5

Harmonic Oscillator Calculations

A The Bargmann Transform

For many calculations involving 1-D harmonic oscillator wave functions, it is useful to introduce the Bargmann transform through the kernel function

$$A(k, x) = \frac{1}{\pi^{\frac{1}{4}}} exp(-\tfrac{1}{2}k^2 + \sqrt{2}kx - \tfrac{1}{2}x^2), \tag{1}$$

where k is a complex number. Given a square-integrable function, $\psi(x)$, its Bargmann transform, $F(k)$, is given by

$$F(k) = \int_{-\infty}^{\infty} dx \psi(x) A(k, x), \tag{2}$$

where

$$\psi(x) = \frac{1}{\pi} \int d^2k e^{-kk^*} A(k^*, x) F(k), \tag{3}$$

and the integral is over the 2-D complex k-plane; i.e., with $k = a + ib$,

$$\int d^2k \equiv \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db. \tag{4}$$

Now, from the definition of $A(k, x)$ and the generating function definition for the Hermite polynomials, with $s = k/\sqrt{2}$,

$$A(k, x) = \sum_{n=0} \left(\frac{H_n(x)}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-\frac{1}{2}x^2} \right) \frac{k^n}{\sqrt{n!}} = \sum_{n=0} \psi_n(x) \frac{k^n}{\sqrt{n!}}. \tag{5}$$

We therefore see

$$\psi_n(x) \qquad \text{has} \quad \text{Bargmann} \quad \text{transform} \qquad \frac{k^n}{\sqrt{n!}}. \tag{6}$$

We can transform a scalar product from x-space into k-space, or from k-space into x-space:

$$\begin{aligned}
&\frac{1}{\pi}\int d^2k e^{-kk^*} F_1^*(k)F_2(k) \\
&= \frac{1}{\pi}\int d^2k e^{-kk^*} \int dx \psi^{(1)}(x)^* A(k^*, x) \int dx' \psi^{(2)}(x') A(k, x') \\
&= \int dx \psi^{(1)}(x)^* \psi^{(2)}(x),
\end{aligned} \tag{7}$$

where we have used, again with $k = a + ib$,

$$\begin{aligned}
&\frac{1}{\pi}\int d^2k e^{-kk^*} A(k^*, x)A(k, x') \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} da e^{-[\sqrt{2}a - \frac{(x-x')}{2}]^2} e^{-[\frac{(x-x')}{2}]^2} \frac{1}{\pi}\int_{-\infty}^{\infty} db e^{i\sqrt{2}b(x'-x)} \\
&= e^{-[\frac{(x-x')}{2}]^2} \frac{1}{2\pi}\int_{-\infty}^{\infty} db' e^{ib'(x'-x)} = \delta(x' - x).
\end{aligned} \tag{8}$$

In the last step, we have used

$$f(x)\delta(x) = f(0)\delta(x). \tag{9}$$

Eq. (7) thus permits us to evaluate a scalar product either in x-space or in k-space. At times, the latter may lead to the easier integral.

B Completeness Relation

The delta-function property of the integral of eq. (8) is also useful to prove the completeness of the set of harmonic oscillator eigenfunctions $\psi_n(x)$. An arbitrary 1-D square-integrable function, $\Psi(x)$, can be expanded in a generalized Fourier series in oscillator eigenfunctions, $\psi_n(x)$,

$$\Psi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{10}$$

with coefficient, c_n, evaluated by the Fourier inversion theorem

$$c_n = \int_{-\infty}^{\infty} dx' \Psi(x'), \ \psi_n^*(x'); \tag{11}$$

so

$$\Psi(x) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx' \Psi(x') \psi_n(x) \psi_n^*(x'). \tag{12}$$

This requires

$$\sum_{n=0}^{\infty} \psi_n(x)\psi_n^*(x') = \delta(x - x'). \tag{13}$$

This is the completeness relation we want to prove. Substituting eq. (5) into the left-hand side of eq. (8), we get

$$\frac{1}{\pi}\int d^2k e^{-kk^*} \sum_{n=0}^{\infty} \psi_n^*(x)\frac{k^{*n}}{\sqrt{n!}} \sum_{m=0}^{\infty} \psi_m(x')\frac{k^m}{\sqrt{m!}} =$$
$$\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\psi_n^*(x)\psi_m(x')}{\sqrt{n!m!}} \int d^2k e^{-kk^*} k^{*n}k^m. \tag{14}$$

