30
The Clebsch–Gordan Series

For a system built from state vectors, $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ express the angular momentum–coupled state vector $|j_1 j_2 j m\rangle$ for the rotated system in terms of the uncoupled state vectors, also in the rotated system

$$
\begin{aligned}
|(j_1 j_2 j m)_{\text{rot.}}\rangle &= \sum_{m_1,(m_2)} |(j_1 m_1)_{\text{rot.}}\rangle |(j_2 m_2)_{\text{rot.}}\rangle \langle j_1 m_1 j_2 m_2 | j m\rangle \\
&= \sum_{\mu} |j_1 j_2 j \mu\rangle D^{j*}_{\mu m} \\
&= \sum_{m_1,(m_2)} \sum_{\mu_1,(\mu_2)} |j_1 \mu_1\rangle |j_2 \mu_2\rangle D^{j_1 *}_{\mu_1 m_1} D^{j_2 *}_{\mu_2 m_2} \langle j_1 m_1 j_2 m_2 | j m\rangle. \qquad (1)
\end{aligned}
$$

Now, expanding

$$
|j_1 \mu_1\rangle |j_2 \mu_2\rangle = \sum_{j'} |j_1 j_2 j' \mu\rangle \langle j_1 \mu_1 j_2 \mu_2 | j' \mu\rangle, \qquad (2)
$$

left-multiplying by $\langle j_1 j_2 j \mu|$, and using the orthonormality of the coupled vectors $\langle j_1 j_2 j \mu|$ and $|j_1 j_2 j' \mu\rangle$, we get (after complex conjugation of this equation, using the reality of the Clebsch–Gordan coefficients)

$$
D^{j}_{\mu m} = \sum_{m_1,(m_2)} \sum_{\mu_1,(\mu_2)} D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} \langle j_1 m_1 j_2 m_2 | j m\rangle \langle j_1 \mu_1 j_2 \mu_2 | j \mu\rangle. \qquad (3)
$$

This relation is the so-called Clebsch–Gordan series. This relation could be used in a build-up process to calculate the D functions for $j = \frac{3}{2}$ from the known D functions for $j = 1$ and $j = \frac{1}{2}$, and so on for D functions of higher j values.

Finally, from the inverse of the process used here, we get the second relation

$$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_j D^j_{\mu m} \langle j_1 m_1 j_2 m_2 | j m \rangle \langle j_1 \mu_1 j_2 \mu_2 | j \mu \rangle. \qquad (4)$$

A Addition Theorem for Spherical Harmonics

Let us replace the angular momentum eigenfunctions $\psi_{jm}(r, \theta, \phi, \vec{\sigma})$ by the spherical harmonics, $Y_{lm}(\theta\ \phi)$ and use the general rotation relation

$$Y_{lm}(\theta', \phi') = \sum_\mu Y_{l\mu}(\theta, \phi) D^l_{\mu m}(\alpha, \beta, \gamma)^*. \qquad (5)$$

Suppose we have two particles, with position vectors $\vec{r}_1 = (r_1, \theta_1, \phi_1)$ and $\vec{r}_2 = (r_2, \theta_2, \phi_2)$, (see Fig. 30.1), we first note

$$\sum_m Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2 \phi_2) = \mathcal{I}, \qquad (6)$$

where $\mathcal{I}$ is a rotationally invariant quantity, depending only on the relative position of the two particles.

$$\sum_m Y^*_{lm}(\theta'_1, \phi'_1) Y_{lm}(\theta'_2, \phi'_2) =$$
$$\sum_m \sum_{\mu\nu} Y^*_{l\mu}(\theta_1, \phi_1) D^l_{\mu m}(\alpha, \beta, \gamma) Y_{l\nu}(\theta_2, \phi_2) D^l_{\nu m}(\alpha, \beta, \gamma)^*. \qquad (7)$$

