# 29

# Invariance of Physical Systems Under Rotations

Before going further with our study of angular momentum, it will be advantageous to study the general behavior of physical systems under rotations in our 3-D space. If a state vector which describes the state of a physical system is specified by $|\psi\rangle$, the state vector for the rotated system will be specified by $|\psi_{\text{rot.}}\rangle = R|\psi\rangle$. (We use the subscript, rot., in place of a prime, which is often used for the rotated state, because primes are also often used on quantum labels.) The operator, $R$, is the operator that rotates the system. Recall from the theory of translation operators, two possible points of view exist for such operators: (1) The active point of view, in which $R$ is used to rotate the system. (2) The passive point of view, in which the system is left unchanged and $R$ is used to rotate the coordinate system (in the opposite sense) to view the system from a rotated reference frame. We shall use the active point of view in this chapter.

The operator $R$ is a linear, unitary operator:

$$R^{-1} = R^{\dagger}, \tag{1}$$

$$R(\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle) = \lambda_1(R|\psi_1\rangle) + \lambda_2(R|\psi_2\rangle). \tag{2}$$

Also, note the following properties.

1. If $|\psi\rangle \to |\psi_{\text{rot.}}\rangle = R|\psi\rangle$, then $\langle\psi| \to \langle\psi_{\text{rot.}}| = \langle\psi|R^{\dagger}$. (3)

2. If $|\chi\rangle = O|\psi\rangle$, then $|\chi_{\text{rot.}}\rangle = R|\chi\rangle = ROR^{\dagger}(R|\psi\rangle)$,
so $O_{\text{rot.}} = ROR^{\dagger}$. (4)

If $[R, O] = 0$, then $O_{\text{rot.}} = O$, and if $O$ is hermitian, $O_{\text{rot.}}$ is hermitian. Also, if $\langle \chi | \psi \rangle$ are observable amplitudes, then

$$3. \qquad \langle \chi_{\text{rot.}} | \psi_{\text{rot.}} \rangle = \langle \chi | R^{\dagger} R | \psi \rangle = \langle \chi | \psi \rangle. \tag{5}$$

Matrix elements of operators are also invariant:

$$4. \qquad \langle \chi_{\text{rot.}} | O_{\text{rot.}} | \psi_{\text{rot.}} \rangle = \langle \chi | O | \psi \rangle. \tag{6}$$

Relations among operators are preserved under rotations:

$$5. \quad \text{If} \quad [A, B] = iC, \qquad \text{then} \quad [A_{\text{rot.}}, B_{\text{rot.}}] = iC_{\text{rot.}}. \tag{7}$$

# A Rotation Operators

We shall begin by studying a single-particle system and assume for the moment that the particle has no spin. We shall construct the rotation operator for a rotation through an angle, $\alpha$, about a specific axis. We shall also take the $z$ axis of our coordinate system along the direction of the rotation axis. Then, in analogy with the translation operator, $T = e^{-\frac{i}{\hbar} c_1 p_x}$, we shall try

$$R_z(\alpha) = e^{-\frac{i}{\hbar}\alpha L_{z\text{phys.}}} = e^{-i\alpha L_z}, \tag{8}$$

where we have converted the physical angular momentum operator ($z$ component) into the dimensionless $L_z$ in the last step. To study the action of $R_z(\alpha)$ on a general $|\psi\rangle$, expand $|\psi\rangle$ in terms of angular momentum eigenfunctions.

$$|\psi\rangle = \sum_{nlm} |nlm\rangle\langle nlm|\psi\rangle \qquad \text{or}$$

$$\langle \vec{r}|\psi\rangle = \psi(r, \theta, \phi) = \sum_{nlm} \langle \vec{r}|nlm\rangle\langle nlm|\psi\rangle$$

$$= \sum_{nlm} R_{nl}(r)\Theta_{lm}(\theta)\frac{e^{im\phi}}{\sqrt{2\pi}} c_{nlm}. \tag{9}$$

Then,

$$|\psi_{\text{rot.}}\rangle = R|\psi\rangle = \sum_{nlm} e^{-i\alpha L_z}|nlm\rangle\langle nlm|\psi\rangle = \sum_{nlm} e^{-i\alpha m}|nlm\rangle\langle nlm|\psi\rangle$$

$$\text{or} \quad \langle \vec{r}|\psi_{\text{rot.}}\rangle = \psi_{\text{rot.}}(r, \theta, \phi) = \sum_{nlm} R_{nl}(r)\Theta_{lm}(\theta)e^{im(\phi-\alpha)}\frac{c_{nlm}}{\sqrt{2\pi}}. \tag{10}$$