Doing the integral in polar coordinates, with $k = \rho e^{i\phi}$,

$$\int d^2k e^{-k^*k} k^{*n}k^m = \int_0^{\infty} d\rho\rho e^{-\rho^2}\rho^{n+m} \int_0^{2\pi} d\phi e^{i(m-n)\phi}$$
$$= \int_0^{\infty} d\rho\rho e^{-\rho^2}\rho^{n+m} 2\pi\delta_{nm} = \pi n!\delta_{nm}. \tag{15}$$

Combining eqs. (8) and (14) leads to the needed completeness relation given by eq. (13).

C A Second Useful Application: The matrix $(x)_{nm}$

As a second example, we will use the Bargmann kernel to calculate the following useful integral

$$\int_{-\infty}^{\infty} dx \psi_n^*(x) x \psi_m(x) =< \psi_n^*, x\psi_m >\equiv (x)_{nm}. \tag{16}$$

With $m = n$ this integral would be needed to calculate the expectation value of x in the n^{th} eigenstate. Because in that case the integrand is an odd function of x, this expectation value is zero. The particle is equally likely to be in the right half or the left half of the x domain. For general n, m, the two-index quantity defined as $(x)_{nm}$ in eq. (16) will be shown to be a matrix in Chapter 6. For general n, m, we can evaluate the needed integral by considering the integral

$$\int_{-\infty}^{\infty} dx A(l^*, x) x A(k, x) \tag{17}$$

as a function of the arbitrary complex parameters k and l^* in two ways

$$\int_{-\infty}^{\infty} dx A(l^*, x) x A(k, x) = \sum_{n,m} \frac{l^{*n}}{\sqrt{n!}}\frac{k^m}{\sqrt{m!}} \int_{-\infty}^{\infty} dx \psi_n^*(x) x \psi_m(x)$$
$$= \frac{1}{\sqrt{\pi}} e^{l^*k} \int_{-\infty}^{\infty} dx x e^{-[x-\frac{1}{\sqrt{2}}(l^*+k)]^2} = \frac{1}{\sqrt{\pi}} e^{l^*k} \int_{-\infty}^{\infty} dx' \big(x' + \frac{1}{\sqrt{2}}(l^* + k)\big) e^{-x'^2}$$

$$= \frac{1}{\sqrt{2}}(l^* + k)e^{l^*k} = \frac{1}{\sqrt{2}}\sum_{n=0} \frac{(l^{*n+1}k^n + l^{*n}k^{n+1})}{n!}$$
$$= \frac{1}{\sqrt{2}}\left(\sum_{n=1} \frac{l^{*n}k^{n-1}}{(n-1)!} + \sum_{n=0} \frac{l^{*n}k^{n+1}}{n!}\right). \tag{18}$$

The only surviving terms are those in which the powers of k, viz., m, differ from n by ± 1. Therefore

$$\int_{-\infty}^{\infty} dx \psi_n^*(x) x \psi_m(x) = 0 \qquad \text{for} \quad m \neq (n \pm 1)$$
$$= \sqrt{\frac{(n+1)}{2}} \quad \text{for} \quad m = (n+1)$$
$$= \sqrt{\frac{n}{2}} \quad \text{for} \quad m = (n-1). \tag{19}$$

Problems

9. For the 1-D harmonic oscillator, calculate all nonzero matrix elements of q^2, p, and p^2 (for the nonzero matrix elements of q, see eq. (19) above). For a general state

$$\Psi(q, t) = \sum_n c_n \psi_n(q) e^{-\frac{i}{\hbar}E_n t},$$

calculate $< q >, < q^2 >, < p >, < p^2 >$, Δq, and Δp as functions of the c_n. Try to determine values of c_n for which the product $(\Delta p)(\Delta q)$ is a minimum. (Hint: Try $c_0 = 1$, all other $c_n = 0$.) For the special case

$$c_0 = \frac{1}{\sqrt{2}}, \qquad c_1 = \frac{i}{\sqrt{2}},$$

calculate $< q >, < q^2 >$, and Δq as functions of the time, t. For this special case, also calculate $\vec{S}$; that is, calculate as a function of q and t the probability per unit time and unit area normal to the displacement, q, that the particle is streaming in the direction of q.

