19

Time-Dependence of State Vectors, Algebraic Techniques, Coherent States

Before examining the time-dependence of state vectors it will be useful to recapitulate and summarize the postulates of quantum theory.

A Recapitulation: The Postulates of Quantum Theory

I. The state of a physical system is specified by a vector, $|\psi\rangle$, of the infinite-dimensional Hilbert space. (We assume $\langle\psi|\psi\rangle = 1$.)

II. Every physically observable quantity is described by a hermitian operator, A.

III. The only possible result of the actual measurement of this physically observable quantity is one of the eigenvalues of the corresponding operator, A. (a) If a_n is a nondegenerate eigenvalue of A (part of the discrete spectrum of A), with

$$A|\alpha_n\rangle = a_n|\alpha_n\rangle, \qquad \text{with} \quad \langle\alpha_n|\alpha_n\rangle = 1, \tag{1}$$

the probability a measurement of the physical observable, A, of the system specified by the state $|\psi\rangle$ will yield the value a_n is given by

$$P(a_n) = |\langle\alpha_n|\psi\rangle|^2. \tag{2}$$

(b) If a_m is a degenerate eigenvalue of A (again part of the discrete spectrum of A), with

$$A|\alpha_m^{(i)}\rangle = a_m|\alpha_m^{(i)}\rangle, \qquad \text{with} \quad i = 1, 2, \dots, g_{a_m}, \tag{3}$$

the corresponding probability a measurement of the observable A will yield the value a_m is

$$P(a_m) = \sum_{i=1}^{i=g_{a_m}} |\langle \alpha_m^{(i)} | \psi \rangle|^2. \tag{4}$$

(c) If A has a continuous spectrum (or if part of the spectrum of A is continuous), with

$$A|\alpha\rangle = \alpha|\alpha\rangle, \tag{5}$$

the probability a measurement of the physical observable, A, will yield a value between α and $\alpha + d\alpha$ is given by

$$P(\alpha)d\alpha = d\alpha |\langle \alpha | \psi \rangle|^2 \tag{6}$$

for the system specified by the state $|\psi\rangle$. Note: In the basis in which A is diagonal

$$A = \sum_{n,i} |\alpha_n^{(i)}\rangle a_n \langle \alpha_n^{(i)}| + \int d\alpha |\alpha\rangle \alpha \langle \alpha| \tag{7}$$

and

$$\begin{aligned}\langle \psi | A | \psi \rangle &= \sum_{n,i} a_n |\langle \alpha_n^{(i)} | \psi \rangle|^2 + \int d\alpha \alpha |\langle \alpha | \psi \rangle|^2 \\ &= \sum_n a_n P(a_n) + \int d\alpha \alpha P(\alpha).\end{aligned} \tag{8}$$

The operators

$$\sum_i |\alpha_m^{(i)}\rangle \langle \alpha_m^{(i)}|, \qquad \text{or} \qquad d\alpha |\alpha\rangle \langle \alpha|,$$

are projection operators onto a_m or α, respectively.

IV. Immediately after a measurement of the physical observable, A, the state of the system is specified by a new state vector. If the system was originally specified by the state vector $|\psi\rangle$, and if the measurement of A performed on the system specified by $|\psi\rangle$ yielded the specific value a_m, immediately after this measurement, the state of the system is specified by the new state vector

$$\frac{\sum_i |\alpha_m^{(i)}\rangle \langle \alpha_m^{(i)} | \psi \rangle}{\sqrt{\sum_j |\langle \alpha_m^{(j)} | \psi \rangle|^2}}.$$

For a nondegenerate a_n, on the other hand, the new state vector is simply $|\alpha_n\rangle$. The measurement of A disturbs the system! In the case of the two successive polarization filters of Chapter 17, after passing through the first filter, the state of the system was specified by $|m\rangle$. If a measurement of the polarization along the new direction z' then yielded the value λ_α, the particle comes out of the second apparatus in the state $|\alpha\rangle$.

B Time Evolution of a state $|\psi\rangle$

If at time t_0 the state of the system is specified by $|\psi(t_0)\rangle$, what is the state of the system at a later time, t; i.e., what is $|\psi(t)\rangle$? If a measurement is made on the system of some observable A (or B, etc.), the time-evolution of the quantum system is noncausal. The measurement of A (between times t_0 and t) disturbs the system. The measurement process itself must be taken into account. If we enlarge the system to include the whole measurement apparatus, this larger system would have to be studied, and it may not be practical to consider this larger system. For an isolated system (undisturbed by an observer and his apparatus in the time interval from t_0 to t), however, the evolution in time from $|\psi(t_0)\rangle$ to $|\psi(t)\rangle$ is completely causal (though we are still tied to the probability description). The time evolution is given by

$$-\frac{\hbar}{i}\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{9}$$

The Hamiltonian will often not be an explicit function of the time, but we have here allowed for the possibility of an explicit time-dependence. We can also think of the $|\psi(t)\rangle$ being produced by the action of a unitary operator acting on $|\psi(t_0)\rangle$

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle, \qquad \text{with} \quad U(t_0,t_0) = 1, \tag{10}$$

so

$$-\frac{\hbar}{i}\frac{d}{dt}U(t,t_0)|\psi(t_0)\rangle = H(t)U(t,t_0)|\psi(t_0)\rangle, \tag{11}$$

and because this is valid for any arbitrary $|\psi(t_0)\rangle$,

$$-\frac{\hbar}{i}\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0). \tag{12}$$

This equation has the solution

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')U(t',t_0). \tag{13}$$

If H is an explicit function of the time, it may be difficult to do the integral. If H is not an explicit function of the time, the equation for U can be integrated and yields

$$U(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)H}. \tag{14}$$

U is unitary if H is hermitian, $H^\dagger = H$. The operator H is then the generator of this unitary operator, which now gives a shift in time or a "time-translation." Recall the operator p_x was the generator, G, for a space translation in the x-direction. Now the time shift, $(t - t_0)$, has taken the place of the space shift, c_1, of Chapter 18. For an H, which is not an explicit function of time,

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}(t-t_0)H}|\psi(t_0)\rangle. \tag{15}$$

The description given here is the so-called *Schrödinger picture* of quantum theory, in which (a) the state vectors $|\psi\rangle$ change with time, but (b) all physical observables A, such as p_x, or x, or functions of $\vec{r}$, $\vec{p}$, such as $H(\vec{p}, \vec{r})$, are assumed to have no explicit time dependence.

An alternative point of view was taken by Heisenberg and is known by the name, the *Heisenberg picture*. Now, (a) state vectors are assumed constant in time, but (b) the physical observables are taken to vary with time. To make the transition from the Schrödinger picture to the Heisenberg picture, consider the expectation value of an operator, $O(\vec{p}, \vec{r})$, where we use the subscript, S, to designate this operator, *not* explicitly a function of t, i.e., $O = O_S$, similarly for the time-dependent or Schrödinger state vector $|\psi(t)\rangle = |\psi_S\rangle$. Then,

$$\begin{aligned} \langle\psi(t)|O|\psi(t)\rangle &\equiv \langle\psi_S|O_S|\psi_S\rangle \\ = \langle\psi(t_0)|U^\dagger(t,t_0)OU(t,t_0)|\psi(t_0)\rangle &\equiv \langle\psi_H|O_H|\psi_H\rangle, \end{aligned} \tag{16}$$

where the subscript H now stands for Heisenberg. The time-independent state vector is $|\psi(t_0)\rangle = |\psi_H\rangle$, and the time-dependent Heisenberg operator, O_H, is given by

$$O_H = U^\dagger(t,t_0)O_SU(t,t_0). \tag{17}$$

Note,

$$\begin{aligned} -\frac{\hbar}{i}\frac{dO_H}{dt} &= \Big(-\frac{\hbar}{i}\frac{dU^\dagger}{dt}\Big)O_SU + U^\dagger O_S\Big(-\frac{\hbar}{i}\frac{dU}{dt}\Big) \\ &= -U^\dagger HO_SU + U^\dagger O_SHU \\ &= -(U^\dagger HU)(U^\dagger O_SU) + (U^\dagger O_SU)(U^\dagger HU) \\ &= -HO_H + O_HH, \end{aligned} \tag{18}$$

where we have used $O_H = U^\dagger O_SU$ and that the Hamiltonian operator, H, commutes with U, which is a function only of pure numbers and the operator H itself. Thus, $U^\dagger HU = HU^\dagger U = H$. The time dependence of O_H is then given by the equation

$$-\frac{\hbar}{i}\frac{dO_H}{dt} = [O_H, H]. \tag{19}$$

This relation is known as the Heisenberg equation. Even in the Schrödinger picture, we sometimes introduce artificially an explicit time dependence, so $O_S = O_S(t)$. In that case, the corresponding equation would be

$$-\frac{\hbar}{i}\frac{dO_H}{dt} = [O_H, H] + \Big(-\frac{\hbar}{i}\frac{\partial O}{\partial t}\Big)_H, \quad \text{with} \quad \Big(-\frac{\hbar}{i}\frac{\partial O}{\partial t}\Big)_H = U^\dagger\Big(-\frac{\hbar}{i}\frac{\partial O_S}{\partial t}\Big)U. \tag{20}$$

In the Heisenberg picture, the kets and bras are time independent. We shall take matrix elements of eq. (19) (assuming that $\frac{\partial O}{\partial t} = 0$) in the energy representation, i.e., between eigenstates of the Hamiltonian, $|n\rangle$, assuming for the moment H has only a discrete spectrum. Then, eq. (19) leads to

$$-\frac{\hbar}{i}\frac{d}{dt}\langle n|O_H(t)|m\rangle = \langle n|O_H(t)|m\rangle(E_m - E_n), \tag{21}$$

$$\frac{d(\langle n|O_H(t)|m\rangle)}{\langle n|O_H(t)|m\rangle} = +\frac{i}{\hbar}(E_n - E_m)dt, \tag{22}$$

$$\langle n|O_H(t)|m\rangle = \langle n|O|m\rangle e^{\frac{i}{\hbar}(E_n - E_m)t}, \tag{23}$$

where we have named the integration constant $\langle n|O|m\rangle$, which is time independent. The Heisenberg matrix elements, $O_{nm} \equiv \langle n|O|m\rangle$, are then just the $\langle n|O_S|m\rangle$. Eq. (23) was essentially the starting point in Heisenberg's thinking. He started with the Fourier time analysis of the classical quantity, O, and replaced the n^{th} overtone of the classical ω with the two-index Bohr quantity $\omega_{nm} = (E_n - E_m)/\hbar$. Similarly, he replaced the n^{th} Fourier coefficient, O_n, in the Fourier time expansion with a two-index quantity he interpreted as the nm^{th} matrix element of O, $\langle n|O|m\rangle$.

