Chapter 1
VaR in High Dimensional Systems-A Conditional Correlation Approach

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Abstract In empirical finance, multivariate volatility models are widely used to capture both volatility clustering and contemporaneous correlation of asset return vectors. In higher dimensional systems, parametric specifications often become intractable for empirical analysis owing to large parameter spaces. On the contrary, feasible specifications impose strong restrictions that may not be met by financial data as, for instance, constant conditional correlation (CCC). Recently, dynamic conditional correlation (DCC) models have been introduced as a means to solve the trade off between model feasibility and flexibility. Here, we employ alternatively the CCC and the DCC modeling framework to evaluate the Value-at-Risk associated with portfolios comprising major U.S. stocks. In addition, we compare their performances with corresponding results obtained from modeling portfolio returns directly via univariate volatility models.

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W.K. Härdle et al. (eds.), *Applied Quantitative Finance*, Statistics and Computing,
DOI 10.1007/978-3-662-54486-0_1

1.1 Introduction

Volatility clustering, i.e. positive correlation of price variations observed on speculative markets, motivated the introduction of autoregressive conditionally heteroskedastic (ARCH) processes by Engle (1982) and its popular generalizations by Bollerslev (1986) (Generalized ARCH, GARCH) and Nelson (1991) (Exponential GARCH). Being univariate in nature, however, these models neglect a further stylized feature of empirical price variations, namely contemporaneous correlation over a cross section of assets, stock or foreign exchange markets (Engle et al. 1990a; Hamao et al. 1990; Hafner and Herwartz 1998; Lee and Long 2009).

The covariance between asset returns is of essential importance in finance. Effectively, many problems in financial theory and practice, such as asset allocation, hedging strategies or Value-at-Risk (VaR) evaluation, require some formalization not merely of univariate risk measures but rather of the entire covariance matrix (Bollerslev et al. 1988; Cecchetti et al. 1988). Similarly, pricing of options with more than one underlying asset will require some (dynamic) forecasting scheme for time varying variances and covariances as well (Duan 1995).

When modeling time dependent second order moments, a multivariate model is a natural framework to take cross sectional information into account. Over recent years, multivariate volatility models have been attracting high interest in econometric research and practice. Popular examples of multivariate volatility models comprise the GARCH model class recently reviewed by Bauwens et al. (2006). Numerous versions of the multivariate GARCH (MGARCH) model suffer from huge parameter spaces. Thus, their scope in empirical finance is limited since the dimension of vector valued systems of asset returns should not exceed five (Ding and Engle 2001). Factor structures (Engle et al. 1990b) and so-called correlation models (Bollerslev 1990) have been introduced to cope with the curse of dimensionality in higher dimensional systems. The latter start from univariate GARCH specifications to describe volatility patterns and formalize in a second step the conditional covariances implicitly via some model for the systems' conditional correlations. Recently, dynamic conditional correlation models have been put forth by Engle (2002), Engle and Sheppard (2001) and Tse and Tsui (2002) that overcome the restrictive CCC pattern (Bollerslev 1990) while retaining its computational feasibility.

Here, we will briefly review two competing classes of MGARCH models, namely the half-vec model family and correlation models. The latter will be applied to evaluate the VaR associated with portfolios comprised by stocks listed in the Dow Jones Industrial Average (DJIA) index. We motivate the idea for VaR backtesting and reference the recent literature on (un)conditional VaR coverage tests. We compare the performance of models building on constant and dynamic conditional correlation. Moreover, it is illustrated how a univariate volatility model performs in comparison with both correlation models.

The remainder of this paper is organized as follows. The next section introduces the MGARCH model and briefly mentions some specifications that fall within the class of so-called half-vec MGARCH models. Correlation models are the focus of Sect. 1.3 where issues like estimation or inference within this model family are discussed in some detail. In Sect. 1.4, we motivate and discuss VaR backtesting by means of (un)conditional coverage. An empirical application of basic correlation models to evaluate the VaR for portfolios comprising U.S. stocks is provided in Sect. 1.5.

1.2 Half-Vec Multivariate GARCH Models

Let $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{Nt})^\top$ denote an N-dimensional vector of serially uncorrelated components with mean zero. The latter could be directly observed or estimated from a multivariate regression model. The process ε_t follows a multivariate GARCH process if it has the representation

$$\varepsilon_t|\mathcal{F}_{t-1} \sim N(0, \Sigma_t), \Sigma_t = [\sigma_{ij,t}], \tag{1.1}$$

where Σ_t is measurable with respect to information generated up to time $t-1$, formalized by means of the filtration $\mathcal{F}_{t-1}$. The $N \times N$ conditional covariance matrix, $\Sigma_t = \mathsf{E}[\varepsilon_t\varepsilon_t^\top|\mathcal{F}_{t-1}]$, has typical elements $\sigma_{ij,t}$ with $i = j$ $(i \neq j)$ indexing conditional variances (covariances). In a multivariate setting, potential dependencies of the second order moments in Σ_t on $\mathcal{F}_{t-1}$ become easily intractable for practical purposes.

The assumption of conditional normality in (1.1) allows to specify the likelihood function for observed processes $\varepsilon_t,\ t = 1, 2, \ldots, T$. In empirical applications of GARCH models, it turned out that conditional normality of speculative returns is more an exception than the rule. Maximizing the misspecified Gaussian log-likelihood function is justified by quasi maximum likelihood (QML) theory. Asymptotic theory on properties of the QML estimator in univariate GARCH models is well developed (Bollerslev and Wooldridge 1992; Lee and Hansen 1994; Lumsdaine 1996 and a few results on consistency Jeantheau 1998) and asymptotic normality Comte and Lieberman (2003); Ling and McAleer (2003) have been derived for multivariate processes.

The so-called half-vec specification encompasses all MGARCH variants that are linear in (lagged) second order moments or squares and cross products of elements in (lagged) ε_t. Let vech(B) denote the half-vectorization operator stacking the elements of a $(m \times m)$ matrix B from the main diagonal downwards in a $m(m+1)/2$ dimensional column vector. We concentrate the formalization of MGARCH models on the MGARCH(1,1) case which is, by far, the dominating model order used in the empirical literature (Bollerslev et al. 1994). Within the half-vec representation of the GARCH(1, 1) model Σ_t is specified as follows:

$$\text{vech}(\Sigma_t) = c + A\,\text{vech}(\varepsilon_{t-1}\varepsilon_{t-1}^\top) + G\,\text{vech}(\Sigma_{t-1}). \tag{1.2}$$

In (1.2), the matrices A and G each contain $\{N(N+1)/2\}^2$ elements. Deterministic covariance components are collected in c, a column vector of dimension $N(N+1)/2$. On the one hand, the half-vec model in (1.2) allows a very general dynamic structure of the multivariate volatility process. On the other hand, this specification suffers from huge dimensionality of the relevant parameter space which is of order $\mathcal{O}(N^4)$. In addition, it might be cumbersome or even impossible in applied work to restrict the admissible parameter space such that the time path of implied matrices Σ_t is positive definite.

