1 Mechanics

This chapter introduces some concepts from mechanics that are of biological or medical interest. We begin with a discussion of sizes important in biology. Then we turn to the forces on an object that is in equilibrium and calculate the forces experienced by various bones and muscles. In Sect. 1.9, we introduce the concept of mechanical work, which will recur throughout the book. The next two sections describe how materials deform when forces act on them. Sections 1.12 through 1.16 discuss the forces in stationary and moving fluids. These concepts are then applied to laminar viscous flow in a pipe, which is a model for the flow of blood and the flow of fluid through pores in cell membranes. The chapter ends with a discussion of the circulatory system.

1.1 Distances and Sizes

In biology and medicine, we study objects that span a wide range of sizes: from giant redwood trees to individual molecules. Therefore, we begin with a brief discussion of length scales. The basic unit of length in the metric system[1] is the meter (m): about the height of a 3-year-old child. For objects much larger or smaller than a meter, we add a prefix as shown in Table 1.1. For example, a kilometer is formed by adding the prefix "kilo," which means times one thousand (10^3 m = 1 km). Living organisms rarely, if ever, reach a size of 1 km; the tallest trees are about 0.1 km (100 m) high. A few animals (whales, dinosaurs) reach the size of tens of meters, but most organisms are a few meters or less in size.

Table 1.1 Common prefixes used in the metric system

Prefix	Abbreviation	Multiply by
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

The diversity of life becomes more obvious as we move down to smaller length scales. One one-hundredth of a meter is called a centimeter (1 cm = 10^{-2} m). The centimeter is still common in the medical literature, although it is going out of style among metric purists who prefer to use only prefixes that are factors of 1000.[2] One one-thousandth of a meter is a millimeter (1 mm = 10^{-3} m), about the thickness of a dime. We can still see objects of this size, but we cannot study their detailed structure with the unaided eye.

The microscope enables us to study objects many times smaller than 1 mm. The natural unit for measuring such objects is 10^{-6} m or 10^{-3} mm, called a micrometer (1 μm = 10^{-6} m). The nickname for the micrometer is the "micron." Figure 1.1 shows the relative sizes of objects in the range of 1 mm–1 μm and encompasses the length scale of cell biology. Many small structures of our body are this size. For instance, our lungs consist of a branching network of tubes through which air flows. These tubes end in small, nearly spherical air sacs called *alveoli* (Fig. 1.1b). Each alveolus has a diameter of about 250 μm, and this size is set by the diffusion properties of air (Chap. 4). *Protozoans* are a type of small one-celled animal. A paramecium is a protozoan about 250 μm long (Fig. 1.1a). The cells in multicellular animals tend to be somewhat smaller than protozoans. For instance, the mammalian *cardiac cell* (a muscle cell found in the heart, Chap. 7) shown in Fig. 1.1c is about

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[1] The metric system is officially called the SI system (systeme internationale). It used to be called the MKS (meter kilogram second) system.

[2] We find that restricting ourselves to prefixes that are a multiple of 1000 makes it easier to remember relative sizes.

R. K. Hobbie, B. J. Roth, *Intermediate Physics for Medicine and Biology*,
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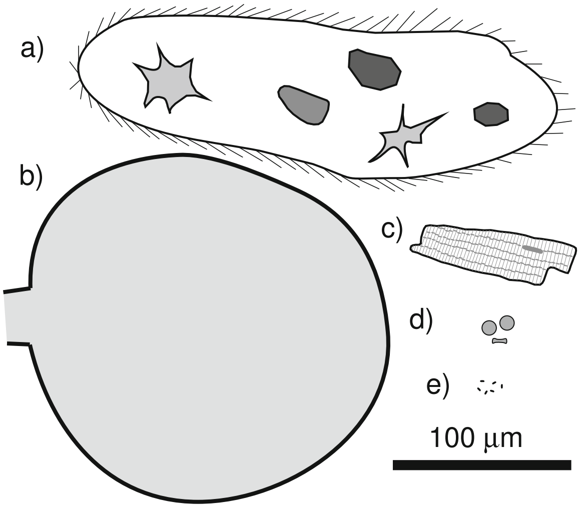

Fig. 1.1 Objects ranging in size from 1 mm down to 1 μm. **a** A paramecium. **b** An alveolus (air sac in the lung). **c** A cardiac cell. **d** Red blood cells. **e** *Escherichia coli* bacteria

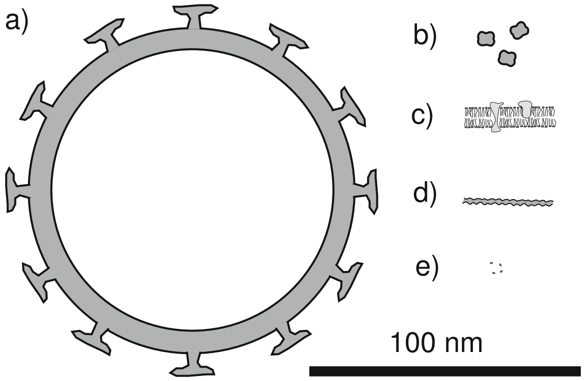

Fig. 1.2 Objects ranging in size from 1 μm down to 1 nm. **a** The human immunodeficiency virus (HIV). **b** Hemoglobin molecules. c A cell membrane. **d** A DNA molecule. **e** Glucose molecules

100-μm long and 20 μm in diameter. Nerve cells have a long fiber-like extension called an *axon*. Axons come in a variety of sizes, from 1-μm diameter up to tens of microns. The squid contains a giant axon nearly 1 mm in diameter. This axon plays an important role in our understanding of how nerves work (Chap. 6). Our red blood cells (*erythrocytes*) carry oxygen to all parts of our body. (Actually, red blood cells are not true cells at all, but rather "corpuscles"). Red blood cells are disk-shaped, with a diameter of about 8 μm and a thickness of 2 μm (Fig. 1.1d). Blood flows through a branching network of vessels (Sect. 1.19), the smallest of which are *capillaries*. Each capillary has a diameter of about 8 μm, meaning that the red blood cells can barely pass through it single-file.

One valuable skill in physics is the ability to make order-of-magnitude estimates, meaning to calculate something approximately right. For instance, suppose we want to calculate the number of cells in the body. This is a difficult calculation, because cells come in all sizes and shapes. But for some purposes we only need an approximate answer (say, within a factor of ten). For example: cells are roughly 10 μm in size, so their volume is about $(10\ \mu\text{m})^3$, or $(10 \times 10^{-6})^3 = 10^{-15}\ \text{m}^3$. An adult is roughly 2 m tall and about 0.3 m wide, so our volume is about $2\ \text{m} \times 0.3\ \text{m} \times 0.3\ \text{m}$, or $0.18\ \text{m}^3$. We are made up almost entirely of cells, so the number of cells in our body is about $\left(0.18\ \text{m}^3\right)/\left(10^{-15}\ \text{m}^3\right)$, or roughly 2×10^{14}. Some problems at the end of the chapter ask you to make similar order-of-magnitude calculations.

Most cells are larger than a few microns. But many cells (called *eukaryotes*) are complex structures that contain *organelles* about this size. *Mitochondria*, organelles where many of the chemical processes providing cells with energy take place, are typically about 2 μm long. *Protoplasts*, organelles found in plant cells where photosynthesis changes light energy to chemical energy, are also about 2 μm long.

The simplest cells are called *prokaryotes* and contain no subcellular structures. *Bacteria* are the most common prokaryotic cells. The bacterium *E. coli* is about 2 μm long (Fig. 1.1e), and has been studied extensively.

To examine structures smaller than bacteria, we must measure lengths that are smaller than a micron. One-thousandth of a micron is called a nanometer ($1\ \text{nm} = 10^{-9}\ \text{m}$). Figure 1.2 shows objects having lengths from 1 nm to 1 μm. *E. coli*, which seemed so tiny compared to cells in Fig. 1.1, are giants on the nanometer length scale, being 20 times longer than the 100-nm scale bar in Fig. 1.2. *Viruses* are tiny packets of genetic material encased in protein. On their own they are incapable of metabolism or reproduction, so some scientists do not even consider them as living organisms. Yet, they can infect a cell and take control of its metabolic and reproductive functions. The length scale of viruses is one-tenth of a micron, or 100 nm. For instance, *The human immunodeficiency virus (HIV)*, the virus that causes AIDS, is roughly spherical with a diameter of about 120 nm (Fig. 1.2a). Some viruses, called *bacteriophages*, infect and destroy bacteria. Most viruses are too small to see in a light microscope. The resolution of a microscope is limited by the wavelength of light, which is about 500 nm (Chap. 14). Thus, with a microscope we can study cells in detail, we can see bacteria without much resolution, and we can barely see viruses, if we can see them at all.

Below 100 nm, we enter the world of individual molecules. *Proteins* are large, complex macromolecules that are vitally important for life. For example, *hemoglobin* is the protein in red blood cells that binds to and carries oxygen. Hemoglobin is roughly spherical, about 6 nm in diameter (Fig. 1.2b). Many biological functions occur in the cell *membrane* (see Chap. 5). Membranes are made up of layers of *lipid* (fat), often with proteins and other molecules embedded in them (Fig. 1.2c). A typical cell membrane is about 10 nm

Table 1.2 Approximate sizes of biological objects

Object	Size
Protozoa	100 μm
Cells	10 μm
Bacteria	1 μm
Viruses	100 nm
Macromolecules	10 nm
Molecules	1 nm
Atoms	100 pm

thick. The molecule *adenosine triphosphate* (ATP), crucial for energy production and distribution in cells, is about 2 nm long (Chap. 3). Chemical energy is stored in molecules called *carbohydrates*. A common (and relatively small) carbohydrate is *glucose* ($C_6H_{12}O_6$), which is about 1 nm long (Fig. 1.2e). Genetic information is stored in long, helical strands of *deoxyribonucleic acid* (DNA). DNA is about 2.5 nm wide, and the helix completes a turn every 3.4 nm along its length (Fig. 1.2d).

At the 1-nm scale and below, we reach the world of small molecules and individual atoms. Water is the most common molecule in our body. It consists of two atoms of hydrogen and one of oxygen. The distance between adjacent atoms in water is about 0.1 nm. The distance 0.1 nm (100 pm) is used so much at atomic length scales that it has earned a nickname: the angstrom (A). Like the centimeter, this unit is going out of fashion as the use of nanometer becomes more common. Individual atoms have diameters of 100 or 200 pm.

Below the level of 100 pm, we leave the realm of biology and enter the world of subatomic physics. The nuclei of atoms (Chap. 17) are very small, and their sizes are measured in femtometers (1 fm $= 10^{-15}$ m).

One cannot possibly memorize the size of all biological objects: there are simply too many. The best one can do is remember a few mileposts along the way. Table 1.2 contains a rough guide to how large a few important biological objects are. Think of these as rules of thumb. Given the diversity of life, one can certainly find exceptions to these rules, but if you memorize Table 1.2 you will have a rough framework to organize your thinking about size. To examine the relative sizes of objects in more detail, see Morrison et al. (1994) or Goodsell (2009).

1.2 Models

Throughout this book we construct *mathematical models* of physical and biological systems. We start with general principles such as Newton's laws and apply them to a simplified model such as the leg in Fig. 1.3. The forces acting on the leg are much more complicated, but we model them with just three forces.

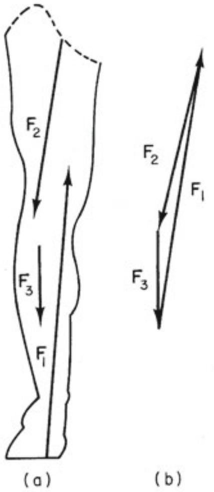

Fig. 1.3 Forces on the leg in equilibrium. Each force is exerted by some other object. **a** The points of application are widely separated. **b** The sum of the forces is zero

Biologists and physicists tend to make models differently (Blagoev et al. 2013). Biologists are used to dealing with complexity and diversity in biological systems. Physicists seek to explain as many phenomena with as few overarching principles as possible. Modeling a process is second nature to physicists. They willingly ignore some features of the biological system while seeking these principles. It takes experience and practice to decide what can be simplified and what can not.

A model incorporates some biological information, such as the ion currents in the Hodgkin–Huxley model (Chap. 6). The HH model can be extended by incorporating more ions and ion permeabilities measured in different nerves and muscles and other species. In other cases, the model has no specific details about the physiologic process, but captures an important feature in simplified form, that may have widespread applicability. We call this a *toy model*. The *radial isochron clock* (page 281) is a good example.

In many cases, simple models are developed in the homework problems at the end of each chapter. Working these problems will provide practice in the art of modeling.

1.3 Forces and Translational Equilibrium

There are several ways that we could introduce the idea of force, depending on the problem at hand and our philosophical bent. For our present purposes, it will suffice to say that a force is a *push* or a *pull*, that forces have both a magnitude and a direction, and that they give rise to accelerations through Newton's second law, $\mathbf{F} = m\mathbf{a}$. Experiments show that forces add like *displacements*, so they can be represented by *vectors*. (Some of the properties of vectors are reviewed in Appendix B; others are introduced as needed.) Vectors will be denoted by boldfaced characters. The force is measured in newtons (N). A newton is a $\mathrm{kg\,m\,s^{-2}}$.

One finds experimentally that an object is in *translational equilibrium* if the vector sum of all the forces acting on the body is zero. *Equilibrium* means that the object either remains at rest or continues to move with a constant velocity. That is, it is not accelerated. *Translational* means that only changes of position are being considered; changes of orientation of the object with respect to the axes are ignored.

We must consider *all* the forces that act *on* the object. If the object is a person standing on both feet, the forces are the upward force of the floor on each foot and the downward force of gravity on the person (more accurately, the vector sum of the gravitational force on every cell in the person). We do *not* consider the downward force that the person's feet exert *on the floor*. It is also possible to replace the sum of the gravitational force on each cell of the body with a single downward gravitational force acting at one point, the *center of gravity* of the body.

The forces that add to zero to give translational equilibrium need not all act at one point on the object. If the object is a person's leg and the leg is at rest, there are three forces exerted on the leg by other objects (Fig. 1.3). Force $\mathbf{F}_1$ is the push of the floor up on the bottom of the foot. The various pushes and pulls of the rest of the body on the leg through the hip joint and surrounding muscles have been added together to give $\mathbf{F}_2$. The gravitational pull of the earth downward on the leg is $\mathbf{F}_3$. Force $\mathbf{F}_1$ acts on the bottom of the leg, $\mathbf{F}_2$ acts on the top, and $\mathbf{F}_3$ acts somewhere in between. If the leg is in equilibrium the sum of these forces is zero, as shown in Fig. 1.3b. Although the points of application of the forces can be ignored in considering translational equilibrium, they are important in determining whether or not the object is in rotational equilibrium. This is discussed shortly.

The Greek letter Σ (capital sigma) is usually used to mean a sum of things. With this notation, the condition for translational equilibrium can be written as

$$\sum_i \mathbf{F}_i = 0. \tag{1.1}$$

The subscript i is used to label the different forces acting on the body. A notation this compact has a lot hidden in it. This is a vector equation, standing for three equations:

$$\sum_i F_{ix} = 0,$$
$$\sum_i F_{iy} = 0, \tag{1.2}$$
$$\sum_i F_{iz} = 0.$$

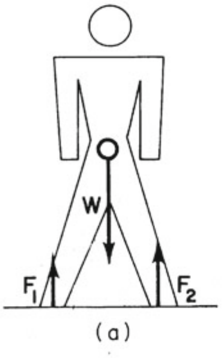

Fig. 1.4 A person standing. **a** The forces on the person. **b** A free-body or force diagram

Often the subscript i is omitted and the equations are written as $\sum F_x = 0$, $\sum F_y = 0$, and $\sum F_z = 0$. In this notation, a component is positive if it points along the positive axis and negative if it points the other way.

Sometimes, as in the next example, we draw forces in particular directions and assume that these directions are positive. If the subsequent algebra happens to give a solution that is negative, the force points opposite the direction assumed.

As an example, consider a person standing on both feet as in Fig. 1.4. The earth pulls down with force $\mathbf{W}$. The floor pushes up on the right foot with force $\mathbf{F}_1$ and on the left foot with force $\mathbf{F}_2$. To determine what the condition for translational equilibrium tells us about the forces, draw the force diagram or *free-body diagram* of Fig. 1.4b. This diagram is an abstraction that ignores the points at which the forces are applied to the body. We can get away with this abstraction because we are considering only translation. When we consider rotational equilibrium, we will have to redraw the diagram showing the points at which the various forces act on the person. If all the forces are vertical, then there is only one component of each force to worry about, and the equilibrium condition gives $F_1 + F_2 - W = 0$ or $F_1 + F_2 = W$. The total force of the floor pushing up on both feet is equal to the pull of the earth down.

If there is a sideways force on each foot, translational equilibrium provides two conditions: $F_{1x} + F_{2x} = 0$ and $F_{1y} + F_{2y} - W = 0$.

This is all that can be learned from the condition for translational equilibrium. If the person stands on one foot, then $F_1 = 0$ and $F_2 = W$. If the person stands with equal force on each foot, then $F_1 = F_2 = W/2$.

