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Another Example: Successive Polarization Filters for Beams of Spin $s = \frac{1}{2}$ Particles

So far, our first example of a unitary transformation from one basis to another involved a finite-dimensional unitary submatrix. Let us consider one more example of this type, an even simpler example involving spin $s = \frac{1}{2}$ particles, hence, a 2×2-dimensional transformation. Suppose we have a beam of spin $s = \frac{1}{2}$ particles. They can be prepared, so all are in a state of definite spin orientation, say, with $m_s = +\frac{1}{2}$, or with $m_s = -\frac{1}{2}$, along some specific z-direction in 3-D space by passing the beam through a polarization filter. The historically first such filter is that employed by Stern and Gerlach involving a set of three magnets, with nonuniform magnetic fields, placed in succession along the beam line, so a set of baffles can eliminate the particles with one of the two spin orientations. Other types of sophisticated polarization filters exist. (For a reference to modern polarization filters, see, e.g., *Polarized Beams and Polarized Gas Targets*, Hans Paetz gen. Schieck and Lutz Sydow, eds. World Scientific, 1996). We will assume the filter is perfect and prepares particles in a pure state of very definite m_s along a specific z-direction. Suppose the first such filter is followed with a second filter, identical to the first, but now with its new z' axis oriented along some new direction, given by polar and azimuth angles, θ and ϕ, relative to the original x, y, z axes, and set for some definite m'_s along the new direction. What fraction of the $s = \frac{1}{2}$-particles will pass through the second filter?

The first filter prepares particles in eigenstates $|m = \pm\frac{1}{2}\rangle$, which are eigenstates of $\vec{S}^2$ and S_z:

$$\vec{S}^2|\tfrac{1}{2}m\rangle = \tfrac{3}{4}|\tfrac{1}{2}m\rangle, \qquad S_z|\tfrac{1}{2}m\rangle = m|\tfrac{1}{2}m\rangle. \tag{1}$$

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The second filter passes particles in the eigenstates $|\alpha = \pm\frac{1}{2}\rangle$, which are eigenstates of $\vec{S}^2$ and $S_{z'}$:

$$\vec{S}^2|\tfrac{1}{2}\alpha\rangle = \tfrac{3}{4}|\tfrac{1}{2}\alpha\rangle, \qquad S_{z'}|\tfrac{1}{2}\alpha\rangle = \alpha|\tfrac{1}{2}\alpha\rangle. \tag{2}$$

The answer to our problem is as follows: The probability a particle in the beam with definite $|\frac{1}{2}m\rangle$ will pass through the second filter set to pass a specific $|\frac{1}{2}\alpha\rangle$ is

$$P(m,\alpha) = |\langle\tfrac{1}{2}\alpha|\tfrac{1}{2}m\rangle|^2, \tag{3}$$

so we need to calculate the transformation coefficient $\langle\frac{1}{2}\alpha|\frac{1}{2}m\rangle = U_{\alpha m}$. It will be advantageous to switch to the operator $\vec{\sigma}$, via $\vec{S} = \frac{1}{2}\vec{\sigma}$, and to omit the common s quantum number of $\frac{1}{2}$ in all equations. Here, $\vec{\sigma}$ is the Pauli spin operator, whose components $\sigma_x, \sigma_y, \sigma_z$ lead to the three Pauli spin matrices, we have already met through eq. (38) of Chapter 14. Thus,

$$\sigma_z|m\rangle = \lambda_m|m\rangle, \qquad \sigma_{z'}|\alpha\rangle = \lambda_\alpha|\alpha\rangle, \tag{4}$$

with $\lambda_m = \pm 1$ for states with $m = \pm\frac{1}{2}$, and $\lambda_\alpha = \pm 1$ for states with $\alpha = \pm\frac{1}{2}$. Now, we shall rewrite the relation $\sigma_{z'}|\alpha\rangle = \lambda_\alpha|\alpha\rangle$ as

$$\sum_m \sigma_{z'}|m\rangle\langle m|\alpha\rangle = \lambda_\alpha|\alpha\rangle. \tag{5}$$

Left-multiplying by a specific $\langle m'|$ leads to

$$\sum_m \langle m'|\sigma_{z'}|m\rangle\langle m|\alpha\rangle = \lambda_\alpha\langle m'|\alpha\rangle. \tag{6}$$

Now we can express $\sigma_{z'}$ in terms of the original x, y, z components of $\vec{\sigma}$

$$\sigma_{z'} = \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z, \tag{7}$$

and use the 2×2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix},$$

to evaluate

$$\langle m'|\sigma_{z'}|m\rangle = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