To reduce the dimensionality of MGARCH models, numerous avenues have been followed that can be nested in the general class of half-vec models. Prominent examples in this vein of research are the Diagonal model (Bollerslev et al. 1988), the BEKK model (Baba et al. 1990; Engle and Kroner 1995), the Factor GARCH (Engle et al. 1990b), the orthogonal GARCH (OGARCH) (Alexander 1998, 2001) or the generalized OGARCH model put forth by Van der Weide (2002). Evaluating the merits of these proposals requires to weight model parsimony and computational issues against the implied loss of generality. For instance, the BEKK model is convenient to allow for cross sectional dynamics of conditional covariances, and weak restrictions have been formalized keeping Σ_t positive definite over time (Engle and Kroner 1995). Implementing the model will, however, involve simultaneous estimation of $\mathcal{O}(N^2)$ parameters such that the BEKK model has been rarely applied in higher dimensional systems ($N > 4$). Factor models build upon univariate factors, such as an observed stock market index (Engle et al. 1990b) or underlying principal components (Alexander 1998, 2001). The latter are assumed to exhibit volatility dynamics which are suitably modeled by univariate GARCH-type models. Thereby, factor models drastically reduce the number of model parameters undergoing simultaneous estimation. Model feasibility is, however, paid with restrictive correlation dynamics implied by the (time invariant) loading coefficients. Moreover, it is worthwhile mentioning that in case of factor specifications still $\mathcal{O}(N)$ parameters have to be estimated jointly when maximizing the Gaussian (quasi) likelihood function.

1.3 Correlation Models

1.3.1 Motivation

Correlation models comprise a class of multivariate volatility models that is not nested within the half-vec specification. Similar to factor models, correlation models circumvent the curse of dimensionality by separating the empirical analysis in two steps. First, univariate volatility models are employed to estimate volatility dynamics of each asset specific return process ε_{it}, $i = 1, \ldots, N$. In a second step Σ_t is obtained imposing some parsimonious structure on the correlation matrix (Bollerslev 1990). Thus, in the framework of correlation models we have

$$\Sigma_t = V_t(\theta)R_t(\phi)V_t(\theta), \tag{1.3}$$

where $V_t = \text{diag}(\sqrt{\sigma_{11,t}}, \ldots, \sqrt{\sigma_{NN,t}})$ is a diagonal matrix having as typical elements the square roots of the conditional variances estimates $\sigma_{ii,t}$. The latter could be obtained from some univariate volatility model specified with parameter vectors θ_i stacked in $\theta = (\theta_1^\top, \ldots, \theta_N^\top)^\top$. If univariate GARCH(1,1) models are used for the conditional volatilities $\sigma_{ii,t}$, θ_i will contain 3 parameters such that θ is of length $3N$. Owing to its interpretation of a correlation matrix, the diagonal elements in $R(\phi)$ are unity ($r_{ii} = 1,\ i = 1, \ldots, N$). From the general representation in (1.3) it is apparent that alternative correlation models particularly differ with regard to the formalization of the correlation matrix $R_t(\phi)$ specified with parameter vector ϕ.

In this section, we will highlight a few aspects of correlation models. First, a log-likelihood decomposition is given that motivates the stepwise empirical analysis. Then, two major variants of correlation models are outlined, the early CCC model (Bollerslev 1990) and the DCC approach introduced by Engle (2002) and Engle and Sheppard (2001). Tools for inference in correlation models that have been applied in the empirical part of the paper are collected in an own subsection. Also, a few remarks on recent generalizations of the basic DCC specification are provided.

1.3.2 Log-Likelihood Decomposition

The adopted separation of volatility and correlation analysis is motivated by a decomposition of the Gaussian log-likelihood function (Engle 2002) applying to the model in (1.1) and (1.3):

$$\begin{aligned} l(\theta, \phi) &= -\frac{1}{2}\left\{\sum_{t=1}^{T} N\log(2\pi) + \log(|\Sigma_t|) + \varepsilon_t^\top \Sigma_t^{-1}\varepsilon_t\right\} \\ &= -\frac{1}{2}\left\{\sum_{t=1}^{T} N\log(2\pi) + 2\log(|V_t|) + \log(|R_t|) + \varepsilon_t^\top \Sigma_t^{-1}\varepsilon_t\right\} \\ &= \sum_{t=1}^{T} l_t(\theta, \varphi), \end{aligned}$$

$$l_t(\theta, \phi) = l_t^V(\theta) + l_t^C(\theta, \phi), \tag{1.4}$$

$$l_t^V(\theta) = -\frac{1}{2}\left\{N\log 2\pi + 2\log(|V_t(\theta)|) + \varepsilon_t' V_t(\theta)^{-2}\varepsilon_t\right\} \tag{1.5}$$

$$l_t^C(\theta, \phi) = -\frac{1}{2}\left(\log|R_t(\phi)| + v_t^\top R_t(\phi)^{-1} v_t - v_t^\top v_t\right). \tag{1.6}$$

According to (1.5) and (1.6), the maximization of the log-likelihood function may proceed in two steps. First, univariate volatility models are used to maximize the volatility component, $l_t^V(\theta)$, and conditional on first step estimates $\hat{\theta}$, the correlation part $l_t^C(\theta, \phi)$ is maximized in a second step. To perform a sequential estimation procedure efficiently, it is required that the volatility and correlation parameters are variation free (Engle et al. 1983) meaning that there are no cross relationships linking single parameters in θ and ϕ when maximizing the Gaussian log-likelihood function. In the present case, the parameters in θ will impact on $v_t = V_t^{-1}\varepsilon_t,\ v_t = (v_{1t}, v_{2t}, \ldots, v_{Nt})^\top$, and, thus, the condition necessary to have full information and limited information estimation equivalent is violated. Note, however, that univariate GARCH estimates $(\hat{\theta})$ will be consistent. Thus, owing to the huge number of available observations which is typical for empirical analyses of financial data, the efficiency loss involved with a sequential procedure is likely to be smaller in comparison with the gain in estimation feasibility.