1.4 Rotational Equilibrium

If the object is in *rotational equilibrium*, then another condition must be placed upon the forces. Rotational equilibrium

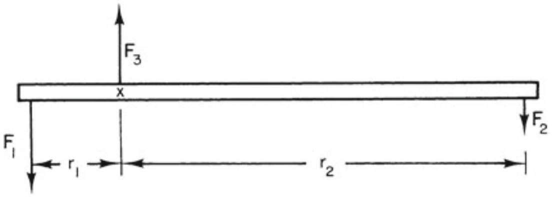

Fig. 1.5 A rigid rod free to rotate about a pivot at point X

means that the object either does not rotate or continues to rotate at a constant rate (with a constant number of rotations per second). Consider the object of Fig. 1.5, which is a rigid rod pivoted at point X so that it can rotate in the plane of the paper. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the rod in the plane of the paper at distances r_1 and r_2 from the pivot and perpendicular to the rod. The pivot exerts the force $\mathbf{F}_3$ on the rod needed to maintain translational equilibrium. If both $\mathbf{F}_1$ and $\mathbf{F}_2$ are perpendicular to the rod, they are parallel to each other. They must also be parallel to $\mathbf{F}_3$, and translational equilibrium requires that $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$.

Experiment shows that there is no rotation of the rod if $F_1 r_1 = F_2 r_2$. The condition for rotational equilibrium can be stated in a form analogous to that for translational equilibrium if we define the *torque*, τ, to be

$$\tau_i = r_i F_i. \tag{1.3}$$

With this definition goes an algebraic sign convention: the torque is positive if it tends to produce a counterclockwise rotation. The rod is in rotational equilibrium if the algebraic sum of all the torques is zero:

$$\sum_i \tau_i = \sum_i r_i F_i = 0. \tag{1.4}$$

Note that $\mathbf{F}_3$ contributes nothing to the torque because r_3 is zero.

The torque is defined about a certain point, X. It depends on the distance from the point of application of each force to X.[3] As long as the object is in translational equilibrium, the torque can be evaluated around any point. This theorem, which we will not prove, often allows calculations to be simplified, because taking torques about certain points can cause some forces not to contribute to the torque equation.

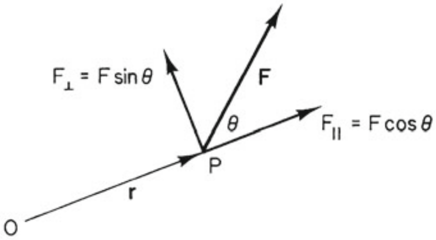

Fig. 1.6 A force **F** is applied to an object at point P. The object can rotate about point O. Vectors **r** and **F** determine the plane of the paper

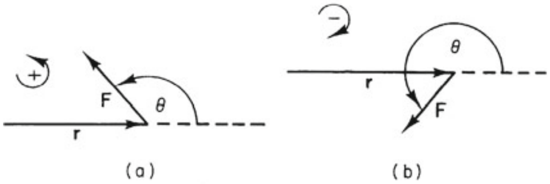

Fig. 1.7 **a** When θ is between 0 and 180 °, both $\sin\theta$ and the torque are positive. **b** When θ is between 180 and 360 °, both $\sin\theta$ and the torque are negative

The torque can also be calculated if the force is not at right angles to the rod. Imagine an object free to rotate about point O in Fig. 1.6. Force **F** lies in the plane of the paper but is applied in some arbitrary direction at point P. The vectors **r** and **F** determine the plane of the paper if they are not parallel. Force **F** can be resolved into two components: one parallel to **r**, $F_\parallel = F\cos\theta$, and the other perpendicular to **r**, $F_\perp = F\sin\theta$. The component parallel to **r** will not cause any rotation about point O. (Pull on an open door parallel to the plane of the door; there is no rotation.) The torque is therefore

$$\tau = r F_\perp = r F \sin\theta. \tag{1.5}$$

The perpendicular distance from the line along which the force acts to point O is $r \sin\theta$. It is often called the *moment arm*, and the torque is the magnitude of the force multiplied by the moment arm.

The angle θ is the angle of rotation from the direction of **r** to the direction of **F**. It is called positive if the rotation is counterclockwise. For the angle shown in Fig. 1.6, $\sin\theta$ has a positive value, and the torque is positive. Figure 1.7a shows an angle between 90 and 180 ° for which the torque and $\sin\theta$ are still positive. Figure 1.7b shows an angle between 180 and 360 °, for which both the torque and $\sin\theta$ are negative. In all cases, Eq. 1.5 gives the correct sign for the torque.

To summarize: the torque due to force **F** applied to a body at point P must be calculated about some point O. If **r** is the vector from O to P, the magnitude of the torque is equal to the magnitude of **r** times the magnitude of **F** times the sine of the angle between **r** and **F**. The angle is measured counterclockwise from **r** to **F**.

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[3] The discussion associated with Fig. 1.5 suggests that torque is taken about an axis, rather than a point. In a three-dimensional problem the torque is taken about a point.

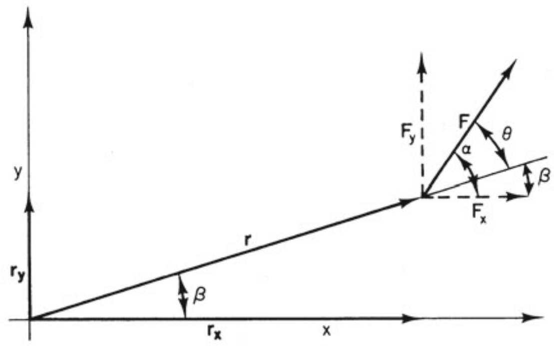

Fig. 1.8 The cross product $\mathbf{r} \times \mathbf{F}$ is calculated by resolving $\mathbf{r}$ and $\mathbf{F}$ into components

1.5 Vector Product

Torque can be thought of as a vector, $\boldsymbol{\tau}$. Its magnitude is $Fr \sin\theta$. The only direction uniquely defined by vectors $\mathbf{r}$ and $\mathbf{F}$ is perpendicular to the plane in which they lie. This is also the direction of an axis about which the torque would cause a rotation. However, there is ambiguity about which direction along this line to assign to the torque. The convention is to say that a positive torque points in the direction of the thumb of the right hand when the fingers curl in the direction of positive rotation from $\mathbf{r}$ to $\mathbf{F}$.[4] When $\mathbf{r}$ and $\mathbf{F}$ point in the same direction, so that no plane is defined, the magnitude of the torque is zero.

The product of two vectors according to the foregoing rules is called the *cross product* or *vector product* of the two vectors. One can use a shorthand notation,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{1.6}$$

There is another way to write the cross product. If both $\mathbf{r}$ and $\mathbf{F}$ are resolved into components, as shown in Fig. 1.8, then the cross product can be calculated by applying the rules above to the components. Since $\mathbf{F}_y$ is perpendicular to $\mathbf{r}_x$ and parallel to $\mathbf{r}_y$, its only contribution is a counterclockwise torque $r_x F_y$. The only contribution from $\mathbf{F}_x$ is a clockwise torque, $-r_y F_x$. The magnitude of the cross product is therefore

$$\tau = r_x F_y - r_y F_x. \tag{1.7}$$

Note that this is the (signed) sum of each component of the force multiplied by its moment arm.

The equivalence of this result to Eq. 1.5 can be verified by writing Eq. 1.7 as

$$\tau = (r \cos\beta)(F \sin\alpha) - (r \sin\beta)(F \cos\alpha),$$

$$\tau = rF\,(\sin\alpha \cos\beta - \cos\alpha \sin\beta)\,.$$

There is a trigonometric identity that

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta.$$

Since $\theta = \alpha - \beta$ (from Fig. 1.8), this is equivalent to $\tau = rF \sin\theta$.

When vectors $\mathbf{r}$ and $\mathbf{F}$ lie in the xy plane, $\boldsymbol{\tau}$ points along the z axis. If $\mathbf{r}$ and $\mathbf{F}$ point in arbitrary directions, Eq. 1.7 gives the z component of $\boldsymbol{\tau}$. One can apply the same reasoning for other components and show that

$$\begin{aligned} \tau_x &= r_y F_z - r_z F_y, \\ \tau_y &= r_z F_x - r_x F_z, \\ \tau_z &= r_x F_y - r_y F_x. \end{aligned} \tag{1.8}$$

If you are familiar with the rules for evaluating determinants, you will see that this is equivalent to the notation,

$$\boldsymbol{\tau} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}. \tag{1.9}$$

1.6 Force in the Achilles Tendon

The equilibrium conditions can be used to understand many problems in clinical orthopedics. Two are discussed in this book: forces that sometimes cause the *Achilles tendon* at the back of the heel to break, and forces in the hip joint.

The Achilles tendon connects the calf muscles (the gastrocnemius and the soleus) to the calcaneus at the back of the heel (Fig. 1.9). To calculate the force exerted by this tendon on the calcaneus when a person is standing on the ball of one foot, assume that the entire foot can be regarded as a rigid body. This is our first example of creating a *model* of the actual situation. We try to simplify the real situation to make the calculation possible while keeping the features that are important to what is happening. In this model, the internal forces within the foot are being ignored.

Figure 1.10 shows the force exerted by the tendon on the foot ($\mathbf{F}_T$), the force of the leg bones (tibia and fibula) on the foot ($\mathbf{F}_B$), and the force of the floor upward, which is equal to the weight of the body ($\mathbf{W}$). The weight of the foot is small compared to these forces and will be neglected. Measurements on a few people suggest that the angle the Achilles tendon makes with the vertical is about 7°.

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[4] This arbitrariness in assigning the sense of $\boldsymbol{\tau}$ means that it does not have quite all the properties that vectors usually have. It is called an axial vector or a pseudovector. It will not be necessary in this book to worry about the difference between a real vector and an axial vector.

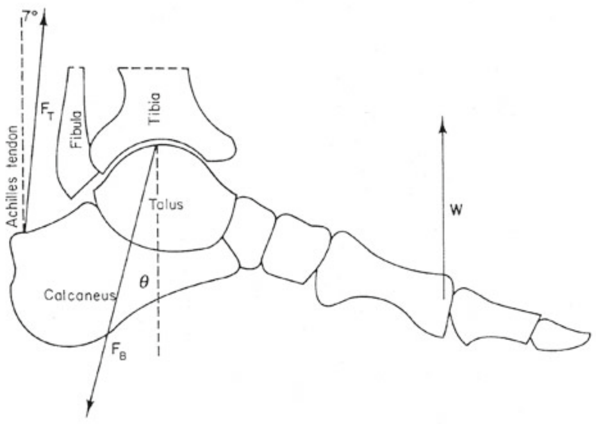

Fig. 1.9 Simplified anatomy of the foot

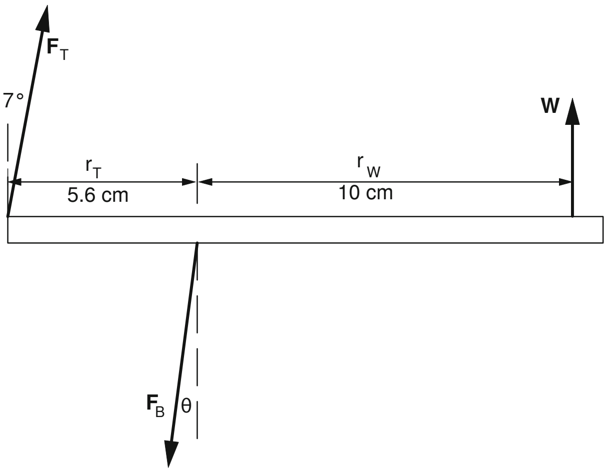

Fig. 1.10 Forces on the foot, neglecting its own weight

Translational equilibrium requires that

$$F_T \cos(7^\circ) + W - F_B \cos\theta = 0, \qquad (1.10)$$
$$F_T \sin(7^\circ) - F_B \sin\theta = 0.$$

To write the condition for rotational equilibrium, we need to know the lengths of the appropriate vectors $\mathbf{r}_T$ and $\mathbf{r}_W$, assuming that the torques are taken about the point where $\mathbf{F}_B$ is applied to the foot. In our simple model, we ignore the contributions of the horizontal components of any forces to the torque equation. This is not essential (if we are willing to make more detailed measurements), but it simplifies the equations and thereby makes the process clearer. The horizontal distances measured by one of the authors are $r_T = 5.6$ cm and $r_W = 10$ cm, as shown in Fig. 1.10. The torque equation is

$$10W - 5.6 F_T \cos 7^\circ = 0. \qquad (1.11)$$

This equation can be solved for the tension in the tendon:

$$F_T = \frac{10W}{5.6 \cos 7^\circ} = 1.8W. \qquad (1.12)$$

This result can now be used in Eq. 1.10 to find $F_{By} = F_B \cos\theta$:

$$\begin{aligned}(1.8)(W)(0.993) + W &= F_B \cos\theta,\\ 2.8W &= F_B \cos\theta. \qquad (1.13)\end{aligned}$$

From Eqs. 1.10 and 1.12, we get

$$\begin{aligned}(1.8)(W)(0.122) &= F_B \sin\theta,\\ 0.22W &= F_B \sin\theta. \qquad (1.14)\end{aligned}$$

Equations 1.13 and 1.14 are squared and summed and the square root taken to give $F_B = 2.8W$, while they can be divided to give

$$\tan\theta = \frac{0.22}{2.8} = 0.079,$$
$$\theta = 4.5^\circ.$$

The tension in the Achilles tendon is nearly twice the person's weight, while the force exerted on the leg by the talus is nearly three times the body weight. One can understand why the tendon might rupture.

1.7 Forces on the Hip

The forces in the hip joint can be several times a person's weight, and the use of a cane can be very effective in reducing them.

As a person walks, there are moments when only one foot is on the ground. There are then two forces acting on the body as a whole: the downward pull of the earth W and the upward push of the ground on the foot N. The pull of the earth may be regarded as acting at the center of gravity of the body (Serway and Jewett 2013, p. 219). The center of gravity is located on the midline (if the limbs are placed symmetrically), usually in the lower abdomen (Williams and Lissner 1962, Chap. 5.) If torques are taken about the foot, then the center of gravity must be directly over the foot so that there will be no torque from either force. This situation is shown in Fig. 1.11. The condition for translational equilibrium requires that $N = W$.

The anatomy of the pelvis, hip, and leg is shown schematically in Fig. 1.12. Fourteen muscles and several ligaments connect the pelvis to the femur. Extensive measurements of the forces exerted by the abductor[5] muscles in the hip have

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[5] *To abduct* means to move away from the midline of the body.

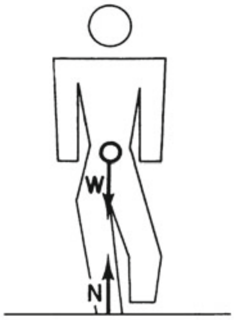

Fig. 1.11 A person standing on one foot must place the foot under the center of gravity, which is on or near the midline

been made by Inman (1947). If the leg is considered an isolated system as in Fig. 1.12, the following forces act:

F: The net force of the abductor muscles, acting on the greater trochanter. These muscles are primarily the gluteus medius and gluteus minimus, shown as a single band of muscle in Fig. 1.12.

R: The force of the acetabulum (the socket of the pelvis) on the head of the femur.

N: The upward force of the floor on the bottom of the foot (in this case, equal to W).

$\mathbf{W}_L$: The weight of the leg, acting vertically downward at the center of gravity of the leg. $W_L \approx W/7$ (Williams and Lissner 1962, Chap. 5).

Inman found that **F** acts at about a 70° angle to the horizontal. In a typical adult, the distance from the greater trochanter to the midline is about 18 cm, the horizontal distance from the greater trochanter to the center of gravity of the leg is about 10 cm, and the distance from the greater trochanter to the middle of the head of the femur is about 7 cm.