Thus,

$$\psi_{\text{rot.}}(r, \theta, \phi) = \psi(r, \theta, \phi_{\text{rot.}}) = \psi(r, \theta, \phi - \alpha). \tag{11}$$

We see (Fig. 29.1), if the original $\psi(r, \theta, \phi)$ has a maximum at some angle $\phi = \phi_0$, the rotated wave function, $\psi_{\text{rot.}}$, has a maximum where $(\phi - \alpha) = \phi_0$, that is, where $\phi = \phi_0 + \alpha$. In other words, the physical system has been rotated in the positive sense through an angle $\alpha$. Note: The prime is often used to designate $\phi_{\text{rot.}}$, i.e., $\phi_{\text{rot.}} \equiv \phi' = (\phi - \alpha)$, and note the last minus sign.

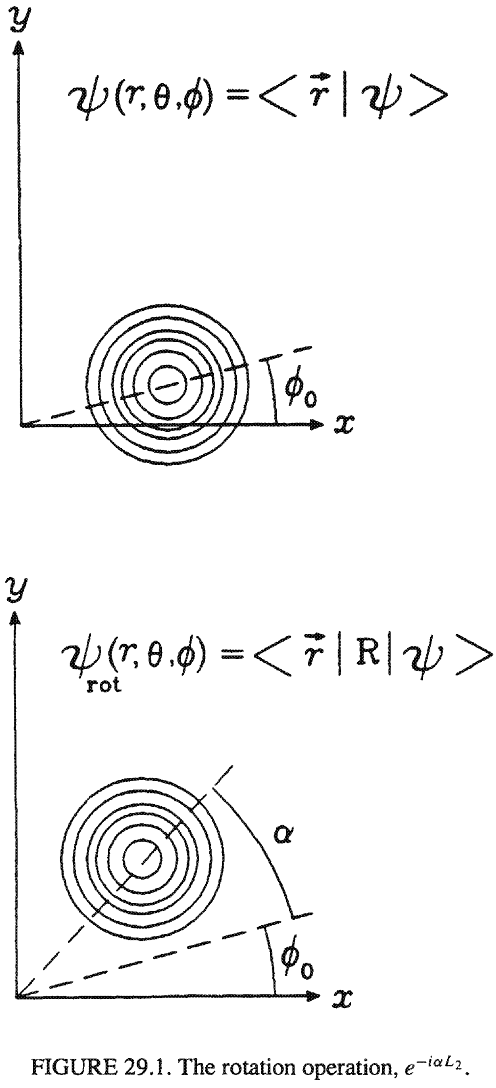


FIGURE 29.1. The rotation operation, $e^{-i\alpha L_z}$.

Next, we shall look at a single-particle system, but now we assume the particle is a spin $\frac{1}{2}$-particle, like the electron. Because spin and orbital operators commute, we shall try

$$R = R_L R_S, \tag{12}$$

with

$$R_S(\alpha) = e^{-i\alpha S_z} = e^{-i\frac{\alpha}{2}\sigma_z}. \tag{13}$$

Now, using $\sigma_z^2 = 1$, we get

$$R_S(\alpha) = \cos(\frac{\alpha}{2}) \times 1 - i\sin(\frac{\alpha}{2})\sigma_z, \tag{14}$$

or

$$R_S(\alpha) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{+i\frac{\alpha}{2}} \end{pmatrix}, \tag{15}$$

leading to

$$(\sigma_z)_{\text{rot.}} = R_S(\alpha)\sigma_z R_S^{\dagger}(\alpha) = \sigma_z, \tag{16}$$

because $R_S$ commutes with $\sigma_z$. Similarly,

$$(\sigma_x)_{\text{rot.}} = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{+i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{+i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix}. \tag{17}$$

Carrying out the matrix multiplication, this equation leads to

$$(\sigma_x)_{\text{rot.}} = \cos\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\alpha\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \tag{18}$$

or

$$(\sigma_x)_{\text{rot.}} = \cos\alpha\sigma_x + \sin\alpha\sigma_y, \tag{19}$$

and, similarly,

$$(\sigma_y)_{\text{rot.}} = -\sin\alpha\sigma_x + \cos\alpha\sigma_y. \tag{20}$$