1.3.3 Constant Conditional Correlation Model

Bollerslev (1990) proposes a constant conditional correlation (CCC) model

$$\sigma_{ij,t} = r_{ij}\sqrt{\sigma_{ii,t}\sigma_{jj,t}},\ i, j = 1, \ldots, N,\ i \neq j. \tag{1.7}$$

Given positive time paths of the systems' volatilities, positive definiteness of Σ_t is easily guaranteed for the CCC model $(|r_{ij}| < 1,\ i \neq j)$. As an additional objective of this specification, it is important to notice that the estimation of the correlation pattern may avoid iterative QML estimation of the $\{N(N-1)/2\}$ correlation parameters r_{ij} comprising $R_t(\phi) = R$. Instead, one may generalize the idea of variance targeting (Engle and Mezrich 1996) towards the case of correlation targeting. Then, $D = E[v_t v_t^\top]$ is estimated as the unconditional covariance matrix of standardized returns, $v_t = V_t^{-1}\varepsilon_t$, and R is the correlation matrix implied by D. With $'\odot'$ denoting matrix multiplication by element, we have formally

$$\hat{R} = \hat{D}^{*-1/2}\hat{D}\hat{D}^{*-1/2},\ \hat{D} = \frac{1}{T}\sum_{t=1}^{T} v_t v_t^\top,\ \hat{D}^* = \hat{D} \odot I_N. \tag{1.8}$$

The price paid for the feasibility of CCC is, however, the assumption of a rather restrictive conditional correlation pattern which is likely at odds with empirical systems of speculative returns. Applying this model in practice therefore requires at least some pretest for constant correlation (Tse 2000; Engle 2002).

1.3.4 Dynamic Conditional Correlation Model

The dynamic conditional correlation model introduced by Engle (2002) and Engle and Sheppard (2001) preserves the analytic separability of the models' volatilities and correlations, but allows a richer dynamic structure for the latter. For convenience, we focus the representation of the DCC model again on the DCC(1,1) case formalizing the conditional correlation matrix $R_t(\phi)$ as follows:

$$R_t(\phi) = \{Q_t^*(\phi)\}^{-1/2} Q_t(\phi) \{Q_t^*(\phi)\}^{-1/2}, \;\; Q_t^*(\phi) = Q_t(\phi) \odot I_N, \tag{1.9}$$

with

$$Q_t(\phi) = R(1 - \alpha - \beta) + \alpha v_{t-1} v_{t-1}^\top + \beta Q_{t-1}(\phi) \tag{1.10}$$

and R is a positive definite (unconditional) correlation matrix of v_t.

Sufficient conditions guaranteeing positive definiteness of the time path of conditional covariance matrices Σ_t implied by (1.3), (1.9) and (1.10) are given in Engle and Sheppard (2001). Apart from well known positivity constraints to hold for the univariate GARCH components, the DCC(1,1) model will deliver positive definite covariances if $\alpha > 0, \; \beta > 0$ while $\alpha + \beta < 1$ and λ_{min}, the smallest eigenvalue of R, is strictly positive, i.e. $\lambda_{min} > \delta > 0$. It is worthwhile to point out that the DCC framework not only preserves the separability of volatility and correlation estimation, but also allows to estimate the nontrivial parameters in R via correlation targeting described in (1.8).

Given consistent estimates of unconditional correlations $r_{ij}, \; i \neq j$, the remaining parameters describing the correlation dynamics are collected in the two-dimensional vector $\varphi = (\alpha, \beta)^\top$. Note that making use of correlation targeting the number of parameters undergoing nonlinear iterative estimation in the DCC model is constant $(= 2)$, and, thus, avoids the curse of dimensionality even in case of very large systems of asset returns.

Instead of estimating the model in three steps, one could alternatively estimate the unconditional correlation parameters in R and the coefficients in φ jointly. Note that the number of unknown parameters in R is $\mathcal{O}(N^2)$. Formal representations of first and second order derivatives to implement the two step estimation and inference can be found in Hafner and Herwartz (2008). We prefer the three step approach here, since it avoids iterative estimation procedures in large parameter spaces.

1.3.5 Inference in the Correlation Models

QML-inference on significance of univariate GARCH parameter estimates is discussed in Bollerslev and Wooldridge (1992). Analytical expressions necessary to evaluate the asymptotic covariance matrix are given in Bollerslev (1986). In the empirical part of the chapter, we will not provide univariate GARCH parameter estimates at all to economize on space. Two issues of evaluating parameter significance remain, inference for the correlation estimates given in (1.8) and for the estimated DCC parameters $\hat{\varphi}$. We consider these two issues in turn:

1. Inference for unconditional correlations

   Conditional on estimates $\hat{\theta}$, we estimate R from standardized univariate GARCH residuals as formalized in (1.8). The elements in $\hat{R}$ are obtained as a nonlinear and continuous transformation of the elements in $\hat{D}$, i.e. $\hat{R} = \hat{D}^{*^{-1/2}} \hat{D} \hat{D}^{*^{-1/2}}$. Denote with vechl$(B)$ an operator stacking the elements below the diagonal of a symmetric $(m \times m)$ matrix B in a $\{m(m-1)/2\}$ dimensional column vector $b_l =$ vechl(B). Thus, $\hat{r}_l =$ vechl$(\hat{R})$ collects the nontrivial elements in $\hat{R}$. Standard errors for the estimates in $\hat{r}_l$ can be obtained from a robust estimator of the covariance of the (nontrivial) elements in $\hat{D}$, $\hat{d} =$ vech$(\hat{D})$, via the delta method. To be precise, we estimate the covariance of $\hat{r}_l$ by means of the following result (Ruud 2000):
$$\sqrt{T}(\hat{r}_l - r_l) \stackrel{\mathcal{L}}{\to} N\left(0, \mathcal{H}(\hat{r})\mathcal{G}\mathcal{H}(\hat{r})^\top\right), \tag{1.11}$$

   where $\mathcal{G}$ is an estimate of the covariance matrix of the elements in d, $\mathcal{G} = \widehat{\text{Cov}}(\hat{d})$, and $\mathcal{H}(\hat{r})$ is a $\{N(N-1)/2 \times (N(N+1)/2)\}$ dimensional matrix collecting the first order derivatives $\partial r_l / \partial d^\top$ evaluated at $\hat{d}$. We determine $\mathcal{G}$ by means of the covariance estimator

$$\mathcal{G} = \frac{1}{T}\sum_{t=1}^{T}(vv)_t(vv)_t^\top, \;\; (vv)_t = \text{vech}(v_t v_t^\top) - \hat{d}. \tag{1.12}$$