Then, eq. (6) can be rewritten in matrix form, where we shall also use the shorthand notation $\langle m|\alpha\rangle = c_\pm$ for c_m with $m = \pm\frac{1}{2}$,

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \lambda_\alpha\begin{pmatrix} c_+ \\ c_- \end{pmatrix},$$

leading to the two linear equations

$$\begin{aligned} (\cos\theta - \lambda_\alpha)c_+ + \sin\theta e^{-i\phi}c_- &= 0, \\ \sin\theta e^{i\phi}c_+ - (\cos\theta + \lambda_\alpha)c_- &= 0. \end{aligned} \tag{8}$$

For $\lambda_\alpha = +1$, these equations lead to

$$\frac{c_+}{c_-} = \frac{\sin\theta e^{-i\phi}}{(1-\cos\theta)} = \frac{(1+\cos\theta)}{\sin\theta e^{i\phi}} = \frac{\cos(\frac{\theta}{2})e^{-i\frac{\phi}{2}}}{\sin(\frac{\theta}{2})e^{+i\frac{\phi}{2}}}, \tag{9}$$

so, for $\lambda = +1$:

$$c_+ = \cos(\frac{\theta}{2})e^{-i\frac{\phi}{2}}, \qquad c_- = \sin(\frac{\theta}{2})e^{+i\frac{\phi}{2}}, \tag{10}$$

where the undetermined normalization factor has been chosen, so $\sum_m |c_m|^2 = 1$. These two numbers give us the first column of the unitary 2×2 matrix $\langle m|\alpha\rangle$, with $\alpha = +\frac{1}{2}$. In the same way, putting $\lambda_\alpha = -1$, we get the second column of the $\langle m|\alpha\rangle$ matrix with $\alpha = -\frac{1}{2}$ to give

$$\langle m|\alpha\rangle = \begin{pmatrix} \cos(\frac{\theta}{2})e^{-i\frac{\phi}{2}} & -\sin(\frac{\theta}{2})e^{-i\frac{\phi}{2}} \\ \sin(\frac{\theta}{2})e^{i\frac{\phi}{2}} & \cos(\frac{\theta}{2})e^{i\frac{\phi}{2}} \end{pmatrix}.$$

In our notation, this is the matrix for $U^\dagger$, viz., $U^\dagger_{m\alpha}$. To obtain the matrix for U, $\langle\alpha|m\rangle = U_{\alpha m}$, we need to transpose and complex conjugate the above matrix to get

$$\langle\alpha|m\rangle = \begin{pmatrix} \cos(\frac{\theta}{2})e^{+i\frac{\phi}{2}} & \sin(\frac{\theta}{2})e^{-i\frac{\phi}{2}} \\ -\sin(\frac{\theta}{2})e^{i\frac{\phi}{2}} & \cos(\frac{\theta}{2})e^{-i\frac{\phi}{2}} \end{pmatrix}.$$

Finally, this U matrix can be written as

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} = e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}.$$

The last operator form of this unitary transformation follows, for the z component, from

$$\mathbf{1} = \sigma_z^2 = \sigma_z^4 = \cdots = \sigma_z^{2n},$$

$$\sigma_z = \sigma_z^3 = \sigma_z^5 = \cdots = \sigma_z^{2n+1}, \tag{11}$$

and, for the y component, from the similar relation

$$\mathbf{1} = \sigma_y^2 = \sigma_y^4 = \cdots \sigma_y^{2n},$$

$$\sigma_y = \sigma_y^3 = \sigma_y^5 = \cdots \sigma_y^{2n+1}, \tag{12}$$

so

$$e^{i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \left(\frac{\theta}{2}\right)^2 \frac{1}{2!} + \left(\frac{\theta}{2}\right)^4 \frac{1}{4!} + \cdots\right)$$

$$+ i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \left(\left(\frac{\theta}{2}\right) - \left(\frac{\theta}{2}\right)^3 \frac{1}{3!} + \left(\frac{\theta}{2}\right)^5 \frac{1}{5!} + \cdots\right).$$

Finally, $U^\dagger$ can be written in similar operator form as

$$U^\dagger = e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y}. \tag{13}$$

Also, note the appearance of the half-angles, associated with the $s = \frac{1}{2}$ character of the particles. Thus, e.g., if both polarization filters are set for the spin-projection $m = +\frac{1}{2}$, the fraction of the incoming particles that will pass through the second filter is

$$P(m = +\tfrac{1}{2}, \alpha = +\tfrac{1}{2}) = \cos^2(\frac{\theta}{2}). \tag{14}$$