C The Heisenberg Treatment of the One-Dimensional Harmonic Oscillator: Oscillator Annihilation and Creation Operators

Let us now briefly follow Heisenberg's analysis of one of his simplest examples, the 1-D harmonic oscillator. Again, we introduce scale factors to transform the physical coordinate, momentum, energy, and so on, to dimensionless x, p_x, ..., $x_{phys.} = x\sqrt{\hbar/m\omega_0}$, $p_{phys.} = p_x\sqrt{\hbar m\omega_0}$, $H_{phys.} = H\hbar\omega_0$, so

$$H = \frac{1}{2}(p_x^2 + x^2). \tag{24}$$

Introduce the two operators

$$a = \frac{1}{\sqrt{2}}(x + ip_x), \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - ip_x). \tag{25}$$

From the commutation relation, $[p_x, x] = -i$, we get the commutator

$$[a, a^\dagger] = 1, \tag{26}$$

and the Hamiltonian can be expressed as

$$H = \frac{1}{2}\big(a^\dagger a + aa^\dagger\big) = \big(a^\dagger a + \frac{1}{2}\big). \tag{27}$$

It will be useful to name the operator $a^\dagger a = N$. Then, we have the family of three commutation relations

$$[N, a^\dagger] = +a^\dagger, \qquad [N, a] = -a, \qquad [a, a^\dagger] = 1. \tag{28}$$

These relations should be compared and contrasted with the standard angular momentum commutation relations

$$[L_0, L_+] = +L_+, \qquad [L_0, L_-] = -L_-, \qquad [L_-, L_+] = -2L_0. \tag{29}$$

The commutator algebra of a, $a^\dagger$, N, is known as the Heisenberg algebra. Note, in particular, the difference from the angular momentum [or SO(3)] algebra in the last entry. If the eigenvectors of the operator, N, are named $|n\rangle$, and its eigenvalues N_n, these $|n\rangle$ are also the eigenvectors of H, because $H = N + \frac{1}{2}$,

$$N_{\mathrm{op.}}|n\rangle = N_n|n\rangle, \qquad \text{and} \qquad E_n = \hbar\omega_0(N_n + \frac{1}{2}), \tag{30}$$

or

$$(1) \quad a^\dagger a|n\rangle = N_n|n\rangle, \qquad (2) \quad aa^\dagger|n\rangle = (N_n + 1)|n\rangle. \tag{31}$$

Acting with a on (1), and using $aa^\dagger = N_{\mathrm{op.}} + 1$, we get

$$aa^\dagger\big(a|n\rangle\big) = N_n\big(a|n\rangle\big), \qquad \text{or} \quad N_{\mathrm{op.}}\big(a|n\rangle\big) = (N_n - 1)\big(a|n\rangle\big). \tag{32}$$

Similarly, acting with $a^\dagger$ on (2), we get

$$a^\dagger a\big(a^\dagger|n\rangle\big) = (N_n + 1)\big(a^\dagger|n\rangle\big), \qquad \text{or} \quad N_{\mathrm{op.}}\big(a^\dagger|n\rangle\big) = (N_n + 1)\big(a^\dagger|n\rangle\big). \tag{33}$$

Eq. (32) tells us

$$\text{either} \qquad (a|n\rangle) = \text{const.}|(n-1)\rangle, \qquad \text{or} \qquad (a|n\rangle) = 0. \tag{34}$$

Similarly, eq. (33) tells us

$$\text{either} \qquad (a^\dagger|n\rangle) = \text{const.}'|(n+1)\rangle, \qquad \text{or} \qquad (a^\dagger|n\rangle) = 0, \tag{35}$$

where $|(n \pm 1)\rangle$ are shorthand notation for the eigenvectors of $N_{\mathrm{op.}}$ with eigenvalues $(N_n \pm 1)$. Now we take the diagonal matrix element of the operator, $N_{\mathrm{op.}}$ (note the similarity of the procedure for the angular momentum algebra!),

$$\begin{aligned} N_n &= \langle n|a^\dagger a|n\rangle \\ &= \sum_k \langle n|a^\dagger|k\rangle\langle k|a|n\rangle \\ &= \sum_k |\langle k|a|n\rangle|^2 \\ &\geq 0. \end{aligned} \tag{36}$$

Thus, N_n is positive definite. Now, if $(a|n\rangle)$ exists, repeat this process by taking the diagonal matrix element of $N_{\mathrm{op.}}$ between states $|(n-1)\rangle$. We conclude $(N_n - 1) \geq 0$. We can repeat this process j times to conclude $(N_n - j) \geq 0$, if $(a|n - j + 1\rangle) \neq 0$. If N_n is positive, however, an integer j will eventually come, which is big enough such that $(N_n - j)$ would be negative unless we hit a state $|n_{\mathrm{min.}}\rangle$, such that $(a|n_{\mathrm{min.}}\rangle) = 0$. For this state, eq. (31) tells us $N_{n_{\mathrm{min.}}} = 0$. Now, if we act on this state, with eigenvalue $N_{n_{\mathrm{min}}} = 0$, n times in succession with $a^\dagger$, we get a state with eigenvalue $(0 + n)$, i.e., with $N_n = n$. Also, continued operation with $a^\dagger$ leaves the eigenvalue of $N_{\mathrm{op.}}$ positive, no upper bound to this discrete set of eigenvalues exists. Thus,

$$E_n = \hbar\omega_0(n + \frac{1}{2}). \tag{37}$$

All that remains is the job of calculating the matrix elements of x and p_x. From eq. (36)

$$N_n = n = \sum_k |\langle k|a|n\rangle|^2$$
$$= |\langle (n-1)|a|n\rangle|^2, \tag{38}$$

where we have assumed the states for this one-degree-of-freedom problem are nondegenerate, so there is just one state, with $k = (n-1)$. Except, for an arbitrary phase, we have determined the matrix element of the operator, a. Choosing the simplest (positive, real) value for this matrix element, we have

$$\langle (n-1)|a|n\rangle = \sqrt{n}. \tag{39}$$

Hermitian conjugation gives us

$$\langle n|a^\dagger|(n-1)\rangle = \sqrt{n}, \qquad \text{or} \qquad \langle (n+1)|a^\dagger|n\rangle = \sqrt{n+1}. \tag{40}$$

$a^\dagger$ is an oscillator quantum creation operator, and a is an oscillator quantum annihilation operator. Now, using

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger), \qquad \text{and} \qquad p_x = \frac{i}{\sqrt{2}}(-a + a^\dagger), \tag{41}$$

we also get

$$\langle m|x|n\rangle = \frac{1}{\sqrt{2}}\Big(\delta_{m(n-1)}\sqrt{n} + \delta_{m(n+1)}\sqrt{n+1}\Big), \tag{42}$$

$$\langle m|p_x|n\rangle = \frac{1}{\sqrt{2}}\Big(-i\delta_{m(n-1)}\sqrt{n} + i\delta_{m(n+1)}\sqrt{n+1}\Big). \tag{43}$$

These relations are of course results we have obtained before; but the Heisenberg method of calculation required no knowledge of wave functions or differential equations. As before, we can build more complicated operators from powers of x and p_x, and then express any operator in Heisenberg (time-dependent) form

$$O_H(t) = \sum_{n,m} |n\rangle\langle n|O|m\rangle\langle m| e^{\frac{i}{\hbar}(E_n - E_m)t}. \tag{44}$$

For x and p_x, we could write the Heisenberg (time-dependent) form of these operators in terms of the Schrödinger (time-independent) oscillator quantum annihilation and creation operators, a and $a^\dagger$, as

$$x_H(t) = \frac{1}{\sqrt{2}}\big(ae^{-i\omega_0 t} + a^\dagger e^{+i\omega_0 t}\big),$$
$$(p_x)_H(t) = \frac{1}{\sqrt{2}}\big(-iae^{-i\omega_0 t} + ia^\dagger e^{+i\omega_0 t}\big), \tag{45}$$

where we have used eq. (44) to get the time dependence, but have subsequently left off the unit operators, $\sum_n |n\rangle\langle n| = 1$ and $\sum_m |m\rangle\langle m| = 1$.

D Oscillator Coherent States

Remembering the physical displacement and momentum coordinates of the harmonic oscillator are related to the dimensionless x and p_x of the last section by $q = \sqrt{\hbar/m\omega_0}x$ and $p = \sqrt{\hbar m\omega_0}p_x$, we can express the time-dependent displacement and momentum operators through eqs. (41) and (45) by

$$
\begin{aligned}
q(t) &= \sqrt{\frac{\hbar}{2m\omega_0}}\Big(ae^{-i\omega_0 t} + a^\dagger e^{+i\omega_0 t}\Big),\\
p(t) &= m\omega_0\sqrt{\frac{\hbar}{2m\omega_0}}\Big(-iae^{-i\omega_0 t} + ia^\dagger e^{+i\omega_0 t}\Big).
\end{aligned} \tag{46}
$$

If we compare this with the classical solution for the harmonic oscillator, $q(t) = q_0\cos(\omega_0 t + \phi)$, with

$$
\begin{aligned}
q(t) &= \Big(\frac{q_0 e^{-i\phi}}{2}e^{-i\omega_0 t} + \frac{q_0 e^{i\phi}}{2}e^{+i\omega_0 t}\Big),\\
p(t) &= m\omega_0\Big(-i\frac{q_0 e^{-i\phi}}{2}e^{-i\omega_0 t} + i\frac{q_0 e^{i\phi}}{2}e^{-i\omega_0 t}\Big),
\end{aligned} \tag{47}
$$

we are led to the idea that it might be very useful to replace the eigenvectors, $|n\rangle$, of the harmonic oscillator hamiltonian (or the oscillator quantum number operator), with eigenvectors of the operator, a (or alternatively $a^\dagger$), if we want to study the transition from the quantum oscillator to the classical oscillator. Moreover, we might expect the eigenvalue of the operator, a, to be given by a complex number, where the square of the absolute value of this number is related to the energy or the number of oscillator quanta, and the argument of this complex number is related to the classical phase. The new oscillator representation in terms of the eigenvectors of the oscillator annihilation operator, a, might be particularly useful if the physics of interest involves a statistical distribution of states with different numbers of oscillator quanta, particularly, if the average oscillator excitation number is large, as in the classical limit. Later, when we quantize the electromagnetic field (see Chapter 60), we shall write the hamiltonian of the electromagnetic field

$$
H = \sum_{\vec{k},\mu}\hbar\omega(a^\dagger_{\vec{k}\mu}a_{\vec{k}\mu} + \tfrac{1}{2})
$$

in terms of an infinite number of oscillators with annihilation and creation operators, $a_{\vec{k}\mu}$ and $a^\dagger_{\vec{k}\mu}$. These operators are interpreted as photon annihilation and creation operators, where the vector $\vec{k}$ is the wave vector, corresponding to the circular frequency $\omega = kc = |\vec{k}|c$, and μ is a polarization index for the photon. The generalization of the single-mode coherent state to be studied in this section to the multimode coherent state of the electromagnetic field will be useful in the study of optical beams whose proper quantum-mechanical description is given by a statistical distribution of quantum states with different numbers of oscillator quanta. (The seminal papers by R. J. Glauber on quantum optics and optical coherent states are all reprinted in an introduction to coherent states by John R.