   The derivatives in $\mathcal{H}(r)$ are derived from a result in Hafner and Herwartz (2008) as

$$\frac{\partial r_l}{\partial d^\top} = P_{N,-}^\top (D^* \otimes D^*) P_N + P_{N,-}^\top (DD^* \otimes I_N + I_N \otimes DD^*) P_N \frac{\partial \text{vech}(D^*)}{\partial \text{vech}(D)^\top}$$

   and
$$\frac{\partial \text{vech}(D^*)}{\partial \text{vech}(D)^\top} = -\frac{1}{2}\,\text{diag}\left[\text{vech}\left\{(I_N \odot D)^{-3/2}\right\}\right],$$

   where the matrices $P_{N,-}$ and P_N serve as duplication matrices (Lütkepohl 1996) such that $(B) = P_{N,-}$vechl(B) and $(B) = P_N$vech(B).
2. Inference for correlation parameters
   The correlation parameters are estimated by maximizing the correlation part, $l^C(\theta, \phi)$, of the Gaussian (quasi) log-likelihood function. When evaluating the estimation uncertainty associated with $\hat{\varphi} = (\hat{\alpha}, \hat{\beta})^\top$, the sequential character of the estimation procedure has to be taken into account. To provide standard errors

for QML estimates $\hat{\varphi}$, we follow a GMM approach introduced in Newey and McFadden (1994), which works in case of sequential GMM estimation under typical regularity conditions. In particular, it is assumed that all steps of a sequential estimation procedure are consistent. The following result on the asymptotic behavior of $\hat{\gamma} = (\hat{\theta}^\top, \hat{\varphi}^\top)^\top$ applies:

$$\sqrt{T}(\hat{\gamma} - \gamma) \xrightarrow{\mathcal{L}} N(0, \mathcal{N}^{-1}\mathcal{M}(\mathcal{N}^{-1})^\top). \tag{1.13}$$

In (1.13), $\mathcal{M}$ is the (estimated) expectation of the outer product of the scores of the log-likelihood function evaluated at $\hat{\gamma}$,

$$\mathcal{M} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\partial l_t}{\partial \gamma}\right)\left(\frac{\partial l_t}{\partial \gamma}\right)^\top, \quad \frac{\partial l_t}{\partial \gamma} = \left(\frac{\partial l_t^V}{\partial \theta^\top}, \frac{\partial l_t^C}{\partial \varphi^\top}\right)^\top. \tag{1.14}$$

Compact formal representations for the derivatives in (1.14) can be found in Hafner and Herwartz (2008) and Bollerslev (1986). The matrix $\mathcal{N}$ in (1.13) has a lower block diagonal structure containing (estimates) of expected second order derivatives, i.e.

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_{11} & 0 \\ \mathcal{N}_{21} & \mathcal{N}_{22} \end{pmatrix},$$

with

$$\mathcal{N}_{11} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_t^V}{\partial\theta\partial\theta^\top}, \quad \mathcal{N}_{21} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_t^C}{\partial\varphi\partial\theta^\top}, \quad \mathcal{N}_{22} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_t^C}{\partial\varphi\partial\varphi^\top}.$$

Formal representations of the latter second order quantities are provided in Hafner and Herwartz (2008).

1.3.6 Generalizations of the DCC Model

Generalizing the basic DCC(1,1) model in (1.9) and (1.10) towards higher model orders is straightforward and in analogy to the common GARCH volatility model. In fact, it turns out that the DCC(1,1) model is often sufficient to capture empirical correlation dynamics (Engle and Sheppard 2001). Tse and Tsui (2002) propose a direct formalization of the dynamic correlation matrix R_t as a weighted average of unconditional correlation, lagged correlation and a local correlation matrix estimated over a time window comprising the M most recent GARCH innovation vectors ξ_{t-i}, $i = 1, ..., M$, $M \geq N$. As discussed so far, dynamic correlation models are restrictive in the sense that asset specific dynamics are excluded. Hafner and Franses (2003) discuss a generalized DCC model where the parameters α and β in (1.10) are replaced by outer products of N-dimensional vectors, e.g. $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)^\top$, obtaining

$$Q_t = R(1 - \tilde{\alpha}\tilde{\alpha}^\top - \tilde{\beta}\tilde{\beta}^\top) + \tilde{\alpha}\tilde{\alpha}^\top \odot v_{t-1}v_{t-1}^\top + \tilde{\beta}\tilde{\beta}^\top \odot Q_{t-1}. \tag{1.15}$$

From (1.15) it is apparent that implied time paths of conditional correlations show asset specific characteristics. Similar to the generalization of the basic GARCH volatility model towards threshold specifications (Glosten et al. 1993), one may also introduce asymmetric dependencies of Q_t on $\mathrm{vech}(v_t v_t^\top)$ as in Cappiello et al. (2006). A semiparametric conditional correlation model is provided by Hafner et al. (2006). In this model, the elements in Q_t are determined via local averaging where the weights entering the nonparametric estimates depend on a univariate factor as, for instance, market volatility or market returns.

1.4 Value-at-Risk

Financial institutions and corporations can suffer financial losses in their portfolios or treasury department due to unpredictable and sometimes extreme movements in the financial markets. The recent increase in volatility in financial markets and the surge in corporate failures are driving investors, management and regulators to search for ways to quantify and measure risk exposure. One answer came in the form of Value-at-Risk (VaR) being the minimum loss a portfolio will not exceed with a given probability over a specific time horizon (Jorion 2007; Christoffersen et al. 2001). For a critical review of the VaR approach see Acerbi and Tasche (2002). They also discuss the merits of an important and closely related risk measure, the expected shortfall. It is defined as the expected tail return conditional on a specific VaR level and provides further sensitive insights into the loss distribution, i.e. the expected portfolio loss when the portfolio value exceeds the VaR.