Thus, the rotation operator $R_S(\alpha)$ rotates the $\vec{\sigma}$ vector properly. Note: This $\vec{\sigma}$ vector is part of the physical system. Finally, if we combine the orbital and spin operators, we get

$$R_z(\alpha) = R_L(\alpha)R_S(\alpha) = e^{-i\alpha(L_z+S_z)} = e^{-i\alpha J_z} \tag{21}$$

for a single particle with spin. The generator of the rotation about an axis is the component of the total angular momentum operator along that axis. This result holds equally well for a many-particle system or any general system, provided $J_z$ is the $z$ component of the *total* angular momentum vector.

## B General Rotations, $R(\alpha, \beta, \gamma)$

The most general rotation will be parameterized by the three Euler angles, $\alpha$, $\beta$, and $\gamma$, and will be built from the three successive rotations as follows.

The first rotation through the angle $\alpha$ about the original space-fixed $z$ axis will take the $(x, y, z)$ coordinate system to a rotated $(x_1, y_1, z_1)$ system, with $z_1 = z$.

The second rotation through an angle $\beta$ about the new $y_1$ axis will take the $(x_1, y_1, z_1)$ coordinate system to a new $(x_2, y_2 z_2)$ system, with $y_2 = y_1$.

The third rotation through an angle $\gamma$ about the $z_2$ axis will take the $(x_2, y_2, z_2)$ coordinate system to the final rotated $(x', y', z')$ system, with $z' = z_2$.

We shall think of the coordinate systems as being attached to our physical system. Thus, the general rotation can be expressed through

$$R(\alpha, \beta, \gamma) = e^{-i\gamma J_{z'}} e^{-i\beta J_{y_1}} e^{-i\alpha J_z}, \tag{22}$$

where this is not a very handy form because the three generators of the unitary transformations, $R(\alpha)$, $R(\beta)$, and $R(\gamma)$, are expressed in terms of angular momentum components along three different coordinate systems. Using $O_{\text{rot.}} = ROR^{\dagger}$, however, and noting the operator $J_{y_1}$ is reached from $J_y$ via the rotation $R(\alpha)$, we have

$$e^{-i\beta J_{y_1}} = R(\alpha) e^{-i\beta J_y} R(\alpha)^{\dagger} = e^{-i\alpha J_z} e^{-i\beta J_y} e^{+i\alpha J_z}. \tag{23}$$

Similarly, noting the operator $J_{z'} = J_{z_2}$ is reached from $J_{z_1}$ via the rotation $R(\beta)$, we have

$$e^{-i\gamma J_{z'}} = R(\beta) e^{-i\gamma J_{z_1}} R(\beta)^{\dagger}. \tag{24}$$

Thus, we can write

$$R(\alpha, \beta, \gamma) = R(\beta) e^{-i\gamma J_{z_1}} R(\beta)^{-1} R(\beta) e^{-i\alpha J_z} = R(\beta) e^{-i\gamma J_{z_1}} e^{-i\alpha J_z}. \tag{25}$$

Now, noting $J_{z_1} = J_z$, the two rotation operators on the extreme right commute with each other, and we can write

$$R(\alpha, \beta, \gamma) = e^{-i\beta J_{y_1}} R(\alpha) e^{-i\gamma J_z} = R(\alpha) e^{-i\beta J_y} R(\alpha)^{-1} R(\alpha) e^{-i\gamma J_z}. \tag{26}$$

This process leads to the final result,

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}. \tag{27}$$

Now, all generators are expressed with respect to components along the original axes, but seemingly the order of the rotations is "backwards," we start on the right with the $\gamma$ rotation, followed by $\beta$, and last the $\alpha$-term.

