13 Sound and Ultrasound

Sound (or *acoustics*) plays two important roles in our study of physics in medicine and biology. First, animals hear sound and thereby sense what is happening in their environment. Second, physicians use high-frequency sound waves (*ultrasound*) to image structures inside the body. This chapter provides a brief introduction to the physics of sound and the medical uses of ultrasound for imaging and therapy. A classic textbook by Morse and Ingard (1968) provides a more thorough coverage of theoretical acoustics, and books such as Hendee and Ritenour (2002) describe the medical uses of ultrasound in more detail.

In Sect. 13.1, we derive the fundamental equation governing the propagation of sound: the wave equation. Section 13.2 discusses some properties of the wave equation, including the relationship between frequency, wavelength, and the speed of sound. The acoustic impedance and its relevance to the reflection of sound waves are introduced in Sect. 13.3. Section 13.4 describes the intensity of a sound wave and introduces the decibel intensity scale. The ear and hearing are described in Sect. 13.5. Section 13.6 discusses the attenuation of sound waves. Physicians use ultrasound imaging for medical diagnosis, as described in Sect. 13.7.

13.1 The Wave Equation

In Chap. 1, we assumed that solids and liquids are incompressible. If a long rod was truly incompressible, a displacement of one end would instantly result in an identical displacement of the other end. In fact, the displacement does not propagate instantaneously. It travels at the speed of sound in the rod.

The propagation of sound involves small displacements of each volume element of the medium from its equilibrium position. In this section, we consider sound waves propagating along the x-axis. The results can be generalized to three dimensions (see Morse and Ingard 1968). We first consider an elastic rod, then a fluid in which viscous effects are not important, and finally, shear waves.

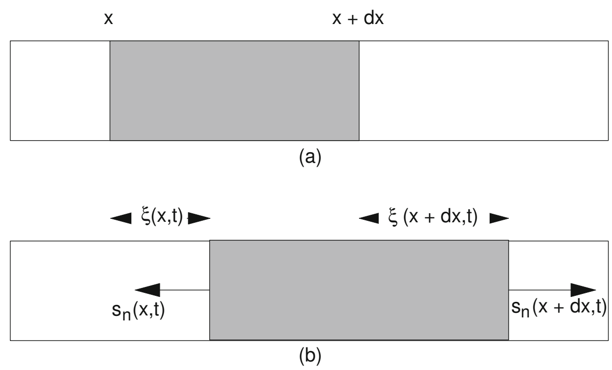

Fig. 13.1 An elastic rod. **a** The rod in its equilibrium position. **b** Each point on the rod has been displaced from its equilibrium position by an amount ξ which depends on x and t. As a result there is a normal stress s_n which also depends on x and t

13.1.1 Plane Waves in an Elastic Rod

The simplest case to consider is an elastic rod which is forced to move longitudinally at one end. This results in the propagation of a sound wave along the rod. We set up a coordinate system where x measures distance along the rod from a fixed origin when no sound wave is traveling along the rod. We also assume that the disturbance of the rod depends only on the position along the rod, x, and not on y or z, which are perpendicular to x. A wave in three dimensions that depends only on one dimension is called a *plane wave*.

When the sound travels along the rod, the material at point x is displaced from its undisturbed position by a small amount $\xi(x,t)$, as shown in Fig. 13.1. The material originally at $x + dx$ is displaced by amount $\xi(x + dx, t)$. Since

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$\xi(x+dx,t)$ is in general different from $\xi(x,t)$, there is a strain in the rod (Eq. 1.24)

$$\epsilon_n(x,t) = \frac{\Delta l}{l} = \frac{\xi(x+dx,t) - \xi(x,t)}{dx} = \frac{\partial \xi}{\partial x}. \quad (13.1)$$