Now, using the unitarity of the D functions,

$$\sum_m D^l_{\mu m} D^{l*}_{\nu m} = \delta_{\mu\nu}, \qquad (8)$$

we have

$$\sum_m Y^*_{lm}(\theta'_1, \phi'_1) Y_{lm}(\theta'_2, \phi'_2) = \sum_\mu Y^*_{l\mu}(\theta_1, \phi_1) Y_{l\mu}(\theta_2, \phi_2) = \mathcal{I}. \qquad (9)$$

To evaluate the invariant, choose the x', y', z' coordinate system, such that $\theta'_1 = 0$; i.e., choose the z' axis along $\vec{r}_1$. Then,

$$Y_{lm}(0, \phi'_1) = \delta_{m0} \sqrt{\frac{(2l+1)}{4\pi}} \Big(P_l(\cos 0) \Big) = \delta_{m0} \sqrt{\frac{(2l+1)}{4\pi}}, \qquad (10)$$

so

$$\mathcal{I} = \sqrt{\frac{(2l+1)}{4\pi}} Y_{l0}(\theta'_2 = \theta_{12}, 0) = \frac{(2l+1)}{4\pi} P_l(\cos\theta_{12}), \qquad (11)$$

and

$$\sum_m Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) = \frac{(2l+1)}{4\pi} P_l(\cos\theta_{12}). \qquad (12)$$

That is, this invariant is expressed in terms of the Legendre polynomial, expressed in terms of the angle θ_{12} between the two vectors $\vec{r}_1$ and $\vec{r}_2$.

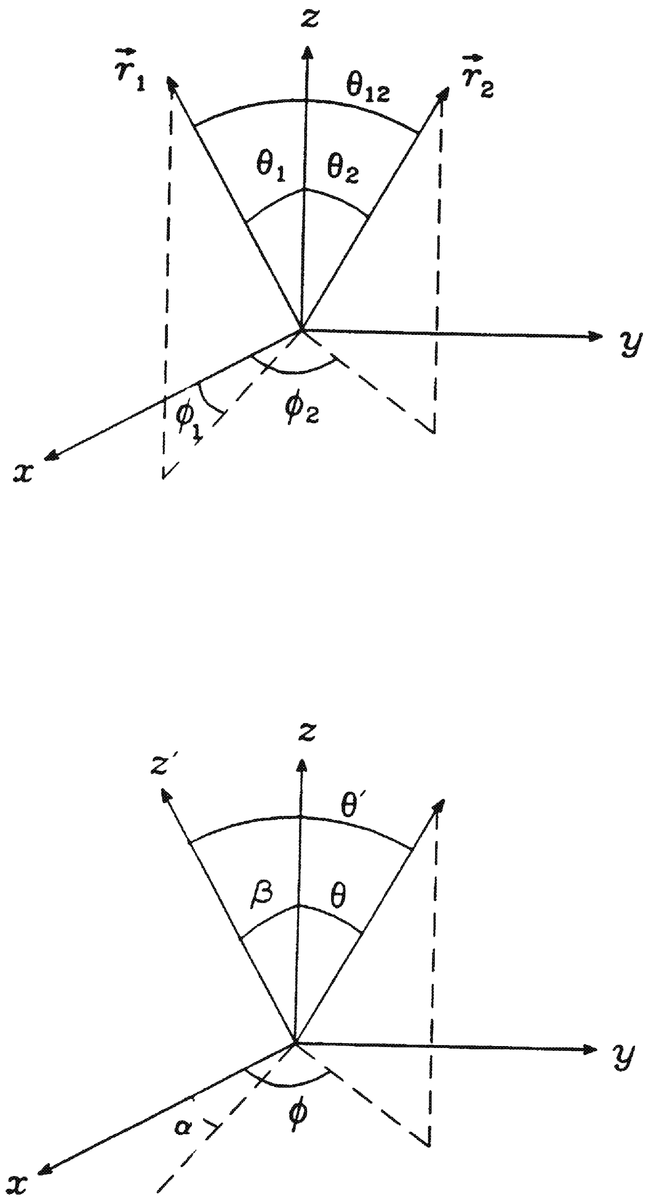

FIGURE 30.1.