# C Transformation of Angular Momentum Eigenvectors or Eigenfunctions

Having derived a useful expression for the most general rotation operator, we can now give an expression for a rotated state vector that is an eigenvector of both $\vec{J}^2$ and $J_z$ (and other operators commuting with these two), in terms of the original eigenvectors of this type

$$|(JM)_{\text{rot.}}\rangle = R(\alpha, \beta, \gamma)|JM\rangle = \sum_{\mu} |J\mu\rangle\langle J\mu|R(\alpha, \beta, \gamma)|JM\rangle, \tag{28}$$

where, for simplicity of notation, we have omitted all quantum numbers other than $J$ and $M$ (associated with the remaining operators). Also, the matrix elements of operators, $J_z$ and $J_y$ are diagonal in the quantum number $J$. Thus, the $J$ sum disappears from the unit operator, $\sum_{J,\mu} |J\mu\rangle\langle J\mu|$. The rotation matrix is usually denoted by the symbol $D$ (for the German word "Darstellung," or representation), because this rotation matrix is an irreducible representation matrix of the rotation group $SO(3)$ for integral angular momenta or the unitary group $SU(2)$ for $\frac{1}{2}$-integral spins.

$$\langle J\mu|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|JM\rangle = D^J_{\mu M}(\alpha,\beta,\gamma)^*. \tag{29}$$

The * is added so the simple exponential factors have + signs; (but this notation is not universal!)

$$D^J_{\mu M}(\alpha,\beta,\gamma) = e^{+i\mu\alpha}d^J_{\mu M}(\beta)e^{+iM\gamma}, \tag{30}$$

where

$$d^J_{\mu M}(\beta) = \langle J\mu|e^{-i\beta J_y}|JM\rangle \tag{31}$$

is a *real* function of $\beta$, because the matrix elements of $iJ_y$ are always real in the standard angular momentum conventions. (Therefore, no * was needed on the $d$ function.)

We can also convert eq. (28) into an equation for angular-momentum eigenfunctions

$$\begin{aligned}\langle\vec{r}|(JM)_{\text{rot.}}\rangle &= (\psi_{\text{rot.}})_{JM}(r,\theta,\phi,\vec{\sigma}) = \psi_{JM}(r,\theta',\phi',\vec{\sigma}')\\ &= \sum_\mu \psi_{J\mu}(r,\theta,\phi,\vec{\sigma})D^J_{\mu M}(\alpha,\beta,\gamma)^*,\end{aligned} \tag{32}$$

where we have now used primed angles for the angles in the rotated angular momentum eigenfunction. Recall $\theta',\phi' = \theta,(\phi-\alpha)$ for the simple $z$ rotation with $\beta = 0, \gamma = 0$. In general, $\theta',\phi'$ are complicated functions of $\theta,\phi,\alpha,\beta,\gamma$. We can write the inverse of this transformation

$$\psi_{JM}(r,\theta,\phi,\vec{\sigma}) = \sum_\mu \psi_{J\mu}(r,\theta',\phi',\vec{\sigma}')((D^{-1})^J_{\mu M})^*. \tag{33}$$

Now, making use of the unitary property of the $D$ matrix,

$$\psi_{JM}(r,\theta,\phi,\vec{\sigma}) = \sum_\mu D^J_{M\mu}(\alpha,\beta,\gamma)\psi_{J\mu}(r,\theta',\phi',\vec{\sigma}'). \tag{34}$$

The job of calculating the $d^J_{\mu M}(\beta)$ of eq. (31) remains. For the smallest $J$ values, this process is quite straightforward. For example, for $J = \frac{1}{2}$, with

$$\mathbf{1} = \sigma_y^2 = \sigma_y^4 = \cdots, \qquad \sigma_y = \sigma_y^3 = \sigma_y^5\cdots, \tag{35}$$

so

$$e^{-i\frac{\beta}{2}\sigma_y} = \sum_n\left(\frac{-i\beta}{2}\right)^n\frac{(\sigma_y)^n}{n!} = \mathbf{1}\cos(\frac{\beta}{2}) - i\sigma_y\sin(\frac{\beta}{2}). \tag{36}$$

Substituting the matrices

$$\mathbf{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \text{and} \qquad \sigma_y = \begin{pmatrix}0 & -i\\ +i & 0\end{pmatrix}, \tag{37}$$

we get the 2 × 2 $d$ matrix

$$d^{\frac{1}{2}}_{m'm}(\beta) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ +\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{38}$$

Similarly, for $J = 1$, using the 3 × 3 matrix relations in this case,

$$J_y = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{+i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{+i}{\sqrt{2}} & 0 \end{pmatrix} = J_y^3 = J_y^5 = \cdots, \tag{39}$$

and

$$J_y^2 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = J_y^4 = J_y^6 = \cdots, \tag{40}$$

we get

$$e^{-i\beta J_y} = \mathbf{1} - iJ_y \sin\beta + J_y^2(\cos\beta - 1), \tag{41}$$

leading to

$$d^{J=1}_{M'M} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}. \tag{42}$$

For higher values of $J$, this direct method of calculating the $d$ matrices will of course become more and more difficult, and we shall have to find a better method.