Klauder and Bo-Sture Skagerstam, *Coherent States. Applications in Physics and Mathematical Physics*, Singapore: World Scientific, 1985.)

For the single-mode harmonic oscillator, two slightly different definitions of the coherent states can be found in the literature, to be denoted by $|z\rangle$, where z is a complex number giving the eigenvalue of the oscillator annihilation operator. In the first definition of the coherent state

$$|z\rangle_{\rm I} = e^{z^* a^\dagger - za}|0\rangle = U(z)|0\rangle, \tag{48}$$

where $|0\rangle$, the "vacuum state," is the oscillator ground state and $U(z)$ is a unitary operator, because

$$U(z)^\dagger = e^{-(z^* a^\dagger - za)} = U(-z) = U(z)^{-1}. \tag{49}$$

The noncommuting operators, $A \equiv z^* a^\dagger$ and $B = -za$, have a very simple commutator, $[A, B] = zz^*$, a c number commuting with both A and B. In this very special case, when

$$[A, [A, B]] = 0 \qquad \text{and} \qquad [B, [A, B]] = 0, \tag{50}$$

we can write the operator relation

$$e^{(A+B)} = e^{-\frac{1}{2}[A,B]} e^A e^B, \tag{51}$$

as can be shown by direct verification. Note the so-called normal order of the operators on the right-hand side, with creation operators, $a^\dagger$, sitting to the left of annihilation operators, a. The above result leads to

$$|z\rangle_{\rm I} = e^{-\frac{1}{2}z^* z} e^{z^* a^\dagger} e^{-za}|0\rangle = e^{-\frac{1}{2}z^* z} e^{z^* a^\dagger}|0\rangle, \tag{52}$$

because $a|0\rangle = 0$. For some purposes, a somewhat simpler second definition of the coherent state may therefore be useful

$$|z\rangle_{\rm II} = e^{z^* a^\dagger}|0\rangle, \tag{53}$$

which differs from the definition $|z\rangle_{\rm I}$ by the simple c number function, $e^{-zz^*/2}$. This second definition may be a particularly useful definition for generalized coherent states, such as the angular momentum coherent states to be introduced in the next section for which the commutator algebra no longer satisfies the simple relations of eq. (50). To avoid confusion, we will denote the complex number z in the two different definitions by two different symbols

$$\begin{aligned}
|z\rangle_{\rm I} &\equiv |\alpha\rangle = e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha^* a^\dagger}|0\rangle = e^{-\frac{1}{2}\alpha^*\alpha} \sum_{n=0} \frac{(\alpha^*)^n}{\sqrt{n!}}|n\rangle, \\
|z\rangle_{\rm II} &\equiv |z\rangle = e^{z^* a^\dagger}|0\rangle = \sum_{n=0} \frac{(z^*)^n}{\sqrt{n!}}|n\rangle.
\end{aligned} \tag{54}$$

Both are eigenvectors of the oscillator annihilation operator, a, with eigenvalue given by the complex number α^* or z^*.

$$\begin{aligned}
a|\alpha\rangle &= \alpha^*|\alpha\rangle, \\
a|z\rangle &= z^*|z\rangle.
\end{aligned} \tag{55}$$

This follows, for the type II coherent state, from

$$a|z\rangle = ae^{z^*a^\dagger}|0\rangle = a\sum_{n=0}\frac{(z^*a^\dagger)^n}{n!}|0\rangle = z^*\sum_{n=1}\frac{(z^*a^\dagger)^{n-1}}{(n-1)!}|0\rangle = z^*e^{z^*a^\dagger}|0\rangle. \quad (56)$$

For the type I coherent state, the c number, $e^{-\alpha\alpha^*/2}$, is merely carried along in the analagous derivation.

The coherent states $|\alpha\rangle$ (or $|z\rangle$) then give us another continuous representation of an arbitrary state vector $|\psi\rangle$ of a physical system. Besides the discrete oscillator quanta representation, $\langle n|\psi\rangle$, we already have two continuous representations, the coordinate representation $\langle x|\psi\rangle$ and the momentum representation $\langle p_x|\psi\rangle$, where the operators x and p_x have a continuous range of eigenvalues from $-\infty \to +\infty$. We can now add the coherent state representation $\langle\alpha|\psi\rangle$, where α is a complex number,

$$\alpha = \xi + i\eta = \rho e^{i\phi},$$

and ξ and η range from $-\infty \to +\infty$, and ρ ranges from $0 \to \infty$ and ϕ from $0 \to 2\pi$. Unlike $\langle x|\psi\rangle$ and $\langle p_x|\psi\rangle$, however, which are orthonormal, continuous representations of $|\psi\rangle$, with

$$\langle x'|x\rangle = \delta(x'-x), \qquad \langle p_x'|p_x\rangle = \delta(p_x'-p_x),$$

the scalar product $\langle\alpha'|\alpha\rangle$ does not lead to δ-functions. Instead,

$$\begin{aligned}\langle\alpha'|\alpha\rangle &= e^{-\frac{1}{2}|\alpha'|^2}e^{-\frac{1}{2}|\alpha|^2}\langle 0|e^{\alpha'^* a}e^{\alpha^* a^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha'|^2}e^{-\frac{1}{2}|\alpha|^2}\sum_{n,m}\frac{\alpha'^m}{\sqrt{m!}}\frac{\alpha^{*n}}{\sqrt{n!}}\langle m|n\rangle\\ &= e^{-\frac{1}{2}|\alpha'|^2}e^{-\frac{1}{2}|\alpha|^2}e^{\alpha'\alpha^*},\end{aligned} \quad (57)$$

so $\langle\alpha'|\alpha\rangle$ is a complicated function of α and α', even though $\langle\alpha|\alpha\rangle = 1$. Also,

$$\langle z'|z\rangle = e^{z'z^*}, \qquad \text{with} \quad \langle z|z\rangle = e^{|z|^2} \neq 1. \quad (58)$$

The coherent states, however, are complete. In fact, with relations (57) or (58), in place of the Dirac δ functions, the coherent states are overcomplete. The completeness can be seen from the existence of the unit operators. For the type I coherent state, the unit operator is

$$\mathbf{1} = \frac{1}{\pi}\int \mathbf{d}^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathbf{d}\xi\int_{-\infty}^{\infty}\mathbf{d}\eta|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int_0^\infty \rho\mathbf{d}\rho\int_0^{2\pi}\mathbf{d}\phi|\alpha\rangle\langle\alpha|. \quad (59)$$

With the use of eq. (54), this unit operator transforms into

$$\begin{aligned}\mathbf{1} &= \frac{1}{\pi}\int d^2\alpha e^{-\alpha\alpha^*}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}|n\rangle\langle m|\\ &= \sum_{n,m}\frac{1}{\pi}\int_0^\infty d\rho\frac{e^{-\rho^2}\rho^{n+m+1}}{\sqrt{n!m!}}\int_0^{2\pi}d\phi e^{i(m-n)\phi}|n\rangle\langle m|\\ &= \sum_{n,m}2\int_0^\infty d\rho\frac{e^{-\rho^2}\rho^{n+m+1}}{\sqrt{n!m!}}\delta_{nm}|n\rangle\langle m| = \sum_{n=0}^{\infty}2\int_0^\infty d\rho\frac{e^{-\rho^2}\rho^{2n+1}}{n!}|n\rangle\langle n|\end{aligned}$$

$$= \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{60}$$

where the completeness relation, $\sum_n |n\rangle\langle n| = 1$, was in fact proved in detail in Chapter 5, eq. (13), via

$$\sum_{n=0}^{\infty} \langle x'|n\rangle\langle n|x\rangle = \sum_{n=0}^{\infty} \psi_n(x')\psi_n^*(x) = \delta(x'-x) = \langle x'|x\rangle. \tag{61}$$

For the type II coherent state, conversely, the unit operator requires the Bargmann weighting factor, e^{-zz^*}, so

$$\mathbf{1} = \frac{1}{\pi}\int d^2z e^{-zz^*}|z\rangle\langle z| = \frac{1}{\pi}\int d^2z e^{-zz^*}\sum_{n,m}\frac{z^{*n}z^m}{\sqrt{n!m!}}|n\rangle\langle m|$$
$$= \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{62}$$

as above.

The type II coherent state realization $\langle z|\psi\rangle$ of an arbitrary state vector is simply the Bargmann transform of $\psi(x)$: $\langle z|\psi\rangle = F(z)$.

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty}\langle n|\frac{z^n}{\sqrt{n!}}|\psi\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\int_{-\infty}^{\infty} dx\psi_n^*(x)\psi(x) = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}c_n, \tag{63}$$

where $\psi(x) = \langle x|\psi\rangle$ is the coordinate representation of $|\psi\rangle$, so

$$\langle z|\psi\rangle = \int dx A(x,z)\psi(x) = F(z),$$

where $A(x,z)$ is the Bargmann kernel function with the simple expansion in terms of the *real* $\psi_n(x)$ [see eq. (5) of chapter 5],

$$A(x,z) = \sum_{n=0}^{\infty}\psi_n(x)\frac{z^n}{\sqrt{n!}}.$$

We therefore again have two possible forms of the scalar product of two state vectors

$$\langle\psi_a|\psi_b\rangle = \int_{-\infty}^{\infty} dx\psi_a^*(x)\psi_b(x) = \frac{1}{\pi}\int d^2z e^{-zz^*}F_a^*(z)F_b(z). \tag{64}$$

The complex k of Chapter 5 has here been renamed z, because k for the multi-mode electromagnetic oscillator is reserved for the wavenumber of the mode. Note, further, the natural appearance of the Bargmann measure, e^{-zz^*}/π, in the scalar product. Finally, the coherent state realizations $F(z) = \langle z|\psi\rangle$ are analytic functions of z: We have mapped the coordinate functions, $\sum_n \psi_n(x)c_n$, into analytic functions, $\sum_n (z^n/\sqrt{n!})c_n$, in a 2-D complex domain.