By renaming $\theta_2, \phi_2 = \theta, \phi$, and $\theta_1, \phi_1 = \beta, \alpha$, and $\theta_{12} = \theta'$, i. e., by thinking of a single particle located at $\vec{r}_2 \equiv (r, \theta, \phi)$ relative to the x, y, z coordinate system, while $\vec{r}_2 \equiv (r, \theta', \phi')$ relative to the x', y', z' coordinate system, we can rewrite eq. (11) as

$$Y_{l0}(\theta', \phi') = \sqrt{\frac{4\pi}{(2l+1)}}\mathcal{I} = \sum_m \sqrt{\frac{4\pi}{(2l+1)}} Y^*_{lm}(\beta, \alpha) Y_{lm}(\theta, \phi). \tag{13}$$

Comparing this equation with

$$Y_{l0}(\theta', \phi') = \sum_m Y_{lm}(\theta, \phi) D^{l*}_{m0}(\alpha, \beta, 0), \tag{14}$$

we get

$$D^l_{m0}(\alpha, \beta, 0) = \sqrt{\frac{4\pi}{(2l+1)}} Y_{lm}(\beta, \alpha). \tag{15}$$

Also, using the unitarity of the $D^l_{\mu m}$, and writing the inverse to the Euler rotation transformation (α, β, γ) as $(-\gamma, -\beta, -\alpha)$,

$$\begin{aligned} D^l_{0m}(0, \beta, \gamma) &= D^{l*}_{m0}(-\gamma, -\beta, 0) = \sqrt{\frac{4\pi}{(2l+1)}} Y^*_{lm}(-\beta, -\gamma) \\ &= (-1)^m \sqrt{\frac{4\pi}{(2l+1)}} Y_{lm}(\beta, \gamma), \end{aligned} \tag{16}$$

so

$$D^l_{0m}(0, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{(2l+1)}} Y_{lm}(\beta, \gamma). \tag{17}$$

B Integrals of D Functions

We shall evaluate the following very useful integral:

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} d\alpha d\beta \sin\beta d\gamma \, D^{j_3*}_{\mu_3 m_3} D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} \\ &= \int\int\int d\Omega_{\alpha\beta\gamma} D^{j_3*}_{\mu_3 m_3} D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2}. \end{aligned} \tag{18}$$

To evaluate this integral, let us use eq. (4)

$$D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_j \langle j_1 m_1 j_2 m_2 | j m \rangle \langle j_1 \mu_1 j_2 \mu_2 | j \mu \rangle D^j_{\mu m} \tag{19}$$

and the relation

$$D^{j_3*}_{\mu_3 m_3} = \left(e^{i\mu_3\alpha} d^{j_3}_{\mu_3 m_3}(\beta) e^{i m_3 \gamma}\right)^* = e^{-i\mu_3\alpha} d^{j_3}_{\mu_3 m_3}(\beta) e^{-i m_3 \gamma}, \tag{20}$$

where we have used the reality of the d function. In addition the d functions have the property

$$d^{j_3}_{\mu_3 m_3}(\beta) = (-1)^{\mu_3 - m_3} d^{j_3}_{-\mu_3, -m_3}(\beta). \tag{21}$$

For the values of $j = \frac{1}{2}$ and $j = 1$, this equation follows by inspection, (see Chapter 29). For values of $j \geq \frac{3}{2}$, this equation follows from the build-up relation, eq. (3), and the fact that sign change under the transformation $m_i \to -m_i$ in the m_i-dependent Clebsch–Gordan coefficient is balanced by the same sign change under the transformation $\mu_i \to -\mu_i$ in the μ_i-dependent Clebsch–Gordan coefficient. Our integral can then be transformed into

$$I = \int\int\int d\Omega_{\alpha\beta\gamma} D^{j_3 *}_{\mu_3 m_3} D^{j_1}_{\mu_1 m_1} D^{j_2}_{\mu_2 m_2} = \sum_j \langle j_1 m_1 j_2 m_2 | j m\rangle \langle j_1 \mu_1 j_2 \mu_2 | j \mu\rangle$$
$$\times (-1)^{\mu_3 - m_3} \int\int\int d\Omega_{\alpha\beta\gamma} D^{j_3}_{-\mu_3, -m_3} D^{j}_{\mu m}. \tag{22}$$