10. A particle of mass, m, in a 1-D harmonic oscillator potential has a probability density amplitude at $t = 0$, specified by the initial value

$$\Psi(q, t = 0) = \left[\frac{m\omega_0}{\hbar\pi}\right]^{\frac{1}{4}} e^{-\frac{1}{2}\frac{m\omega_0}{\hbar}(q-q_0)^2},$$

that is, by the $n = 0$ eigenfunction displaced by a distance q_0. Calculate $P(E_n)$, the probability the particle is in an energy eigenstate with energy E_n at $t = 0$ as a function of $x_0 = q_0/\sqrt{\hbar/m\omega_0}$ and n. [Check that $P(E_n) \rightarrow \delta_{n0}$, as $q_0 \rightarrow 0$.] Calculate $\Psi\Psi^*$ at a later time, t, and discuss the motion of the particle. Note: You may be able to sum an infinite series by using the generating function definition of the Hermite polynomials, $H_n(x)$, or the Bargmann kernel expansion

$$A(k,x) = \sum_{n=0}^{\infty} \psi_n(x) \frac{k^n}{\sqrt{n!}}.$$

11. Repeat problem 10 for the case when

$$\Psi(q, t=0) = \left[\frac{m\omega_0}{\hbar\pi}\right]^{\frac{1}{4}} \sqrt{\frac{2m\omega_0}{\hbar}}(q - q_0) e^{-\frac{1}{2}\frac{m\omega_0}{\hbar}(q-q_0)^2},$$

that is, by the $n = 1$ eigenfunction displaced by a distance q_0, or for the case when

$$\Psi(q, t=0) = \psi_n(q - q_0)$$

for arbitrary n.

Solution for Problem 11

a. The case $n = 1$: Let us write Ψ at $t = 0$ in terms of the dimensionless x and x_0

$$\Psi(x,0) = \frac{2(x-x_0)}{\sqrt{2}\pi^{\frac{1}{4}}} e^{-\frac{1}{2}(x-x_0)^2} = \sqrt{2}(x-x_0) e^{-\frac{1}{4}x_0^2} \frac{e^{-\frac{1}{2}\left(\frac{x_0}{\sqrt{2}}\right)^2 + \sqrt{2}x\frac{x_0}{\sqrt{2}} - \frac{1}{2}x^2}}{\pi^{\frac{1}{4}}}$$
$$= \sqrt{2}(x-x_0) e^{-\frac{1}{4}x_0^2} A(\frac{x_0}{\sqrt{2}}, x) = \sqrt{2}(x-x_0) e^{-\frac{1}{4}x_0^2} \sum_{n=0}^{\infty} \frac{\psi_n(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n, \quad (1)$$

where we have used the expansion of the Bargmann kernel function, $A(k, x)$, with $k = x_0/\sqrt{2}$, in terms of the normalized $\psi_n(x)$. If we further use

$$x\psi_n(x) = \sqrt{\frac{(n+1)}{2}} \psi_{n+1}(x) + \sqrt{\frac{n}{2}} \psi_{n-1}(x), \quad (2)$$

the above yields

$$\Psi(x,0) = e^{-\frac{1}{4}x_0^2} \Bigg(-\sqrt{2}x_0 \sum_{n=0}^{\infty} \frac{\psi_n(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n$$
$$+ \sum_{n=0}^{\infty} \sqrt{(n+1)} \frac{\psi_{n+1}(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n + \sum_{n=1}^{\infty} \sqrt{n} \frac{\psi_{n-1}(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n \Bigg)$$
$$= e^{-\frac{1}{4}x_0^2} \sum_{n=0}^{\infty} \Bigg(-\sqrt{2}x_0 \frac{\psi_n(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n$$
$$+ n \frac{\psi_n(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^{n-1} + \frac{x_0}{\sqrt{2}} \frac{\psi_n(x)}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^n \Bigg), \quad (3)$$

