12

The Darboux Method: Supersymmetric Partner Potentials

Even if the two partner potentials, $V(x, m)$ and $V(x, m-1)$, of the form $[k^2(x, m) \mp k'(x, m) + \mathcal{L}(m)]$ of eqs. (7) and (8) of the last chapter do *not* have the same shape, it may still be possible to say something about the eigenvalue spectrum of the partner potential if the eigenvalues of one of the potentials are known. This will be true whether or not the potentials are functions of a parameter, m. This has been known since 1882 through the work of G. Darboux; (Comptes Rendus Acad. de Sci. (Paris) **94**(1882)1456). This 19^{th} century work has only recently been rediscovered by quantum theorists in connection with work in particle physics on supersymmetry. Hence, the partner potentials are known as supersymmetric partner potentials.

Suppose we have an eigenvalue problem with a potential $V_1(x)$

$$\left(-\frac{d^2}{dx^2} + V_1(x)\right)u_\lambda(x) = \left(A^\dagger A + \text{const.}\right)u_\lambda(x) = \lambda u_\lambda(x), \tag{1}$$

which is a solved problem and can be put in the form

$$\left(\left[-\frac{d}{dx} + k(x)\right]\left[\frac{d}{dx} + k(x)\right] + \text{const.}\right)u_\lambda(x) = \lambda u_\lambda(x). \tag{2}$$

Because we have no parameter, m, we have named the two operators, A, and $A^\dagger$,

$$A = \left[\frac{d}{dx} + k(x)\right]; \qquad A^\dagger = \left[-\frac{d}{dx} + k(x)\right]. \tag{3}$$

In addition, let $u_{\bar{\lambda}}$ be any solution of eq. (1), perhaps *not* a square-integrable solution,

$$\left(-\frac{d^2}{dx^2} + V_1(x)\right)u_{\bar{\lambda}}(x) = \bar{\lambda} u_{\bar{\lambda}}(x). \tag{4}$$

This equation is satisfied if we choose

$$k(x) = -\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right), \qquad \text{and} \qquad \text{const.} = \bar{\lambda}, \tag{5}$$

because

$$A^\dagger A = -\frac{d^2}{dx^2} + \left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 + \frac{d}{dx}\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right) = -\frac{d^2}{dx^2} + \left(\frac{u''_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right), \tag{6}$$

so

$$A^\dagger A u_{\bar{\lambda}} = -\frac{d^2 u_{\bar{\lambda}}}{dx^2} + u''_{\bar{\lambda}} = 0 = \left(\bar{\lambda} - \text{const.}\right) u_{\bar{\lambda}}(x). \tag{7}$$

Therefore, the constant in the original equation must be $\bar{\lambda}$ and the potential $V_1(x)$ is given by

$$V_1(x) = \left(\frac{u''_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right) + \bar{\lambda}. \tag{8}$$

Now, let us look at a different eigenvalue problem, with a different potential, and different eigenfunctions, but with the same eigenvalues λ

$$AA^\dagger w_\lambda(x) = (\lambda - \bar{\lambda}) w_\lambda(x). \tag{9}$$

The order of the operators, A, and $A^\dagger$, is reversed from that in the original equation, which was

$$A^\dagger A u_\lambda(x) = (\lambda - \bar{\lambda}) u_\lambda(x). \tag{10}$$

Now, because

$$\begin{aligned}\left(AA^\dagger + \bar{\lambda}\right) w_\lambda(x) &= \lambda w_\lambda(x) \\ &= \left[-\frac{d^2}{dx^2} + 2\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 - \left(\frac{u''_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right) + \bar{\lambda}\right] w_\lambda(x) \\ &= \left(-\frac{d^2}{dx^2} + V_2(x)\right) w_\lambda,\end{aligned} \tag{11}$$

we have

$$\begin{aligned}V_2(x) &= 2\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 - \left(\frac{u''_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right) + \bar{\lambda} \\ &= 2\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 + 2\bar{\lambda} - V_1(x),\end{aligned} \tag{12}$$