Using the Clebsch–Gordan series once more on the product of two D functions in our integral, we obtain

$$I = \sum_j \sum_{j'} \langle j_1 m_1 j_2 m_2 | j m\rangle \langle j_1 \mu_1 j_2 \mu_2 | j \mu\rangle (-1)^{\mu_3 - m_3} \langle j \mu j_3 - \mu_3 | j' \mu - \mu_3\rangle$$
$$\times \langle j m j_3 - m_3 | j' m - m_3\rangle \int\int\int d\Omega_{\alpha\beta\gamma} D^{j'}_{\mu - \mu_3, m - m_3}. \tag{23}$$

Now, we can use explicitly

$$\int\int\int d\Omega_{\alpha\beta\gamma} D^{L}_{\mathcal{M} M} = \int_0^{2\pi} d\alpha e^{i\mathcal{M}\alpha} \int_0^{2\pi} d\gamma e^{iM\gamma} \int_0^{\pi} d\beta \sin\beta d^{L}_{\mathcal{M} M}$$
$$= 2\pi \delta_{\mathcal{M}0} 2\pi \delta_{M0} \int_0^{\pi} d\beta \sin\beta d^{L}_{00}(\beta). \tag{24}$$

We also use

$$d^{L}_{00}(\beta) = \sqrt{\frac{4\pi}{(2L+1)}} Y_{L0}(\beta, -) \qquad \text{and} \qquad Y_{00} = \frac{1}{\sqrt{4\pi}} \tag{25}$$

to get

$$\int\int\int d\Omega_{\alpha\beta\gamma} D^{L}_{\mathcal{M} M} =$$
$$2\pi \delta_{\mathcal{M}0} \delta_{M0} \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta \Big(\sqrt{\frac{4\pi}{(2L+1)}} Y_{L0}\Big)\Big(\sqrt{4\pi} Y^*_{00}\Big)$$
$$= 8\pi^2 \delta_{\mathcal{M}0} \delta_{M0} \delta_{L0}, \tag{26}$$

where we have made use of the orthonormality of the spherical harmonics. Thus, in eq. (23), we must have $j' = 0$, $\mu - \mu_3 = 0$, and $m - m_3 = 0$. This result simplifies two of the Clebsch–Gordan coefficients. From the symmetry property

(123) → (3 − 21) of the Clebsch–Gordan coefficients, we get

$$\langle jmj_3 - m_3|00\rangle = \frac{(-1)^{j-m_3}\langle 00j_3m_3|jm\rangle}{\sqrt{(2j+1)}} = \frac{(-1)^{j_3-m_3}}{\sqrt{(2j_3+1)}}\delta_{jj_3}\delta_{mm_3}. \tag{27}$$

Similarly,

$$\langle j\mu j_3 - \mu_3|00\rangle = \frac{(-1)^{\mu_3-j_3}}{\sqrt{(2j_3+1)}}\delta_{jj_3}\delta_{\mu\mu_3}. \tag{28}$$

With all of these relations, we get our final very simple result

$$\int\!\!\int\!\!\int d\Omega_{\alpha\beta\gamma} D^{j_3*}_{\mu_3m_3} D^{j_1}_{\mu_1m_1} D^{j_2}_{\mu_2m_2} = 8\pi^2 \frac{\langle j_1\mu_1 j_2\mu_2|j_3\mu_3\rangle\langle j_1m_1j_2m_2|j_3m_3\rangle}{(2j_3+1)}. \tag{29}$$