The VaR of some portfolio (.) may be defined as a one-sided confidence interval of expected h-periods ahead losses:

$$\mathrm{VaR}^{(.)}_{t+h,\zeta} = \Xi^{(.)}_t (1 + \bar{\xi}_{t+h,\zeta}), \tag{1.16}$$

where $\Xi^{(.)}_t$ is the value of a portfolio in time t and $\bar{\xi}_{t+h,\zeta}$ is a time dependent quantile of the conditional distribution of portfolio returns $\xi^{(.)}_{t+h}$ such that

$$\mathrm{P}[\xi^{(.)}_{t+h} < \bar{\xi}_{t+h,\zeta}] = \zeta, \;\; \bar{\xi}_{t+h,\zeta} = \sigma_{t+h} z_\zeta, \tag{1.17}$$

and z_ζ is a quantile from an unconditional distribution with unit variance. In the light of the assumption of conditional normality in (1.1), we will take the quantiles z_ζ from the Gaussian distribution. As outlined in (1.16) and (1.17), the quantities $\bar{\xi}_{t+h,\zeta}$ and σ_{t+h} generally depend on the portfolio composition. For convenience, however, our notation does not indicate this relationship. Depending on the risk averseness of the agent, the parameter ζ is typically chosen as some small probability, for instance, $\zeta = 0.005, 0.01, 0.05$.

In order to assess the performance of distinct VaR models in-sample and out-of-sample, one can employ VaR backtesting methods. Several contributions in the recent literature exploit the statistical properties of the empirical hit series. A literature review and a comparative simulation study can be found in Campbell (2006). Given ζ, a so-called hit in time $t + h$ is defined by

$$\text{hit}_{t+h}(\zeta) = \mathbf{1}\left(\Xi_{t+h}^{(.)} < \text{VaR}_{t+h,\zeta}^{(.)}\right).$$

The indicator function $\mathbf{1}$ becomes unity if the portfolio value falls below its computed VaR and is zero otherwise. If the model is correctly specified the empirical hit rate, $\hat{\zeta} = 1/T \sum_{t=1}^{T} \text{hit}_{t+h}(\zeta)$, for $T \to \infty$ periods converges to ζ. In the empirical part, we will exploit this fact and compare the unconditional coverage of the estimated VaR series for the discussed volatility models.

Secondly, if the model is correctly specified, the observed hits do not provide any serial information and they are assumed to be independent. To validate the unconditional and conditional VaR coverage, Christoffersen (1998) suggests two likelihood ratio tests. These tests have been widely employed in the literature on multivariate volatility (Chib et al. 2006). A similar idea on testing the conditional coverage, Engle and Manganelli (2004) propose a dynamic quantile test assessing an autoregressive model on the series of centered hits by a Wald test for joint significance of the coefficients. A linear dependency of the hits in time contradicts the VaR model specification. Ready to use software implementations for VaR backtesting are briefly exposed in Chap. 1 Appendix.

1.5 An Empirical Illustration

1.5.1 Equal and Value Weighted Portfolios

We analyze portfolios comprised by all 30 stocks listed in the Dow Jones Industrial Average (DJIA) over the period Jan, 2nd, 1990 to Jan, 31st, 2005. The asset returns were computed using historical closing prices provided by Yahoo Finance. Measured at the daily frequency, 3803 observations are used for the empirical analysis. Two alternative portfolio compositions are considered. In the first place, we analyze a portfolio weighting each asset equally. Returns of this equal weight portfolio (EWP) are obtained from asset specific returns ($\varepsilon_{it},\ i = 1, \ldots, N$) as

$$\xi_t^{(e)} = \sum_{i=1}^{N} w_{it}^{(e)} \varepsilon_{it}, \quad w_{it}^{(e)} = N^{-1}.$$

Secondly, we consider value weighted portfolios (VWP) determined as:

$$\xi_t^{(v)} = \sum_{i=1}^{N} w_{it}^{(v)} \varepsilon_{it}, \; w_{it}^{(v)} = w_{it-1}(1+\varepsilon_{it-1})/w_t^{(v)}, \; w_t^{(v)} = \sum_i w_{it-1}(1+\varepsilon_{it-1}).$$

Complementary to an analysis of EWP and VWP, dynamics of minimum variance portfolios (MVP) could also be of interest. The MVP, however, will typically depend on some measure of the assets' volatilities and covariances. The latter, in turn, depend on the particular volatility model used for the analysis. Since the comparison of alternative measures of volatility in determining VaR is a key issue of this investigation, we will not consider MVP to immunize our empirical results from impacts of volatility specific portfolio compositions.

Our empirical comparison of alternative approaches to implement VaR concentrates on the relative performance of one step ahead ex-ante evaluations of VaR ($h = 1$). Note, that the (M)GARCH model specifies covariance matrices Σ_t or univariate volatilities σ_t^2 conditional on $\mathcal{F}_{t-1}$. Therefore, we practically consider the issue of two step ahead forecasting when specifying

$$\text{VaR}_{t+1,\zeta}^{(.)}|\mathcal{F}_{t-1} = \text{VaR}^{(.)}(\hat{\sigma}_{t+1}^2), \; \hat{\sigma}_{t+1}^2|\mathcal{F}_{t-1} = E[(\xi_{t+1}^{(.)})^2|\mathcal{F}_{t-1}].$$

The performance of alternative approaches to forecast VaR is assessed by means of the relative frequency of actual hits observed over the entire sample period, i.e.

$$\text{hf}_\zeta^{(.)} = \frac{1}{3800} \sum_{t=3}^{3802} \mathbf{1}(\xi_t^{(.)} < \bar{\xi}_{t,\zeta}), \tag{1.18}$$

where $\mathbf{1}(.)$ is an indicator function. To determine the forecasted conditional standard deviation entering the VaR, we adopt three alternative strategies. As a benchmark, we consider standard deviation forecasts obtained from univariate GARCH processes fitted directly to the series of portfolio returns $\xi_t^{(.)}$. For the two remaining strategies, we exploit forecasts of the covariance matrix, $\hat{\Sigma}_{t+1} = E[\varepsilon_{t+1}\varepsilon_{t+1}^\top|\mathcal{F}_{t-1}]$, to determine VaR. Note that given portfolio weights $w_t = (w_{1t}, w_{2t}, \ldots, w_{Nt})^\top$, the expected conditional variance of the portfolio is $\hat{\sigma}_{t+1}^2 = w^\top \hat{\Sigma}_{t+1} w$. Feasible estimates for the expected covariance matrix are determined alternatively by means of the CCC and DCC model.