Let us examine some further properties of the coherent state $|\alpha\rangle$. From eq. (54), $\langle n|\alpha\rangle = e^{-\frac{1}{2}\alpha^*\alpha}(\alpha^*)^n/\sqrt{n!}$. The probability of finding the oscillator in the n^{th} level in the coherent state $|\alpha\rangle$ is, therefore,

$$P_n(\alpha) = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}. \tag{65}$$

The expectation value of the oscillator quantum number operator in the state $|\alpha\rangle$ is

$$< N >= \langle\alpha|a^\dagger a|\alpha\rangle = \alpha\alpha^* = |\alpha|^2, \tag{66}$$

so the probability

$$P_n(\alpha) = \frac{e^{-<N>}(< N >)^n}{n!} \tag{67}$$

is given by the familiar Poisson distribution. We also have the expectation values

$$\begin{aligned}
\langle\alpha|x|\alpha\rangle &= \sqrt{\tfrac{1}{2}}(\langle\alpha|a + a^\dagger|\alpha\rangle = \sqrt{\tfrac{1}{2}}(\alpha^* + \alpha) = \sqrt{2}\xi, \\
\langle\alpha|x^2|\alpha\rangle &= \tfrac{1}{2}\langle\alpha|(aa + a^\dagger a^\dagger + 2a^\dagger a + 1)|\alpha\rangle = \tfrac{1}{2}[(\alpha^* + \alpha)^2 + 1], \\
\langle\alpha|p_x|\alpha\rangle &= -i\sqrt{\tfrac{1}{2}}\langle\alpha|a - a^\dagger|\alpha\rangle = -i\sqrt{\tfrac{1}{2}}(\alpha^* - \alpha) = -\sqrt{2}\eta, \\
\langle\alpha|p_x^2|\alpha\rangle &= \tfrac{1}{2}\langle\alpha|(-aa - a^\dagger a^\dagger + 2a^\dagger a + 1)|\alpha\rangle = \tfrac{1}{2}[1 - (\alpha^* - \alpha)^2],
\end{aligned} \tag{68}$$

so

$$(\Delta x)^2_\alpha = \tfrac{1}{2}, \qquad (\Delta p_x)^2_\alpha = \tfrac{1}{2}. \tag{69}$$

In the coherent state, $|\alpha\rangle$, we therefore have

$$\Delta p_x \Delta x = \tfrac{1}{2} \qquad \text{and, thus,} \quad \Delta p \Delta q = \frac{\hbar}{2}, \tag{70}$$

where dimensionless x and p_x have been converted to physical q and p. The coherent state is therefore a state with the minimum possible uncertainty, despite the seemingly complicated probability distribution, $P_n(\alpha)$, spread over a range of states $|n\rangle$ about the most likely $n =< N >= |\alpha|^2$. To understand this, we note that the unitary operator, $U(\alpha) = e^{(\alpha^* a^\dagger - \alpha a)}$, shifts the operators, a and $a^\dagger$, hence, x and p_x, according to

$$U(\alpha)aU^{-1}(\alpha) = a - \alpha^*, \qquad U(\alpha)a^\dagger U^{-1}(\alpha) = a^\dagger - \alpha, \tag{71}$$

where we have used

$$\sum_{n=0}^{\infty}[a, \frac{(\alpha a - \alpha^* a^\dagger)^n}{n!}] = -\alpha^* \sum_{n=1}^{\infty} \frac{(\alpha a - \alpha^* a^\dagger)^{n-1}}{(n-1)!}.$$

Eq. (71) thus leads to

$$\begin{aligned}
U(\alpha)xU^{-1}(\alpha) &= x - \sqrt{\tfrac{1}{2}}(\alpha + \alpha^*) = (x - \sqrt{2}\xi), \\
U(\alpha)p_xU^{-1}(\alpha) &= p_x + i\sqrt{\tfrac{1}{2}}(\alpha^* - \alpha) = (p_x + \sqrt{2}\eta).
\end{aligned} \tag{72}$$

$U(\alpha)$ is a displacement operator shifting coordinate, x, and momentum, p_x,

$$x \to x - \sqrt{2}\Re(\alpha), \qquad p_x \to p_x + \sqrt{2}\Im(\alpha).$$

Also, the eigenvalue equation, $a|\alpha\rangle = \alpha^*|\alpha\rangle$, leads to the following simple differential equations in coordinate and momentum space:

$$\frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big)\langle x|\alpha\rangle = \alpha^*\langle x|\alpha\rangle,$$
$$\frac{i}{\sqrt{2}}\Big(\frac{d}{dp_x} + p_x\Big)\langle p_x|\alpha\rangle = \alpha^*\langle p_x|\alpha\rangle, \tag{73}$$

with solutions

$$\langle x|\alpha\rangle = \mathcal{N}e^{-\frac{1}{2}(x-\sqrt{2}\alpha^*)^2}, \qquad \text{with } \mathcal{N} = e^{-\eta^2}/\pi^{\frac{1}{4}},$$
$$\langle p_x|\alpha\rangle = \mathcal{N}'e^{-\frac{1}{2}(p_x+i\sqrt{2}\alpha^*)^2}, \qquad \text{with } \mathcal{N}' = e^{+\xi^2}/\pi^{\frac{1}{4}}, \tag{74}$$

so

$$|\langle x|\alpha\rangle|^2 = \frac{1}{\sqrt{\pi}}e^{-(x-\sqrt{2}\xi)^2}, \qquad |\langle p_x|\alpha\rangle|^2 = \frac{1}{\sqrt{\pi}}e^{-(p_x+\sqrt{2}\eta)^2}. \tag{75}$$

The minimum uncertainties for the coherent state can now be understood because these are the probabilities for the space and momentum distributions of a displaced oscillator in its $n = 0$ lowest state, with a displacement $+\sqrt{2}\Re(\alpha)$ in coordinate space and $-\sqrt{2}\Im(\alpha)$ in momentum space. Moreover, such an oscillator wave packet will move with time *without* change of shape (see problems 10 and 11). This is the motion of a coherent wave packet.

Just as state vectors can be given in the coherent state representation, $\langle\alpha|\psi\rangle$, physical quantities represented by operators, O, can also be given in the coherent state representation. In the oscillator quantum number representation, we multiplied an operator, O, by unit operators on the left and on the right to yield, $O = \sum_{n,m}|n\rangle\langle n|O|m\rangle\langle m|$. Similarly, we could represent O by

$$O = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta |\alpha\rangle\langle\alpha|O|\beta\rangle\langle\beta|, \tag{76}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states. Using eq. (54), we have

$$O = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}|\alpha\rangle\sum_{n,m}\Big(\frac{\alpha^n}{\sqrt{n!}}\langle n|O|m\rangle\frac{\beta^{*m}}{\sqrt{m!}}\Big)\langle\beta|. \tag{77}$$

The operator, O, in this form is two-sided, made up of operators, $|\alpha\rangle\langle\beta|$, involving two different coherent states. In our earlier continuous representations, such as the coordinate representation, e.g., operators were expressible in terms of functions of a single x and its derivative. In coordinate representation, e.g., an operator, $O(x, p_x)$, which is a function of the basic operators, x and p_x, was expressible as $O(x, \frac{1}{i}\frac{\partial}{\partial x})$, where we were able to use

$$\langle x'|O(x, p_x)|x\rangle = \delta(x' - x)O(x, \frac{1}{i}\frac{\partial}{\partial x}),$$

so O becomes expressible as a function of a single x and its derivative through the δ function relation, $\langle x'|x\rangle = \delta(x' - x)$. Because this relation does not exist in the coherent state representation, the analagous operator relations must be handled with some care. For the harmonic oscillator, an operator, O, can be expressed as a function of the basic operators, $a^\dagger$ and a, $O(a^\dagger, a)$. We want to express such an operator as a function of z and $\partial/\partial z$. We have purposely chosen z in place of α because a type II coherent state will be somewhat simpler for this purpose. For type II coherent states, z-space realizations of state vectors, $\langle z|\psi\rangle$, are given by the Bargmann transforms, $F(z) = \sum_n c_n z^n/\sqrt{n!}$. The scalar product in the complex z-space involves the Bargmann measure, e^{-zz^*}/π. In this scalar product, the operator $\partial/\partial z$ is the adjoint of the operator z. Given two Bargmann-space functions,

$$F_a(z) = \langle z|\psi_a\rangle = \sum_n a_n \frac{z^n}{\sqrt{n!}} \quad \text{and} \quad F_b(z) = \langle z|\psi_b\rangle = \sum_n b_n \frac{z^n}{\sqrt{n!}},$$

we have (using the orthonormality of the $z^n/\sqrt{n!}$)

$$\begin{aligned}\frac{1}{\pi}\int d^2z e^{-zz^*} F_a^*(z)\Big(\frac{\partial}{\partial z} F_b(z)\Big) &= \sum_{n=0} a_n^* b_{n+1}\sqrt{(n+1)}\\ &= \frac{1}{\pi}\int d^2z e^{-zz^*}\Big(zF_a(z)\Big)^* F_b(z). \end{aligned} \tag{78}$$

Also, from

$$z\Big(\frac{z^n}{\sqrt{n!}}\Big) = \sqrt{(n+1)}\Big(\frac{z^{n+1}}{\sqrt{(n+1)!}}\Big) \quad \text{and} \quad \frac{\partial}{\partial z}\Big(\frac{z^n}{\sqrt{n!}}\Big) = \sqrt{n}\Big(\frac{z^{n-1}}{\sqrt{(n-1)!}}\Big), \tag{79}$$

we have

$$\begin{aligned}\langle n'|z|n\rangle &= \sqrt{(n+1)}\delta_{n'(n+1)} = \langle n'|a^\dagger|n\rangle,\\ \langle n'|\frac{\partial}{\partial z}|n\rangle &= \sqrt{n}\delta_{n'(n-1)} = \langle n'|a|n\rangle.\end{aligned} \tag{80}$$

The z-space realizations of the operators, $a^\dagger$ and a, are therefore

$$\gamma(a^\dagger) = z \quad \text{and} \quad \gamma(a) = \frac{\partial}{\partial z}. \tag{81}$$

A more complicated operator, such as $x^2 = \frac{1}{2}(aa + a^\dagger a^\dagger + 2a^\dagger a + 1)$, e.g., has the z-space realization

$$\gamma(x^2) = \frac{1}{2}\Big(\frac{\partial^2}{\partial z^2} + z^2 + 2z\frac{\partial}{\partial z} + 1\Big).$$