Young's modulus, E, relates the stress in the rod, s_n, to the strain, ϵ_n (Eq. 1.25):

$$s_n(x,t) = E\epsilon_n(x,t) = E\frac{\partial \xi}{\partial x}. \quad (13.2)$$

The difference between the stress at each end, multiplied by the cross-sectional area of the rod, S, provides a net force that accelerates the shaded volume element in Fig. 13.1. The net force on the volume element is

$$F_{\text{net}} = S\left[s_n(x+dx,t) - s_n(x,t)\right] = S\frac{\partial s_n}{\partial x}dx = SE\frac{\partial \epsilon_n}{\partial x}dx,$$

$$F_{\text{net}} = SE\frac{\partial^2 \xi}{\partial x^2}dx. \quad (13.3)$$

The mass of the shaded volume is $\rho S dx$, where ρ is the density, and the acceleration of the volume is $\partial^2\xi/\partial t^2$. [Since we are not subtracting a value at one end from the value at the other, and since we are taking the limit as $dx \to 0$, we can ignore changes in ξ in the interval $(x, x+dx)$]. Therefore, Newton's second law becomes

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 \xi}{\partial t^2}. \quad (13.4)$$

This is the *wave equation*, and it is seen in many contexts, from the vibrations of a string to the propagation of electromagnetic waves. It is usually written as

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}, \quad (13.5)$$

where c is the speed of propagation of sound in the rod. In this case

$$c = \sqrt{\frac{E}{\rho}}. \quad (13.6)$$

As Young's modulus becomes very large or the density of the rod becomes very small, the speed with which a disturbance travels from one end of the rod to the other becomes larger and larger.

13.1.2 Plane Waves in a Fluid

Now we consider a sound wave propagating in a fluid, where shear can be neglected. We also neglect viscous effects. Changes in the fluid caused by the sound wave depend only on x and t.[1] To make it easier to imagine the situation, suppose the fluid is confined in a tube. Then, we can construct a figure very similar to Fig. 13.1. A small volume of fluid at rest extends from position x to $x+dx$, with cross-sectional area S as shown in Fig. 13.2a. The force pushing on the left side of the volume is SP_0, and the force on the right is $-SP_0$.[2] Here, P_0 is the pressure when the fluid is undisturbed by a sound wave. In equilibrium, there is no net force on the volume element.

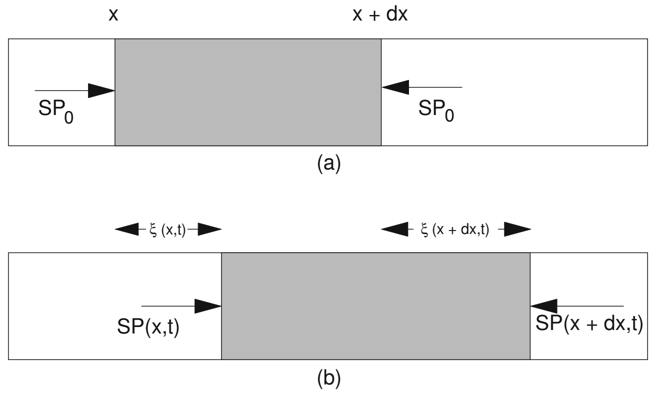

Fig. 13.2 Sound propagates in one dimension in a fluid in a tube of cross-sectional area S. **a** In equilibriuim the pressure is p_0 and the force on the shaded volume of fluid has magnitude $p_0 S$ on each end. **b** When the sound is propagating, the forces on each end are as shown

When the fluid element is displaced, as in Fig. 13.2b, the net force to the right on the fluid element is

$$F_{\text{net}} = S\left[P(x,t) - P(x+dx,t)\right] = -S\frac{\partial P}{\partial x}dx. \quad (13.7)$$

The change of pressure from the equilibrium value P_0 is related to the change of volume of the fluid by the compressibility, κ (Eq. 1.32):

$$P - P_0 = p = -\frac{1}{\kappa}\frac{dV}{V_0} = -\frac{1}{\kappa}\frac{d\xi}{dx}, \quad (13.8)$$

from which

$$F_{\text{net}} = \frac{S}{\kappa}\frac{\partial^2 \xi}{\partial x^2}dx. \quad (13.9)$$