where we have shifted indices $n \to (n-1)$ and $n \to (n+1)$ in the last two sums above. (The second sum is proportional to a factor n and thus begins at $n = 1$.) We have therefore expanded our $\Psi(x, 0)$ in terms of the $\psi_n(x)$

$$\Psi(x,0) = \sum_n c_n \psi_n(x), \quad \text{with } c_n = \frac{e^{-\frac{1}{4}x_0^2}}{\sqrt{n!}} \left(\frac{x_0}{\sqrt{2}}\right)^{n-1} \left(n - \tfrac{1}{2}x_0^2\right). \quad (4)$$

At a later time, we have

$$\begin{aligned}\Psi(x,t) &= \sum_n c_n \psi_n(x) e^{-i\omega_0(n+\frac{1}{2})t} \\ &= e^{-\frac{i}{2}\omega_0 t} \sum_n e^{-\frac{1}{4}x_0^2} \frac{\sqrt{2}}{x_0}\left(n - \frac{x_0^2}{2}\right)\left(\frac{x_0 e^{-i\omega_0 t}}{\sqrt{2}}\right)^n \frac{\psi_n(x)}{\sqrt{n!}}. \end{aligned} \tag{5}$$

We can now sum these infinite series, using the expansion of the Bargmann kernel function, through

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\psi_n(x)}{\sqrt{n!}} k^n &= A(k,x) = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{1}{2}k^2 + \sqrt{2}kx - \frac{1}{2}x^2} \\ \sum_{n=1}^{\infty} n \frac{\psi_n(x)}{\sqrt{n!}} k^{n-1} &= \frac{dA(k,x)}{dk} = (-k + \sqrt{2}x) A(k,x), \end{aligned} \tag{6}$$

now with $k = (x_0 e^{-i\omega_0 t}/\sqrt{2})$. This yields

$$\begin{aligned}\Psi(x,t) &= e^{-\frac{i}{2}\omega_0 t}\left(\left[-\left(\frac{x_0}{\sqrt{2}}e^{-i\omega_0 t}\right) + \sqrt{2}x\right] e^{-i\omega_0 t} - \frac{x_0}{\sqrt{2}}\right) \\ &\times e^{-\frac{1}{4}x_0^2}\left[\frac{e^{[-\frac{1}{4}x_0^2 e^{-2i\omega_0 t} + x x_0 e^{-i\omega_0 t} - \frac{1}{2}x^2]}}{\pi^{\frac{1}{4}}}\right] \\ &= e^{-\frac{i}{2}\omega_0 t}\left(e^{-i\omega_0 t}\left[\frac{2(x - x_0\cos\omega_0 t)}{\sqrt{2}}\right]\right) \\ &\times \frac{e^{-\frac{1}{2}(x - x_0\cos\omega_0 t)^2} e^{-i x_0 \sin\omega_0 t (x - \frac{1}{2}x_0\cos\omega_0 t)}}{\pi^{\frac{1}{4}}} \\ &= e^{-i\frac{3}{2}\omega_0 t} \frac{2(x - x_0\cos\omega_0 t)}{\sqrt{2}\pi^{\frac{1}{4}}} e^{-\frac{1}{2}(x - x_0\cos\omega_0 t)^2} e^{-i x_0 \sin\omega_0 t (x - \frac{1}{2}x_0\cos\omega_0 t)}. \end{aligned} \tag{7}$$

Therefore,

$$|\Psi(x,t)|^2 = 2\frac{(x - x_0\cos\omega_0 t)^2}{\sqrt{\pi}} e^{-(x - x_0\cos\omega_0 t)^2} = |\psi_1(x - x_0\cos\omega_0 t)|^2; \tag{8}$$

that is, the probability density is that of the $n = 1$ state, but it oscillates about the origin with the oscillator frequency, (ω_0), with amplitude x_0, and without change of shape.