A special case follows from the above by setting $j_2 = \mu_2 = m_2 = 0$. This leads to the orthonormality integral for the D functions

$$\int\!\!\int\!\!\int d\Omega_{\alpha\beta\gamma} D^{j*}_{\mu m} D^{j'}_{\mu'm'} = \frac{8\pi^2}{(2j+1)}\delta_{jj'}\delta_{\mu\mu'}\delta_{mm'}. \tag{30}$$

Another important special case follows by setting $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, and now setting $j_i = l_i$, where l_i denotes the angular momentum quantum number is an integer. Using our result,

$$D^l_{m0}(\alpha,\beta,\gamma) = \sqrt{\frac{4\pi}{(2l+1)}}Y_{lm}(\beta,\alpha),$$

we get as a special case of the above

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta Y^*_{l_3m_3} Y_{l_1m_1} Y_{l_2m_2}$$
$$= \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l_3+1)4\pi}}\langle l_1m_1l_2m_2|l_3m_3\rangle\langle l_10l_20|l_30\rangle. \tag{31}$$

We could also write this formula as

$$\langle nl'm'|Y_{l_0m_0}|nlm\rangle = \sqrt{\frac{(2l+1)(2l_0+1)}{(2l'+1)4\pi}}\langle lml_0m_0|l'm'\rangle\langle l0l_00|l'0\rangle. \tag{32}$$

In earlier chapters, we worked hard to calculate these matrix elements for the special case with $l_0 = 1$. Now, everything follows from a knowledge of Clebsch–Gordan coefficients. Eqs. (31) and (32) tell us in particular these matrix elements are zero, unless l, l_0, l' satisfy the triangle condition of angular momentum addition, i.e., unless $l' = |l - l_0|, |l - l_0| + 1, \cdots, (l + l_0)$. Also, these matrix elements are zero unless $l + l_0 - l' =$ even integer. This parity selection rule follows because

$$\langle l_10l_20|l_30\rangle = (-1)^{l_1+l_2-l_3}\langle l_10l_20|l_30\rangle \tag{33}$$

a special case of the symmetry property

$$\langle j_1 - m_1 j_2 - m_2|j_3 - m_3\rangle = (-1)^{j_1+j_2-j_3}\langle j_1m_1j_2m_2|j_3m_3\rangle. \tag{34}$$

Because the parity of the spherical harmonics, Y_{lm}, is given by $(-1)^l$, operators of odd parity can connect levels of even parity only to levels of odd parity and vice versa, whereas operators of even parity must preserve the parity of the states. In terms of the polar and azimuth angles, θ, ϕ, the space inversion operation can be achieved by the transformation $\theta, \phi \rightarrow (\pi - \theta), (\phi + \pi)$. The parity of the spherical harmonics follows from

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi). \qquad (35)$$

The result of eq. (31) or (32) can be generalized to a much wider class of operators, which transform under rotations in our three-space like the spherical harmonics. These operators are named spherical tensor operators and will be the subject of the next chapter. Their matrix elements between the appropriate angular momentum eigenstates can be shown to have a form similar to that of eq. (31).

Problems

39. Calculate the Clebsch–Gordan coefficients, $\langle j_1 m_1 1 m_2 | jm\rangle$, for the three values, $m_2 = +1, 0, -1$, and $j = (j_1 + 1), j_1, (j_1 - 1)$, as functions of j_1 and m. Calculate first the coefficients $\langle j_1 m_1 j - m | 1 - m_2\rangle$ with $m_2 = 0, +1$ from the known coefficients with $m_2 = -1$. [See eq. (20) of Chapter 27 for the latter.] Then use symmetry properties of the Clebsch–Gordan coefficients to relate these coefficients to $\langle j_1 m_1 1 m_2 | jm\rangle$.

40. Calculate $D^{\frac{3}{2}}_{\mu m}(\alpha, \beta, \gamma)$ from a knowledge of $d^{\frac{1}{2}}_{\mu_1 m_1}(\beta)$ and $d^{1}_{\mu_2 m_2}(\beta)$ and the Clebsch–Gordan series.