A free body diagram is shown in Fig. 1.13. The middle of the head of the femur will turn out to be very close to the intersection of the line along which **R** acts and a horizontal line drawn from the point where **F** acts. This means that if torques are taken about this intersection point (point O), there will be no contributions from **R** or from the horizontal component of **F**. The intersection is about 7 cm toward the midline from the point of application of **F**. Since $N = W$ and $W_L \approx W/7$, the equilibrium equations are

$$\sum F_y = F\sin(70^\circ) - R_y - W/7 + W = 0, \qquad (1.15)$$

$$\sum F_x = F\cos(70^\circ) - R_x = 0, \qquad (1.16)$$

$$\sum \tau = -F\sin(70^\circ)(7) - (W/7)(10-7) + W(18-7) = 0.$$

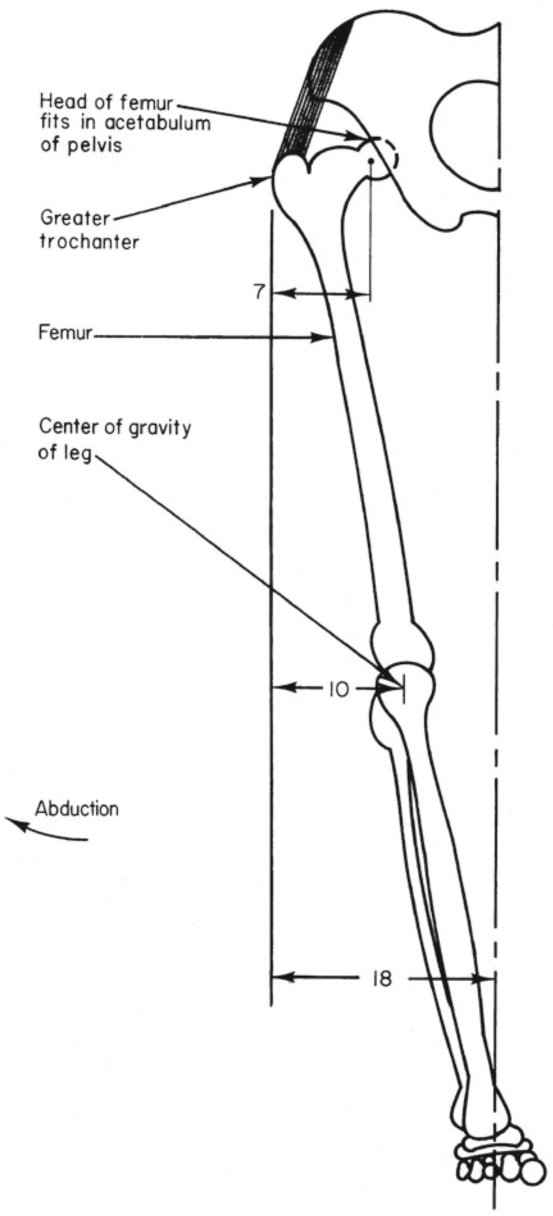

Fig. 1.12 Pertinent features of the anatomy of the leg

The last of these equations can be written as $11W - \frac{3}{7}W - 6.6F = 0$, from which $F = 1.6W$. The magnitude of the force in the abductor muscles is about 1.6 times the body weight.

Equations 1.15 and 1.16 can now be used to find R_x and R_y:

$$R_x = F\cos(70^\circ) = (1.6W)(0.342) = 0.55W,$$

$$R_y = F\sin(70^\circ) + \frac{6}{7}W = (1.6W)(0.94) + 0.86W = 2.36W.$$

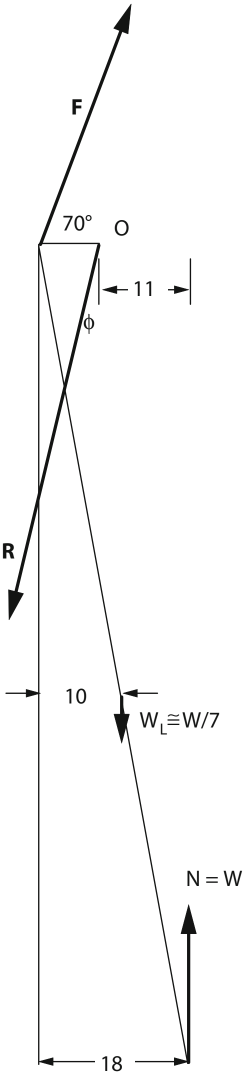

Fig. 1.13 A free-body diagram of the forces acting on the leg. Torques are taken about point O, which is the intersection of a line along which **R** acts and a horizontal line through the point at which **F** is applied. This point is 7 cm toward the midline (medially) from the greater trochanter

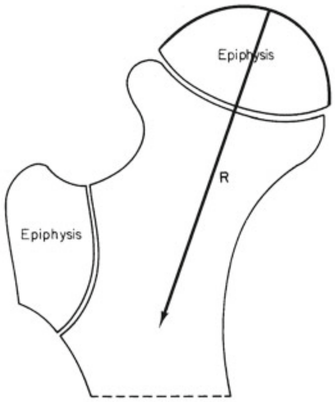

Fig. 1.14 The femoral epiphysis and the direction of **R**

Fig. 1.15 X-ray of a slipped femoral epiphysis in an adolescent male. (Courtesy of the Department of Diagnostic Radiology, University of Minnesota)

The angle that **R** makes with the vertical is given by

$$\tan\phi = \frac{R_x}{R_y} = 0.23,$$

$$\phi = 13^\circ.$$

The magnitude of **R** is $R = (R_x^2 + R_y^2)^{1/2} = 2.4W$.

If the patient did not have to put the foot under the center of gravity of the body, the moment arm of the only positive torque, $11W$, could have been much less, and this would have been balanced by a smaller value of F. This can be done by having the patient use a cane on the *opposite* side, so that the foot need not be right under the center of gravity. This will be explored in the next section. Conversely, if the patient were carrying a suitcase in the opposite hand, the center of mass would be moved away from the midline, the foot would still have to be placed under the center of mass, and the moment arm, and hence F, would be even larger (Problem 11).

One very interesting conclusion of Inman's study was that the force **R** always acts along the neck of the femur in such a direction that the femoral epiphysis has very little sideways force on it. The epiphysis is the growing portion of the bone (Fig. 1.14) and is not very well attached to the rest of the bone. If there were an appreciable sideways force, the epiphysis would slip sideways, and indeed it sometimes does (Fig. 1.15). This is a serious problem, since if the blood supply to the epiphysis is compromised, there will be no more bone growth.

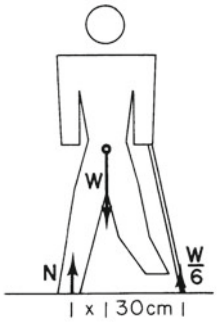

Fig. 1.16 A person using a cane on the left side (*front view*) to favor the right hip

Suppose that, for some reason, the gluteal muscles are severed. The patient can no longer apply force **F** to the greater trochanter; Eq. 1.16 shows that then R_x must be zero. This change in the direction of **R** causes a rotation of the epiphyseal plate and a gradual reshaping of the femur.

1.8 The Use of a Cane

A cane is beneficial if used on the side opposite to the affected hip (Fig. 1.16). We ignore the fact that the arm holding the cane has moved, thereby shifting slightly the center of mass, and we assume that the force of the ground on the cane is vertical. If we assume that the tip of the cane is about 30 cm (12 in.) from the midline and supports one-sixth of the body weight, then we can apply the equilibrium conditions to learn that $N + \frac{1}{6}W - W = 0$, so $N = \frac{5}{6}W$. Torques taken about the center of mass give $(30)(\frac{W}{6}) - x(\frac{5}{6})W = 0$, $x = 6$ cm. (Figure 1.16 is not to scale.)

Having the foot 6 cm from the midline reduces the force in the muscle and the joint. To find out how much, consider the force diagram in Fig. 1.17. The most difficult part of the problem is working out the various moment arms. Assume that the slight movement of the leg has not changed the point about which we take torques (point O). Again, **R** contributes no torque about this point. The horizontal distance of **F** from this point is still 7 cm. The force of the ground on the leg is now $5W/6$, and its moment arm is $18-6-7=5$ cm. The weight of the leg, $W/7$, acts at the center of mass of the leg, which is still $\frac{10}{18}$ of the distance from the greater trochanter to the foot. Its horizontal position is therefore $\frac{10}{18}$ of the horizontal distance from the greater trochanter to the foot: (10)(12)/18 = 6.67 cm. The moment arm is 7−6.67 cm = 0.33 cm. The torque equation is

$$-F\sin(70^\circ)(7) + \left(\frac{W}{7}\right)(0.33) + \left(\frac{5W}{6}\right)(5) = 0.$$

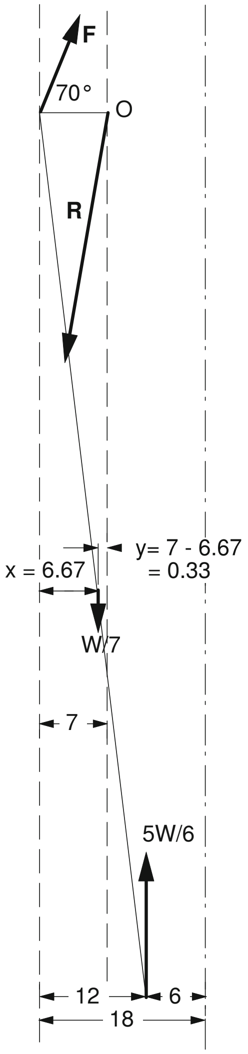

Fig. 1.17 A force diagram for the leg when a cane is being used and the leg is 6 cm from the midline

It is solved by writing it as

$$-6.58F + 0.047W + 4.17W = 0,$$
$$F = 0.64W.$$

Even though the cane supports only one-sixth of the body weight, F has been reduced from $1.6W$ to $0.64W$ by the change in the moment arm.

The force of the acetabulum on the head of the femur can be determined from the conditions for translational equilibrium:

$$F\cos(70^\circ) - R_x = 0,$$
$$R_x = 0.22W,$$
$$F\sin(70^\circ) - R_y - \frac{W}{7} + \frac{5}{6}W = 0,$$
$$R_y = 1.29W.$$

The resultant force **R** has magnitude $(R_x^2 + R_y^2)^{1/2} = 1.3W$. This compares to the value $2.4W$ without the cane. The force in the joint has been reduced by slightly more than the body weight. It is interesting to read what an orthopedic surgeon had to say about the use of a cane. The following is from the presidential address of W. P. Blount, M.D., to the Annual Meeting of the American Academy of Orthopedic Surgeons, January 30, 1956:

> The patient with a wise orthopedic surgeon walks with crutches for six months after a fracture of the neck of the femur. He uses a stick for a longer time—the wiser the doctor, the longer the time. If his medical adviser, his physical therapist, his friends, and his pride finally drive him to abandon the cane while he still needs one, he limps. He limps in a subconscious effort to reduce the strain on the weakened hip. If there is restricted motion, he cannot shift his body weight, but he hurries to remove the weight from the painful hip joint when his pride makes him reduce the limp to a minimum. The excessive force pressing on the aging hip takes its toll in producing degenerative changes. He should not have thrown away the stick.[6]

1.9 Work

So far this chapter has considered only those situations in which a mass m is in equilibrium. If the total force on the object is not zero, the object experiences an acceleration **a** given by Newton's second law:

$$\mathbf{F} = m\mathbf{a}.$$

The study of how forces produce accelerations is called *dynamics*. It is an extensive field that will be discussed only briefly here.

Suppose the object moves along the x axis with velocity v_x. If it is subject to a force in the x direction F_x, it will be accelerated, and the velocity will change according to $F_x = ma_x = m\,(dv_x/dt)$. If F_x is known as a function of time, then this equation can be written as $dv_x = (1/m)\,F_x(t)dt$, and it can be integrated, at least numerically.

In this context it is useful to define the *kinetic energy*

$$E_k = \frac{1}{2}mv_x^2. \qquad (1.17)$$

As long as F_x acts, the object is accelerated and the kinetic energy changes. We can gain some understanding of how it changes by noting that

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2\right) = mv_x\frac{dv_x}{dt} = F_xv_x. \qquad (1.18)$$

Therefore F_xv_x is the rate at which the kinetic energy is changing with time. It is called the *power* due to force F_x. The units of kinetic energy are kg m^2 s^{-2} or joules (J); the units of power are J s^{-1} or watts (W).

If v_x and F_x are both positive, the acceleration increases the object's velocity, the kinetic energy increases, and the power is positive. If v_x and F_x are both negative, v_x decreases—becomes more negative—but the magnitude of the velocity increases. The kinetic energy increases with time, and the power is positive. If v_x and F_x point in opposite directions, then the effect of the acceleration is to reduce the magnitude of v_x, the kinetic energy decreases, and the power is negative.

Equation 1.18 can be written as

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2\right) = F_x\frac{dx}{dt}.$$

Both sides of this equation can be integrated with respect to t:

$$\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{1}{2}mv_x^2\right)dt = \int_{t_1}^{t_2}F_x\,(t)\,\frac{dx}{dt}\,dt.$$

The indefinite integral corresponding to the left-hand side is the integral with respect to time of the derivative of $\frac{1}{2}mv_x^2$ and is therefore $\frac{1}{2}mv_x^2$. If F_x is known not as a function of t but as a function of x, it is convenient to write the right-hand side as

$$\int_{x_1}^{x_2}F_x(x)\,dx = W.$$

This quantity is called the *work* done by force F_x on the object as it moves from x_1 to x_2. The complete equation is therefore

$$\left[\frac{1}{2}mv_x^2\right]_2 - \left[\frac{1}{2}mv_x^2\right]_1 = \int_{x_1}^{x_2}F_x(x)\,dx = W. \qquad (1.19)$$

The increase in kinetic energy of the body as it moves from position 1 (at time 1) to position 2 (at time 2) is equal to the work done *on* the body *by* the force F_x. The work done on the body by force F_x is the area under the curve of F_x versus x, between points x_1 and x_2. This is shown in Fig. 1.18.

If several forces act on the body, then the acceleration is given by Newton's second law, where **F** is the *total* force on the body. The change in kinetic energy is therefore the work done by the total force or the sum of the work done by each individual force.

When the force and displacement vectors point in any direction, the kinetic energy is defined to be

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2). \qquad (1.20)$$

Differentiating this expression with respect to time shows that the power is given by an extension of Eq. 1.18:

$$\frac{dE_k}{dt} = F_xv_x + F_yv_y + F_zv_z.$$

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[6] Quoted with permission from Blount (1956). Copyright © 1956 *J Bone Joint Surg.* This article was first quoted to the physics community by Benedek and Villars (1973)

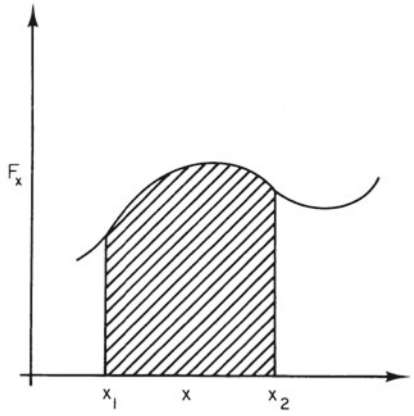

Fig. 1.18 The work done by F_x is the shaded area under the curve between x_1 and x_2

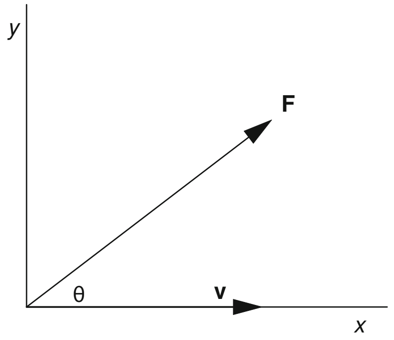

Fig. 1.19 Aligning the axes so that **v** is along the x axis and **F** in the xy plane shows that an alternative expression for $\mathbf{F} \cdot \mathbf{v}$ is $Fv \cos\theta$

This particular combination of vectors **F** and **v** is called the *scalar product* or *dot product*. It is written as $\mathbf{F} \cdot \mathbf{v}$.

There is another way to write the scalar product. If **F** and **v** are not parallel, they define a plane. Align the x axis with **v** so that v_y and v_z are zero, and choose the direction of y so that **F** is in the xy plane (Fig. 1.19). Then it is easy to see that $\mathbf{F} \cdot \mathbf{v} = F_x v_x = Fv \cos\theta$, where θ is the angle between **F** and **v**.

To summarize, the power is

$$P = \frac{dE_k}{dt} = \mathbf{F} \cdot \mathbf{v} = Fv \cos\theta = F_x v_x + F_y v_y + F_z v_z. \tag{1.21}$$

Equation 1.21 can be integrated in the same manner as above to obtain

$$\Delta E_k = \int F_x \, dx + \int F_y \, dy + \int F_z \, dz = \int \mathbf{F} \cdot d\mathbf{s}. \tag{1.22}$$

This is the general expression for the work done by force **F** on a point mass that undergoes displacement **s**.

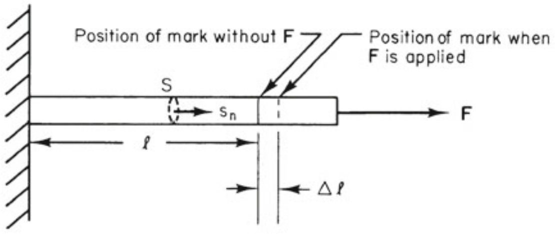

Fig. 1.20 A rod subject to a force **F** along it

1.10 Stress and Strain

Whenever a force acts on an object, it undergoes a change of shape or *deformation*. Often these deformations can be ignored, as they were in the previous sections. In other cases, such as the contraction of a muscle, the expansion of the lungs, or the propagation of a sound wave, the deformation is central to the problem and must be considered. This book will not develop the properties of deformable bodies extensively; nevertheless, deformable body mechanics is important in many areas of biology (Fung 1993). We will develop the subject only enough to be able to consider viscous forces in fluids.

Consider a rod of cross-sectional area S. One end is anchored, and a force F is exerted on the other end parallel to the rod (Fig. 1.20). Effects of weight will be ignored. A *surface force* is transmitted across any surface defined by an imaginary cut perpendicular to the axis of the rod. A surface force is exerted by the substance to the right of the cut on the substance to the left (and vice versa, in accordance with Newton's third law: when object A exerts a force on object B, object B exerts an equal and opposite force on object A). The surface force per unit area is called the *stress*. In this case, when the surface is perpendicular to the axis of the rod and the force is along the axis of the rod, it is called a *normal stress*:

$$s_n = \frac{F}{S}. \tag{1.23}$$

In the general case there can also be a component of stress parallel to the surface.