where problem 1, with potential $V_1(x)$, is given by

$$A^\dagger A u_\lambda = (\lambda - \bar{\lambda}) u_\lambda, \tag{13}$$

whereas problem 2, with potential $V_2(x)$, is given by

$$A A^\dagger w_\lambda = (\lambda - \bar{\lambda}) w_\lambda. \tag{14}$$

Now, acting on eq. (13) from the left with A yields

$$A A^\dagger (A u_\lambda) = (\lambda - \bar{\lambda})(A u_\lambda). \tag{15}$$

Thus, we see: If u_λ is an eigenfunction of problem 1, with eigenvalue λ, (Au_λ) is an eigenfunction of problem 2, with the same eigenvalue, λ. The question remains: Is w_λ square-integrable, if u_λ is square-integrable ? To answer this question, calculate

$$\int_{-\infty}^{+\infty} dx w_\lambda^* w_\lambda = \int_{-\infty}^{+\infty} dx (A u_\lambda)^* A u_\lambda = \int_{-\infty}^{+\infty} dx u_\lambda^* (A^\dagger A u_\lambda)$$
$$= (\lambda - \bar{\lambda}) \int_{-\infty}^{+\infty} dx u_\lambda^* u_\lambda. \tag{16}$$

Thus, if u_λ is square-integrable over the domain from $-\infty$ to $+\infty$ (as assumed here, or over some domain from a to b), and *if* the value $\bar{\lambda}$ lies below the lowest allowed eigenvalue λ of the original problem, the right-hand side is positive, and

$$w_\lambda(x) = \left[\frac{d}{dx} - \left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}} \right) \right] u_\lambda(x) \tag{17}$$

will also be square-integrable, even if $u_{\bar{\lambda}}$ is not. A word of caution is needed here. The above derivation required the property $A^\dagger = (A)^\dagger$, which required an integration by parts over the domain from $-\infty$ to $+\infty$. For the needed integrals to exist, the logarithmic derivative, $(u'_{\bar{\lambda}}/u_{\bar{\lambda}})$, which arises through the function $k(x)$, must not have any infinities; i.e., the function $u_{\bar{\lambda}}$ must not have any zeros. This will be true in general if $V(x)$ has both a left and a right classical turning point, and if $\bar{\lambda}$ lies below the lowest eigenvalue λ. For the lowest possible eigenvalue, the eigenfunction u_λ will have just enough curvature away from the x-axis in the classically forbidden regions so both u_λ and its first derivative will go to zero together as $x \to \pm\infty$, as required for a square-integrable function. Moreover, the lowest allowed eigenfunction will have no zeros. For a $\bar{\lambda}$ below the lowest allowed λ, the curvature away from the x-axis in the classically forbidden regions will be too great and the function $u_{\bar{\lambda}}(x)$ will approach ∞ for both $x \to \pm\infty$ before $u_{\bar{\lambda}}(x)$ can reach the value zero (see Fig. 12.1). Thus, for every eigenvalue λ of the potential $V_1(x)$, a square-integrable (calculable) eigenfunction of the potential $V_2(x)$ exists. The potential $V_2(x)$, however, has an additional eigenvalue, $\bar{\lambda}$, below the lowest λ. Two candidates exist for square-integrable eigenfunctions associated with this additional eigenvalue

$$w_{\bar{\lambda}}^{(1)} = \frac{1}{u_{\bar{\lambda}}}, \qquad w_{\bar{\lambda}}^{(2)} = \frac{1}{u_{\bar{\lambda}}} \int_0^x d\xi \, [u_{\bar{\lambda}}(\xi)]^2. \tag{18}$$