A beam of particles with spin $s = \frac{3}{2}$ is polarized by a filter so the particles are in a state with $m_s = +\frac{3}{2}$, with respect to the beam axis. If there are N (particles/cm^2 sec.) in the beam, how many particles will pass through a second filter set for $m_s = +\frac{1}{2}$, but with its axis rotated about a direction perpendicular to the beam through an angle of 30^o?

41. The 2-D isotropic harmonic oscillator and the D functions.

The eigenvectors $|n_1 n_2\rangle$ of the 2-D isotropic harmonic oscillator Hamiltonian

$$H = \hbar\omega_0(\tfrac{1}{2}p_x^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}p_y^2 + \tfrac{1}{2}y^2), \quad \text{with} \quad H|n_1 n_2\rangle = \hbar\omega_0(n_1 + n_2 + 1)|n_1 n_2\rangle,$$

can be expressed in terms of oscillator creation operators

$$a_x^\dagger = \sqrt{\tfrac{1}{2}}(x - ip_x); \qquad a_y^\dagger = \sqrt{\tfrac{1}{2}}(y - ip_y),$$

by

$$|n_1 n_2\rangle = \frac{(a_x^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_y^\dagger)^{n_2}}{\sqrt{n_2!}} |00\rangle.$$

Show that the operators, Λ_i, defined by

$$\Lambda_i = \tfrac{1}{2}\,(a_x^\dagger \quad a_y^\dagger)\,(\sigma_i)\begin{pmatrix} a_x \\ a_y \end{pmatrix},$$

where σ_i are 2×2 Pauli spin matrices, satisfy standard angular momentum commutation relations, such that, if we define

$$\Lambda_{\pm} = (\Lambda_1 \pm i\Lambda_2), \qquad \Lambda_0 = \Lambda_3,$$

$$[\Lambda_0, \Lambda_{\pm}] = \pm\Lambda_{\pm}, \qquad [\Lambda_+, \Lambda_-] = 2\Lambda_0,$$

with

$$\Lambda_+ = a_x^\dagger a_y, \qquad \Lambda_- = a_y^\dagger a_x, \qquad \Lambda_0 = \tfrac{1}{2}(a_x^\dagger a_x - a_y^\dagger a_y).$$

Show that the $|n_1 n_2\rangle$ are eigenvectors of the operators $\vec{\Lambda}^2$ and Λ_0, $|\Lambda M_\Lambda\rangle$, with

$$\Lambda = \tfrac{1}{2}(n_1 + n_2), \qquad M_\Lambda = \tfrac{1}{2}(n_1 - n_2).$$

We therefore have

$$a_x^\dagger|0\rangle = |\Lambda = \tfrac{1}{2}, M_\Lambda = +\tfrac{1}{2}\rangle, \qquad a_y^\dagger|0\rangle = |\Lambda = \tfrac{1}{2}, M_\Lambda = -\tfrac{1}{2}\rangle,$$

and

$$e^{-i\beta\Lambda_2}a_x^\dagger|0\rangle = \sum_{\bar{M}_\Lambda}|\tfrac{1}{2}\bar{M}_\Lambda\rangle d^{\frac{1}{2}}_{\bar{M}_\Lambda,+\frac{1}{2}}(\beta) = \Big(a_x^\dagger \cos(\frac{\beta}{2}) + a_y^\dagger \sin(\frac{\beta}{2})\Big)|0\rangle,$$

$$e^{-i\beta\Lambda_2}a_y^\dagger|0\rangle = \sum_{\bar{M}_\Lambda}|\tfrac{1}{2}\bar{M}_\Lambda\rangle d^{\frac{1}{2}}_{\bar{M}_\Lambda,-\frac{1}{2}}(\beta) = \Big(-a_x^\dagger \sin(\frac{\beta}{2}) + a_y^\dagger \cos(\frac{\beta}{2})\Big)|0\rangle.$$