The *strain* ϵ_n is the fractional change in the length of the rod:

$$\epsilon_n = \frac{\Delta l}{l}. \tag{1.24}$$

If increasing stress is applied to a typical substance, the strain increases linearly with the stress for small stresses. Then it increases even more rapidly. At higher strains it may be necessary to reduce the stress to maintain the same strain. If the stress is not reduced, the rod elongates further and breaks. Finally, at a high enough strain, the sample

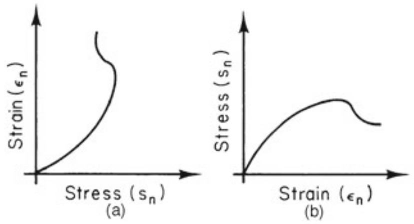

Fig. 1.21 A typical stress–strain relationship. On the left, stress is the independent variable. On the right, strain is the independent variable. Strain is usually used as the independent variable because it is often a double-valued function of the stress

Table 1.3 Young's modulus, tensile strength, and compressive strength of various materials in pascal

Material	E	Tensile strength	Compressive strength
Steel (approx.)[a]	20×10^{10}	50×10^{7}	–
Femur (wet)[b]	1.4×10^{10}	8.3×10^{7}	1.8×10^{7}
Walnut[c]	0.8×10^{10}	4.1×10^{7}	5.2×10^{7}

[a] *American Institute of Physics Handbook* (1957). New York, McGraw-Hill, p. 2–70

[b] cf. Kummer (1972)

[c] cf. U.S. Department of Agriculture (1955)

breaks. This is plotted in Fig. 1.21. Because of the double-valuedness of the strain as a function of stress, the strain is usually plotted as the independent variable, as on the right in Fig. 1.21.

In the linear region, the relationship between stress and strain is written as

$$s_n = E\epsilon_n. \tag{1.25}$$

The proportionality constant E is called *Young's modulus*. Since the strain is dimensionless, E has the dimensions of stress. Various units are N m^{-2} or pascal (Pa), dyn cm^{-2}, psi (pound per square inch), and bar (1 bar = 14.5 psi = 10^5 Pa = 10^6 dyn cm^{-2}).

If the stress is increased enough, the bar breaks. The value of the stress when the bar breaks under tension is called the *tensile strength*. The material will also rupture under compressive stress; the rupture value is called the *compressive strength*. Table 1.3 gives values of Young's modulus, the tensile strength, and the compressive strength for steel, long bone (femur), and wood (walnut).

In some materials, the stress depends not only on the strain, but on the rate at which the strain is produced. It may take more stress to stretch the material rapidly than to stretch it slowly, and more stress to stretch it than to maintain a fixed strain. Such materials are called *viscoelastic*. They are often important biologically but will not be discussed here (Fung 1993).

Still other materials exhibit *hysteresis*. The stress–strain relationship is different when the material is being stretched than when it is allowed to return to its unstretched state. This difference is observed even if the strain is changed so slowly that viscoelastic effects are unimportant.

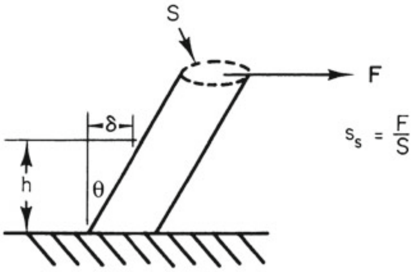

Fig. 1.22 Shear stress and strain

1.11 Shear

In a *shear stress*, the force is parallel to the surface across which it is transmitted.[7] In a *shear strain*, the deformation increases as one moves in a direction perpendicular to the deformation. An example of shear stress and strain is shown in Fig. 1.22. The shear stress is

$$s_s = \frac{F}{S}, \tag{1.26}$$

and the shear strain is

$$\epsilon_s = \frac{\delta}{h}. \tag{1.27}$$

It is possible to define a *shear modulus G* analogous to Young's modulus when the shear strain is small:

$$s_s = G\epsilon_s. \tag{1.28}$$

1.12 Hydrostatics

We now turn to some topics in the mechanics of fluids that will be useful for understanding several phenomena, including the circulation and fluid movement through membranes in Chap. 5. *Hydrostatics* is the description of fluids at rest. A fluid is a substance that will not support a shear when it is at rest. When the fluid is in motion, there can be a shear force arising from *viscosity*.

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[7] This discussion of stress and strain has been made simpler than is often the case. In general, the force **F** across any surface is a vector. It can be resolved into a component perpendicular to the surface and two components parallel to the surface. One can speak of nine components of stress: $s_{xx}, s_{xy}, s_{xz}, s_{yx}, s_{yy}, s_{yz}, s_{zx}, s_{zy}, s_{zz}$. The first subscript denotes the direction of the force and the second denotes the normal to the surface across which the force acts. Components s_{xx}, s_{yy}, and s_{zz} are normal stresses; the others are shear stresses. It can be shown that $s_{xy} = s_{yx}$, and so forth.

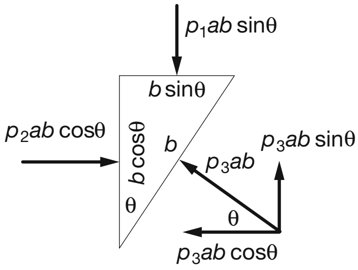

Fig. 1.23 A volume element of fluid used to show that the pressure in a fluid at rest is the same in all directions

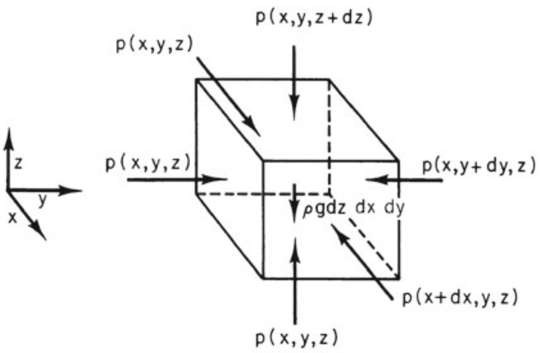

Fig. 1.24 The fluid in volume $dxdydz$ is in equilibrium

An immediate consequence of the definition of a fluid is that when the fluid is at rest, all the stress is normal. The normal stress is called the *pressure*. The pressure at any point in the fluid is the same in all directions. This can be demonstrated experimentally, and it can be derived from the conditions for equilibrium. Consider the small volume of fluid shown in Fig. 1.23. It has a length a perpendicular to the page. This volume is in equilibrium. Since the fluid at rest cannot support a shear, the pressure is perpendicular to each face, and there is no other force across each face. To prove this, assume that the pressures perpendicular to the three faces can be different, and call them p_1, p_2, and p_3. The force exerted across face 1 is $p_1ab\sin\theta$, acting downward. The force across face 2 is $p_2ab\cos\theta$, acting to the right. Across face 3 it is p_3ab, with vertical component $p_3ab\sin\theta$ and horizontal component $p_3ab\cos\theta$. The vertical components sum to zero only if $p_1 = p_3$, while the horizontal components sum to zero only if $p_2 = p_3$. Since this result is independent of the value of θ, the pressure must be the same in every direction.

Next, consider how the pressure changes with position. Suppose that p depends on the coordinates $p = p(x, y, z)$ and that the *density* of the fluid is ρ kg m^{-3}. The only external force acting is gravity in the direction of the $-z$ axis. The fluid in the volume $dxdydz$ of Fig. 1.24 is in equilibrium. In the y direction, there is a force to the right across the left-hand face equal to $p(x, y, z)dxdz$ and to the left across the right-hand face equal to $-p(x, y + dy, z)dxdz$. These are the only forces in the y direction, and their magnitudes must be the same. Therefore, p does not change in the y direction. A similar argument shows that p does not change in the x direction. In the z direction there are three terms: the upward force across the bottom face, the downward force across the top face, and the pull of gravity. The weight of the fluid is its mass ($\rho\, dxdydz$) times the gravitational acceleration g ($g = 9.8\,\mathrm{m\,s^{-2}}$). The three forces must add to zero:

$$p(x, y, z)\, dxdy - p(x, y, z + dz)\, dxdy - \rho g\, dxdydz = 0.$$

For small changes in height, dz, it is possible to approximate[8] $p(x, y, z + dz)$ by $p(x, y, z) + (dp/dz)\, dz$. With this approximation, the equilibrium equation is

$$dxdydz\left(-\frac{dp}{dz} - \rho g\right) = 0.$$

This equation can be satisfied only if

$$\frac{dp}{dz} = -\rho g. \tag{1.29}$$

This is a *differential equation* for $p(z)$. It is a particularly simple one, since the right-hand side is constant if ρ and g are constant: $dp = -\rho g dz$. Integrating this gives

$$\int dp = -\rho g \int dz,$$

$$p = -\rho g z + c.$$

The constant of integration is determined by knowing the value of p for some value of z. If $p = p_0$ when $z = 0$, then $p_0 = c$ and

$$p = p_0 - \rho g z. \tag{1.30}$$

With a constant gravitational force per unit volume acting on the fluid, the pressure decreases linearly with increasing height. The SI unit of pressure is N m^{-2} or pascal (Pa). The density is expressed in kg m^{-3}, so that ρg has units of N m^{-3} and $\rho g z$ is in N m^{-2}. Pressures are often given as equivalent values of z in some substance, for example, in millimeters of mercury (torr) or centimeters of water. In such cases, the value of z must be converted to an equivalent value of $\rho g z$ before calculations involving anything besides pressure are done. The density of water is 1 g cm^{-3} or 10^3 kg m^{-3}. The density of mercury is 13.6×10^3 kg m^{-3}, so 1 torr = 133 Pa. Another common unit for pressure is the atmosphere (atm), equal to 1.01×10^5 Pa. One atmosphere is approximately the atmospheric pressure at sea level.

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[8] See Appendix D on Taylor series for a more complete discussion of this approximation.

1.13 Buoyancy

Buoyancy is important when an object is immersed in a fluid. We are all familiar with buoyant effects when swimming; they are also important in instruments such as the centrifuge. Consider an object of density ρ immersed in a fluid of density ρ_{fluid}. The net force on such an object is the sum of the gravitational force and a force arising from the pressure gradient in the fluid. To visualize this, consider a small object with sides dx, dy, and dz. We have just seen that the pressure on the bottom face is greater than the pressure on the top face. Therefore, there is an upward force on the cube. The total force on the object is then

$$F = \left(-\frac{dp}{dz} - \rho g\right) dx\, dy\, dz.$$

Since the pressure gradient in the fluid is $-\rho_{\text{fluid}} g$, the total force is

$$F = (\rho_{\text{fluid}} - \rho)\, gV, \tag{1.31}$$

where V is the volume of the object. The second term is the object's weight, directed downward. The first term is called the buoyant force and is directed upward. The buoyant force reduces the "effective weight" of the object and depends on the difference of densities of the object and the surrounding fluid.

Animals are made up primarily of water, so their density is approximately $10^3\ \text{kg m}^{-3}$. The buoyant force depends on the animal's environment. Terrestrial animals live in air, which has a density of $1.2\ \text{kg m}^{-3}$. The buoyant force on terrestrial animals is very small compared to their weight. Aquatic animals live in water, and their density is almost the same as the surrounding fluid. The buoyant force almost cancels the weight, so the animal is essentially "weightless." Gravity plays a major role in the life of terrestrial animals, but only a minor role for aquatic animals. Denny (1993) explores the differences between terrestrial and aquatic animals in more detail.

1.14 Compressibility

Increasing the pressure on a fluid causes a deformation and a decrease in volume. The *compressibility* κ is defined as

$$\frac{\Delta V}{V} = -\kappa \Delta p. \tag{1.32}$$

Since $\Delta V/V$ is dimensionless, κ has the units of inverse pressure, $\text{N}^{-1}\,\text{m}^2$ or Pa^{-1}. In many liquids, the compressibility is quite small (e.g., $5 \times 10^{-10}\,\text{Pa}^{-1}$ for water), and for many purposes, such as flow through pipes, compressibility can be ignored. Other effects, such as the transmission of sound through a fluid, depend on deformation, and compressibility cannot be ignored. The bulk modulus is the reciprocal of the compressibility.

1.15 Diving

Air is easily compressible, so swimming at large depths can be dangerous as the volume of the air in the lungs decreases. One can swim safely for depths of tens of meters (several atmospheres of pressure) using a self-contained underwater breathing apparatus (SCUBA). Compressed air tanks are used to supply air to the lungs, and the pressure of the air is adjusted to match the pressure of the surrounding water.

One physiological effect of breathing high-pressure air is that nitrogen dissolves into the blood, which can lead to a mental impairment known as nitrogen narcosis. Moreover, if the swimmer returns rapidly to the surface after a long deep dive, the lowered pressure allows the dissolved nitrogen to form bubbles in the blood that block blood flow and cause decompression sickness, often called "the bends" (Benedek and Villars 2000). To avoid the bends, swimmers must return to the surface slowly, or replace nitrogen by other gasses, such as helium, that are less soluble in blood.

1.16 Viscosity

A fluid at rest does not support a shear. If the fluid is moving, a shear force can exist. At large velocities the flow of the fluid is turbulent and may be difficult or impossible to calculate. We will consider only those cases in which the velocity is low enough so that the flow is smooth. This means that particles of dye that are introduced into the fluid to monitor its motion flow along smooth lines called *streamlines*. A streamline is tangent to the velocity vector of the fluid at every point along its path. There is no mixing of fluid across streamlines; the flow is *laminar* (in layers). Laminar flow is often used in rooms where dirt or bacterial contamination is to be avoided, such as operating rooms or manufacturing clean rooms. Clean air enters and passes through the room without mixing. Any contaminants picked up are carried out in the air.

A fluid can support a viscous shear stress if the shear strain is changing. One way to create such a situation is to immerse two parallel plates, each of area S, in the fluid, and to move one parallel to the other as in Fig. 1.25. If the fluid in contact with each plate sticks to the plate[9],

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[9] This is called the "no-slip" boundary condition. There are exceptions.

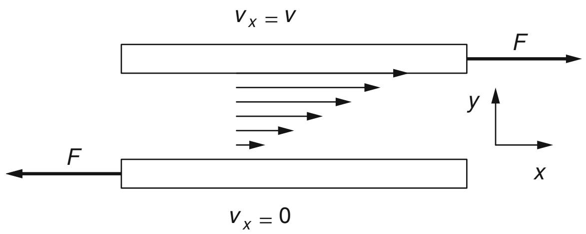

Fig. 1.25 Forces **F** and −**F** are needed to make the top plate move in a viscous fluid while the bottom plate remains stationary. The velocity profile is also shown

the fluid in contact with the lower plate is at rest and that in contact with the upper plate moves with the same velocity as the plate. Between the plates the fluid flows parallel to the plates, with a speed that depends on position as shown in Fig. 1.25. The variation of velocity between the plates gives rise to a velocity gradient dv_x/dy. Note that this is the rate of change of the shear strain.

In order to keep the top plate moving and the bottom one stationary, it is necessary to exert a force of magnitude F on each plate: to the right on the upper plate and to the left on the lower plate. The resulting shear stress or force per unit area is in many cases proportional to the velocity gradient:

$$\frac{F}{S} = \eta \frac{dv_x}{dy}. \tag{1.33}$$

Often this equation is written with a minus sign, in which case **F** is the force of the fluid on the plate rather than the plate on the fluid. The constant η is called the *coefficient of viscosity*. The units of η are $\mathrm{N\,s\,m^{-2}}$ or $\mathrm{kg\,m^{-1}\ s^{-1}}$ or Pa s. Older units are the $\mathrm{dyn\,cm^{-2}}$ or poise, the centipoise, and the micropoise. 1 poise = 0.1 Pa s. Water has a viscosity of about 10^{-3} Pa s at room temperature. Equation 1.33 gives the force exerted by fluid above the plane at height y on the fluid below the plane. In the case of the parallel plates, the force from above on fluid in the slab between y and $y + dy$ is the same in magnitude as (and opposite in direction to) the force exerted by the fluid below the slab. Therefore, there is no net force on the fluid in the slab, and the fluid moves with constant velocity. Fluids that are described by Eq. 1.33 are called *Newtonian fluids*. Many fluids are not Newtonian.

Since dv_x/dy is the rate of change of the shear strain, Eqs. 1.27 and 1.33 can be written as

$$s_s = \frac{F}{S} = \eta \frac{d\epsilon_s}{dt}.$$

The rate of change of the shear strain is also called the *shear rate*.

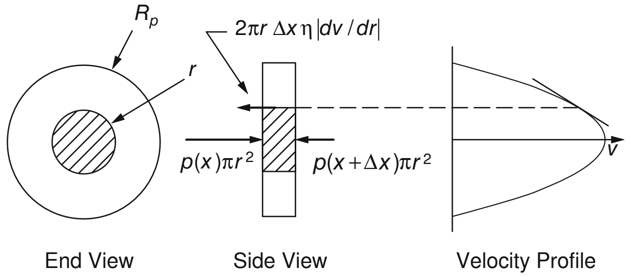

Fig. 1.26 Longitudinal and transverse cross sections of the tube. Newton's first law is applied to the shaded volume

1.17 Viscous Flow in a Tube

Biological fluid dynamics is a well-developed area of study (Lighthill 1975; Mazumdar 1992; Vogel 1994). External biological fluid dynamics is concerned with locomotion—from single-celled organisms to swimming fish and flying birds. Internal biological fluid dynamics deals with mass transport within the organism. Two obvious examples are flow in the airways and the flow of blood.