The empirical exercises first cover a joint analysis of all assets comprising the DJIA. Moreover, we consider 1000 portfolios composed of 5 securities randomly drawn from all assets listed in the DJIA. Implementing the volatility parts of both the CCC and the DCC model, we employ alternatively the symmetric GARCH(1,1) and the threshold GARCH(1,1) model as introduced by Glosten et al. (1993). Opposite to the symmetric GARCH model, the latter accounts for a potential leverage effect (Black 1976) stating that volatility is larger in the sequel of bad news (negative returns) in comparison with good news (positive returns).

Table 1.1 Estimation results and performance of VaR estimates. G and TG are short for GARCH(1,1) and TGARCH(1,1) models for asset specific volatilities, respectively. D, C and U indicate empirical results obtained from DCC, CCC and univariate GARCH(1,1) models applied to evaluate forecasts of conditional variances of equal weight (EWP) and value weighted portfolios (VWP). Entries in hf and s(hf) are relative frequencies of extreme losses and corresponding standard errors, respectively

$\zeta \cdot 1000$		$N = 30$		$N = 5$			
		G	TG	G		TG	
		hf	hf	$\bar{\text{hf}}$	s($\bar{\text{hf}}$)	$\bar{\text{hf}}$	s($\bar{\text{hf}}$)
EWP							
D	5.00	8.15	7.36	7.56	.033	7.13	.034
	10.0	13.2	12.4	11.7	.041	11.2	.042
	50.0	41.6	41.8	40.4	.075	40.3	.078
C	5.00	10.8	9.73	7.78	.034	7.36	.035
	10.0	14.2	14.2	11.9	.040	11.5	.042
	50.0	42.6	41.8	40.8	.074	40.7	.077
U	5.00	11.6	11.6	8.70	.036	8.36	.037
	10.0	14.7	14.7	13.2	.045	12.9	.045
	50.0	47.3	47.3	43.5	.076	44.0	.077
VWP							
D	5.00	6.58	7.10	7.86	.033	7.55	.033
	10.0	12.9	11.8	11.9	.043	11.6	.041
	50.0	41.6	40.5	40.3	.076	40.4	.078
C	5.00	9.21	9.21	8.18	.036	7.90	.035
	10.0	14.5	13.4	12.3	.043	12.1	.043
	50.0	42.6	41.8	41.1	.072	41.3	.071
U	5.00	9.99	9.99	8.71	.037	8.62	.035
	10.0	15.5	15.5	13.0	.048	12.9	.048
	50.0	43.7	43.7	42.6	.095	43.2	.098
Estimation results							
D	$\hat{\alpha}$	2.8e-03	2.8e-03	6.6e-03	4.5e-05	6.7e-03	4.8e-05
	t_α	17.5	17.3				
	$\hat{\beta}$	.992	.992	.989	8.3e-05	.989	9.5e-05
	t_β	1.8e+03	1.8e+03				

1.5.2 Estimation Results

A few selected estimation results are given in Table 1.1. Since we investigate 30 assets or 1000 random portfolios each containing $N = 5$ securities, we refrain from providing detailed results on univariate GARCH(1,1) or TGARCH(1,1) estimates. Moreover, we leave estimates of the unconditional correlation matrix R undocumented since the number of possible correlations in our sample is $N(N-1)/2 = 435$.

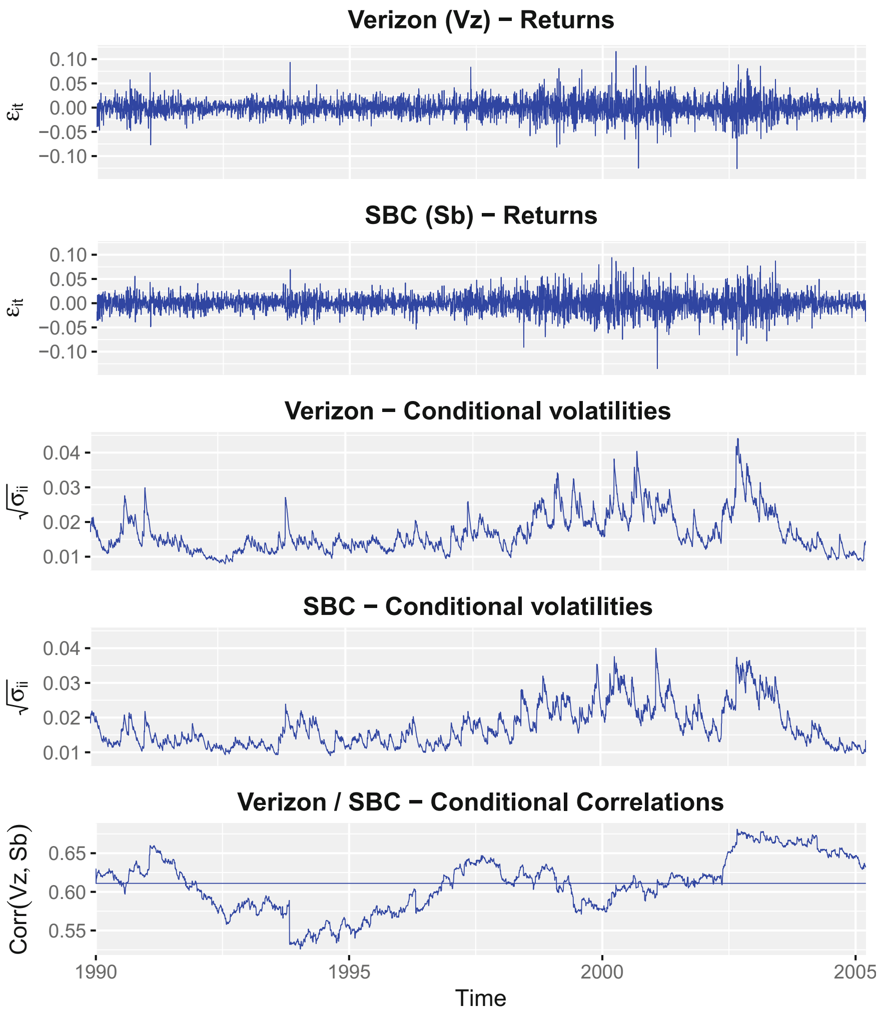

Fig. 1.1 Returns, conditional volatilities and correlations for Verizon and SBC communications