Finally, with

$$\langle z|\psi\rangle = \langle 0|e^{za}|\psi\rangle = F(z)$$

$\langle z|\psi'\rangle$, with $|\psi'\rangle = O|\psi\rangle$, is given by $\langle 0|e^{za}O|\psi\rangle$, which we want to write in the form $\gamma(O)F(z)$, where $\gamma(O)$ is to be determined as a function of operators, z, and $\partial/\partial z$. For this purpose, we rewrite $\langle z|O|\psi\rangle$ with the use of the unit operator, $e^{-za}e^{za} = 1$, as

$$\langle 0|e^{za}O|\psi\rangle = \langle 0|(e^{za}Oe^{-za})e^{za}|\psi\rangle$$

$$= \langle 0|\Big(O + z[a, O] + \frac{z^2}{2!}[a, [a, O]] + \cdots\Big)e^{za}|\psi\rangle, \tag{82}$$

where we have used eq. (23) of Chapter 16 (renaming $i\epsilon \equiv z$) for the expansion of $e^{za}Oe^{-za}$. In particular, with $O = a$, we have

$$\begin{aligned}\langle z|a|\psi\rangle = \langle 0|ae^{za}\psi\rangle = \langle 0|\frac{\partial}{\partial z}e^{za}|\psi\rangle = \frac{\partial}{\partial z}\langle 0|e^{za}|\psi\rangle \\ = \frac{\partial}{\partial z}\langle z|\psi\rangle,\end{aligned} \tag{83}$$

leading to

$$\gamma(a) = \frac{\partial}{\partial z}. \tag{84}$$

Similarly, with $O = a^\dagger$, we have

$$\langle z|a^\dagger|\psi\rangle = \langle 0|(a^\dagger + z)e^{za}|\psi\rangle = \langle 0|ze^{za}|\psi\rangle = z\langle z|\psi\rangle, \tag{85}$$

where we have used $\langle 0|a^\dagger = (a|0\rangle)^\dagger = 0$, leading again to

$$\gamma(a^\dagger) = z. \tag{86}$$

E Angular Momentum Coherent States

Because we can make an analogy between the operators

$$a, a^\dagger, 1 = [a, a^\dagger] \qquad \text{of the oscillator algebra and}$$

$$J_-, J_+, J_0 = -\tfrac{1}{2}[J_-, J_+] \qquad \text{of the angular momentum algebra,}$$

it is possible to define angular momentum coherent states in analogy with the oscillator coherent states. Again, we will distinguish between two slightly different definitions. For the type I coherent state, using a complex variable α, we define

$$|\alpha\rangle_{\mathrm{I}} \equiv |\alpha\rangle = e^{\alpha^* J_+ - \alpha J_-}|J, M = -J\rangle. \tag{87}$$

For the type II coherent state, using the complex variable z, we define

$$|z\rangle_{\mathrm{II}} \equiv |z\rangle = e^{z^* J_+}|J, M = -J\rangle. \tag{88}$$

A generic $|J, M\rangle$ has been used for the angular momentum eigenvectors, and the oscillator ground state, $|0\rangle$, has been replaced with the angular momentum eigenvector with the lowest possible eigenvalue of J_0, $M = -J$, so $J_-|J, M = -J\rangle = 0$ in analogy with $a|0\rangle = 0$. For the type I coherent state, it will be useful to relate the complex number α to the real angle variables θ and ϕ, via

$$-\alpha = \frac{\theta}{2}e^{i\phi},$$

where θ, ϕ are polar and azimuth angles giving the standard orientation of a unit vector $\vec{r}/r$ in our 3-D world. With this choice of parameterization of the complex variable, α, the type I coherent state becomes

$$\begin{aligned} |\alpha\rangle &= e^{-i\theta(-\sin\phi J_x+\cos\phi J_y)}|M=-J\rangle \\ &= e^{-i\theta(\vec{n}\cdot\vec{J})}|M=-J\rangle, \end{aligned} \tag{89}$$

where $\vec{n}$ is a unit vector in the x, y-plane making an angle ϕ with the y-axis and $(\frac{\pi}{2}-\phi)$ with the negative x-axis; i.e., $\vec{n}$ is a unit vector in the direction of the y' axis after a rotation about the z-axis through an angle ϕ. We shall return to the type I coherent state in chapter 29 after studying rotation operators in our 3-D world in greater generality.

For the type II coherent state, we can expand $|z\rangle$ in terms of angular momentum eigenstates $|JM\rangle$, via

$$\begin{aligned} |z\rangle = e^{z^*J_-}|M=-J\rangle &= \sum_{n=0}^{2J}\frac{z^{*n}}{n!}(J_+)^n|M=-J\rangle \\ &= \sum_{n=0}^{2J}\frac{z^{*n}}{\sqrt{n!}}\sqrt{\frac{(2J)!}{(2J-n)!}}|M=-J+n\rangle. \end{aligned} \tag{90}$$

An arbitrary state vector $|\psi\rangle$ in the subspace of Hilbert space appropriate to our angular momentum operator, $\vec{J}$, can now be specified through its z-space realization, $\langle z|\psi\rangle$,

$$\begin{aligned} \langle z|\psi\rangle = \langle M=-J|e^{zJ_-}|\psi\rangle &= \sum_{n=0}^{2J}\frac{z^n}{\sqrt{n!}}\sqrt{\frac{(2J)!}{(2J-n)!}}\langle M=-J+n|\psi\rangle \\ &= \sum_{n=0}^{2J}\frac{z^n}{\sqrt{n!}}K_n\langle M=-J+n|\psi\rangle. \end{aligned} \tag{91}$$

Except for a new numerical factor, K_n, we have expanded the coherent state in terms of the orthonormal z-space oscillator basis. The orthonormality of the $z^n/\sqrt{n!}$ requires the Bargmann weighting function e^{-zz^*}/π in the complex z-plane. We will therefore find it convenient to use the unit operator in the Bargmann form

$$1 = \frac{1}{\pi}\int d^2z e^{-zz^*}|z\rangle\langle z|$$

for z-space scalar products. We have thus mapped the angular momentum states onto oscillator states in the complex z-space realization. The oscillator excitation, however, is now limited to $n \leq 2J$. Also, the angular momentum coherent state $|z\rangle$ is *not* an eigenvector of the operator J_- because of the additional n-dependent numerical factors, K.

To get the z-space realizations of operators, we use

$$\begin{aligned} \langle z|O|\psi\rangle &= \langle -J|e^{zJ_-}O|\psi\rangle = \langle -J|(e^{zJ_-}Oe^{-zJ_-})e^{zJ_-}|\psi\rangle \\ &= \langle -J|\Big(O + z[J_-,O] + \frac{z^2}{2!}[J_-,[J_-,O]] + \cdots\Big)e^{zJ_-}|\psi\rangle, \end{aligned} \tag{92}$$

where we have used the abbreviation, $|J, M = -J\rangle \equiv |-J\rangle$, for the state of lowest possible M. Let us choose $O = J_-, J_0, J_+$ in turn to get the z-space realizations of the angular momentum operators themselves.

$$\begin{aligned}
\langle z|J_-|\psi\rangle &= \langle -J|J_- e^{zJ_-}\psi\rangle = \langle -J|\frac{\partial}{\partial z}e^{zJ_-}|\psi\rangle = \frac{\partial}{\partial z}\langle -J|e^{zJ_-}|\psi\rangle \\
&= \frac{\partial}{\partial z}\langle z|\psi\rangle, \\
\langle z|J_0|\psi\rangle &= \langle -J|(J_0 + zJ_-)e^{zJ_-}|\psi\rangle = \langle -J|(-J + z\frac{\partial}{\partial z})e^{zJ_-}|\psi\rangle, \\
&= (-J + z\frac{\partial}{\partial z})\langle z|\psi\rangle; \\
\langle z|J_+|\psi\rangle &= \langle -J|(J_+ - 2J_0 z - z^2 J_-)e^{zJ_-}|\psi\rangle = \langle -J|\Big(2Jz - z^2\frac{\partial}{\partial z}\Big)e^{zJ_-}|\psi\rangle \\
&= \Big(2Jz - z^2\frac{\partial}{\partial z}\Big)\langle z|\psi\rangle, \qquad (93)
\end{aligned}$$

where we have used $\langle -J|J_0 = (J_0| -J\rangle)^\dagger = -J\langle -J|$, and $\langle -J|J_+ = 0$, via the hermitian conjugate of $J_-|-J\rangle = 0$. We have thus found z-space realizations of the operators, J_-, J_0, J_+,

$$\begin{aligned}
\Gamma(J_-) &= \frac{\partial}{\partial z}, \\
\Gamma(J_0) &= (-J + z\frac{\partial}{\partial z}), \\
\Gamma(J_+) &= z(2J - z\frac{\partial}{\partial z}). \qquad (94)
\end{aligned}$$

It is easy to verify these $\Gamma(J_i)$ satisfy the angular momentum commutation rules, which of course is just a check of our arithmetic. In addition,

$$\Gamma(\vec{J}^2) = \tfrac{1}{2}\big[\Gamma(J_+)\Gamma(J_-) + \Gamma(J_-)\Gamma(J_+)\big] + \Gamma(J_0)^2 = J(J+1). \qquad (95)$$

$\Gamma(J_+)$, however, is not the adjoint of $\Gamma(J_-)$ with respect to the Bargmann measure, where, as we have seen, $\partial/\partial z$ is the adjoint of z. This is related to the fact that our z-space realization of the angular momentum operators is a nonunitary one. It is the reason why we have used $\Gamma(J_i)$ for the above z-space realization of the J_i, reserving $\gamma(J_i)$ for the unitary one. To calculate matrix elements of an operator, O, through its z-space realization, built from operators z and $\partial/\partial z$, we see from the expansion of $\langle z|\psi\rangle$ of eq. (91) that such a $\Gamma(O)$, acting on the n^{th} term of the expansion will in general create Bargmann space orthonormal $(z^{n'}/\sqrt{n'!})$ not multiplied by the proper $K_{n'}$. To attain the proper $(K_{n'}z^{n'}/\sqrt{n'!})$, we can multiply the resultant obtained from the action of $\Gamma(O)$ by $(K_{n'}^{-1} \times K_{n'}) = 1$ and thereby transform the nonunitary form of the operator, $\Gamma(O)$, into a unitary form, to be denoted by $\gamma(O)$, where

$$\gamma(O) = K^{-1}\Gamma(O)K = \Big(\gamma(O^\dagger)\Big)^\dagger, \qquad (96)$$

thus making $\gamma(O)$ unitary. The operators, K, are merely the command: Multiply an orthonormal Bargmann space function $z^n/\sqrt{n!}$ by the appropriate factor, K_n.