# D General Expression for the Rotation Matrices

Although we know the matrix elements of $J_y = -\frac{i}{2}(J_+ - J_-)$, the calculation of $(J_y)^n$ (in the expansion of the exponential $e^{-i\beta J_y}$) is complicated because of the noncommutability of the operators $J_+$ and $J_-$. The calculation of the $d$ matrix would be straightforward if we could restructure the rotation operator in the form,

$$e^{\beta_+ J_+} e^{\beta_0 J_0} e^{\beta_- J_-}, \qquad \text{or} \quad e^{\gamma_- J_-} e^{\gamma_0 J_0} e^{\gamma_+ J_+}.$$

In principle, the transformation of an operator product of the form $e^A e^B$ into the form $e^C$, where $A$ and $B$ are noncommuting operators (or their matrix realizations) can be achieved by the so-called Baker–Campbell–Hausdorff relation

$$e^A e^B = e^C, \qquad \text{with}$$
$$C = A + B + \tfrac{1}{2}[A, B] + \tfrac{1}{12}\Big([A, [A, B]] + [B, [B, A]]\Big) + \cdots, \tag{43}$$

where the $\cdots$ involves triple and quadruple and ever higher commutators of the operators $A$ and $B$. The Baker–Campbell–Hausdorff expansion is very useful in cases in which the multiple commutators are all zero, say, after the second or third term. For the angular momentum algebra, unfortunately, the series is an infinite one, with ever more complicated coefficients. The desired final result, however, depends only on the angular momentum commutator algebra. The coefficients, $\beta_\pm, \beta_0$, or $\gamma_\pm, \gamma_0$, depend only on the commutator algebra of $J_+$, $J_-$, and $J_0$, not on the quantum number, $J$. It will therefore be sufficient to use the simplest nontrivial representation of the rotation operator, viz., the representation for $J = \frac{1}{2}$, where we deal with extremely simple $2 \times 2$ matrices. It will be useful to solve a slightly more general problem, and "disentangle" the more general operator, $(a_+ J_+ + a_0 J_0 + a_- J_-)$ through

$$e^{(a_+J_+ + a_0J_0 + a_-J_-)} = e^{b_+J_+} e^{(lnb_0)J_0} e^{b_-J_-} = e^{c_-J_-} e^{(lnc_0)J_0} e^{c_+J_+}. \tag{44}$$

(We have renamed $\beta_0 = lnb_0$, $\beta_\pm = b_\pm$, for convenience, similarly for the $\gamma_\pm, \gamma_0$. Now,

for $J = \frac{1}{2}$,

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{45}$$

so $(J_+)^n$ and $(J_-)^n$, with $n \geq 2$, are all null matrices. With

$$(a_+J_+ + a_0J_0 + a_-J_-) = \begin{pmatrix} \frac{1}{2}a_0 & a_+ \\ a_- & -\frac{1}{2}a_0 \end{pmatrix} \equiv \mathbf{a}, \qquad \text{we have} \tag{46}$$

$$\mathbf{a}^2 = \begin{pmatrix} (\frac{1}{4}a_0^2 + a_+a_-) & 0 \\ 0 & (\frac{1}{4}a_0^2 + a_+a_-) \end{pmatrix} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a^2\mathbf{1}, \tag{47}$$

and with $\mathbf{a}^{2n} = a^{2n}\mathbf{1}$ and $\mathbf{a}^{2n+1} = a^{2n}\mathbf{a}$, we have

$$e^{(a_+J_+ + a_0J_0 + a_-J_-)} = \begin{pmatrix} \cosh a + \frac{1}{2}\frac{a_0}{a}\sinh a & \frac{a_+}{a}\sinh a \\ \frac{a_-}{a}\sinh a & \cosh a - \frac{1}{2}\frac{a_0}{a}\sinh a \end{pmatrix}$$
$$\equiv \cosh a\mathbf{1} + \frac{\sinh a}{a}\mathbf{a}. \tag{48}$$