Our derivation so far has made use of some simple properties of harmonic oscillator eigenfunctions, [see eq. (2)], and the expansion of the Bargmann kernel function in terms of the $\psi_n(x)$, or, what would be equivalent, the generating function definition of the Hermite polynomials.

b. The case of arbitrary n: To generalize our result to a function $\Psi(x,0) = \psi_n(x - x_0)$ with arbitrary n, it may prove more convenient to work with the Bargmann transform of $\psi_n(x - x_0)$:

$$F_n(k) = \int_{-\infty}^{\infty} dx\, \psi_n(x - x_0) A(k,x) = \int_{-\infty}^{\infty} dx' \psi_n(x') A(k, x' + x_0)$$

$$= e^{\sqrt{2}kx_0 - \frac{1}{2}x_0^2} \int_{-\infty}^{\infty} dx' \psi_n(x') e^{-\frac{1}{2}k^2 - \frac{1}{2}x'^2 + \sqrt{2}(k - \frac{x_0}{\sqrt{2}})x'}$$
$$= e^{-\frac{1}{4}x_0^2 + \frac{1}{\sqrt{2}}kx_0} \int_{-\infty}^{\infty} dx' \psi_n(x') A\big((k - \frac{x_0}{\sqrt{2}}), x'\big)$$
$$= e^{-\frac{1}{4}x_0^2 + \frac{1}{\sqrt{2}}kx_0} \int_{-\infty}^{\infty} dx' \psi_n(x') \sum_{N=0}^{\infty} \frac{\psi_N(x')}{\sqrt{N!}} \left(k - \frac{x_0}{\sqrt{2}}\right)^N$$
$$= e^{-\frac{1}{4}x_0^2 + \frac{1}{\sqrt{2}}kx_0} \frac{\left(k - \frac{x_0}{\sqrt{2}}\right)^n}{\sqrt{n!}}, \tag{9}$$

where we have used the reality and the orthonormality of the harmonic oscillator eigenfunctions. To obtain the expansion coefficients, $c_m^{(n)}$,

$$c_m^{(n)} = \int_{-\infty}^{\infty} dx \psi_m^*(x) \psi_n(x - x_0) = \frac{1}{\pi} \int d^2k e^{-kk^*} F_n(k) \frac{k^{*m}}{\sqrt{m!}}, \tag{10}$$

it is sufficient to expand $F_n(k)$ in powers of k and use the k-space orthonormality relation

$$\frac{1}{\pi} \int d^2k e^{-kk^*} \frac{k^m}{\sqrt{m!}} \frac{k^{*l}}{\sqrt{l!}} = \delta_{ml}. \tag{11}$$

For this purpose, therefore, we expand

$$F_n(k) = e^{-\frac{1}{4}x_0^2 + \frac{x_0}{\sqrt{2}}k} \frac{\left(k - \frac{x_0}{\sqrt{2}}\right)^n}{\sqrt{n!}}$$
$$= e^{-\frac{1}{4}x_0^2} \frac{1}{\sqrt{n!}} \sum_{b=0}^{\infty} \sum_{a=0}^{n} \frac{1}{b!} \left(\frac{x_0}{\sqrt{2}}\right)^b \frac{n!}{a!(n-a)!} \left(-\frac{x_0}{\sqrt{2}}\right)^{n-a} k^{a+b}$$
$$= \sum_{m=0}^{\infty} \left[\frac{1}{\sqrt{n!}} e^{-\frac{1}{4}x_0^2} \sum_{a=0}^{n} \frac{n!}{a!(n-a)!} \frac{(-1)^{n-a}}{(m-a)!} \left(\frac{x_0}{\sqrt{2}}\right)^{n+m-2a}\right] k^m, \tag{12}$$

so

$$c_m^{(n)} = \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} e^{-\frac{1}{4}x_0^2} \sum_{a=0}^{n} (-1)^{n-a} \frac{n!}{a!(n-a)!} \frac{m!}{(m-a)!} \left(\frac{x_0}{\sqrt{2}}\right)^{n+m-2a}. \tag{13}$$

For the time-dependent function, we then get

$$\Psi(x, t) = \sum_m c_m^{(n)} \psi_m(x) e^{-i\omega_0(m+\frac{1}{2})t}, \tag{14}$$

and we could perform the m sums via

$$\sum_m \frac{m!}{(m-a)!} \psi_m(x) \frac{k^m}{\sqrt{m!}} = \frac{d^a}{dk^a} A(k, x), \qquad \text{now with } k = \frac{x_0 e^{-i\omega_0 t}}{\sqrt{2}}. \tag{15}$$