For the most general, O, eq. (95) can be put in the form

$$\gamma(O) = K^{-1}\Gamma(O)K = \big(\gamma(O^{\dagger})\big)^{\dagger} = \big(K^{-1}\Gamma(O^{\dagger})K\big)^{\dagger} = K^{\dagger}\big(\Gamma(O^{\dagger})\big)^{\dagger}(K^{-1})^{\dagger}, \tag{97}$$

or, via left-multiplication by K and right-multiplication by $K^{\dagger}$,

$$\Gamma(O)KK^{\dagger} = KK^{\dagger}\big(\Gamma(O^{\dagger})\big)^{\dagger}. \tag{98}$$

For the specific operator, $O = J_{+}$, of the angular momentum algebra, eq. (96) becomes

$$\gamma(J_{+}) = K^{-1}\Gamma(J_{+})K = \big(\gamma(J_{-})\big)^{\dagger} = K^{\dagger}\big(\Gamma(J_{-})\big)^{\dagger}(K^{-1})^{\dagger}, \tag{99}$$

and eq. (98) becomes

$$z\Big(2J - z\frac{\partial}{\partial z}\Big)KK^{\dagger} = KK^{\dagger}\Big(\frac{\partial}{\partial z}\Big)^{\dagger} = KK^{\dagger}z. \tag{100}$$

In eq. (91), the factor K_n was evaluated from the known matrix elements of J_{+} acting n times in succession on the state $|J, -J\rangle$. If we had not had prior knowledge of these matrix elements, we could now evaluate these by using eq. (100) to first evaluate $(KK^{\dagger})_n$. In particular, both the operator z and the operator $\Gamma(J_{+}) = z(2J - z\partial/\partial z)$ convert a z-space function, z^n, into a z-space function, z^{n+1}, so eq. (100) becomes

$$z(2J - n)(KK^{\dagger})_n = (KK^{\dagger})_{n+1}z.$$

On the right-hand side of the equation, the action of $KK^{\dagger}$ follows the action of the operator z and $z\partial/\partial z(z^n) = n(z^n)$. We therefore have

$$\frac{(KK^{\dagger})_{n+1}}{(KK^{\dagger})_n} = (2J - n).$$

Because the Bargmann state with $n = 0$ has the same normalization as the angular momentum eigenstate, $|J, M = -J\rangle$, we have $(KK^{\dagger})_0 = 1$. Iterating the above recursion relation for $KK^{\dagger}$, starting with $n = 0$, we obtain

$$(KK^{\dagger})_n = 2J(2J-1)\cdots(2J+1-n) = \frac{(2J)!}{(2J-n)!}.$$

The hermitian operator, $KK^{\dagger}$, must have real eigenvalues. In our special case, K is the simple command: Multiply $(z^n/\sqrt{n!})$ by an n-dependent factor. We can make this renormalization factor real without loss of generality, so K_n becomes the real number

$$K_n = (K^{\dagger})_n = \sqrt{\frac{(2J)!}{(2J-n)!}}. \tag{101}$$

We can therefore rederive the matrix elements of J_{-}, J_0, and J_{+}. Eqs. (94) and (96) lead to

$$\langle n-1|J_{-}|n\rangle = (K^{-1})_{n-1}\Big(n-1\Big|\frac{\partial}{\partial z}\Big|n\Big)K_n = \sqrt{(2J+1-n)n}$$

$$
\begin{aligned}
&= \sqrt{(J+1-M)(J+M)}, \\
\langle n|J_0|n\rangle n &= (K^{-1})_n \big(n|(-J + z\frac{\partial}{\partial z}|n\big)K_n = (-J+n) \\
&= M, \\
\langle n+1|J_+|n\rangle &= (K^{-1})_{n+1}\big(n+1|z(2J - z\frac{\partial}{\partial z})|n\big)K_n \\
&= \sqrt{\frac{(2J-n-1)!}{(2J-n)!}}\sqrt{n+1}(2J-n) = \sqrt{(2J-n)(n+1)} \\
&= \sqrt{(J-M)(J+M+1)},
\end{aligned}
\tag{102}
$$

where the Bargmann space matrix elements

$$
(n'|\Gamma(O)|n) = \frac{1}{\pi}\int d^2 z e^{-zz^*}\frac{z^{*n'}}{\sqrt{n'!}}\Gamma(O)\frac{z^n}{\sqrt{n!}}
$$

have been denoted by round parentheses and we have used $M = -J + n$ to express all matrix elements in their standard form.

Final Notes:

(1). The technique used here to derive the matrix elements of the angular momentum operators can be used to derive the matrix elements of more complicated families of operators with more complicated commutator algebras. For coherent state techniques of such generalized coherent states, see, e.g., A. Perelomov, *Generalized Coherent States and Their Applications*. Springer-Verlag, 1986, or K. T. Hecht, *The Vector Coherent State Method and its Application to Problems of Higher Symmetries*. Lecture Notes in Physics **290**. Springer-Verlag, 1987.

(2). The technique used here, which involved a mapping of angular momentum eigenstates onto orthonormal harmonic oscillator z-space Bargmann eigenstates, $z^n/\sqrt{n!}$, is useful if we have no a priori knowledge of the numerical values of the K operator or the $KK^\dagger$ eigenvalues. Alternatively, we could have used a different z-space measure to make the $(z^n/\sqrt{n!})K_n$ into an orthonormal set in the complex z-space domain. This would have involved a change of measure

$$
\frac{1}{\pi}e^{-zz^*} \to \frac{(2J+1)}{\pi}\frac{1}{(1+zz^*)^{2J+2}},
$$

as can be seen from the orthonormality integral

$$
\begin{aligned}
&\frac{(2J+1)}{\pi}\int d^2 z \frac{1}{(1+zz^*)^{2J+2}}\frac{z^{*m}K_m^*}{\sqrt{m!}}\frac{z^n K_n}{\sqrt{n!}} \\
&= \frac{(2J+1)}{\pi}\int_0^\infty \frac{d\rho\rho^{n+m+1}}{(1+\rho^2)^{2J+2}}\int_0^{2\pi} d\phi e^{i(n-m)\phi}\frac{K_m^* K_n}{\sqrt{m!n!}} \\
&= \delta_{nm}\frac{(2J+1)(2J)!}{n!(2J-n)!}2\int_0^\infty \frac{d\rho\rho^{2n+1}}{(1+\rho^2)^{2J+2}} = \delta_{nm},
\end{aligned}
\tag{103}
$$

via the integral

$$
2\int_0^\infty \frac{d\rho\rho^{2n+1}}{(1+\rho^2)^{2J+2}} = \int_0^\infty \frac{d\tau\tau^n}{(1+\tau)^{2J+2}} = B(n+1, 2J+1-n)
$$

$$= \frac{\Gamma(n+1)\Gamma(2J+1-n)}{\Gamma(2J+2)} = \frac{n!(2J-n)!}{(2J+1)!}, \tag{104}$$

where $B(p, q)$ is the Beta function expressed by Γ functions and in terms of factorials because $2J$ must be an integer.

Problems

23. Given three hermitian operators, T_1, T_2, T_3, with commutation relations

$$[T_2, T_3] = iT_1, \qquad [T_3, T_1] = iT_2, \qquad [T_1, T_2] = -iT_3,$$

which differ from the angular momentum commutator algebra because of the minus sign in the last commutation relation! Show that the three T_j all commute with the operator

$$\mathcal{T}^2 = T_3^2 - T_1^2 - T_2^2.$$

Again, note the minus signs and the difference from the angular momentum case. Convert the operators, T_j, to the new set

$$T_\pm = (T_1 \pm iT_2), \qquad T_3 = T_0,$$

and show these equations satisfy the commutation relations

$$[T_0, T_\pm] = \pm T_\pm, \qquad [T_+, T_-] = -2T_0.$$

Again, note the minus sign in the last commutation relation. Solve the simultaneous eigenvalue problem

$$\begin{aligned} \mathcal{T}^2|\lambda m\rangle &= \lambda|\lambda m\rangle = j(j+1)|\lambda m\rangle = j(j+1)|jm\rangle, \\ T_0|\lambda m\rangle &= m|\lambda m\rangle = m|jm\rangle, \end{aligned} \tag{1}$$

where we have named $\lambda = j(j+1)$. (No implication exists that j be an integer or half-integer.) Show, in particular, that now:

(1) If T_3 has positive eigenvalues, a minimum possible, $m_{\text{min.}}$ exists, such that

$$T_-|\lambda m_{\text{min.}}\rangle = 0, \qquad m = m_{\text{min.}} + n, \quad \text{with } n = 0, 1, 2, \ldots, \to \infty,$$

where $m_{\text{min.}} = (j+1)$.

(2) If T_3 has negative eigenvalues, a maximum possible $m_{\text{max.}} = -|m_{\text{max.}}|$ exists, such that

$$T_+|\lambda m_{\text{max.}}\rangle = 0, \qquad m = m_{\text{max.}} - n = -(|m_{\text{max.}}| + n), \quad \text{with } n = 0, 1, \ldots \to \infty,$$

where now $m_{\text{max.}} = -(j+1)$ and we assume j is positive.

Find the nonzero matrix elements of T_+ and T_-,

$$\langle jm'|T_+|jm\rangle = \langle j(m+1)|T_+|jm\rangle,$$

$$\langle jm'|T_-|jm\rangle = \langle j(m-1)|T_-|jm\rangle,$$

and note the differences and similarities with the corresponding angular momentum case.

Note: The angular momentum operators, J_k, generate the group SO(3) (the special orthogonal transformations in three dimensions, with determinant = +1), in the case when j are integers, or SU(2) (the special unitary transformations in two dimensions) and in the case when j are half-integers. The three operators, T_k, on the other hand, generate the group SO(2,1). Note the two minus signs and the one plus sign in the operator $\mathcal{T}^2$.

Solution for Problem 23: The SO(2,1) Algebra

The various commutator relations follow from the given commutation relations by simple commutator algebra. For example,

$$\begin{aligned}[T_3, \mathcal{T}^2] &= -T_1[T_3, T_1] - [T_3, T_1]T_1 - T_2[T_3, T_2] - [T_3, T_2]T_2 \\ &= -iT_1T_2 - iT_2T_1 + iT_2T_1 + iT_1T_2 = 0. \end{aligned} \tag{2}$$

We are interested in the simultaneous eigenvectors of the two commuting hermitian operators, T_3 and $\mathcal{T}^2$,

$$\begin{aligned}\mathcal{T}^2|\lambda m\rangle &= \lambda|\lambda m\rangle, \\ T_3|\lambda m\rangle &= m|\lambda m\rangle. \end{aligned} \tag{3}$$

Let us consider the new vectors, $T_\pm|\lambda m\rangle$. Acting on either of these with both $\mathcal{T}^2$ and T_3, we get (with the use of the commutation relations),

$$\begin{aligned}\mathcal{T}^2\Big(T_+|\lambda m\rangle\Big) &= T_+\mathcal{T}^2|\lambda m\rangle = \lambda\Big(T_+|\lambda m\rangle\Big), \\ T_3\Big(T_+|\lambda m\rangle\Big) &= T_+T_3|\lambda m\rangle + T_+|\lambda m\rangle = (m+1)\Big(T_+|\lambda m\rangle\Big). \end{aligned} \tag{4}$$

Thus, if $|\lambda m\rangle$ is simultaneously an eigenvector of $\mathcal{T}^2$ and T_3, with eigenvalues λ and m, either $\Big(T_+|\lambda m\rangle\Big)$ is simultaneously an eigenvector of $\mathcal{T}^2$ and T_3 with eigenvalues λ and $(m+1)$ or $\Big(T_+|\lambda m\rangle\Big) = 0$. Similarly,

$$\begin{aligned}\mathcal{T}^2\Big(T_-|\lambda m\rangle\Big) &= T_-\mathcal{T}^2|\lambda m\rangle = \lambda\Big(T_-|\lambda m\rangle\Big), \\ T_3\Big(T_-|\lambda m\rangle\Big) &= T_-T_3|\lambda m\rangle - T_-|\lambda m\rangle = (m-1)\Big(T_-|\lambda m\rangle\Big). \end{aligned} \tag{5}$$

