

## Symmetry Properties of Clebsch–Gordan Coefficients

Clebsch–Gordan coefficients in which the three angular momenta,  $j_1$ ,  $j_2$ , and  $j \equiv j_3$ , are reordered may be simply related to each other. The most trivial case involves the exchange of the order of the quantum numbers,  $j_1 m_1$  and  $j_2 m_2$ . The state vector  $|j_1 m_1 j_2 m_2\rangle$  is a direct product of two vectors involving separate subspaces of the full Hilbert space, or in terms of the coordinate representation, the wave function  $\psi_{j_1 m_1} \psi_{j_2 m_2}$  is a product of functions involving different variables. For example,  $\psi_{j_1 m_1}$  might be a function of orbital variables and  $\psi_{j_2 m_2}$  might be a function of spin variables. Thus, the product of these two functions should not depend on the order in which we write the two functions. Therefore, when we expand this product function in terms of the total angular momentum eigenfunctions  $\Psi_{j_1 j_2 j m}$ , the result must be independent of the order in which we write the original product function,  $\psi_{j_1 m_1} \psi_{j_2 m_2}$ , or  $\psi_{j_2 m_2} \psi_{j_1 m_1}$ , with the possible exception of an overall phase factor. This phase factor comes in because our phase convention fixing the overall sign of the Clebsch–Gordan coefficients gives preference to the angular momenta sitting in the number 1 and number 3 positions of the Clebsch–Gordan coefficient. Thus,  $\langle j_1 j_1 j_2 m_2 | j_3 j_3 \rangle$  must be positive by our phase convention. Similarly,  $\langle j_2 j_2 j_1 m_1 | j_3 j_3 \rangle$  must also be positive. On the contrary, the Clebsch–Gordan coefficient  $\langle j_1 m_1 j_2 j_2 | j_3 j_3 \rangle$  has the sign  $(-1)^{j_1 - m_1}$  with  $m_1 = j_3 - j_2$ . Hence, its sign is  $(-1)^{j_1 + j_2 - j_3}$ . Thus, the coefficients in which the order of  $j_1$  and  $j_2$  is exchanged will differ by this phase factor for all possible  $m$ 's. Thus, we have our first symmetry property:

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 + j_2 - j_3} \langle j_2 m_2 j_1 m_1 | j_3 m_3 \rangle. \quad (1)$$

Next, if we rearrange the vector addition equation,

$$\vec{J}_1 + \vec{J}_2 = \vec{J}_3, \tag{2}$$

to read

$$\vec{J}_3 - \vec{J}_2 = \vec{J}_1, \tag{3}$$

we can see that the Clebsch–Gordan coefficient  $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$  must be related to the coefficient  $\langle j_3 m_3 j_2 - m_2 | j_1 m_1 \rangle$ . In particular, if we make the substitution  $j_1 m_1 \leftrightarrow j_3 m_3$  and  $m_2 \rightarrow -m_2$  in recursion formula I (or II), we obtain recursion formula II (or I), provided the transformed coefficients are related to the original ones via a phase factor proportional to  $(-1)^{m_2}$  and an  $m$ -independent factor; i.e., we expect

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{m_2} K(j_1, j_2, j_3) \langle j_3 m_3 j_2 - m_2 | j_1 m_1 \rangle, \tag{4}$$

where  $K(j_1, j_2, j_3)$  is the  $m$ -independent overall factor. This factor can be determined via the orthonormality of the Clebsch–Gordan coefficients

$$\begin{aligned} \sum_{m_3} \sum_{m_1, (m_2)} |\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle|^2 &= \sum_{m_3} 1 = (2j_3 + 1) \\ &= \sum_{m_1} \sum_{m_3, (m_2)} |K(j_1, j_2, j_3)|^2 |\langle j_3 m_3 j_2 - m_2 | j_1 m_1 \rangle|^2 = \sum_{m_1} |K(j_1, j_2, j_3)|^2 \\ &= (2j_1 + 1) |K(j_1, j_2, j_3)|^2. \end{aligned} \tag{5}$$

Thus,

$$K(j_1, j_2, j_3) = (-1)^{\phi(j_1, j_2, j_3)} \sqrt{\frac{(2j_3 + 1)}{(2j_1 + 1)}}, \tag{6}$$

where the  $j_i$ -dependent phase  $\phi$  can be determined because the coefficients with both  $m_1 = j_1$  and  $m_3 = j_3$  and hence  $m_2 = j_3 - j_1$  must both be positive, and hence  $\phi = j_1 - j_3$ . Thus, we get a second symmetry property