The lower left part of Table 1.1 provides estimates of the DCC parameters α and β and corresponding t-ratios for the analysis of all assets comprising the DJIA. Although the estimated α parameter governing the impact of lagged GARCH innovations on the conditional correlation matrix is very small (around $2.8 \cdot 10^{-3}$ for both implementations of the DCC model), it is significant at any reasonable significance level. The relative performance of the CCC and DCC model may also be evaluated in terms of the models' log-likelihood difference. Using symmetric and asymmetric volatility models for the diagonal elements of Σ_t, the log-likelihood difference between DCC and CCC is 645.66 and 622.00, respectively. Since the DCC specification has only two additional parameters, it apparently provides a substantial improvement of fitting multivariate returns. It is also instructive to compare, for the DCC case say, the log-likelihood improvement achieved when employing univariate TGARCH instead of a symmetric GARCH. Interestingly, implementing the DCC model with asymmetric GARCH the improvement of the log-likelihood is only 236.27, which is to be related to the number of $N = 30$ additional model parameters. Reviewing the latter two results, one may conclude that dynamic correlation is a more striking feature of U.S. stock market returns than leverage.

The sum of both DCC parameter estimates, $\hat{\alpha} + \hat{\beta}$, is slightly below unity and, thus, the estimated model of dynamic covariances is stationary. The lower right part of Table 1.1 gives average estimates obtained for the DCC parameters when modeling 1000 portfolios randomly composed of five securities contained in the DJIA. We also provide an estimator of the empirical standard error associated with the latter average. Irrespective of using a symmetric or asymmetric specification of univariate volatility models, estimates for α are small throughout. According to the reported standard error estimates, however, the true α parameter is apparently different from zero at any reasonable significance level.

The maximum over all 435 unconditional correlations is obtained for two firms operating on the telecommunication market, namely Verizon Communications and SBC Communications. To illustrate the performance of the DCC model and compare it with the more restrictive CCC counterpart, Fig. 1.1 provides the return processes for these two assets, the corresponding time paths of conditional standard deviations as implied by TGARCH(1,1) models and the estimated time paths of conditional correlations implied by the DCC model fitted over all assets contained in the DJIA. Facilitating the interpretation of the results, we also give the level of unconditional correlation.

Apparently, the univariate volatility models provide accurate descriptions of the return variability for both assets. Not surprisingly, estimated volatility turns out to be larger over the last third of the sample period in comparison with the first half. Although conditional correlation estimates vary around their unconditional level, the time path of correlation estimates exhibits only rather slow mean reversion. Interestingly, over the last part of the sample period, the conditional correlation measured between Verizon and SBC increases with the volatilities of both securities.

As mentioned, Verizon and SBC provide the largest measure of unconditional correlation within the DJIA over the considered sample period. To illustrate that time varying conditional correlation with slow mean reversion is also an issue for bivariate returns exhibiting medium or small correlation, we provide the conditional correlation estimates for Verizon and General Electric (medium unconditional correlation) and Verizon and Boeing (small unconditional correlation) in Fig. 1.2. For completeness, Fig. 1.3 provides empirical return processes for General Electric and Boeing.

The upper part of Table 1.1 shows relative frequencies of realized losses exceeding the one step ahead ex-ante VaR forecasts. We provide average relative frequencies when summarizing the outcome for 1000 portfolios with random composition. To

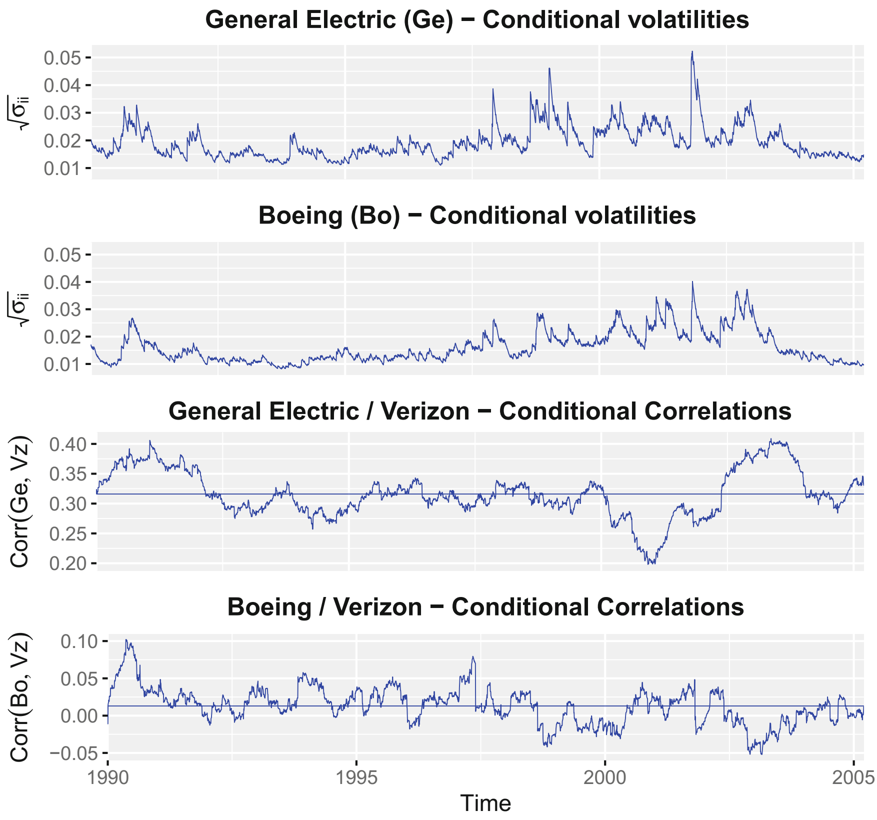

Fig. 1.2 Conditional volatilities for General Electric and Boeing and conditional correlations with Verizon

facilitate the discussion of the latter results, all frequencies given are multiplied with a factor of 1000.

The relative frequency of empirical hits of dynamic VaR estimates at the 5% level is uniformly below the nominal probability, indicating that dynamic VaR estimates are too conservative on average. For the remaining probability levels $\zeta = 0.5\%$ and $\zeta = 1\%$, the empirical frequencies of hitting the VaR exceed the nominal probability. We concentrate the discussion of empirical results on the latter cases. With regard to the performance of alternative implementations of VaR it is worthwhile to mention that the basic results are qualitatively similar for EWP in comparison with VWP. Similarly, employing an asymmetric GARCH model instead of symmetric GARCH has only minor impacts on the model comparison between the univariate benchmark and the CCC and DCC model, respectively. For the latter reason, we focus our discussion of the relative model performance on VaR modeling for EWP with symmetric GARCH(1,1) applied to estimate conditional variances.