If we now rename

$$\Lambda \to j, \qquad M_\Lambda \to m, \qquad \text{so} \qquad n_1 = j + m; \; n_2 = j - m,$$

we have

$$e^{-i\beta\Lambda_2}|n_1 n_2\rangle = \sum_{\bar{n}_1,\bar{n}_2}|\bar{n}_1\bar{n}_2\rangle d^{\frac{1}{2}(n_1+n_2)}_{\frac{1}{2}(\bar{n}_1-\bar{n}_2),\frac{1}{2}(n_1-n_2)}(\beta),$$

or

$$e^{-i\beta\Lambda_2}|jm\rangle = \sum_{m'}|jm'\rangle d^j_{m',m}(\beta)$$

$$= \frac{\Big(a_x^\dagger \cos(\frac{\beta}{2}) + a_y^\dagger \sin(\frac{\beta}{2})\Big)^{j+m}}{\sqrt{(j+m)!}} \frac{\Big(-a_x^\dagger \sin(\frac{\beta}{2}) + a_y^\dagger \cos(\frac{\beta}{2})\Big)^{j-m}}{\sqrt{(j-m)!}}|0\rangle.$$

Use this expression to derive a general expression for the d function:

$$d^j_{m'm}(\beta) = \sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}}\sum_\alpha \frac{(-1)^{j-m-\alpha}(j+m)!(j-m)!}{(m+m'+\alpha)!(j-m'-\alpha)!\alpha!(j-m-\alpha)!}$$

$$\times\Big(\sin(\frac{\beta}{2})\Big)^{2j-2\alpha-m-m'}\Big(\cos(\frac{\beta}{2})\Big)^{2\alpha+m+m'}.$$

42. Show that the Hamiltonian for an electron in a uniform external magnetic field, $\vec{B}_0$, in the z direction, with $A_x = -\frac{1}{2}y_{\text{phys.}}B_0$, $A_y = \frac{1}{2}x_{\text{phys.}}B_0$, can be written in terms of dimensionless x and y, defined by

$$x_{\text{phys.}} = x\sqrt{\frac{\hbar}{m\omega_L}}, \qquad y_{\text{phys.}} = y\sqrt{\frac{\hbar}{m\omega_L}}, \qquad \text{with} \quad \omega_L = \frac{|e|B_0}{2mc},$$

as

$$H = \hbar\omega_L\Big((a_x^\dagger a_x + a_y^\dagger a_y + 1) + i(a_y^\dagger a_x - a_x^\dagger a_y)\Big) + \frac{p_z^2}{2m},$$

where

$$a_x = \sqrt{\tfrac{1}{2}}(x + ip_x), \qquad a_x^\dagger = \sqrt{\tfrac{1}{2}}(x - ip_x),$$

$$a_y = \sqrt{\tfrac{1}{2}}(y + ip_y), \qquad a_y^\dagger = \sqrt{\tfrac{1}{2}}(y - ip_y),$$

are 1-D harmonic oscillator annihilation and creation operators. Use the commutator algebra of the operators, $\Lambda_+ = a_x^\dagger a_y$, $\Lambda_- = a_y^\dagger a_x$, and $\Lambda_0 = \frac{1}{2}(a_x^\dagger a_x - a_y^\dagger a_y)$ (see problem 41) and standard angular momentum theory results to find the energy eigenvalues for $H(x, y)$. In particular, find the degeneracy, g_n, for the n^{th} level, with energy E_n.

For this purpose, use a basis in which the operators $\vec{\Lambda}^2$ and Λ_2, rather than the conventional $\vec{\Lambda}^2$ and Λ_3, are diagonal. Also, the full energy includes a contribution from the free particle motion in the z direction,

$$E_{\text{total}} = E_n + \frac{\hbar^2 k^2}{2m}, \qquad \text{with } k = 0 \to \infty,$$

where $\hbar^2 k^2$ gives the eigenvalue of p_z^2. Also, try to give an explanation for the value of the degeneracy, g_n, in terms of the possible classical orbits of the electron.