To obtain the mass, we use the volume Sdx times the equilibrium density ρ_0. We multiply by the acceleration of the fluid element, $\partial^2\xi/\partial t^2$, to obtain

$$\frac{\partial^2 \xi}{\partial x^2} = \rho_0 \kappa \frac{\partial^2 \xi}{\partial t^2}. \quad (13.10)$$

[1] We might be looking at a wave whose properties depend on all three coordinates, x, y, and z, but where, in the region we are studying, the dependence on y and z is very slight. This is like the 1-D electrostatic approximations in Chap. 6.

[2] See Sect. 1.12; we ignore any forces arising from viscosity, gravity, or surface tension.

This is the wave equation, Eq. 13.5, with

$$c = \sqrt{\frac{1}{\kappa \rho_0}}. \tag{13.11}$$

In both of these cases, the wave equation has been written in terms of the displacement of elements of the rod or the fluid from their equilibrium positions. It is also possible to show that the pressure, fluid density, and velocity of the fluid element also satisfy the wave equation. The pressure is discussed in Problem 2. The velocity of the fluid due to the sound wave is

$$v = \frac{d\xi}{dt}. \tag{13.12}$$

Another important relationship is obtained by combining Eq. 13.12 with Eq. 13.8 and interchanging the order of differentiation (Appendix N):

$$\frac{\partial v}{\partial x} = -\kappa \frac{\partial p}{\partial t}. \tag{13.13}$$

Equation 13.8 and 13.10 can also be used to show that

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}. \tag{13.14}$$

Finally, since the density is $\rho = M/V$, we can show that

$$\frac{d\rho}{\rho_0} = \kappa dp. \tag{13.15}$$

In this section, we have considered Young's modulus E and compressibility κ. Remember from Chap. 3 that we can compress a gas at a constant temperature, and we can also do it adiabatically, in which case there is no heat flow and the temperature rises as the gas is compressed. The compressibility is different in these two cases. When static measurements of these parameters are made, there is usually time for the system being studied to remain isothermal. The pressure changes in a sound wave usually occur so rapidly that there is not time for heat to flow, and the adiabatic compressibility must be used. Values of Young's modulus are also different for isothermal and adiabatic stresses and strains.

13.1.3 Shear Waves

Sound in a fluid is a *longitudinal wave*, which means that the fluid moves in the same direction that the wave propagates. A fluid cannot support a shear stress, but shear stresses can exist in tissue, which results in another type of acoustic wave, called a *transverse wave* or *shear wave*, where the tissue moves perpendicular to the direction the wave propagates. Consider the tissue in Fig. 13.3. When a shear wave travels through the tissue, the material at point x is displaced by a small amount $\eta(x,t)$ in the transverse direction. The shear strain is (Eq. 1.27)

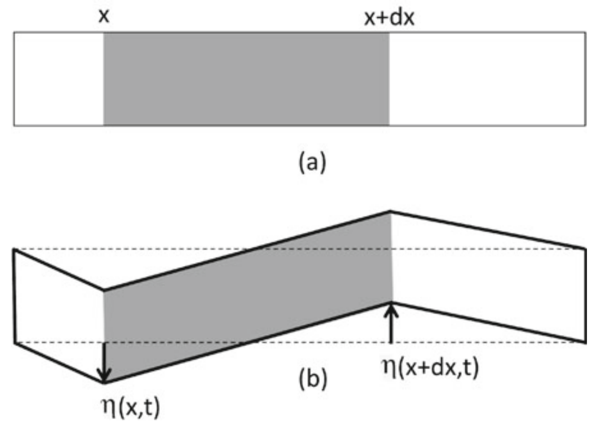

Fig. 13.3 A shear wave in a rod viewed from the top. The deflection is in the plane of the paper

$$\epsilon_s = \frac{\eta(x+dx,t) - \eta(x,t)}{dx} = \frac{\partial \eta}{\partial x},$$

and the shear modulus, G, relates the stress and strain (Eq. 1.28)

$$s_s = G\epsilon_s = G\frac{\partial \eta}{\partial x}.$$

The difference between the stress at each end of the shaded volume in Fig. 13.3, multiplied by the cross-sectional area S, provides the net force

$$F_{net} = S[s_s(x+dx,t) - s_s(x,t)] = S\frac{\partial s_s}{\partial x}dx = SG\frac{\partial^2 \eta}{\partial x^2}dx.$$