Thus, if $|\lambda m\rangle$ is simultaneously an eigenvector of $\mathcal{T}^2$ and T_3, with eigenvalues λ and m, either $\Big(T_-|\lambda m\rangle\Big)$ is simultaneously an eigenvector of $\mathcal{T}^2$ and T_3 with eigenvalues λ and $(m-1)$; or $\Big(T_-|\lambda m\rangle\Big) = 0$. To investigate these two possibilities, let us rewrite $\mathcal{T}^2$

$$\begin{aligned}\mathcal{T}^2 = T_3^2 - T_1^2 - T_2^2 = T_0^2 - \tfrac{1}{2}(T_+T_- + T_-T_+) &= T_0^2 - T_+T_- - T_0 \\ &= T_0^2 - T_-T_+ + T_0, \end{aligned} \tag{6}$$

$$\text{so} \qquad T_+T_- = T_0^2 - T_0 - \mathcal{T}^2$$

$$\text{and} \qquad T_-T_+ = T_0^2 + T_0 - \mathcal{T}^2. \tag{7}$$

Now, let us take the diagonal matrix element of these two relations between states with the same λ, m, assuming a state $|\lambda m\rangle$ exists, viz., it leads to a square-integrable eigenfunction. First,

$$\begin{aligned}\langle\lambda m|T_+T_-|\lambda m\rangle &= \Big((m^2 - m) - \lambda\Big) \\ &= \sum_{m'}\langle\lambda m|T_+|\lambda m'\rangle\langle\lambda m'|T_-|\lambda m\rangle \\ &= \sum_{m'}|\langle\lambda m'|T_-|\lambda m\rangle|^2 \geq 0,\end{aligned} \tag{8}$$

where we have used $\langle\lambda m|T_+|\lambda m'\rangle = \langle\lambda m'|T_-|\lambda m\rangle^*$ and have summed over a complete set of intermediate states. We must have $m' = (m-1)$. By including a sum over states with $m' = (m-1)$, we have allowed for the possibility more than one independent state with that restriction exists. Similarly, we have

$$\begin{aligned}\langle\lambda m|T_-T_+|\lambda m\rangle &= \Big((m^2 + m) - \lambda\Big) \\ &= \sum_{m'}|\langle\lambda m'|T_+|\lambda m\rangle|^2 \geq 0.\end{aligned} \tag{9}$$

We have the two patently positive quantities of eqs. (7) and (8) only if the two functions, $f(m) = \big((m^2-m)-\lambda\big)$ or $\big((m^2+m)-\lambda\big)$ are equal to or greater than zero. The two functions of m, $(m^2 \mp m)$, have minima at $m = \pm\frac{1}{2}$, both with a minimum value of $-\frac{1}{4}$. Unlike the corresponding operator of the angular momentum algebra, with its slightly different commutation relations, the operator, $\mathcal{T}^2$, is no longer a sum of positive hermitian operators. Thus, the eigenvalue, λ, could be either positive or negative. In particular, if $\lambda < -\frac{1}{4}$, the quantities $[(m^2 \mp m) - \lambda]$ are positive for all values of m, positive or negative. Thus, all values of $\pm m$ are possible, and for any λ, such that $\lambda < -\frac{1}{4}$, we have a continuous spectrum of allowed values for both λ and m. Conversely, if $\lambda > 0$, and if an eigenvalue, $m_0 > 0$ exists, such that $((m_0^2 - m_0) - \lambda) > 0$, the step-down action of n operations with T_- could eventually lead to an $(m_0 - n)$ such that $\big((m_0 - n)(m_0 - n - 1) - \lambda\big) < 0$, and eq. (7) would lead to an inconsistency. A patently positive quantity on one side of eq. (7) would be equal to a negative quantity on the other side. Hence, our assumption of the existence of a square-integrable $|\lambda m_0\rangle$ must have been incorrect. If m_0, however, is such that an integer n exists such that $(m_0 - n) \equiv m_{\text{min.}}$, so

$$\Big(T_-|\lambda m_{\text{min.}}\rangle\Big) = 0 \qquad \text{and} \quad [m_{\text{min.}}(m_{\text{min.}} - 1) - \lambda] = 0, \tag{10}$$

an inconsistency never exists. If we name

$$\lambda = j(j+1), \tag{11}$$

with $j \geq 0$, to be as close as possible to the language of the angular momentum algebra, the solution to $[m_{\text{min.}}(m_{\text{min.}} - 1) - \lambda] = 0$ gives us $m_{\text{min.}} = (j+1)$. Only the positive root has meaning in this case. Also, the quantum number j is only a language to give us the eigenvalue λ. In this case, j may not be an integer or half-integer. The actual values of $m_{\text{min.}} = (j+1)$ and $\lambda = j(j+1)$ will depend on the detailed properties of the operators $\mathcal{T}^2$ and T_3, i.e., on the specific nature of the physics of the problem. If the nature of the problem is such that the eigenvalues of $\mathcal{T}^2$ and T_3 must all be positive definite, we are done. The spectrum is given by

$$m = m_{\text{min.}}, (m_{\text{min.}} + 1), \ldots, (m_{\text{min.}}, +n), \ldots, \to \infty,$$

$$m = (j+1), (j+2), \ldots, (j+1+n), \ldots, \to \infty, \qquad \text{with } \lambda = j(j+1).$$

[The possible m-values can go to ∞. This follows because for $m_{\text{min.}} > 0$, the quantity $(m^2_{\text{min.}} + m_{\text{min.}} - \lambda) = 2m_{\text{min.}} > 0$ and thus $[(m_{\text{min.}}+n)(m_{\text{min.}}+n+1) - \lambda] > 0$ for any positive integer n, so the state $(T_+|\lambda(m_{\text{min.}} + n)\rangle)$ exists; i.e., the state $|\lambda(m_{\text{min.}} + n + 1)\rangle$ also leads to a square-integrable eigenfunction.]

Let us next examine the possibility the operators $\mathcal{T}^2$ and T_3 are such that $\lambda > 0$, but $m < 0$. Now let us suppose some $m_0 = -|m_0|$ exists; i.e., the state $|\lambda, -|m_0|\rangle$ leads to square-integrable eigenfunctions. Now, if eq. (8) is satisfied for $m = -|m_0|$, i.e., $|m_0|(|m_0| + 1) - \lambda > 0$, the state with $m = (-|m_0| - n)$ leads to an $m(m-1) - \lambda = (|m_0| + n)(|m_0| + n + 1) - \lambda$ also > 0, so n actions with T_- would lead to another allowed state. However, n actions with the step-up operator, T_+, would lead to a state with $m = -|m_0| + n$, for which the function $[(m^2 + m) - \lambda]$ of eq. (9) would lead to the value $[(|m_0| - n)(|m_0| - n - 1) - \lambda]$, which for large enough n could now be negative. Thus, eq. (9) would say that a patently positive quantity is equal to a function that can become a negative quantity for a large enough n. Now, a state $|\lambda, -|m_0|\rangle$ can be an allowed state only if an integer n exists, such that $m = (-|m_0| + n) \equiv m_{\text{max.}}$ and

$$\Big(T_+|\lambda m_{\text{max.}}\rangle\Big) = 0, \qquad \text{so that} \quad (m_{\text{max.}}(m_{\text{max.}} + 1) - \lambda) = 0, \qquad (12)$$

where now $m_{\text{max.}}$ must be the negative root of the equation: $(m_{\text{max.}}(m_{\text{max.}} + 1) - j(j+1)) = 0$; that is, $m_{\text{max.}} = -\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}} = -(j+1)$. In this case, therefore, the spectrum of possible m-values is

$$-\infty, \ldots, -(j+1+n), \ldots, -(j+2), -(j+1) = m_{\text{max.}}, \quad \text{again with } \lambda = j(j+1).$$

Now, however, for general $\lambda = j(j+1) > 0$, the two branches of allowed m values are unconnected, so the commutator algebra of the T_i does not lead to additional restrictions on λ and, hence, j. In particular, j need not be an integer or a half-integer. The physics of the operators T_3, $\mathcal{T}^2$, dictate the nature of the eigenvalues m and λ. Thus, for some physical applications for which the eigenvalues of T_3 can be positive only, only the positive branch of allowed m values can exist.

Let us now finally use eqs. (7) and (8) to find the matrix elements of the operators $T_\pm$. We shall look at the simple case, in which the states

$$|\lambda m_{\text{min.}}\rangle \qquad \text{or} \qquad |\lambda m_{\text{max.}}\rangle$$

are nondegenerate; i.e., the two relations

$$T_-|\lambda m_{\text{min.}}\rangle = 0 \qquad \text{and} \qquad T_+|\lambda, -|m_{\text{max.}}|\rangle = 0$$

are assumed to have only one allowed solution. This would of course be automatic if these two relations lead to first-order differential equations. In this case, action with T_+ or T_-, respectively, would lead to a single (nondegenerate) state with the appropriate m value. Thus, all states in the ladder of either positive or negative m values might be expected to be nondegenerate, and the sums over m' in eqs. (8) and (9) would collapse to a single term with $m' = (m-1)$ or $m' = (m+1)$, respectively. For branches of allowed m values, eq. (9) tells us

$$|\langle j(m+1)|T_+|jm\rangle|^2 = m(m+1) - j(j+1) = (m-j)(m+j+1). \quad (13)$$

This relation leads to no upper limit for positive m values with $m \geq (j+1)$, but leads to a zero matrix element for $m = -(j+1)$ within the branch of negative m values. Similarly, eq. (8) leads to

$$|\langle j(m-1)|T_-|jm\rangle|^2 = m(m-1) - j(j+1) = (m+j)(m-j-1). \quad (14)$$

Now, no zero matrix elements exist for the negative branch with $m \leq -(j+1)$, but this matrix element is automatically zero if $m = (j+1)$. As for the corresponding angular momentum problem, eqs. (13) and (14) do not fix the phases of these matrix elements. If we choose these phases such that the matrix elements of T_+, T_- are real, we have

$$\begin{aligned} \langle j(m+1)|T_+|jm\rangle &= \sqrt{(m-j)(m+j+1)}, \\ \langle j(m-1)|T_-|jm\rangle &= \sqrt{(m+j)(m-j-1)}. \end{aligned} \quad (15)$$

(In particular, these equations satisfy $\langle jm'|T_-|jm\rangle = \langle jm|T_+|jm'\rangle^*$.)

Final remark: The operators $J_x = J_1$, $J_y = J_2$, $J_z = J_3$ of the angular-momentum algebra, which commute with the operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$ are the generators of infinitesimal rotations in three-space about the x, y, and z axes and thus connected with the group SO(3), the "special orthogonal group in three dimensions" (where the "special" means the 3×3 orthogonal rotation matrices have determinant $+1$, leading to pure rotations and not including rotation reflections). The operators T_1, T_2, T_3, which commute with the operator, $\mathcal{T}^2 = T_3^2 - T_1^2 - T_2^2$, conversely, are related in a similar way with a 3-D space with two space-like and one time-like dimension and thus connected with the group SO(2,1). As for the SO(3) group, most of the properties of the group SO(2,1) follow from the matrix elements of the T_i, which are now known to us.