[The scalar $a$ is defined through $a^2 = (\frac{1}{4}a_0^2 + a_+a_-)$.] Similarly,

$$e^{b_+J_+} e^{(lnb_0)J_0} e^{b_-J_-} = \frac{1}{\sqrt{b_0}} \begin{pmatrix} (b_0 + b_+b_-) & b_+ \\ b_- & 1 \end{pmatrix}, \tag{49}$$

and

$$e^{c_-J_-} e^{(lnc_0)J_0} e^{c_+J_+} = \sqrt{c_0} \begin{pmatrix} 1 & c_+ \\ c_- & (\frac{1}{c_0} + c_+c_-) \end{pmatrix}. \tag{50}$$

Now, the coefficients $b_\pm, b_0$ or $c_\pm, c_0$ can be evaluated in terms of the $a_\pm, a_0$ by comparing eqs. (49) or (50) with eq. (48). We are interested in the rotation operator, $e^{-i\beta J_y} = e^{-\frac{1}{2}\beta(J_+ - J_-)}$. In our special case, therefore, $a_\pm = \mp\frac{1}{2}\beta$, $a_0 = 0$. For this case, we have

$$b_\pm = \mp\tan\frac{\beta}{2}, \quad b_0 = \frac{1}{\cos^2(\frac{\beta}{2})}, \qquad c_\pm = \mp\tan\frac{\beta}{2}, \quad c_0 = \cos^2(\tfrac{\beta}{2}), \tag{51}$$

so

$$e^{-i\beta J_y} = e^{+\tan\frac{\beta}{2}J_-}\left(\cos^2(\tfrac{\beta}{2})\right)^{J_0} e^{-\tan\frac{\beta}{2}J_+}$$
$$= e^{-\tan\frac{\beta}{2}J_+}\frac{1}{\left(\cos^2(\frac{\beta}{2})\right)^{J_0}} e^{+\tan\frac{\beta}{2}J_-}. \quad (52)$$

With the first form of this result, we can write

$$d^J_{M'M}(\beta) = \langle JM'|e^{+\tan\frac{\beta}{2}J_-}\left(\cos^2(\tfrac{\beta}{2})\right)^{J_0} e^{-\tan\frac{\beta}{2}J_+}|JM\rangle$$
$$= \sum_n \frac{\left(\tan\frac{\beta}{2}\right)^{M+n-M'}\left(\cos\frac{\beta}{2}\right)^{2(M+n)}\left(-\tan\frac{\beta}{2}\right)^n}{n!(M+n-M')!}$$
$$\times \langle JM'|(J_-)^{M+n-M'}(J_+)^n|JM\rangle, \quad (53)$$

where both $n$ and $M+n-M'$ must be positive integers (including zero) and restrictions on magnetic quantum numbers limit the sum over $n$ to a sum from $n \geq (M'-M)$ to $n \leq (J-M)$ for the case $M'-M \geq 0$, and to a sum from $n=0$ to $n \leq (J-M)$ for the case $M'-M < 0$. Using the simple known matrix elements of $J_{\pm}$, with

$$(J_+)^n|JM\rangle = \sqrt{\frac{(J-M)!(J+M+n)!}{(J-M-n)!(J+M)!}}|J(M+n)\rangle \qquad \text{and} \quad (54)$$

$$(J_-)^{M+n-M'}|J(M+n)\rangle = \sqrt{\frac{(J+M+n)!(J-M')!}{(J+M')!(J-M-n)!}}|JM'\rangle. \quad (55)$$

We therefore get

$$d^J_{M'M}(\beta) = \left[\frac{(J-M)!(J-M')!}{(J+M)!(J+M')!}\right]^{\frac{1}{2}}\sum_n(-1)^n$$
$$\times \frac{(J+M+n)!}{(J-M-n)!(M+n-M')!n!}(\cos\tfrac{\beta}{2})^{M+M'}(\sin\tfrac{\beta}{2})^{M-M'+2n}, \quad (56)$$

where the sum over $n$ ranges from $n = \max.[0, M'-M]$ to $n=(J-M)$. In the very special case $M=J$, the integer $n$ is restricted to $n=0$, and

$$d^J_{M'J}(\beta) = \left[\frac{(2J)!}{(J+M')!(J-M')!}\right]^{\frac{1}{2}}(\cos\tfrac{\beta}{2})^{J+M'}(\sin\tfrac{\beta}{2})^{J-M'}. \quad (57)$$