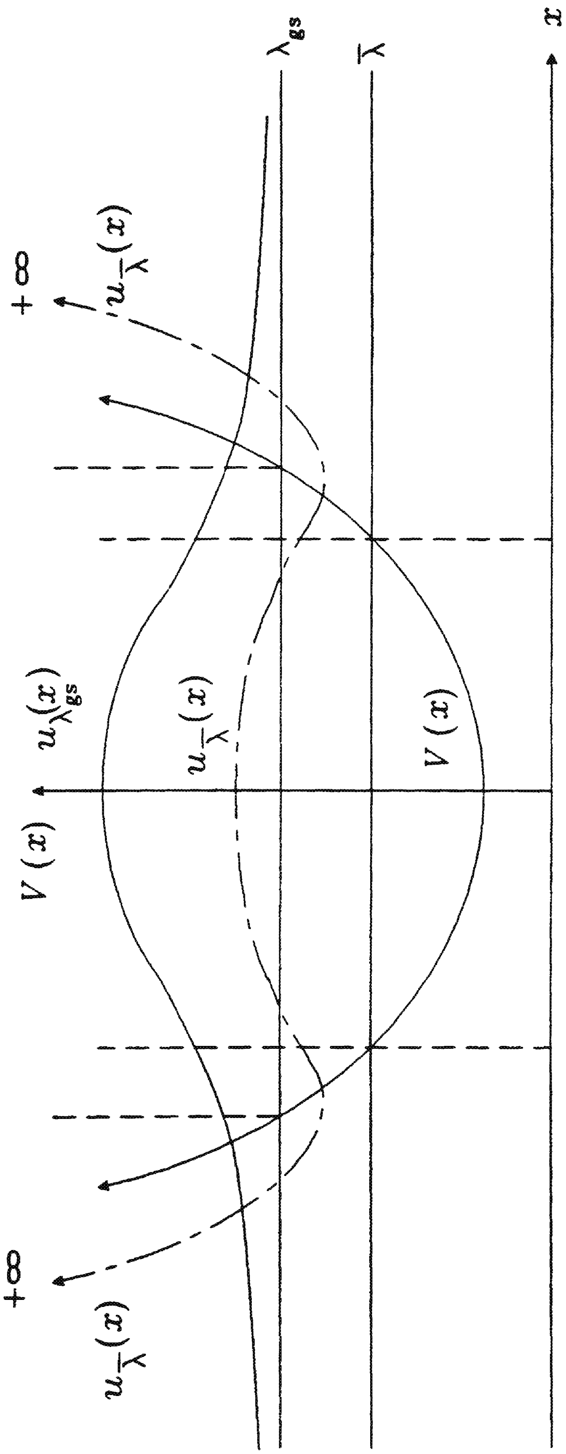

FIGURE 12.1. A $V(x)$ with a single deep minimum showing the allowed ground-state eigenfunction with $\lambda = \lambda_{gs}$ and a solution for $\bar{\lambda} < \lambda_{gs}$.

To show these equations are solutions of eq. (11) with eigenvalue $\lambda = \bar{\lambda}$, note

$$A^{\dagger}\left(\frac{1}{u_{\bar{\lambda}}}\right) = \left[-\frac{d}{dx} - \left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)\right]\left(\frac{1}{u_{\bar{\lambda}}}\right) = 0. \tag{19}$$

Also,

$$\begin{aligned} AA^{\dagger}w^{(2)}_{\bar{\lambda}} &= A\left(-\frac{d}{dx} - \frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)\frac{1}{u_{\bar{\lambda}}}\int_0^x d\xi\,[u_{\bar{\lambda}}(\xi)]^2 \\ &= A\left[\left(\frac{u'_{\bar{\lambda}}}{(u_{\bar{\lambda}})^2} - \frac{u'_{\bar{\lambda}}}{(u_{\bar{\lambda}})^2}\right)\int_0^x d\xi\,[u_{\bar{\lambda}}(\xi)]^2 - \frac{1}{u_{\bar{\lambda}}}[u_{\bar{\lambda}}]^2\right] \\ &= -Au_{\bar{\lambda}} = -\left[\frac{d}{dx} - \frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right]u_{\bar{\lambda}} = 0. \end{aligned} \tag{20}$$

If either $w^{(1)}_{\bar{\lambda}}$ or $w^{(2)}_{\bar{\lambda}}$ are square-integrable, we have a valid eigenfunction for the additional $\bar{\lambda}$ of the spectrum. The arguments given in connection with Fig. 12.1 show the new eigenfunction , $w^{(1)}_{\bar{\lambda}} = 1/u_{\bar{\lambda}}$ will in general be square-integrable if the potential $V_1(x)$ has both left and right classical turning points, and if $\bar{\lambda} < \lambda$. Thus, we have a prescription for finding an infinite number of new potentials $V_2(x)$ with an eigenvalue spectrum given by the new $\bar{\lambda}$ and the original full spectrum of λ's. The eigenfunctions for the new potential are given by eq. (17) and (18). This is the method of supersymmetric partner potentials.

In problem 17, we shall use the 1-D harmonic oscillator to find double minimum potentials with a known spectrum of eigenvalues and eigenfunctions. Because the process of finding a $V_2(x)$ from a $V_1(x)$ with a known spectrum can in principle be iterated, we can find a potential with a spectrum of eigenvalues such that a few low-lying eigenvalues are placed arbitrarily, but with a spectrum of higher eigenvalues of the initial $V_1(x)$.