Finally, the angular momentum operators, $J_\pm$ are related to the ladder operators of case 3 $\mathcal{L}(m)$ of the factorization method, whereas the $T_\pm$ of the SO(2,1) algebra are related to the ladder operators of case 4 $\mathcal{L}(m)$ of the factorization method.

Problems 24–26 give some actual physical examples of the SO(2,1) algebra. The signs of λ and m are usually dictated by the nature of the problem.

24. The 3-D harmonic oscillator via the SO(2,1) algebra. For the 3-D harmonic oscillator, define dimensionless quantities, $\vec{r}$, H, etc., via

$$\vec{r}_{\text{phys.}} = \sqrt{\hbar/m\omega_0}\,\vec{r} \qquad \text{and} \qquad H_{\text{phys.}} = \hbar\omega_0 H; \ldots .$$

Show that the three operators

$$T_1 = \tfrac{1}{4}(r^2 - \vec{p}^{\,2}) = \tfrac{1}{4}(r^2 + \nabla^2),$$

$$T_2 = -\tfrac{1}{4}(\vec{r}\cdot\vec{p} + \vec{p}\cdot\vec{r}) = \frac{i}{2}\Big(r\frac{\partial}{\partial r} + \tfrac{3}{2}\Big),$$

$$T_3 = \tfrac{1}{4}(r^2 + \vec{p}^{\,2}) = \tfrac{1}{4}(r^2 - \nabla^2),$$

where

$$\nabla^2 = \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big) - \frac{\vec{L}^2}{r^2} = -p_r^2 - \frac{\vec{L}^2}{r^2}$$

[with $\vec{L}^2$ given in terms of θ, ϕ-dependent operators through eq. (13) of Chapter 8] are hermitian with respect to the conventional volume element and satisfy the SO(2,1) commutation relations of problem 23.

Note that, with $\psi = R(r)Y_{lm}(\theta,\phi)$, the operator ∇^2 is hermitian with respect to the usual measure, $r^2 \sin\theta$,

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr r^2 \psi_2^*(\nabla^2\psi_1) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr r^2 \psi_1^*(\nabla^2\psi_2).$$

Also,

$$H = \frac{H_{\text{phys}}}{\hbar\omega_0} = \tfrac{1}{2}(r^2 - \nabla^2) = 2T_3 = 2T_0$$

with known (positive) eigenvalues, $(N + \frac{3}{2})$. Show that the functions

$$r^l e^{-\frac{1}{2}r^2} Y_{lm}(\theta,\phi)$$

satisfy the equations

$$T_-\Big(r^l e^{-\frac{1}{2}r^2} Y_{lm}(\theta,\phi)\Big) = 0 \quad \text{and} \quad T_0\Big(r^l e^{-\frac{1}{2}r^2} Y_{lm}(\theta,\phi)\Big) = \tfrac{1}{2}(l+\tfrac{3}{2})\Big(r^l e^{-\frac{1}{2}r^2} Y_{lm}(\theta,\phi)\Big),$$

so

$$m_{\text{min.}} = \tfrac{1}{2}(l + \tfrac{3}{2}) \equiv (j+1),$$

where m and j refer to the quantum numbers as defined in problem 23. (m and j are $\frac{1}{4}$-integers here!)

Use these results to show

$$E = \hbar\omega_0(l + 2n + \tfrac{3}{2}),$$

and the dimensionless operator r^2 is given by

$$r^2 = 2T_0 + T_+ + T_-.$$

Find the nonzero matrix elements of r^2 as functions of n and l or N and l. (Consult the results of problem 15.)

25. (a) For the SO(2,1) algebra with operators, T_-, T_+, T_0, satisfying the commutation relations

$$[T_0, T_\pm] = \pm T_\pm \qquad \text{and} \qquad [T_+.T_-] = -2T_0,$$

with eigenvalues $(T_0)_{\text{eigen}} = m = m_{\text{min.}} + n = (j+1+n)$, where $n = 0, 1, \dots, \to \infty$, show that coherent state z-space realizations of these operators can be given by

$$\Gamma(T_-) = \frac{\partial}{\partial z}, \qquad \Gamma(T_0) = j + 1 + z\frac{\partial}{\partial z},$$

$$\Gamma(T_+) = 2(j+1)z + z^2\frac{\partial}{\partial z}.$$

Find the eigenvalues, K_n, of the operator, K, which converts the above nonunitary $\Gamma(T_i)$ into unitary $\gamma(T_i)$ for this algebra, and rederive the general expressions found in problem 23 for the matrix elements of T_-, T_+, T_0, in a $|jm\rangle$ basis.

(b) For the 1-D harmonic oscillator, the oscillator annihilation and creation operators, $a_x, a_x^\dagger$, expressed in terms of the dimensionless x and p_x, are

$$a_x = \tfrac{1}{\sqrt{2}}(x + ip_x) \qquad \text{and} \qquad a_x^\dagger = \tfrac{1}{\sqrt{2}}(x - ip_x).$$

Show that the three operators

$$T_+ = \tfrac{1}{4}a_x^\dagger a_x^\dagger, \qquad T_- = \tfrac{1}{4}a_x a_x, \qquad T_0 = \tfrac{1}{2}(a_x^\dagger a_x + \tfrac{1}{2}),$$

satisfy the SO(2,1) commutation relations

$$[T_0, T_\pm] = \pm T_\pm \qquad \text{and} \qquad [T_+, T_-] = -2T_0.$$

Show that the two oscillator states, $|0\rangle$, and $|1\rangle = a_x^\dagger|0\rangle$, satisfy

$$(1): T_-|0\rangle = 0, \qquad T_0|0\rangle = \tfrac{1}{4}|0\rangle,$$

$$(2): T_-|1\rangle = 0, \qquad T_0|1\rangle = \tfrac{3}{4}|1\rangle,$$

so $m_{\text{min.}} = \frac{1}{4}$ for case (1) and $m_{\text{min.}} = \frac{3}{4}$ for case (2). Use these results together with the results of problem 23, to calculate the matrix elements

$$\langle n'|a_x^\dagger a_x^\dagger|n\rangle,$$

$$\langle n'|a_x a_x|n\rangle.$$

26. For the hydrogen atom in stretched parabolic coordinates, μ, ν, ϕ (see problem 6), the following operators are useful

$$T_1 = \frac{1}{4}\left(\frac{\partial^2}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial}{\partial\mu}\right) - \frac{m^2}{4\mu^2} + \frac{\mu^2}{4},$$

$$T_2 = \frac{i}{2}\left(\mu\frac{\partial}{\partial\mu} + 1\right),$$

$$T_3 = -\frac{1}{4}\left(\frac{\partial^2}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial}{\partial\mu}\right) + \frac{m^2}{4\mu^2} + \frac{\mu^2}{4},$$

$$T_1' = \frac{1}{4}\left(\frac{\partial^2}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial}{\partial\nu}\right) - \frac{m^2}{4\nu^2} + \frac{\nu^2}{4},$$

$$T_2' = \frac{i}{2}\left(\nu\frac{\partial}{\partial\nu} + 1\right),$$

$$T_3' = -\frac{1}{4}\left(\frac{\partial^2}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial}{\partial\nu}\right) + \frac{m^2}{4\nu^2} + \frac{\nu^2}{4}.$$

Show that both the T_j and the T_j' satisfy the SO(2,1) commutation relations of problem 23. Show that the operators are hermitian with respect to the scalar product

$$\int_0^\infty d\mu\mu U_1^*(\mu)U_2(\mu) \qquad \text{for the } T_j,$$

and with respect to the scalar product

$$\int_0^\infty d\nu\nu V_1^*(\nu)V_2(\nu) \qquad \text{for the } T_j'.$$

Note: With $\psi(\mu, \nu, \phi) = U(\mu)V(\nu)\Phi(\phi)$, with $\Phi_m(\phi) = e^{im\phi}/\sqrt{2\pi}$, the standard scalar product would have been

$$\int d\vec{r}\psi_1^*\psi_2 = \int_0^{2\pi} d\phi \int_0^\infty d\mu \int_0^\infty d\nu \frac{(\mu^2+\nu^2)\mu\nu}{[-2\epsilon]^{\frac{3}{2}}}$$

$$U_1^*(\mu)V_1^*(\nu)\Phi_1^*(\phi)U_2(\mu)V_2(\nu)\Phi_2(\phi).$$

Show how the Schrödinger equation for the hydrogenic atom can be rewritten in terms of the operators, T_j, T_j'. For this purpose, rewrite the Schrödinger equation

$$(H - \epsilon)\psi = 0,$$

or

$$-\frac{1}{2}\frac{(-2\epsilon)}{(\mu^2+\nu^2)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial}{\partial\nu} + \left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right)\frac{\partial^2}{\partial\phi^2}\right)\psi$$

$$-\frac{2\sqrt{(-2\epsilon)}}{(\mu^2+\nu^2)}\psi - \epsilon\psi = 0,$$

by left-multiplying with $\frac{1}{2}n^2(\mu^2+\nu^2)$ to gain

$$-\frac{1}{4}\left(\frac{\partial^2}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial}{\partial\nu} + \left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right)\frac{\partial^2}{\partial\phi^2}\right)\psi$$

$$+\frac{1}{4}(\mu^2 + \nu^2)\psi - \frac{1}{\sqrt{(-2\epsilon)}}\psi = 0.$$

Show

$$\mathcal{T}^2 = T_3^2 - T_1^2 - T_2^2 = (T_3 - T_1)(T_3 + T_1) + [T_1, T_3] - T_2^2,$$

or

$$\mathcal{T}^2 = (T_3 - T_1)(T_3 + T_1) - iT_2 - T_2^2 = \frac{(m^2 - 1)}{4}.$$

Similarly,

$$\mathcal{T}'^2 = \frac{(m^2 - 1)}{4},$$

where m is the eigenvalue of the operator

$$\frac{1}{i}\frac{\partial}{\partial\phi}.$$

Show that for positive values of m:

$$(T_3)_{\text{eigen}} = \tfrac{1}{2} + \tfrac{1}{2}m + n_1, \qquad n_1 = 0, 1, 2, \ldots, \to \infty,$$

$$(T_3')_{\text{eigen}} = \tfrac{1}{2} + \tfrac{1}{2}m + n_2, \qquad n_2 = 0, 1, 2, \ldots, \to \infty.$$

Find the corresponding ranges for $(T_3)_{\text{eigen}}$ and $(T_3')_{\text{eigen}}$, valid for negative values of m.

Find the energy, ϵ, for the hydrogenic atom as a function of the quantum numbers, m, n_1, n_2.

In an $|mn_1n_2\rangle$ basis, find expressions for the nonzero matrix elements of the dimensionless variables

$$(r + z) = \frac{\mu^2}{[-2\epsilon]^{\frac{1}{2}}} \qquad \text{and} \qquad (r - z) = \frac{\nu^2}{[-2\epsilon]^{\frac{1}{2}}}.$$