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_3 + m_2} \sqrt{\frac{(2j_3 + 1)}{(2j_1 + 1)}} \langle j_3 m_3 j_2 - m_2 | j_1 m_1 \rangle. \tag{7}$$

By combining this symmetry property with the first one, (1  $\leftrightarrow$  2) exchange, we get

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_2 + m_2} \sqrt{\frac{(2j_3 + 1)}{(2j_1 + 1)}} \langle j_2 - m_2 j_3 m_3 | j_1 m_1 \rangle. \tag{8}$$

This process is a cyclic exchange of the type 123  $\rightarrow$  (–2)31. If we follow this by the cyclic exchange (–2)31  $\rightarrow$  (–3)1(–2) and subsequently by the cyclic exchange (–3)1(–2)  $\rightarrow$  (–1)(–2)(–3), we obtain

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 + j_2 - j_3} \langle j_1 - m_1 j_2 - m_2 | j_3 - m_3 \rangle, \tag{9}$$

where we have used the identity  $j_1 + m_1 + j_2 + m_2 + j_3 + m_3 = j_1 + j_2 + j_3 + 2m_3$ , and  $j_1 + j_2 + j_3 + 2m_3 = j_1 + j_2 - j_3 + \text{even integer}$ , because  $2j_3 + 2m_3$  is always an even integer. (Either  $j_3$  and  $m_3$  are both integers or are both  $\frac{1}{2}$ -integers.)

We have now derived several symmetry properties, involving interchanges such as  $123 \rightarrow 213$ , or cyclic interchanges such as  $123 \rightarrow (-2)31$ , or changes of sign in all  $m$ 's,  $123 \rightarrow (-1)(-2)(-3)$ . Twelve such symmetry properties exist altogether. These properties are much easier to remember by introducing the 3- $j$  symbol, defined by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 + m_3}}{\sqrt{(2j_3 + 1)}} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle. \tag{10}$$

This 3- $j$  symbol has the following symmetry properties: The 3- $j$  symbol is invariant under any even permutation of columns. The 3- $j$  symbol changes sign by the factor  $(-1)^{j_1 + j_2 + j_3}$  under either an odd permutation of columns or under the transformation  $m_i \rightarrow -m_i$  for all  $i = 1, 2, 3$ .

Although the symmetry properties are easier to remember in terms of the 3- $j$  symbol, this symbol does not have simple orthonormality properties. The orthonormality relations for the Clebsch–Gordan coefficients are so useful most authors prefer to use the Clebsch–Gordan coefficients.

References: several little books on angular momentum coupling in quantum mechanics exist: (1) D. M. Brink and G. R. Satchler. *Angular Momentum*. Oxford: Clarendon Press, 1968; (2) M. E. Rose. *Elementary Theory of Angular Momentum*. New York: John Wiley, 1957; (3) A. R. Edmonds. *Angular Momentum in Quantum Mechanics*. Princeton University Press, 1974.

Tables: *The 3-j and 6-j Symbols*. M. Rotenberg, R. Bivins, N. Metropolis, and J. K. Wooten. Cambridge, Mass.: MIT Press, 1959. *Tables of Clebsch-Gordan Coefficients*, Peking: Science Press, 1965.

TABLE 28.1.  $\langle j_1 m_1 \frac{1}{2} m_2 | j m \rangle$

$j =$	$m_2 = +\frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{(j_1 + m + \frac{1}{2})}{(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m + \frac{1}{2})}{(2j_1 + 1)}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{(j_1 - m + \frac{1}{2})}{(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m + \frac{1}{2})}{(2j_1 + 1)}}$

TABLE 28.2.  $\langle j_1 m_1 1 m_2 | j m \rangle$

$j =$	$m_2 = +1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\sqrt{\frac{(j_1 + m)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{(2j_1 + 1)(2j_2 + 2)}}$
$j_1$	$-\sqrt{\frac{(j_1 + m)(j_1 - m + 1)}{2j_1(j_1 + 1)}}$	$\frac{m}{\sqrt{j_1(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 + m + 1)}{2j_1(j_1 + 1)}}$
$j_1 - 1$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{2j_1(2j_1 + 1)}}$	$-\sqrt{\frac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)}}$

The most useful Clebsch–Gordan coefficients are those in which one of the angular momenta, say,  $j_2$ , are small. Such coefficients, with  $j_2 \leq 4$ , can be found in general algebraic form in the last reference (Peking: Science Press, 1965). Coefficients with  $j_2 = \frac{1}{2}$  and  $j_2 = 1$  are appended.