Consider laminar viscous flow of fluid through a pipe of constant radius R_p and length Δx. Ignore for now the gravitational force. The pressure at the left end of a segment of pipe is $p(x)$; at the right end it is $p(x + \Delta x)$. For now consider the special case in which none of the fluid is accelerated, so the total force on any volume element of the fluid is zero. The velocity profile must be as shown in Fig. 1.26: zero at the walls and a maximum at the center. Our problem is to determine $v(r)$.

Let us determine the forces acting on the shaded cylinder of fluid of radius r shown in Fig. 1.26. Since gravity is ignored, there are only three forces acting on the volume. The fluid on the left exerts a force $\pi r^2 p(x)$ acting to the right in the direction of the positive x axis. The fluid on the right exerts a force $-\pi r^2 p(x + \Delta x)$ (the minus sign because it points to the left). The slower moving fluid outside the shaded region exerts a viscous drag force across the cylindrical surface at radius r. The area of the surface is $2\pi r \Delta x$. The force points to the left. Its magnitude is $2\pi r \Delta x\, \eta\, |dv/dr|$. Since dv/dr is negative, we obtain the correct sign by writing it as $2\pi r\, \Delta x\, \eta\, (dv/dr)$. Since the fluid is not accelerating, the forces sum to zero:

$$\pi r^2 [p(x) - p(x + \Delta x)] + 2\pi r\, \Delta x\, \eta\, (dv/dr) = 0, \tag{1.34}$$

which can be rearranged to give

$$\frac{dv}{dr} = \frac{r}{2\eta} \left(\frac{p(x + \Delta x) - p(x)}{\Delta x} \right) = \frac{dp}{dx} \frac{r}{2\eta}. \tag{1.35}$$

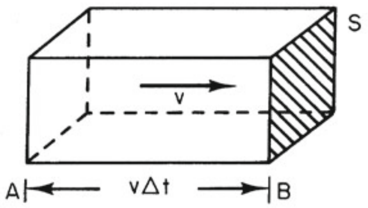

Fig. 1.27 Flow of fluid across the plane at B

This can be integrated:

$$\int dv = \frac{1}{2\eta}\left(\frac{dp}{dx}\right)\int r\,dr,$$

$$v(r) = \frac{1}{4\eta}\left(\frac{dp}{dx}\right)r^2 + A. \tag{1.36}$$

For flow to the right, dp/dx is negative. Therefore it is convenient to write Δp as the pressure drop from x to $x + dx$: $\Delta p = p(x) - p(x + \Delta x)$. Then the first term in Eq. 1.36 is $-(1/4\eta)(\Delta p/\Delta x)r^2$. The constant of integration can be determined assuming the "no-slip" boundary condition: the velocity of the fluid immediately adjacent to a solid is the same as the velocity of the solid itself. Because the wall is at rest, the velocity of the fluid is zero at the wall ($r = R_p$). The final result is

$$v(r) = \frac{1}{4\eta}\frac{\Delta p}{\Delta x}(R_p^2 - r^2). \tag{1.37}$$

The total flow rate or *volume flux* or *volume current* i is the volume of fluid per second moving through a cross section of the tube. Its units are $\mathrm{m^3\,s^{-1}}$. The *volume fluence rate* or *volume flux density*[10] or *current density* j_v is the volume per unit area per unit time across some small area in the tube. The units of j_v are $\mathrm{m^3\,s^{-1}\,m^{-2}}$ or $\mathrm{m\,s^{-1}}$.

In fact, j_v is just the velocity of the fluid at that point. To see this, consider the flow of an incompressible fluid during time Δt. In Fig. 1.27, the fluid moves to the right with velocity v. At $t = 0$, the fluid just to the left of plane B crosses the plane; at $t = \Delta t$, the fluid that was at A at $t = 0$ crosses plane B. All the fluid between plane A and plane B crosses plane B during the time interval Δt. The volume fluence rate is

$$j_v = \frac{\text{(volume transported)}}{\text{(area)(time)}} = \frac{Sv\Delta t}{S\Delta t} = v. \tag{1.38}$$

It may seem unnecessarily confusing to call the fluence rate or flux density j_v instead of v; however, this notation corresponds to a more general notation in which j means the fluence rate or flux density of anything per unit area per unit time, and the subscript v, s, or q tells us whether it is the fluence rate of volume, solute particles, or electric charge.

To find the volume current i, j_v must be integrated over the cross-sectional area of the pipe. The volume of fluid crossing the washer-shaped area $2\pi r dr$ is $j_v 2\pi r dr = v 2\pi r dr$. The total flux through the tube is

$$i = \int_0^{R_p} j_v(r)2\pi r\,dr,$$

$$i = \frac{2\pi}{4\eta}\frac{\Delta p}{\Delta x}\int_0^{R_p}\left(R_p^2 - r^2\right)r\,dr. \tag{1.39}$$

To integrate this, let $u = R_p^2 - r^2$. Then $du = -2rdr$ and the integral is $R_p^4/4$. Therefore

$$i = \frac{\pi R_p^4}{8\eta}\frac{\Delta p}{\Delta x} \tag{1.40}$$

is the flux of a viscous fluid through a pipe of radius R_p due to a pressure gradient $(\Delta p/\Delta x)$ along the pipe. The dependence of i on R_p^4 means that small changes in diameter cause large changes in flow.

This relationship was determined experimentally in painstaking detail by a French physician, Jean Leonard Marie Poiseuille, in 1835. He wanted to understand the flow of blood through capillaries. His work and knowledge of blood circulation at that time have been described by Herrick (1942).

As an example of the use of Eq. 1.40, consider a pore of the following size, which might be found in the basement membrane of the glomerulus of the kidney:

$$\begin{gathered} R_p = 5\,\mathrm{nm}, \\ \Delta p = 15.4\,\mathrm{torr}, \\ \eta = 1.4 \times 10^{-3}\,\mathrm{kg\,m^{-1}\,s^{-1}}, \\ \Delta x = 50\,\mathrm{nm}. \end{gathered} \tag{1.41}$$

It is first necessary to convert 15.4 torr to Pa using Eq. 1.30 and the value of ρ for mercury, $13.55\times10^3\,\mathrm{kg\,m^{-3}}$:

$$\begin{aligned} \Delta p &= \rho g \Delta z = (13.55 \times 10^3)(9.8)(15.4 \times 10^{-3}) \\ &= 2.04 \times 10^3\,\mathrm{Pa}. \end{aligned}$$

Then Eq. 1.40 can be used:

$$i = \frac{(3.14)(5 \times 10^{-9})^4(2.04 \times 10^3)}{(8)(1.4 \times 10^{-3})(50 \times 10^{-9})} = 7.2 \times 10^{-21}\,\mathrm{m^3\,s^{-1}}.$$

Now consider the general case in which we have not only viscosity, but the fluid may be accelerated and gravity is important. We continue to write Δp as the pressure drop and consider four contributions, each of which will be discussed:

$$\begin{aligned} \Delta p = p_1 - p_2 &= -\int_{x_1}^{x_2}(dp/dx)\,dx \\ &= \Delta p_{\text{visc}} + \Delta p_{\text{grav}} + \Delta p_{\text{accel1}} + \Delta p_{\text{accel2}}. \end{aligned} \tag{1.42}$$

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[10] Some authors call j_v the flux. The nomenclature used here is consistent throughout the book.

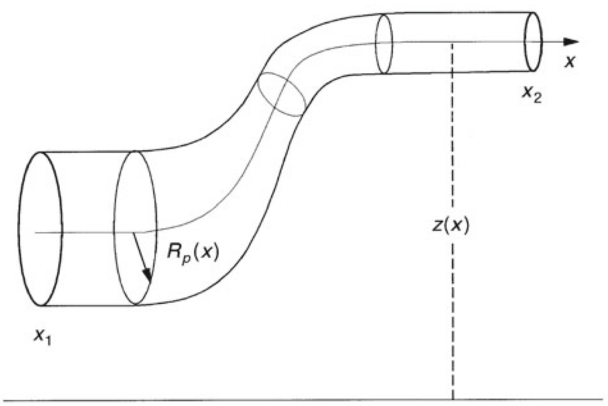

Fig. 1.28 A pipe of circular cross section with radius and height varying along the pipe

For simplicity, we restrict the derivation to an incompressible fluid and a pipe of circular cross section where the radius can change. Distance along the pipe is x, the radius of the pipe is $R_p(x)$, and the cross-sectional area is $S(x) = \pi R_p^2(x)$. Gravitational force acts on the fluid, and the height of the axis of the pipe above some reference plane is $z(x)$, as shown in Fig. 1.28.

Because the fluid is incompressible, the total current i is independent of x. We define the average velocity by

$$i = \overline{v}(x)S(x), \tag{1.43}$$

so

$$\overline{v}(x) = \frac{i}{S(x)} = \frac{i}{\pi R_p^2(x)}. \tag{1.44}$$

If the area decreases, the average velocity increases, and vice versa. This is a special case of the continuity equation, which is discussed further in Chap. 4.

Assume that changes in pipe radius occur slowly enough so that the velocity profile remains parabolic at every point in the pipe and that we can treat x as though it were distance along the axis of the cylinder. We can use Eq. 1.37 to rewrite the velocity profile as

$$v(r,x) = 2\overline{v}\left[1 - \frac{r^2}{R_p^2(x)}\right] = \frac{2i}{\pi R_p^2(x)}\left[1 - \frac{r^2}{R_p^2(x)}\right]. \tag{1.45}$$

The first term in Eq. 1.42 is the drop in pressure because of viscous drag. We can rewrite Eq. 1.35 as

$$\frac{dp_{\text{visc}}}{dx} = \frac{2\eta}{r}\frac{dv}{dr}.$$

Using Eq. 1.45, we can write

$$\frac{dp_{\text{visc}}}{dx} = -\frac{8\eta\, i}{\pi R_p^4(x)}. \tag{1.46}$$

We saw this earlier, solved for i in a pipe of constant radius, as Eq. 1.40. The pressure drop is obtained by integration:

$$\begin{aligned}\Delta p_{\text{visc}} &= -\int_{x_1}^{x_2} dp_{\text{visc}} = -\int_{x_1}^{x_2}\left(\frac{dp_{\text{visc}}}{dx}\right)dx \\ &= +\frac{8\eta i}{\pi}\int_{x_1}^{x_2}\frac{dx}{R_p^4(x)}.\end{aligned} \tag{1.47}$$

To go further requires knowing $R_p(x)$.

The next term p_{grav} is the hydrostatic pressure change that we saw in Eq. 1.30:

$$\Delta p_{\text{grav}} = -\int_{x_1}^{x_2} dp_{\text{grav}} = -\int \frac{dp_{\text{grav}}}{dz}\,dz = \rho g(z_2 - z_1). \tag{1.48}$$

The last two terms of Eq. 1.42 are pressure differences required to accelerate the fluid. When the flow is steady—that is, the velocity depends only on position, and the velocity at a fixed position does not change with time—there can still be an acceleration if the cross section of the pipe changes. The third term, Δp_{accel1}, is the pressure drop required to cause this acceleration. It can be derived as follows: Imagine a streamline in the fluid. No fluid crosses the streamline. Consider a small length of streamline ds and a small area dA perpendicular to it. Note that ds is a small displacement along a streamline, while dx is along the axis of the pipe. The edge of dA defines another set of streamlines that form a tube of flow, and $dA ds$ defines a small volume of fluid. Make ds and dA small enough so that v is nearly the same at all points within the volume. The mass of fluid in the volume is $dm = \rho dA ds$. We ignore viscosity and gravity, so the only pressure difference is due to acceleration. The net force on the volume is

$$dF = -\frac{dp}{ds}\,ds\,dA. \tag{1.49}$$

This is equal to the mass times the acceleration dv/dt. The acceleration of the fluid in the element is then

$$\frac{dv}{dt} = \frac{dF}{dm} = \frac{-\left(\frac{dp}{ds}\right)dsdA}{\rho dsdA} = -\frac{1}{\rho}\left(\frac{dp}{ds}\right). \tag{1.50}$$

We are considering only velocity changes that occur because the fluid moves along a streamline to a different position. We use the chain rule to write

$$\frac{dv}{dt} = \left(\frac{dv}{ds}\right)\left(\frac{ds}{dt}\right) = v\left(\frac{dv}{ds}\right).$$

Combining these gives

$$\frac{dp_{\text{accel1}}}{ds} = -\rho v\left(\frac{dv}{ds}\right). \tag{1.51}$$

This can be integrated along the streamline to give

$$\begin{aligned}\Delta p_{\text{accel1}} &= -\int_{s_1}^{s_2}\left(\frac{dp_{\text{accel1}}}{ds}\right)ds = +\rho\int_{x_1}^{x_2} v\left(\frac{dv}{ds}\right)ds\\ &= \frac{\rho v_2^2}{2}-\frac{\rho v_1^2}{2}.\end{aligned} \tag{1.52}$$

This is sometimes called the dynamic pressure.

The final term Δp_{accel2} is the pressure drop required to accelerate the fluid between points 1 and 2 if the velocity of the fluid at a fixed position is changing with time (unsteady flow). This happens, for example, to blood that is accelerated as it is ejected from the heart during systole. To derive this term, again imagine a small length of streamline ds and a small area dA perpendicular to it. In addition to ignoring gravity and viscosity, we ignore changes in velocity because of changes in cross section. There is acceleration only if the velocity at a fixed location is changing. The acceleration is $\partial v/\partial t$. The derivative is written with ∂s to signify the fact that we are considering only changes in the velocity with time that occur at a fixed position. The net force required to accelerate this mass is provided by the pressure difference Eq. 1.49:

$$dF = -dA\,dp_{\text{accel2}} = dm\left(\frac{\partial v}{\partial t}\right) = \rho\left(\frac{\partial v}{\partial t}\right)dA\,ds,$$

$$dp_{\text{accel2}} = -\rho\left(\frac{\partial v}{\partial t}\right)ds,$$

$$\Delta p_{\text{accel2}} = -\int_{s_1}^{s_2} dp_{\text{accel2}} = \rho\int_{s_1}^{s_2}\left(\frac{\partial v}{\partial t}\right)ds. \tag{1.53}$$

All of these effects can be summarized in the *generalized Bernoulli equation*:

$$\begin{aligned}p_1 - p_2 = \Delta p = &\underbrace{\rho\int_{s_1}^{s_2}\frac{\partial v}{\partial t}\,ds}_{\Delta p_{\text{accel2}}} + \underbrace{\int_{s_1}^{s_2}\left(-\frac{dp_{\text{visc}}}{ds}\right)ds}_{\Delta p_{\text{visc}}}\\ &\underbrace{+\frac{\rho v_2^2}{2}-\frac{\rho v_1^2}{2}}_{\Delta p_{\text{accel1}}} + \underbrace{\rho g\left(z_2 - z_1\right)}_{\Delta p_{\text{grav}}}.\end{aligned} \tag{1.54}$$

Equation 1.54 is valid for nonuniform viscous flow that may be laminar or turbulent if the integral is taken along a streamline (see, for example, Synolakis and Badeer 1989).

1.18 Pressure–Volume Work

An important example of work is that done in a biological system when the volume of a container (such as the lungs or the heart or a blood vessel) changes while the fluid within the container is exerting a force on the walls.

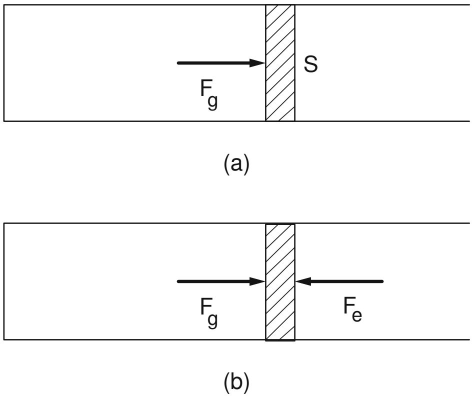

Fig. 1.29 **a** A cylinder containing gas has a piston of area S at one end. b The force exerted on the piston by the gas is balanced by an external force if the piston is at rest

To deduce an expression for pressure–volume work, consider a cylinder of gas fitted with a piston, Fig. 1.29a. If the piston has area S, the gas exerts a force $F_g = pS$ on the piston. If no other force is exerted on the piston to restrain it, it will be accelerated to the right and gain kinetic energy as the gas does work on it:

$$(\text{work done by gas}) = F_g dx = pSdx = pdV. \tag{1.55}$$

If the piston is prevented from accelerating by an external force $\mathbf{F}_e$, equal and opposite to that exerted by the gas (Fig. 1.29b), then the external force does work on the piston:

$$\begin{aligned}(\text{work done by external force}) &= -F_e dx\\ &= -pSdx = -pdV,\end{aligned} \tag{1.56}$$

which is the negative of the work done on the piston by the expanding gas. The result is that the kinetic energy of the piston does not change. The gas does work on the surroundings as it expands, increasing the energy of the surroundings; the surroundings, through the external force, do *negative* work on the gas; that is, they decrease the energy of the gas. (The meaning of "energy of the gas" and "energy of the surroundings" is discussed in Chap. 3.) If the gas is compressed, the situation is reversed: the surroundings do positive work on the gas and the gas does negative work on the surroundings.