In the further special case $M=-J$, it will be useful to rename the summation index $n = n' + J + M'$, so eq. (56) yields

$$d^J_{M',-J}(\beta) = (-1)^{J+M'}\left[\frac{2J)!}{(J+M')!(J-M')!}\right]^{\frac{1}{2}}(\cos\tfrac{\beta}{2})^{-J+M'}(\sin\tfrac{\beta}{2})^{J+M'}$$
$$\times \sum_{n'=0}^{J-M'}\frac{(J-M')!(-1)^{n'}}{(J-M'-n')!n'!}(\sin^2\tfrac{\beta}{2})^{n'}$$

$$= (-1)^{J+M'} \left[ \frac{(2J)!}{(J+M')!(J-M')!} \right]^{\frac{1}{2}} (\cos \tfrac{\beta}{2})^{J-M'} (\sin \tfrac{\beta}{2})^{J+M'}, \tag{58}$$

where we have used the binomial expansion of $(1 - \sin^2(\beta/2))^{J-M'}$.

# E Rotation Operators and Angular Momentum Coherent States

In Chapter 19, we defined two slightly different types of angular momentum coherent states $|\alpha\rangle$ and $|z\rangle$, giving us two slightly different continuous representations of state vectors $|\psi\rangle$ of our 3-D world in terms of functions of the complex variables, $\alpha$ and $z$, where these complex numbers are related to the orientation of the physical system in our 3-D world. We are now in a position to see the relationship between these type I and type II coherent states. Such angular coherent states will again be very useful for physical systems best described through a statistical distribution of states with different orientations in our laboratory.

The type I coherent state was defined through

$$|\alpha\rangle = e^{\alpha^* J_+ - \alpha J_-} |J, M = -J\rangle = e^{-i\theta(\vec{J}\cdot\vec{n})} |J, M = -J\rangle, \tag{59}$$

where the complex variable, $\alpha$, is related to the angles $\theta$, $\phi$ through $\alpha = -(\theta/2)e^{i\phi}$, so the unit vector, $\vec{n}$, lies in the $x$, $y$-plane and is rotated forward from the laboratory $y$ axis through an angle $\phi$. The coherent state $|\alpha\rangle$ can thus be expressed through a rotation operator, with Euler angles, $\phi$, $\theta$, and $\gamma = 0$:

$$|\alpha\rangle \equiv |\alpha(\theta, \phi)\rangle = R(\phi, \theta, 0) |J, M = -J\rangle. \tag{60}$$

We can therefore expand the type I angular coherent state through

$$\begin{aligned}
|\alpha(\theta,\phi)\rangle &= \sum_{M=-J}^{+J} |JM\rangle D^J_{M,-J}(\phi,\theta,0)^* \\
&= \sum_{M=-J}^{+J} |JM\rangle c_{J,M} (-1)^{J+M} \left(\sin \tfrac{\theta}{2}\right)^{J+M} \left(\cos \tfrac{\theta}{2}\right)^{J-M} e^{-iM\phi} \\
&= \sum_{M=-J}^{+J} |JM\rangle c_{J,M} \frac{(-1)^{J+M} \left(\tan \tfrac{\theta}{2}\right)^{J+M}}{\left(1 + \tan^2 \tfrac{\theta}{2}\right)^J} e^{-iM\phi}, \\
\text{with} \quad c_{J,M} &= \sqrt{\frac{(2J)!}{(J+M)!(J-M)!}},
\end{aligned} \tag{61}$$

where we have used eq. (58) of the last section and the trivial identity,

$$\cos^2 \tfrac{\theta}{2} = (1 + \tan^2 \tfrac{\theta}{2})^{-1}.$$

The above expansion suggests the unit operator appropriate for this type of coherent state is

$$\begin{aligned}
1 &= \frac{(2J+1)}{4\pi}\int\!\!\int d\Omega|\alpha(\theta,\phi)\rangle\langle\alpha(\theta,\phi)| \\
&= \sum_{M,M'}\frac{(2J+1)}{4\pi}\int\!\!\int d\Omega D^{J}_{M,-J}(\phi,\theta,0)^{*}D^{J}_{M',-J}(\phi,\theta,0)|JM\rangle\langle JM'| \\
&= \sum_{M}|JM\rangle\langle JM|, \qquad (62)
\end{aligned}$$