Problems

17. Supersymmetric partner potentials. Use a solution $u_{\bar{\lambda}}(x)$ of the 1-D harmonic oscillator equation

$$-\frac{d^2u_{\bar{\lambda}}}{dx^2} + x^2u_{\bar{\lambda}}(x) = \bar{\lambda}u_{\bar{\lambda}}(x)$$

to find the eigenvalues and eigenfunctions for the wave equation for a particle moving in the potential $V_2(x)$, the supersymmetric partner potential, where

$$V_2(x) = 2\bar{\lambda} + 2\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 - V_1(x) = 2\bar{\lambda} + 2\left(\frac{u'_{\bar{\lambda}}}{u_{\bar{\lambda}}}\right)^2 - x^2$$

in the case $u_{\bar{\lambda}}$ is an even function of x, and $\bar{\lambda} < 1$, so $u_{\bar{\lambda}}$ has no zeros. Show first that the infinite series for $u_{\bar{\lambda}}$ can be put in the form

$$u_{\bar{\lambda}}(x) = {}_1F_1\left(\frac{(1-\bar{\lambda})}{4}; \frac{1}{2}; x^2\right)e^{-\frac{1}{2}x^2},$$

where

$$ {}_1F_1(a;b;x^2) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^{2n}}{n!}, $$

$$ \text{and } (a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1, $$

and show that

$$ u'_{\bar{\lambda}} = x\left((1-\bar{\lambda})\,{}_1F_1(\frac{(5-\bar{\lambda})}{4};\frac{3}{2};x^2) - {}_1F_1(\frac{(1-\bar{\lambda})}{4};\frac{1}{2};x^2)\right)e^{-\frac{1}{2}x^2}. $$

The case with two nearly degenerate levels near $\lambda = 1$ is of particular interest. Plot the potential $V_2(x)$ together with the eigenvalue spectrum for the two cases:

$$ \bar{\lambda} = 1 - \tfrac{1}{3}, \qquad \bar{\lambda} = 1 - \tfrac{1}{256}. $$

Also, plot the eigenfunctions for the two lowest energy eigenvalues.

18. Find the hydrogenic expectation values of $(1/r,\ 1/r^2,\ 1/r^3)$:

(a) Use

$$ \frac{dO}{dt} = \frac{i}{\hbar}[H,O] + \frac{\partial O}{\partial t} $$

to derive the quantum-mechanical form of the virial theorem, for an N-particle system, including $N = 1$:

$$ \frac{1}{4}\frac{d}{dt}\langle\psi, \sum_{i=1}^{N}(\vec{r}_i\cdot\vec{p}_i + \vec{p}_i\cdot\vec{r}_i)\psi\rangle = \langle\psi, \sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{2m}\psi\rangle - \frac{1}{2}\langle\psi, \sum_{i=1}^{N}(\vec{r}_i\cdot\vec{\nabla}_i V)\psi\rangle. $$

Use this theorem to find the expectation value of $1/r$ for a state ψ_{nlm} of the hydrogen atom.

(b) Derive the Hellmann–Feynman theorem which applies to a system whose Hamiltonian is a function of a parameter, ν, and states

$$ \frac{\partial E_n}{\partial \nu} = \langle\psi_n, \frac{\partial H}{\partial \nu}\psi_n\rangle. $$

Use this theorem to calculate the expectation value of $1/r^2$ for a state ψ_{nlm} of the hydrogen atom. Use l as the parameter, ($l \equiv \nu$). The quantum number, n, depends on this parameter through $n = (n_r + l + 1)$; ($n_r = 0, 1, 2, \ldots$).

(c) The radial functions, $u_{nl}(r)$, $u_{n'l'}(r)$, with $l' \neq l$, are by themselves not orthogonal to each other

$$ \int_0^\infty dr u_{nl}^*(r)u_{n'l'}(r) = \int_0^\infty dr r^2 R_{nl}^*(r)R_{n'l'}(r) \neq 0, $$

but show that, with $n' = n$,

$$ \int_0^\infty dr \frac{1}{r^2}u_{nl}^*(r)u_{nl'}(r) = \int_0^\infty dr R_{nl}^*(r)R_{nl'}(r) = 0, \qquad \text{with } l' \neq l. $$

Hint: Use the radial equation to evaluate

$$\frac{\hbar^2}{2\mu}[l'(l'+1) - l(l+1)] \int_0^\infty dr \frac{1}{r^2} u_{nl}^* u_{nl'}.$$

Now eliminate $\frac{d}{dr}$ from the hydrogen step-up and down operators, $O_+(l+1)$, $O_-(l)$, and combine the resulting expression for $\frac{1}{r}u_{nl}(r)$ with the above to evaluate the expectation value of $1/r^3$ in the state ψ_{nlm}.