For a large change in volume from V_1 to V_2, the pressure may change as the volume changes. In that case the work done by the gas on the surroundings is

$$W_{\text{by gas}} = \int_{V_1}^{V_2} p\,dV. \tag{1.57}$$

This work is the shaded area in Fig. 1.30. If the gas is compressed, the change in volume is negative and the work done *by* the gas is negative.

Let us apply this model to the heart. Suppose that the left ventricle of the heart contracts at constant pressure, so that

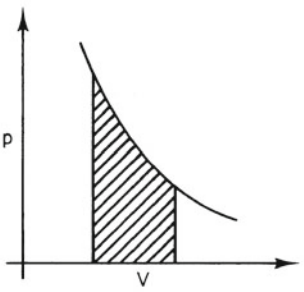

Fig. 1.30 A plot of p versus V, showing the work done by the gas as it expands

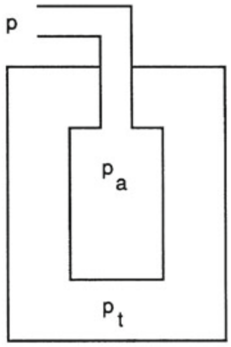

Fig. 1.31 A model of the thorax, lungs, and airways that can be used to understand some features of breathing

it changes volume by $\Delta V = V_2 - V_1$. (Since $V_2 < V_1$, the quantity ΔV is negative. A volume of blood $-\Delta V$ is ejected into the aorta.) The work done by the heart wall on the blood is $-p\Delta V$ and is positive, since ΔV is negative.

As another example of pressure–volume work, we can develop a model to estimate the work necessary to breathe. Consider the model of the lungs and airways shown in Fig. 1.31. The pressure at the nose is the atmospheric pressure p. In the alveoli (air sacs), the pressure is p_a. If there is no flow taking place, $p_a = p$. For air to flow in, p_a must be less than p; for it to flow out, p_a must be greater than atmospheric. The work done by the walls of the alveoli on the gas in them is $-\int p_a \, dV$. The net value of this integral for a respiratory cycle is positive. Perhaps the easiest way to see this is to imagine an inspiration, in which the alveolar pressure is $p_a = p - \Delta p$ and the volume change is ΔV. The work done on the gas is $-(p - \Delta p)\Delta V$. This is followed by an expiration at pressure $p_a = p + \delta p$, for which the work is $-(p + \delta p)(\Delta V)$. The net work done on the gas is $(\Delta p + \delta p)\Delta V$. The energy imparted to the gas shows up as a mixture of heating because of frictional losses and kinetic energy of the exhaled air.

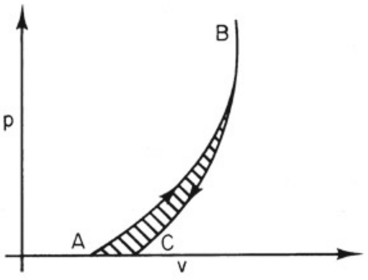

Fig. 1.32 A hypothetical plot of the pressure–volume relationship for inhalation and exhalation

There is another mechanism by which work is done in breathing. Refer again to Fig. 1.31. The pressure in the chest cavity (thorax) is p_t. (The pressure measured in mid-esophagus is a good estimate of p_t.) Because of contractile forces in the lung tissue, $p_a > p_t$. The gas in the alveoli and the fluid in the thorax both do work on the lung tissue. The latter has opposite sign, since a positive displacement dx of a portion of the alveolar wall is in the direction of the force exerted by the alveolar gas but is opposite to the direction of the force exerted by the thoracic fluid. The elastic recoil pressure, multiplied by dV, gives the net work done by both forces on the wall of the lung.

Figure 1.32 shows the elastic recoil pressure versus lung volume. The elastic recoil pressure is the difference between the pressure in the alveoli (air sacs) of the lung and the pressure in the thorax just outside the lung. During inspiration (curve AB), the elastic recoil pressure $p_a - p_t$ is greater than that during expiration (curve BC). The net work done on the lung wall during the respiratory cycle goes into frictional heating of the lung tissue.

1.19 The Human Circulatory System

The human circulatory system is responsible for pumping blood and its life-sustaining nutrients to all parts of the body (Vogel 1992). The circulatory system has two parts: the *systemic circulation* and the *pulmonary circulation*, as shown in Fig. 1.33. The left heart pumps blood into the systemic circulation: organs, muscles, etc. The right heart pumps blood through the lungs. As the heart beats, the pressure in the blood leaving the heart rises and falls. The maximum pressure during the cardiac cycle is the *systolic pressure*. The minimum is the *diastolic pressure*. (A blood pressure reading is in the form systolic/diastolic, measured in torr. A typical blood pressure might be 110/70.)

A *sphygmomanometer* is used to measure blood pressure. Air is pumped into a cuff placed around the forearm. The applied pressure is measured using either a column of mercury or a mechanical pressure transducer. The cuff is inflated

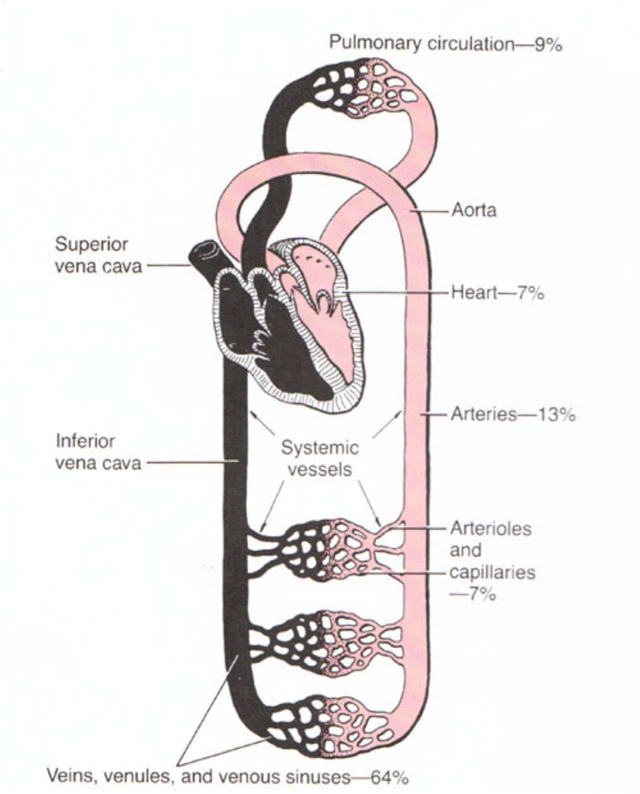

Fig. 1.33 The human circulatory system. The subject is facing you, so the left chambers of the heart are on the right in the picture. The left heart pumps oxygenated blood (red), and the right heart pumps deoxygenated blood (black). (Reprinted from Guyton 1991 © Elsevier Inc. Used with permission of Elsevier)

until flow in the brachial artery ceases. The cuff pressure is then slowly reduced until flow returns during *systole*. The flow can be detected by listening with a stethoscope for the sounds associated with the starting and stopping of flow (*Korotkoff sounds*), or with a *pulse oximeter* (see page 392). The cuff pressure is then further reduced until flow occurs continuously throughout the cardiac cycle including *diastole*.

The blood flows from the aorta to several large arteries, to medium-sized arteries, to small arteries, to arterioles, and finally to the capillaries, where exchange with the tissues of oxygen, carbon dioxide, and nutrients takes place. The blood emerging from the capillaries is collected by venules, flows into increasingly larger veins, and finally returns to the heart through the vena cava.

At any given time, blood is flowing in only a fraction of the capillaries. The state of flow in the capillaries is continually changing to provide the amount of oxygen required by each organ. In skeletal muscle, terminal arterioles constrict and dilate to control distribution of blood to groups of capillaries. In smooth muscle and skin, a precapillary sphincter muscle controls the flow to each capillary (Patton et al. 1989, p. 860). Since the blood is incompressible and is conserved,[11] the total volume flow i remains the same at all generations of branching in the vascular tree. Table 1.4 shows average values for the pressure and vessel sizes at different generations of branching. Most of the pressure drop occurs in the arterioles.

We define the *vascular resistance* R in a pipe or a segment of the circulatory system as the ratio of pressure difference across the pipe or segment to the flow through it:

$$R = \frac{\Delta p}{i}. \tag{1.58}$$

The units are Pa m^{-3} s. Physiologists use the *peripheral resistance unit (PRU)*, which is torr ml^{-1} min. For Poiseuille flow, the resistance can be calculated from Eq. 1.40:

$$R = \frac{8\eta \Delta x}{\pi R_p^4}. \tag{1.59}$$

The resistance decreases rapidly as the radius of the vessel increases.

If vessels of different diameters are connected in series so that the flow i is the same through each one and the total pressure drop is the sum of the drops across each vessel, then the total resistance is the sum of the resistances of each vessel:

$$R_{\text{tot}} = R_1 + R_2 + R_3 + \cdots . \tag{1.60}$$

If there is branching so that several vessels are in parallel with the same pressure drop across each one, the total flow through all the branches equals the flow in the vessel feeding them. The total resistance is then given by

$$\frac{1}{R_{\text{tot}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots . \tag{1.61}$$

For the most part, the capillaries are arranged in parallel. Even though the resistance of an individual capillary is large because of its small radius (Eq. 1.59), the resistance of the capillaries as a whole is relatively small because there are so many of them (see Problem 42).

The pressure in the left ventricle changes during the cardiac cycle. It can be plotted versus time. It can also be plotted versus ventricular volume, as in Fig. 1.34. The p–V relationship moves counterclockwise around the curve during the cycle. Filling occurs at nearly zero pressure until the ventricle begins to distend when the volume exceeds 60 ml[12]. There is then a period of contraction at nearly constant volume

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[11] This is not strictly true. Some fluid leaves the capillaries and returns to the heart through the lymphatic system instead of the venous system. See Chap. 5.

[12] 1 ml = 10^{-3} liter(l) = 10^{-6} m^3.

Table 1.4 Typical values for the average pressure at the entrance to each generation of the major branches of the cardiovascular tree, the average blood volume in certain branches, and typical dimensions of the vessels

Location	Average pressure (torr)	Blood volume[a] (ml)	Diameter[b] (mm)	Length[b] (mm)	Wall thickness[b] (mm)	Avg. velocity[b] ($\mathrm{m\,s^{-1}}$)	Reynolds number at maximum flow[c]
Systemic circulation							
Left atrium	5						
Left ventricle	100						
Aorta	100	156	20	500	2	0.48	9 400
Arteries	95	608	4	500	1	0.45	1 300
Arterioles	86	94	0.05	10	0.2	0.05	
Capillaries	30	260	0.008	1	0.001	0.001	
Venules	10	470	0.02	2	0.002	0.002	
Veins	4	2682	5	25	0.5	0.01	
Vena cava	3	125	30	500	1.5	0.38	3 000
Right atrium	3						
Pulmonary circulation							
Right atrium	3						
Right ventricle	25						
Pulmonary artery	25	52					
Arteries	20	91					7 800
Arterioles	15	6					
Capillaries	10	104					
Veins	5	215					2 200
Left atrium	5						

[a] From Plonsey (1995)
[b] From Mazumdar (1992)
[c] From Milnor (1989)

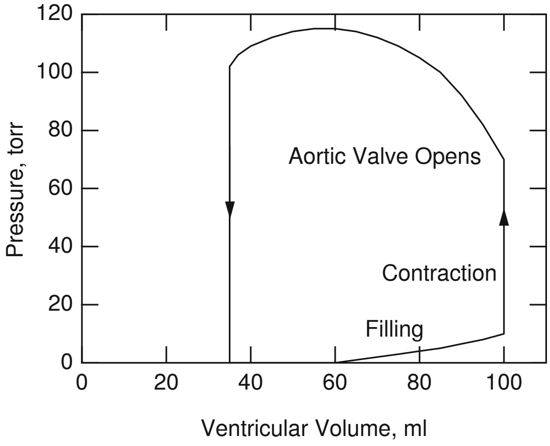


Fig. 1.34 Pressure–volume relationship in the left ventricle. The curve is traversed counterclockwise with increasing time. The stroke volume is 100−35 = 65 ml. Systolic pressure is 118 torr, and diastolic pressure is 70 torr. The ventricular pressure drops below diastolic while the pressure in the arteries remains about 70 torr because the aortic valve is closed and prevents back flow

that causes the ventricular pressure to rise until it exceeds the (diastolic) pressure in the aorta, and the aortic valve opens. The contraction continues, and the pressure rises further, but the ventricular volume decreases as blood flows into the aorta. The ventricle then relaxes. The aortic valve closes when the ventricular pressure drops below that in the aorta. The work done in one cycle is the area enclosed by the curve. For the curve shown, it is 6600 torr ml = 0.88 J. At 80 beats per minute, the power is 1.2 W. In this drawing the stroke volume is 100 − 35 = 65 ml, and the cardiac output is

$$i = (65\,\mathrm{ml\,beat^{-1}})(80\,\mathrm{beats}/60\,\mathrm{s}) = 87 \times 10^{-6}\mathrm{m^3\,s^{-1}}.$$

1.20 Turbulent Flow and the Reynolds Number

Many features of the circulation can be modeled by Poiseuille flow. However, at least four effects—in addition to those in Eq. 1.42—cause departures from Poiseuille flow: (1) there may be *turbulence*; (2) there are departures from a parabolic velocity profile; (3) the vessel walls are elastic; and (4) the apparent viscosity depends on both the fraction of the blood volume occupied by red cells and the size of the vessel.

The importance of turbulence (nonlaminar flow) is determined by a dimensionless number characteristic of the system called the *Reynolds number* N_R. It is defined by

$$N_R = \frac{LV\rho}{\eta}, \tag{1.62}$$

where L is a length characteristic of the problem, V a velocity characteristic of the problem, ρ the density, and η the viscosity of the fluid. When N_R is greater than a few thousand, turbulence usually occurs.

The Reynolds number arises in the following way: If we were to write Newton's second law for a fluid (which we have not done) in terms of dimensionless primed variables such as $\mathbf{r}' = \mathbf{r}/L$, $\mathbf{v}' = \mathbf{v}/V$, and $t' = t/(L/V)$, we would find that the equations depended on the properties of the fluid only through the combination N_R (Mazumdar 1992, p. 14). With appropriate scaling of dimensions and times, flows with the same Reynolds number are identical.

There is ambiguity in defining the characteristic length and the characteristic velocity. Should one use the radius or the diameter of a tube? The maximum velocity or the average velocity? If one is solving the equations of motion, one knows what values of L and V were used to transform the equations. They are used to transform the solution back to "real world" coordinates. However, if one is making a statement such as "turbulence usually occurs for values of N_R greater than a few thousand," there is ambiguity. On the other hand, the statement is not very precise. Sometimes an additional subscript is used to specify how N_R was determined.

When N_R is large, inertial effects are important. External forces accelerate the fluid. This happens when the density is large and the viscosity is small. As the viscosity increases (for fixed L, V, and ρ), the Reynolds number decreases. When the Reynolds number is small, viscous effects are important. The fluid is not accelerated, and external forces that cause the flow are balanced by viscous forces. Since viscosity is a form of internal friction in the fluid, work done on the system by the external forces is transformed into thermal energy. The low-Reynolds-number regime is so different from our everyday experience that the effects often seem counterintuitive. They are nicely described by Purcell (1977).

Here is an example of an estimate expressed in terms of the Reynolds number. A pressure difference Δp acts on a segment of fluid of length Δx undergoing Poiseuille flow. The difference between the force exerted on the segment of fluid by the fluid "upstream" and that exerted by the fluid "downstream" is $\pi R_p^2 \Delta p$. If the average speed of the fluid is $\overline{v}$, then the net work done on the segment by the fluid upstream and downstream in time Δt is $W_{\text{visc}} = \pi R_p^2 \Delta p \overline{v} \Delta t$. Since the fluid is not accelerated, this work is converted into thermal energy. We can solve Eq. 1.40 for Δp and use Eq. 1.44 to write

$$W_{\text{visc}} = \pi R_p^2 \, \Delta p \, \overline{v} \Delta t = 8 \eta \pi \overline{v}^2 \, \Delta x \, \Delta t.$$

The kinetic energy of the moving fluid in a cylinder of length $\overline{v}\Delta t$ is

$$E_k = \frac{m v^2}{2} = \frac{\rho \, \pi R_p^2 \, (\overline{v}\Delta t) \overline{v}^2}{2} = \frac{\rho \, \pi R_p^2 \, \overline{v}^3 \, \Delta t}{2},$$

and the ratio of the kinetic energy to the work done is

$$\frac{E_k}{W_{\text{visc}}} = \frac{\rho \, \overline{v} \, R_p^2}{16 \eta \, \Delta x} = \frac{1}{16\xi} \frac{\rho \, \overline{v} \, R_p}{\eta} = \frac{1}{16\xi} N_R$$

where we write Δx as ξR_p. This result shows that the ratio of kinetic energy to viscous work is proportional to the Reynolds number. Another example is given in the problems.