where $d\Omega = \sin\theta d\theta d\phi$. The angular ranges in the integrals have their usual values, and we have made use of the orthonormality integral

$$\int\!\!\int d\Omega D^{J}_{M,\mu}(\phi,\theta,0)^{*}D^{J'}_{M',\mu'}(\phi,\theta,0) = \delta_{JJ'}\delta_{MM'}\delta_{\mu\mu'}\frac{4\pi}{(2J+1)}. \qquad (63)$$

[The derivation of this integral is given in detail through eq. (30) of the next chapter; note the D functions with Euler angle $\gamma = 0$ are of course independent of this third angle.]

The type II angular momentum coherent state, conversely, was defined by

$$\begin{aligned}
|z\rangle = e^{z^{*}J_{+}}|J,-J\rangle &= \sum_{n=0}^{2J}\frac{z^{*n}}{\sqrt{n!}}\sqrt{\frac{(2J)!}{(2J-n)!}}|J,M=-J+n\rangle \\
&= \sum_{M=-J}^{+J}(z^{*})^{J+M}\sqrt{\frac{(2J)!}{(J+M)!(J-M)!}}|JM\rangle. \qquad (64)
\end{aligned}$$

In Chapter 19, it was shown that the $z$ space functions

$$\frac{z^{n}}{\sqrt{n!}}\sqrt{\frac{(2J)!}{(2J-n)!}}$$

formed an orthonormal set with respect to the measure

$$\frac{(2J+1)}{\pi}\frac{d^{2}z}{(1+zz^{*})^{2J+2}}.$$

We can now complete our discussion of these angular momentum coherent states by showing explicitly the relationship between the two types of coherent states. Eq. (61) could have been obtained directly by putting the operator $e^{\alpha^{*}J_{+}-\alpha J_{-}}$ into normal ordered form through the comparison of eqs. (49) and (48) of the last section, yielding

$$\begin{aligned}
|\alpha\rangle &= e^{b_{+}J_{+}}\frac{1}{\left(\cos^{2}\frac{\theta}{2}\right)^{J_{0}}}e^{b_{-}J_{-}}|-J\rangle \\
&= e^{b_{+}J_{+}}\left(\cos^{2}\tfrac{\theta}{2}\right)^{J}|-J\rangle,
\end{aligned}$$

$$\text{with} \qquad b_{\pm} = \mp\tan\tfrac{\theta}{2}e^{\mp i\phi}. \qquad (65)$$

Expansion of the exponential $e^{b_+ J_+}$ again leads to eq. (61), but the present form suggests a change from the complex variable, $\alpha$, to the new complex variable $z$, where

$$z = \rho e^{i\phi} = -\tan\tfrac{\theta}{2} e^{i\phi},$$

so

$$|\alpha\rangle = e^{z^* J_+} \frac{1}{(1 + zz^*)^J} | -J\rangle. \tag{66}$$

Also, with

$$\rho^2 = zz^* = \tan^2\tfrac{\theta}{2},$$

we have

$$\sin\theta d\theta d\phi = 4\rho d\rho d\phi \frac{1}{(1+\rho^2)^2}.$$

With this relation and eq. (66), the unit operator can be transformed into

$$\begin{aligned} &\frac{(2J+1)}{4\pi} \int\int d\Omega\, |\alpha\rangle\langle\alpha| \\ &= \frac{(2J+1)}{\pi} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{d\rho\rho}{(1+\rho^2)^2} \frac{e^{z^* J_+}}{(1+\rho^2)^J} | -J\rangle\langle -J| \frac{e^{zJ_-}}{(1+\rho^2)^J} \\ &= \frac{(2J+1)}{\pi} \int \frac{d^2 z}{(1+zz^*)^{2J+2}} |z\rangle\langle z|. \end{aligned} \tag{67}$$

This relation is precisely the unit operator needed for the type II coherent states, $|z\rangle$, with a measure making the $z$ space functions

$$\frac{z^n}{\sqrt{n!}} \sqrt{\frac{(2J)!}{(2J-n)!}}$$

into an orthonormal set, as shown in Chapter 19.