19. Commutator algebra for the hydrogen atom.

(a) Use the dimensionless angular momentum and Runge–Lenz vectors, $\vec{L}$ and $\vec{\mathcal{R}}$, as well as the dimensionless $\vec{p}$, $\vec{r}$, H, and ϵ of problem 13 to show these dimensionless operators satisfy the commutation relations

$$[L_j, L_k] = i\epsilon_{jk\alpha} L_\alpha, \qquad [L_j, \mathcal{R}_k] = i\epsilon_{jk\alpha} \mathcal{R}_\alpha,$$

$$[\mathcal{R}_j, \mathcal{R}_k] = (-\vec{p}^2 + \frac{2}{r}) i\epsilon_{jk\alpha} L_\alpha = (-2H) i\epsilon_{jk\alpha} L_\alpha.$$

(b) Define

$$\vec{V} = \frac{\vec{\mathcal{R}}}{\sqrt{(-2\epsilon)}}$$

to show in the subspace of a fixed, n,

$$[L_j, L_k] = i\epsilon_{jk\alpha} L_\alpha, \qquad [L_j, V_k] = i\epsilon_{jk\alpha} V_\alpha, \qquad [V_j, V_k] = i\epsilon_{jk\alpha} L_\alpha.$$

If we define

$$L_{ij} = \frac{1}{i}\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right)$$

so $L_1 = L_{23}$; $L_2 = L_{31}$; $L_3 = L_{12}$, V_j can be defined through

$$V_j = \frac{1}{i}\left(x_j \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_j}\right).$$

Show that these operators satisfy the above commutation relations. That is, show the six operators, $\vec{L}$ and $\vec{V}$ can be related to the six operators, L_{ij}, with $i, j = 1, \ldots, 4$, which are the angular momentum operators in an abstract 4-D space generating rotations in this abstract 4-D space, x_1, x_2, x_3, x_4, where x_1, x_2, x_3 are our 3-D real space.

(c) Show that $\vec{M}$ and $\vec{N}$, defined by

$$\vec{M} = \tfrac{1}{2}(\vec{L} + \vec{V}), \qquad \vec{N} = \tfrac{1}{2}(\vec{L} - \vec{V}),$$

satisfy the commutation relations of two commuting angular momentum operators

$$[M_j, M_k] = i\epsilon_{jk\alpha} M_\alpha, \qquad [N_j, N_k] = i\epsilon_{jk\alpha} N_\alpha, \qquad [M_j, N_k] = 0.$$

(d) Show that

$$\vec{M}^2 - \vec{N}^2 = \tfrac{1}{2}\Big((\vec{L}\cdot\vec{V}) + (\vec{V}\cdot\vec{L})\Big) = 0, \qquad \text{see Problem 13,}$$

$$2(\vec{M}^2 + \vec{N}^2) + 1 = (\vec{L}^2 + \vec{V}^2 + 1) = -\frac{1}{2\epsilon} = n^2.$$

(e) Show that the double angular momentum eigenfunctions, $\psi_{j_1 m_1 j_2 m_2}$, with

$$\vec{M}^2 \psi_{j_1 m_1 j_2 m_2} = j_1(j_1 + 1)\psi_{j_1 m_1 j_2 m_2}, \qquad M_3 \psi_{j_1 m_1 j_2 m_2} = m_1 \psi_{j_1 m_1 j_2 m_2},$$

$$\vec{N}^2 \psi_{j_1 m_1 j_2 m_2} = j_2(j_2 + 1)\psi_{j_1 m_1 j_2 m_2}, \qquad N_3 \psi_{j_1 m_1 j_2 m_2} = m_2 \psi_{j_1 m_1 j_2 m_2},$$

are also eigenvectors of the hydrogen atom H, provided

$$j_1 = j_2 = \frac{(n-1)}{2}.$$