For small values of n, where the a sum contributes only $n + 1$ terms, this method works well, as you could again verify for the special case, $n = 1$. For arbitrary values of n, particularly for large values of n, we could use the summed form of the Bargmann transform, $F_n(k)$. From the above expansion in powers of k,

$$\text{If} \quad \Psi(x,0) \quad \text{has Bargmann transform} \qquad F_n(k),$$

$$\text{then} \quad \Psi(x,t) \quad \text{has Bargmann transform} \quad F_n(k,t) = e^{-\frac{i}{2}\omega_0 t} F_n(ke^{-i\omega_0 t}).$$

Therefore, for us,

$$\begin{aligned} F_n(k,t) &= e^{-\frac{i}{2}\omega_0 t} e^{-\frac{1}{4}x_0^2} e^{[\frac{x_0}{\sqrt{2}} k e^{-i\omega_0 t}]} \frac{\left(ke^{-i\omega_0 t} - \frac{x_0}{\sqrt{2}}\right)^n}{\sqrt{n!}} \\ &= e^{-i(n+\frac{1}{2})\omega_0 t} e^{-\frac{1}{4}x_0^2} e^{[\frac{x_0}{\sqrt{2}} k e^{-i\omega_0 t}]} \frac{\left(k - \frac{x_0 e^{-i\omega_0 t}}{\sqrt{2}}\right)^n}{\sqrt{n!}}. \end{aligned} \tag{16}$$

The function, $\Psi(x,t)$, then follows at once from the inverse Bargmann transform

$$\Psi(x,t) = \frac{1}{\pi} \int d^2k e^{-kk^*} A(k^*, x) F_n(k,t). \tag{17}$$

To do this integral, it will now be convenient to make the substitution

$$k' = \left(k - \frac{x_0 e^{i\omega_0 t}}{\sqrt{2}}\right),$$

so

$$\begin{aligned} \Psi(x,t) &= \frac{e^{-i(n+\frac{1}{2})\omega_0 t}}{\pi} \int d^2k' e^{-(k' + \frac{x_0 e^{i\omega_0 t}}{\sqrt{2}})(k'^* + \frac{x_0 e^{-i\omega_0 t}}{\sqrt{2}})} \\ &\times \frac{e^{-\frac{1}{2}(k'^* + \frac{x_0 e^{-i\omega_0 t}}{\sqrt{2}})^2} e^{\sqrt{2}k'^* x + x x_0 e^{-i\omega_0 t}} e^{-\frac{1}{2}x^2}}{\pi^{\frac{1}{4}}} e^{[\frac{x_0}{\sqrt{2}} k' e^{-i\omega_0 t}]} e^{+\frac{x_0^2}{4}} \left[\frac{k'^n}{\sqrt{n!}}\right] \\ &= \frac{e^{-i(n+\frac{1}{2})\omega_0 t}}{\pi} \int d^2k' e^{-k'^* k'} \frac{e^{[-\frac{1}{2}k'^{*2} + \sqrt{2}k'^*(x - x_0 \cos\omega_0 t) - \frac{1}{2}(x - x_0\cos\omega_0 t)^2]}}{\pi^{\frac{1}{4}}} \\ &\times \left[\frac{k'^n}{\sqrt{n!}}\right] e^{-ix_0 \sin\omega_0 t(x - \frac{x_0}{2}\cos\omega_0 t)} \\ &= e^{-i(n+\frac{1}{2})\omega_0 t} \left[\frac{1}{\pi}\int d^2k' e^{-k'^* k'} A\big(k'^*, (x - x_0\cos\omega_0 t)\big)\left[\frac{k'^n}{\sqrt{n!}}\right]\right] \\ &\times e^{-ix_0 \sin\omega_0 t(x - \frac{x_0}{2}\cos\omega_0 t)} \\ &= \psi_n(x - x_0\cos\omega_0 t) e^{-i(n+\frac{1}{2})\omega_0 t} e^{-ix_0\sin\omega_0 t(x - \frac{x_0}{2}\cos\omega_0 t))}. \end{aligned} \tag{18}$$