The behavior of a sphere moving through a fluid illustrates how flow behavior depends on Reynolds number. At low Reynolds number, the viscous forces tend to make the fluid stick to the sphere, creating a large amount of viscous drag. This flow can be analyzed analytically (Schlichting and Gersten 2000). The drag force is $6\eta R v$, where R is the sphere radius, v is the speed of the sphere, and η is the viscosity, a result known as *Stokes' law*. At high Reynolds number, Bernoulli's equation (see Problem 36) tells us that high pressure is associated with low fluid speeds, and low pressure is associated with fast speeds. There is a region of high pressure in front of and in back of the sphere (where speeds are slow), and low pressure to either the left or right side (where speeds are fast). At very high Reynolds number, viscosity is small but still plays a role because of the no-slip boundary condition at the sphere surface. A thin layer of fluid, called the *boundary layer*, sticks to the solid surface, causing a large velocity gradient and therefore significant viscous drag (Schlichting and Gersten 2000). At extremely high Reynolds number, the flow undergoes *separation*, where eddies and turbulent flow occur downstream from the sphere, lowering the pressure in the sphere's wake, but they do not influence the high pressure in front of the sphere. Thus, pressure drag contributes to the total drag force, in addition to viscous drag. Similarly, if we consider a nonsymmetrical object instead of a sphere, we can make the flow speed and pressure differ on the left and right sides of the object, resulting in lift: a force perpendicular to the direction of the main fluid flow. Vogel (1994) discusses the biological implications of high Reynolds number flow, which is particularly important for flying animals and large swimmers. However, many of the biological fluid dynamics applications we will consider occur at low Reynolds number, where turbulence, separation, pressure drag, and boundary layers are not important, and Stokes' law dominates.

A large range of values of N_R occurs in the circulatory system. Typical values corresponding to the peak flow are given in Table 1.4. Blood flow is laminar except in the ascending aorta and main pulmonary artery, where turbulence may occur during peak flow. The Reynolds number in the capillaries is about 10^{-2}.

There are two main causes of departures from the parabolic velocity profile. First, a red cell is about the same diameter as a capillary. Red cells in capillaries line up single file, each nearly blocking the capillary. The plasma flows in

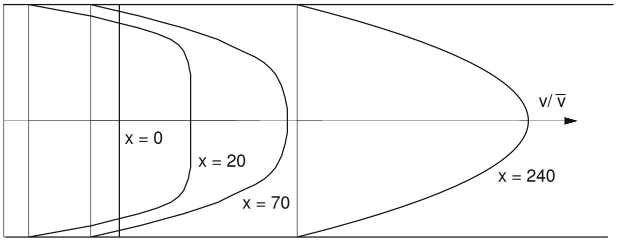


Fig. 1.35 Velocity profiles in steady laminar flow at the entrance to a tube, showing the development of the parabolic velocity profile. The velocity is given as $v/\bar{v}$. At the entrance, $v/\bar{v} = 1$. When the Poiseuille flow is fully developed, $v/\bar{v}$ is 2 at the center of the tube. These curves are calculated from a graph by Cebeci and Bradshaw (1977) for laminar flow in a tube of radius 2 mm and a pressure gradient of 20 torr m^{-1}, carrying a fluid with a viscosity of 3×10^{-3} $\mathrm{N\,s\,m^{-2}}$ and a density of 10^3 $\mathrm{kg\,m^{-3}}$. The scales are different along the axis and radius of the tube; the tube radius is 2 mm and the entrance region is 240-mm long

small volumes between each red cell, with a velocity profile that is nearly independent of radius. Second, the *entry region* causes deviations from Poiseuille flow in larger vessels. Suppose that blood flowing with a nearly flat velocity profile enters a vessel, as might happen when blood flowing in a large vessel enters the vessel of interest, which has a smaller radius. At the wall of the smaller vessel, the flow is zero. Since the blood is incompressible, the *average* velocity is the same at all values of x, the distance along the vessel. (We assume the vessel has constant cross-sectional area.) However, the velocity profile $v(r)$ changes with distance x along the vessel. At the entrance to the vessel ($x = 0$), there is a very abrupt velocity change near the walls. As x increases, a parabolic velocity profile is attained. The transition or entry region is shown in Fig. 1.35. In the entry region, the pressure gradient is different from the value for Poiseuille flow. The velocity profile cannot be calculated analytically in the entry region. Various numerical calculations have been made, and the results can be expressed in terms of scaled variables (see, for example, Cebeci and Bradshaw 1977). The Reynolds number used in these calculations was based on the diameter of the pipe, $D = 2R_p$, and the average velocity. The length of the entry region is

$$L = 0.05 D N_{R,D} = 0.1 R_p N_{R,D} = 0.2 R_p N_{R,R_p}. \quad (1.63)$$

Blood pressure is, of course, pulsatile. This means that the average velocity and $v(r)$ are changing with time and also departing from the parabolic profile. Also, at the peak pressure during systole, the aorta and arteries expand, storing some of the blood and releasing it gradually during the rest of the cardiac cycle. Pulsatile flow and the elasticity of vessel walls are discussed extensively by Caro et al. (1978) and Milnor (1989).

Blood is not a Newtonian fluid. The viscosity depends strongly on the fraction of volume occupied by red cells (the *hematocrit*). In blood vessels of less than 100-µm radius, the apparent viscosity decreases with tube radius. Since a red cell barely fits in a capillary, the velocity profile in capillaries is not parabolic. Flow in arterioles and arteries is often modeled as individual particles surrounded by plasma and transported by laminar flow, each red cell staying at its own distance from the central axis. However, high-speed motion pictures show that the red cells often collide with other red cells and with the wall. (See the articles by Trowbridge (1982, 1983) and Trowbridge and Meadowcroft (1983), and also the Caro et al. and Milnor articles.)

Symbols Used in Chapter 1

Symbol	Use	Units	First used page
$\mathbf{a}$, a	Acceleration	$\mathrm{m\,s^{-2}}$	3
a, b	Small distances	m	14
c	Constant of integration		14
g	Acceleration due to gravity	$\mathrm{m\,s^{-2}}$	14
h	Small distance	m	13
i	Total volume flux or flow rate or current	$\mathrm{m^3\,s^{-1}}$	17
j_v	Volume fluence rate or flux density (flow of volume per unit area per second)	$\mathrm{m\,s^{-1}}$	17
l	Length of rod	m	12
m	Mass	kg	3
p	Pressure	Pa	14
p_t	Pressure in thorax	Pa	20
p_a	Pressure in alveoli	Pa	20
$\mathbf{r}$	Position	m	5
r	Distance from origin (radius) in polar coordinates	m	5
$\mathbf{s}$	Displacement	m	12
s_n	Normal stress	Pa	12
s_s	Shear stress	Pa	13
s	Distance along a streamline	m	18
t	Time	s	11
v, $\mathbf{v}$	Velocity	$\mathrm{m\,s^{-1}}$	11
x, y, z	Coordinates	m	4
$\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$	Unit vectors along the x, y, and z axes		6
A	Constant of integration		17
dA	Small area perpendicular to a streamline	$\mathrm{m^2}$	19
D	Pipe diameter	m	24
E	Young's modulus	Pa	13
E_k	Kinetic energy	J	11
F, $\mathbf{F}$	Force	N	3
G	Shear modulus	Pa	13
L	Characteristic length	m	24
N, $\mathbf{N}$	Force	N	8
N_R	Reynolds number		22
$N_{R,D}$	Reynolds number based on diameter		24
N_{R,R_p}	Reynolds number based on pipe radius		24
P	Power	W	12
R, $\mathbf{R}$	Force	N	8
R_p	Radius of pipe	m	17
R	Vascular resistance	$\mathrm{Pa\,m^{-3}\,s}$	21
S	Cross-sectional area	$\mathrm{m^2}$	12
V	Volume	$\mathrm{m^3}$	15

V	Velocity	$\mathrm{m\,s^{-1}}$	23
W, $\mathbf{W}$	Weight	N	4
W	Work	J	19
δ	A small distance	m	13
ϵ_n	Normal strain		12
ϵ_s	Shear strain		13
η	Viscosity	Pa s	16
$\alpha, \beta, \theta, \phi$	Angle		5
κ	Compressibility	$\mathrm{Pa^{-1}}$	15
ρ	Mass density	$\mathrm{kg\,m^{-3}}$	14
τ, $\boldsymbol{\tau}$	Torque	N m	5
ξ	Dimensionless ratio		23

Problems

Section 1.1

Problem 1. Estimate the number of hemoglobin molecules in a red blood cell. Red blood cells are little more than bags of hemoglobin, so it is reasonable to assume that the hemoglobin takes up all the volume of the cell.

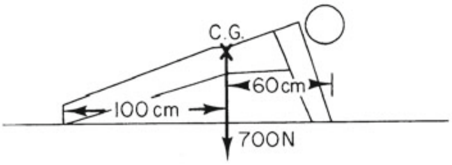
Problem 2. Our genetic information or genome is stored in the parts of the DNA molecule called base pairs. Our genome contains about 3 billion (3×10^9) base pairs, and there are two copies in each cell. Along the DNA molecule, there is one base pair every one-third of a nanometer. How long would the DNA helix from one cell be if it were stretched out in a line? If the entire DNA molecule were wrapped up into a sphere, what would be the diameter of that sphere?

Problem 3. Estimate the size of a box containing one air molecule. (Hint: What is the volume of one mole of gas at standard temperature and pressure?) Compare the size of the box to the size of an air molecule (about 0.1 nm).

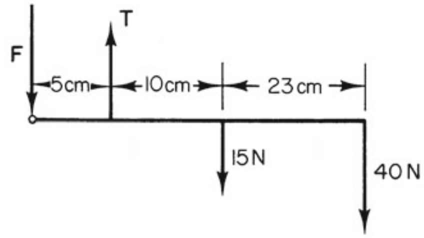
Problem 4. Estimate the density of water (H_2O) in $\mathrm{kg\,m^{-3}}$. Useful information: an oxygen atom contains eight protons and eight neutrons. A hydrogen atom contains one proton and no neutrons. The mass of the electron is negligible.

Section 1.4

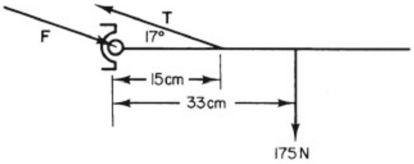
Problem 5. A person with mass $m = 70$ kg has a weight (mg) of about 700 N. If the person is doing push-ups as shown, what are the vertical components of the forces exerted by the floor on the hands and feet?



Problem 6. A person with upper arm vertical and forearm horizontal holds a mass of 4 kg. The mass of the forearm is 1.5 kg. Consider four forces acting on the forearm: **F** by the bones and ligaments of the upper arm at the elbow, **T** by the biceps, 40 N by the mass, and 15 N as the weight of the arm. The points of application are shown in the drawing. Calculate the vertical components of **F** and **T**.

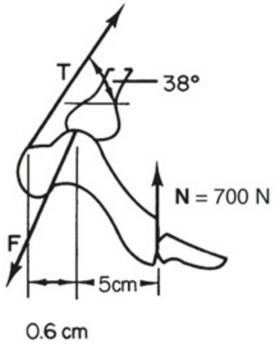


Problem 7. When the arm is stretched out horizontally, it is held by the deltoid muscle. The situation is shown schematically. Determine **T** and **F**.



Section 1.6

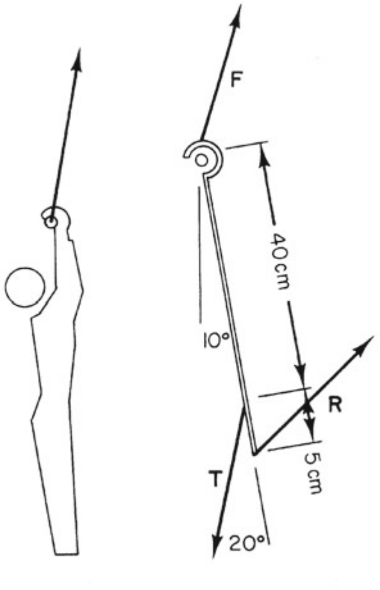
Problem 8. When a person crouches, the geometry of the heel is as shown. Determine **T** and **F**. Assume all the forces act in the plane of the drawing.



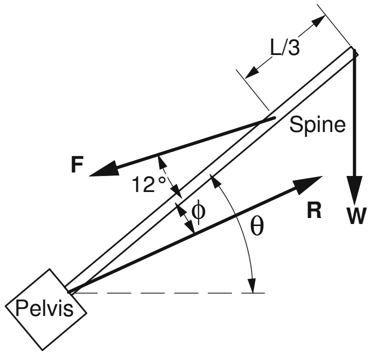
Problem 9. A person of weight W is suspended by both hands from a high bar as shown. The center of mass is directly below the bar.

(a) Find the horizontal and vertical components F_x and F_y, where **F** is the force exerted by the bar on each of the two hands.

(b) Given the additional information about the arm shown in the second drawing, calculate the components of **R**, the force exerted by the humerus on the forearm through the elbow, and the tension **T** in the biceps tendon. Neglect the weight of the arm, and assume that **T** and **R** are the only forces exerted on the forearm by the upper arm.



Problem 10. Consider the forces on the spine when lifting. Approximate the spinal column as a stiff bar of length L that has three forces acting on it. **W** is the downward force acting at the top of the spinal column (via the arms and shoulders), and equals the weight of the object being lifted. **F** is the force applied by the erector spinae muscle, which attaches to the spine about 1/3 of the way from the top of the column. Assume this muscle acts at an angle of 12 ° to the spinal column. **R** is the force the pelvis exerts on the spinal column. The weight of the trunk is neglected. Assume the spinal column makes an angle θ with the horizontal.



(a) Determine **R** and **F** in terms of **W** and θ.
(b) The spinal column may be injured if **R** is too large. Compare **R** when θ is 0 and 90 °. This problem explains why people say, "lift with your legs, not with your back."
(c) Compare the angle ϕ when θ is 0 and 90 °. If ϕ is not close to zero, there will be considerable transverse force at the discs in the lower back, which is not a good situation.

Section 1.8

Problem 11. Suppose that instead of using a cane, a person holds a suitcase of weight $W/4$ in one hand, 0.4 m from the midline. The person is standing on the opposite leg. Calculate the force exerted by the hip abductor muscles and by the acetabulum on that leg.

Section 1.10

Problem 12. Young's modulus for a spider's thread is about 0.2×10^{10} Pa, and the thread breaks when it undergoes a strain of about 50 % (Köhler and Vollrath 1995).
(a) Calculate the tensile strength of the thread and compare it to the tensile strength of steel.
(b) Calculate the strain that steel undergoes when it breaks. (Assume that a linear relationship between stress and strain holds until it breaks.) Compare the breaking strain to the spider's thread.

Problem 13. Assume an object undergoes a normal strain in all three directions: $\epsilon_x = \Delta x/l_x$, $\epsilon_y = \Delta y/l_y$, and $\epsilon_z = \Delta z/l_z$. Relate the three strains to the change in volume of the object. Assume the strains are small.

Section 1.11

Problem 14. Relate the shear strain to angle θ in Fig. 1.22. How does this relationship simplify if θ is small?

Section 1.12

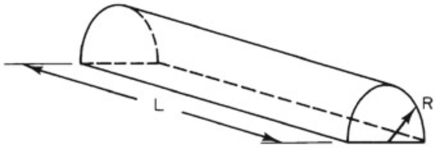
Problem 15. The inspirational pressure difference p_{in} that the lung can generate is about 86 torr. What would be the absolute maximum depth at which a person could breathe through a snorkel device? (A safe depth is only about half this maximum, since the lung ventilation becomes very small at the maximum depth. Assume the lungs are 30 cm below the mouth.)

Problem 16. A person standing erect can in some cases be modeled by a column of water.
(a) Estimate the hydrostatic pressure difference between a person's head and foot in torr.
(b) Explain why blood pressure is measured in the arm at the same vertical height as the heart.
(c) Our body has adapted to having a larger hydrostatic pressure in our feet than in our head. Speculate on why you feel uncomfortable when you "stand on your head."

Problem 17. A medication dissolved in a saline solution is infused into a vein in the patient's arm (IV infusion). The density of saline is the same as water. The pressure of the blood inside the vein is 5 torr above atmospheric pressure. How high above the insertion point must the container be hung so that there is sufficient hydrostatic pressure to force fluid into the vein?

Problem 18. The walls of a cylindrical pipe that has an excess pressure p inside are subject to a tension force per unit length T. (Consider only the force per unit length in the walls of the cylinder, not the force in any end caps of the pipe.) The force per unit length in the walls can be calculated by considering a different pipe made up of two parts as shown in the figure: a semicircular half-cylinder of radius R and length L attached to a flat plate of width $2R$ and length L. What is the force that the excess pressure exerts on the flat plate? Show that the tension force per unit length in the wall of the tube is $T = pR$. This is called the *Law of Laplace*. (Do not worry about any deformation.)

See if you can obtain the same answer by direct integration of the horizontal and vertical components of the force due to the excess pressure.



