z^2 - (\sigma_1 X + \sigma_2)\sigma_1 Z}{Z^2} \, dt \\
&= \frac{Z(c_2 - \sigma_1\sigma_2)}{Z^2} \, dt + \frac{Z\sigma_2}{Z^2} \, dW \\
&= \frac{(c_2 - \sigma_1\sigma_2)}{Z} \, dt + \frac{\sigma_2}{Z} \, dW \, .
\end{aligned}$$

As before, we obtain the desired result by integration and multiplication by $Z(t)$.

References

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy, 81*, 637–654.

Grimmett, G. R., & Stirzaker, D. R. (2001). *Probability and random processes* (3rd ed.). Oxford: Oxford University Press.

Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science, 4*, 141–183.

Mikosch, Th. (1998). *Elementary stochastic calculus with finance in view*. Singapore: World Scientific Publishing.

Øksendal, B. (2003). *Stochastic differential equations: An introduction with applications* (6th ed.). Berlin/New York: Springer.