For arbitrary n,

$$|\Psi(x,t)|^2 = |\psi_n(x - x_0\cos\omega_0 t)|^2, \qquad \text{if} \quad \Psi(x,0) = \psi_n(x - x_0). \tag{19}$$

For arbitrary n, therefore, the probability density oscillates without change of shape about the origin with the oscillator frequency, (ω_0), and with amplitude x_0, if the initial state is the n^{th} oscillator state displaced in the x-direction through a distance x_0. This extremely simple property is unique for the harmonic oscillator and does not follow for the energy eigenstates of more complicated Hamiltonians. Also, the use of the Bargmann transform greatly facilitated the proof for general, n.

12. The 2-D isotropic harmonic oscillator with Hamiltonian

$$H + \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega_0^2}{2}(x^2 + y^2)$$

has eigenfunctions

$$\psi_{n_1 n_2}(x, y) = \psi_{n_1}(x)\psi_{n_2}(y),$$

with eigenvalues, $E_{n_1 n_2} = \hbar\omega_0(n_1 + n_2 + 1)$. Show that H is invariant to rotations

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = -x\sin\theta + y\cos\theta,$$

where θ is a constant. Show by means of the Bargmann kernel that an eigenfunction in which only the x' degree of freedom is excited can be expanded in terms of the above $\psi_{n_1 n_2}$; i.e., find the expansion coefficients, $c_{n_1 n_2}^{(N)}$:

$$\psi_N(x')\psi_0(y') = \sum_{n_1 n_2} c_{n_1 n_2}^{(N)} \psi_{n_1}(x)\psi_{n_2}(y).$$

13. For the conservation laws for the hydrogen atom, the three components of the Runge–Lenz vector are

$$\vec{\mathcal{R}} = \frac{1}{2\mu}\Big([\vec{p} \times \vec{L}] - \vec{L} \times \vec{p}]\Big) - \frac{Ze^2}{r}\vec{r}.$$

Show that they are hermitian when written in the above form. Also, show that they commute with the hydrogen atom Hamiltonian

$$H = \frac{(\vec{p} \cdot \vec{p})}{2\mu} - \frac{Ze^2}{r}.$$

In the above, $\vec{\mathcal{R}}$ and H are expressed in terms of the physical quantities, $\vec{r}_{\text{phys.}}$, $\vec{p}_{\text{phys.}}$, $\vec{L}_{\text{phys.}}$, and $H_{\text{phys.}}$. If these are expressed in terms of dimensionless quantities, $\vec{r}$, $\vec{p}$, $\vec{L}$, H, through

$$\vec{r}_{\text{phys.}} = a_0\vec{r}, \qquad \vec{p}_{\text{phys.}} = \frac{\hbar}{a_0}\vec{p}, \qquad \vec{L}_{\text{phys.}} = \hbar\vec{L},$$

$$H_{\text{phys.}} = \frac{\mu Z^2 e^4}{\hbar^2}H, \qquad \text{with } a_0 = \frac{\hbar^2}{\mu Z e^2},$$

the Runge vector in physical units, as given above, can be expressed in terms of a dimensionless $\vec{\mathcal{R}}$ by

$$\vec{\mathcal{R}}_{\text{phys.}} = Ze^2\vec{\mathcal{R}}.$$

Show that this dimensionless $\vec{\mathcal{R}}$ can also be expressed as

$$\vec{\mathcal{R}} = [\vec{p} \times \vec{L}] - i\vec{p} - \frac{\vec{r}}{r} = \vec{r}(\vec{p} \cdot \vec{p}) - \vec{p}(\vec{r} \cdot \vec{p}) - \frac{\vec{r}}{r}.$$

Also,

$$(\vec{\mathcal{R}} \cdot \vec{\mathcal{R}}) = \left(\vec{p} \cdot \vec{p} - \frac{2}{r}\right)(\vec{L} \cdot \vec{L} + 1) + 1 = 2H(\vec{L} \cdot \vec{L} + 1) + 1,$$

and

$$\vec{\mathcal{R}} \cdot \vec{L} = \vec{L} \cdot \vec{\mathcal{R}} = 0.$$