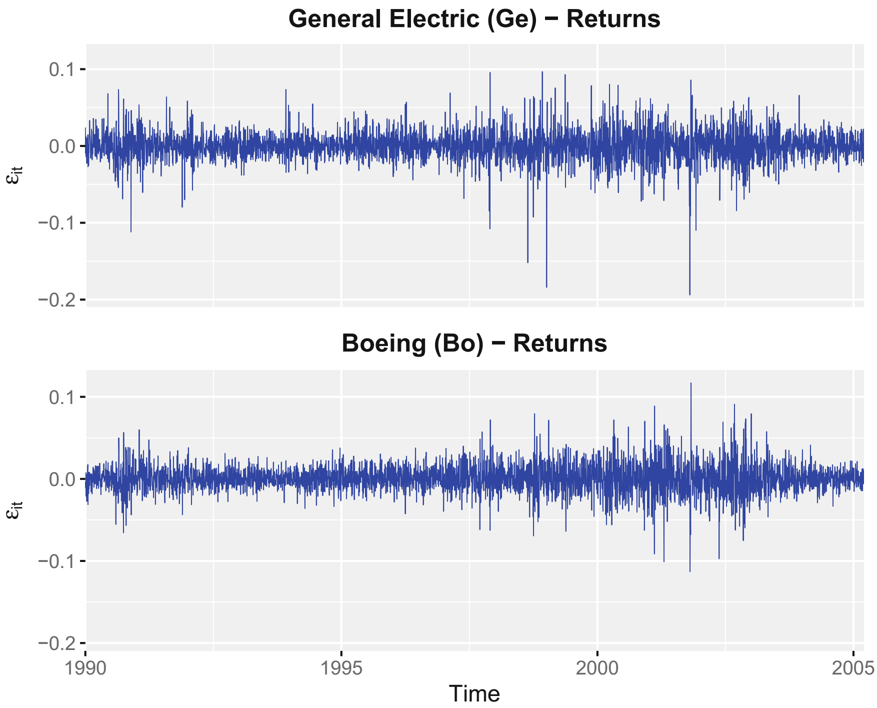

Fig. 1.3 Returns for General Electric and Boeing

Regarding portfolios composed of 30 securities, it turns out that for both probability levels, $\zeta = 1\%$ and $\zeta = 0.5\%$, the empirical frequencies of hitting the dynamic VaR estimates are closest to the nominal level for the DCC model and worst for modeling portfolio returns directly via univariate GARCH. Although it provides the best empirical frequencies of hitting the VaR, the DCC model still underestimates (in absolute value) on average the true quantile. For instance, the 0.5% VaR shows an empirical hit frequency of 0.82% (EWP) and 0.66% (VWP), respectively. Drawing randomly 5 out of 30 assets to form portfolios, and regarding the average empirical frequencies of hitting the VaR estimates, we obtain almost analogous results in comparison with the case $N = 30$. The reported standard errors of average frequencies, however, indicate that the discussed differences of nominal and empirical probabilities are significant at a 5% significance level since the difference between both exceeds twice the standard error estimates.

In summary, using the CCC and DCC model and, alternatively, univariate GARCH specifications to determine VaR, it turns out that the former outperform the univariate GARCH as empirical loss frequencies are closer to the nominal VaR coverage. DCC based VaR estimates in turn outperform corresponding quantities derived under the CCC assumption. Empirical frequencies of large losses, however, exceed the corresponding nominal levels if the latter are rather small, i.e. 0.5 and 1%. This might indicate that the DCC framework is likely to restrictive to hold homogeneously over a sample period of the length (more than 15 years) considered in this work. More general versions of dynamic correlation models are available but allowance of asset specific dynamics requires simultaneous estimation of $O(N)$ parameters.

Appendix: Software Packages

Various numerical programming environments provide built-in or third-party methods for analyzing conditional correlation models and Value-at-Risk backtesting tools. In this section, we briefly point out distinct implementations for the programming languages R, MATLAB and Stata.

Regarding the R Project, the package `rmgarch` (Ghalanos 2015) is suitable for modeling and analyzing the conditional correlation models, such as CCC and DCC. Its comprehensive function set supports the analysis of further multivariate volatility models, such as, for instance, the generalized orthogonal GARCH model by Van der Weide (2002). The package offers a sophisticated design of functions, time-critical procedures are partly implemented in C/C++ and various time series statistics are computed. The code is based on the package `rugarch` by the same author which can be used to study univariate volatility models in a similar sophisticated way. In addition, the latter package includes an implementation of the unconditional and conditional coverage VaR tests according to Christoffersen (1998). As an alternative, the package `ccgarch` Nakatani (2014) might be used for the evaluation of CCC and DCC models. Its functions were used to compute estimates and statistics quickly and correctly in several test applications. In comparison with `rmgarch`, its design and capabilities are less complex and it is restricted to conditional correlation models. Currently, there are no efforts by the authors of both packages to support the BEKK model.

Working with MATLAB, MathWorks' Econometrics Toolbox supports the simulation, estimation, and forecasting of different variants of univariate GARCH-type models. Its Risk Management Toolbox comprises an entire set of functions for assessing market risk, i.e. implementations of common approaches for VaR backtesting, which include the (un)conditional coverage tests described before. However, evaluations of multivariate volatility models including CCC or DCC can be carried out by means of the non-official MFE Toolbox. [1] It is the successor of the UCSD Toolbox by Kevin Sheppard. [2] The MFE project implements various univariate and multivariate volatility models and metrics. Its open source codebase is maintained and augmented by volunteers and particularly well suited as a starting point to study the programming of multivariate time series algorithms. Despite its wide range of functions, the user should always critically question the numerical results because the MFE project is still under development.

[1] Project website: https://www.kevinsheppard.com/MFE_Toolbox.

[2] Project website: https://www.kevinsheppard.com/UCSD_GARCH.

The Stata software package provides the user with comfortable fitting algorithms for conditional correlation models and diagonal half-vec models by means of the function `mgarch`. Its optimized program code proceeds rapidly and, at the same time, computes common metrics. The Stata documentation of the implemented methods is exemplary and might be a good complement while studying publicly available code examples of the volatility model implementations which are investigated in this chapter.

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