Problem 19. Find a relationship among the tension per unit length T across any element of the wall of a spherical soap bubble, the excess pressure inside the bubble, p, and the radius of the bubble, R. (Hint: Use the same technique as for the previous problem.)

Problem 20. The Law of Laplace, $T = pR$, relates the tension in an arterial wall, T, to the pressure p inside the artery and its radius R. Assume the wall obeys *Hooke's law*, $T = k(R - R_0)$, where R_0 is the radius of the artery when $p = 0$ and k is a measure of the wall stiffness.

(a) Derive an expression for R as a function of p. Sketch plots of r versus p and T versus p.

(b) Determine the critical pressure at which R goes to infinity. Physically, this is the pressure that guarantees a burst artery (an *aneurysm*).

(c) Arteries would be unstable if they were to balloon out and burst as the pressure approaches a critical value. They avoid this problem by becoming more stiff as the radius increases. Repeat part (a) using $k = cR$ for the stiffness. In this case is there a critical pressure at which the artery will burst?

The law of Laplace has many applications in biology and medicine (Basford 2002). For a discussion of how arteries become stiffer as R grows, see Vogel (1992).

Section 1.13

Problem 21. Suppose a fish has an average density of 1030 $\mathrm{kg\,m^{-3}}$, compared to the density of the surrounding water, 1000 $\mathrm{kg\,m^{-3}}$. One way the fish can keep from slowly sinking is by using an *air bladder* (the density of air is 1.2 $\mathrm{kg\,m^{-3}}$). What fraction of the fish's total volume must be air in order for the fish to be neutrally buoyant (the buoyant force is equal and opposite to the weight). Assume that the volume V of the fish's tissue is fixed, so in order to increase the volume U of the air bladder, the total volume of the fish $V + U$ must increase.

Problem 22. This problem explores the physics of a *centrifuge*. A cylinder of fluid of density ρ_{fluid} and length L is rotated at an angular velocity ω ($\mathrm{rad\,s^{-1}}$) in a horizontal plane about a vertical axis through one end of the tube. Neglect gravity. An object moving in a circle with constant angular velocity has a *centripetal acceleration* $a = -r\omega^2$ toward the center of the circle. Find the pressure in the fluid as a function of distance from the axis of rotation, assuming the pressure is p_0 at $r = 0$.

Problem 23. Buoyancy plays an important role in the centrifuge. Consider a small cubic particle of density ρ immersed in a fluid of density ρ_{fluid}.

(a) Write Newton's second law for the particle, considering only the centripetal acceleration and the pressure exerted by the fluid (Problem 22). Find an expression for the "effective weight" of the particle (analogous to Eq. 1.31) in terms of ρ, ρ_{fluid}, ω, r, and the particle volume V. Your result is more general than you might expect: it is true for a particle of any shape (Wick and Tooby 1977).

(b) Find the ratio of the "effective weight" derived in (a) to the "effective weight" due to gravity (Eq. 1.31).

(c) If the particle is 10 cm from the axis of a centrifuge spinning at 40,000 revolutions per minute, evaluate the ratio obtained in (b).

(d) The *density gradient* technique uses a sucrose solution of varying concentration to produce a fluid density that varies with r, $\rho_{\text{fluid}}(r)$. Explain how in this case the centrifuge can be used to separate particles of different densities.

Problem 24. For the centrifuge of Problem 23, assume there is one additional force: a viscous force proportional to the speed u of the particle relative to the fluid.

(a) Derive an expression for u, the *sedimentation velocity*, assuming the particle is not accelerating relative to the fluid.

(b) The sedimentation velocity per unit acceleration, S, is a parameter commonly used in centrifuge work. Divide the expression obtained in (a) by the centripetal acceleration to obtain an expression for S. The common unit for S is the svedberg ($1\ \mathrm{Sv} = 10^{-13}\ \mathrm{s}$).

(c) Consider two particles with $S = 50$ and 70 Sv. For the centrifuge of Problem 23(c), how long will it take for the particles to separate by 3 mm if they were initially at the same position? How long would this separation take if gravity were used instead of a centrifuge?

Section 1.14

Problem 25. What is the compressibility of a gas for which $pV = $const.? Compare the compressibility of water to that of air at atmospheric pressure. What are the implications of this for the volume of the lungs of a swimmer diving deep below the water surface?

Problem 26. Figure 1.20, showing a rod subject to a force along its length, is a simplification. Actually, the cross-sectional area of the rod shrinks as the rod lengthens. Let the axial strain and stress be along the z axis. They are related by Eq. 1.25, $s_z = E\epsilon_z$. The lateral strains ϵ_x and ϵ_y are related to s_z by $s_z = -(E/\nu)\epsilon_x = -(E/\nu)\epsilon_y$, where ν is called the *Poisson's ratio* of the material.

(a) Use the result of Problem 13 to relate E and ν to the fractional change in volume $\Delta V/V$.

(b) The change in volume caused by hydrostatic pressure is the sum of the volume changes caused by axial stresses in all three directions. Relate Poisson's ratio to the compressibility.

(c) What value of ν corresponds to an incompressible material?

(d) For an isotropic material, $-1 < \nu < 0.5$. How would a material with negative ν behave?

Elliott et al. (2002) measured Poisson's ratio for articular (joint) cartilage under tension and found $1 < \nu < 2$. This large value is possible because cartilage is *anisotropic*: its properties depend on direction.

Section 1.16

Problem 27. Consider the fluid flowing between two slabs as shown in Fig. 1.25. Since the work done by the external force on the system in time dt is $dW = Fv dt$, the rate of doing work is $P = dW/dt = Fv$, where v is the speed of the moving plate. Find the power dissipated per unit volume of the fluid in terms of the velocity gradient.

Problem 28. Consider a fluid that is flowing in the x direction, but with the velocity v_x changing in the y direction.

(a) Start with Newton's second law. Analyze the forces on a small cube of fluid and derive the equation

$$\rho\frac{\partial v_x}{\partial t} + \rho v_x\frac{\partial v_x}{\partial x} = -\frac{\partial p}{\partial x} + \eta\frac{\partial^2 v_x}{\partial y^2}.$$

This is a simplified version of the *Navier–Stokes equation* that governs fluid flow.

(b) Which term in the equation is nonlinear (that is, if p and v_x are doubled, which term does not double)? A nonlinear equation is needed to describe complicated flows such as turbulence.

Problem 29. Consider the simplified version of the Navier–Stokes equation in Problem 28. Assume the fluid speed is approximately V and all spatial changes occur over distances of order L. Take the ratio of the "inertial term" $\rho v_x(\partial v_x/\partial x)$ to the "viscous term" $\eta(\partial^2 v_x/\partial y^2)$ and show that you get the Reynolds number, Eq. 1.62.

Section 1.17

Problem 30. Consider laminar flow in a pipe of length Δx and radius R_p. Find the total viscous drag exerted by the pipe on the fluid.

Problem 31. The maximum flow rate from the heart is $500\ \mathrm{ml\,s^{-1}}$. If the aorta has a diameter of 2.5 cm and the flow is Poiseuille, what are the average velocity, the maximum velocity at the center of the vessel, and the pressure gradient along the vessel? Plot the velocity versus distance from the center of the vessel. As an approximation to the viscosity of blood, use $\eta = 10^{-3}\ \mathrm{kg\,m^{-1}\,s^{-1}}$.

Problem 32. The glomerular pore described in Eq. 1.41 has a flow $i = 7.2\times10^{-21}\ \mathrm{m^3\,s^{-1}}$. How many molecules of water per second flow through it? What is their average speed?

Problem 33. Organisms may use shear stress to determine the appropriate size of vessels for fluid transport (LaBarbera 1990). Consider a parent vessel of radius R_p that branches into two daughter vessels of radii R_{d1} and R_{d2}.

(a) Find a relationship between the radii R_p, R_{d1}, and R_{d2} such that the shear stress on the vessel wall is the same in each vessel. (Hint: Use conservation of the volume flow.) This relationship is called *Murray's Law*.

(b) If a 100-µm parent vessel branches into two identical daughter vessels, what is the radius of each daughter vessel? What is the cross-sectional area of the parent vessel, and what is the sum of the cross-sectional areas of the daughter vessels?

(c) If the two daughter vessels branch into subsequent generations of even smaller vessels, all obeying Murray's law, and the daughter vessels of any generation are all the same size, then find a relationship between the number of vessels in the nth generation, the radius of the single parent vessel, and the radii of the nth generation's daughter vessels.

(d) We have one aorta of radius 10 mm. Use Murray's law to estimate how many capillaries we have, each of radius 5 µm. Calculate the cross-sectional area of the aorta and the sum of the cross-sectional area of all our capillaries. Warning: Murray's law is a good approximation, but may not be exact for our smallest vessels.

Problem 34. Sap flows up a tree at a speed of about $1\ \mathrm{mm\,s^{-1}}$ through its vascular system (*xylem*), which consists of cylindrical pores of 20-µm radius. Assume the viscosity of sap is the same as the viscosity of water. What pressure difference between the bottom and top of a 100-m tall tree is needed to generate this flow? How does it compare to the hydrostatic pressure difference caused by gravity?

Problem 35.

(a) Consider a small cube of incompressible fluid. Analyze the volume fluence rate for each face of the cube and show that the divergence of **v** is zero. (The divergence is defined in Chap. 4.)

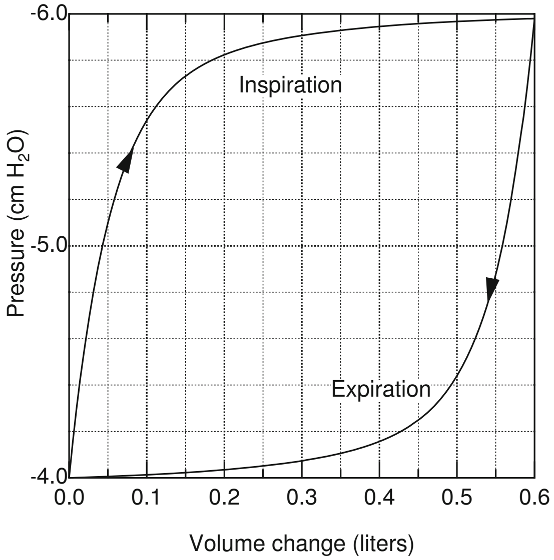
(b) Use the velocity distribution given in Problem 46 and the material in Appendix L to show that for this flow the fluid is incompressible.

Problem 36. Consider Eq. 1.54 when viscosity is negligible and the flow is steady ($\partial v/\partial t = 0$). Show that it reduces to the *Bernoulli equation*

$$p_1 + \rho\frac{v_1^2}{2} + \rho g z_1 = p_2 + \rho\frac{v_2^2}{2} + \rho g z_2.$$

Section 1.18

Problem 37. The accompanying figure shows the negative pressure (below atmospheric) that must be maintained in the thorax during the respiratory cycle by a patient with airway obstruction in order to breathe. Viscous effects are included. Estimate the work in joules done by the body during a breath.



Section 1.19

Problem 38. The volume of blood in a typical person is 5 l, and the volume current through the aorta is about $5\ \mathrm{l\,min^{-1}}$.

(a) What is the total volume current through all the systemic capillaries?

(b) What is the total volume current through all the pulmonary capillaries?

(c) How long does the blood take to make one complete circuit through the circulatory system?

Problem 39. Find the conversion factor between PRU and $\mathrm{Pa\,m^{-3}\,s}$. The total resistance of the systemic circulation was calculated in the text to be $1.66 \times 10^8\ \mathrm{Pa\,m^{-3}\,s}$. Express this in PRU.

Problem 40. Equation 1.59 relates the resistance of a vessel to its radius. In the circulatory system, the resistance of an arteriole increases when the smooth muscle surrounding the arteriole contracts, thereby decreasing its radius. By what factor does the resistance increase if the radius decreases by 10 %?

Problem 41. Derive the equations for resistance in a collection of vessels in series and in parallel. Remember that when several vessels are in series, the current is constant and the total pressure change is the sum of the pressure changes along the length of each vessel. When vessels are in parallel, each has the same pressure drop, but the current before the vessels branch is the sum of the currents in each branch.

Problem 42. The velocity of the blood in the aorta is about $0.5\ \mathrm{m\,s^{-1}}$, and the velocity of the blood in a capillary is about $0.001\ \mathrm{m\,s^{-1}}$. We have only one aorta, with a diameter of 20 mm, but many capillaries in parallel, each with a diameter of 8 µm. Estimate how many capillaries are typically open at any one time.

Problem 43. Suppose a student asked you, "How can blood be moving more slowly in a capillary than in the aorta? For an incompressible fluid, when the cross-sectional area along a pipe decreases, the velocity increases, so that the volume current i is the same. The capillary has a much smaller cross-sectional area than the aorta. Therefore, the blood should move faster in the capillary than in the aorta!" How would you respond to this student?

Problem 44. For Poiseuille flow, find an expression for the maximum shear rate in each vessel from Eq. 1.45. Where in the vessel does it occur? Typical maximum shear rates are $50\ \mathrm{s^{-1}}$ in the aorta, $150\ \mathrm{s^{-1}}$ in the femoral artery, and $400\ \mathrm{s^{-1}}$ in an arteriole.

Problem 45. A sphere of radius a moving through a fluid with speed v is subject to a viscous drag $F_{\mathrm{drag}} = 6\pi\eta a v$. Make an argument similar to that in the text to show that the ratio of kinetic energy of a sphere of fluid of the same size moving at the same speed to the viscous work done to displace the sphere by its own diameter is $N_R/18$.

Problem 46. Consider a stationary sphere of radius a placed in a fluid of viscosity η moving uniformly with speed V. For low Reynolds number flow, the radial and tangential components of the fluid velocity and the pressure surrounding the sphere are

$$v_r = V\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

$$v_\theta = -V\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$

$$p = -\eta V\cos\theta\frac{3a}{2r^2}.$$

(a) Show that the no-slip boundary condition is satisfied.
(b) Integrate the shear force and the pressure force over the sphere surface and find an expression for the net drag force on the sphere (Stoke's law). What fraction of this force arises from pressure drag, and what fraction from viscous drag?

Problem 47. Find an expression for the entry length in terms of the tube size, the pressure gradient, and the properties of the fluid. Estimate the length of the entry region in the aorta, in an artery, and in an arteriole of radius 20 µm. Use $\eta = 10^{-3}\,\mathrm{kg\,m^{-1}\,s^{-1}}$.

Problem 48. Estimate the tension per unit length and the stress in the walls of various blood vessels using the data in Table 1.4.

Problem 49. Compare the magnitude of the four terms in Eq. 1.42 in the following two cases. Ignore branching. Assume the vessels are vertical. Use $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ and $\eta = 10^{-3}\,\mathrm{Pa\,s}$.
(a) The descending aorta. Assume the length is 35 cm, the radius is 1 cm (independent of distance along the aorta), the peak acceleration of the blood is $1800\,\mathrm{cm\,s^{-2}}$, and the peak velocity (during the cardiac cycle) is $70\,\mathrm{cm\,s^{-1}}$ at the entrance and $60\,\mathrm{cm\,s^{-1}}$ at the exit. (These velocities are different because some of the blood leaves the aorta in major arteries.)
(b) An arteriole of radius 50 µm, length 10 mm, and constant velocity of $5\,\mathrm{mm\,s^{-1}}$ at both entrance and exit.

Problem 50. The viscosity of water (and therefore of blood) is a rapidly decreasing function of temperature. Water at 5 °C is twice as viscous as water at 35 °C. Speculate on the implications of this extreme temperature dependence for the circulatory system of cold-blooded animals. (For a further discussion, see Vogel 1994, pp. 27–31.)

Section 1.20

Problem 51. Estimate the Reynolds number for the following flows. In each case, determine whether the Reynolds number is high ($\gg 1$) or low ($\ll 1$).
(a) *E. coli* (length 2 µ) swim in water at speeds of about $0.01\,\mathrm{mm\,s^{-1}}$.
(b) An Olympic swimmer (length 2 m) swims in water at speeds of up to $2\,\mathrm{m\,s^{-1}}$.
(c) A bald eagle (wingspan 2 m) flies in air (density = $1.2\,\mathrm{kg\,m^{-3}}$, viscosity = $1.8 \times 10^{-5}\,\mathrm{Pa\,s}$) at speeds of $20\,\mathrm{km\,hr^{-1}}$.

Problem 52. Estimate the Reynolds number of blood flow in a capillary, using the data in Table 1.4. How does this compare to that in the aorta?

Problem 53. Consider a sphere of radius R moving at speed v through a fluid of density ρ and viscosity η.
(a) If the Reynolds number is low, then viscous effects dominate and the drag force F_{visc} depends on η and not ρ. Assume that F_{visc} depends only on R, η, and v and use dimensional analysis[13] to determine the form of Stokes' law (i.e., the power to which each variable is raised).
(b) At high Reynolds number, the force needed to accelerate the fluid out of the way is important, and the drag force F_{pres} depends on ρ and not η. Find the dependence of F_{pres} on the relevant variables.
(c) Find, to within a dimensionless factor, the critical speed at which $F_{\mathrm{visc}} = F_{\mathrm{pres}}$.

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[13] For a detailed discussion of dimensional analysis, see Jensen (2013).

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