The mass of the shaded volume is $\rho S dx$, and the acceleration of the volume is $\partial^2\eta/\partial t^2$. Newton's second law becomes

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\rho}{G}\frac{\partial^2 \eta}{\partial t^2}$$

so the speed of the shear wave is

$$c_{\text{shear}} = \sqrt{\frac{G}{\rho}}.$$

Shear moduli in soft tissue are in the order of G = 4 kPa, implying that the shear wave speed is about 2 m s^{-1}, compared to 1500 m s^{-1} for longitudinal acoustic waves.

13.2 Properties of the Wave Equation

The parameter c in the wave equation has units of speed. To appreciate its physical interpretation, consider the departure from the undisturbed pressure $p(x,t) = P(x,t) - P_0 =$

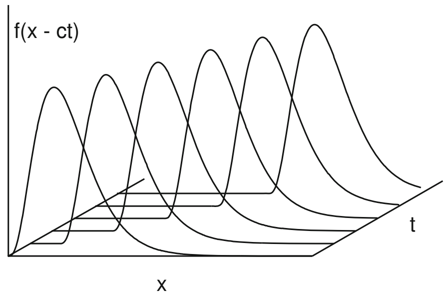

Fig. 13.4 A wave $f(x - ct)$ traveling to the right with speed c

$f(x - ct)$, where f is any function. This solution obeys the wave equation (see Problem 5). It is called a *traveling wave*. A point on $f(x - ct)$, for instance its maximum value, corresponds to a particular value of the argument $x - ct$. To travel with the maximum value of $f(x - ct)$, as t increases, x must also increase in such a way as to keep $x - ct$ constant. This means that the pressure distribution propagates to the right with speed c, as shown in Fig. 13.4. Solutions $p(x, t) = g(x + ct)$, where g is any function, also are solutions to the wave equation, corresponding to a wave propagating to the left

The wave speed c is one of the most important parameters governing the propagation of sound waves. The density of water is about $\rho_0 = 1000$ kg m^{-3} and the compressibility of water is approximately 5×10^{-10} Pa^{-1}, so the speed of sound in water is about 1400 m s^{-1}. The speed of sound in tissues is slightly higher: 1540 m s^{-1} is often taken as an average speed of sound in soft tissue. The speed of sound in air is about 344 m s^{-1}. See Denny (1993) for a more detailed comparison of the speed of sound in air and water.

One very useful traveling wave is $p(x, t) = p_0 \sin\left[\frac{2\pi}{\lambda}(x - ct)\right] = p_0 \sin\left[2\pi(\frac{x}{\lambda} - \frac{t}{T})\right] = p_0 \sin(kx - \omega t)$. The pressure distribution oscillates sinusoidally with frequency

$$f = c/\lambda \tag{13.16}$$

cycles per second (Hz) or angular frequency $\omega = 2\pi f$ (radians) s^{-1}. Equation 13.16 relates the frequency and wavelength. For instance, middle C has a frequency of 261.63 Hz. In air, the wavelength is (344 m s^{-1})/(261.63 Hz) = 1.315 m. The *wave number* is

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}. \tag{13.17}$$

Standing waves such as

$$p(x, t) = p \cos(\omega t) \sin(kx) \tag{13.18}$$

are also solutions to the wave equation. An example is shown in Fig. 13.5. The standing wave in Eq. 13.18 has nodes fixed in space where $\sin(kx)$ is zero. Standing waves can occur, for example, in an organ pipe and in the ear canal (Problem 7).

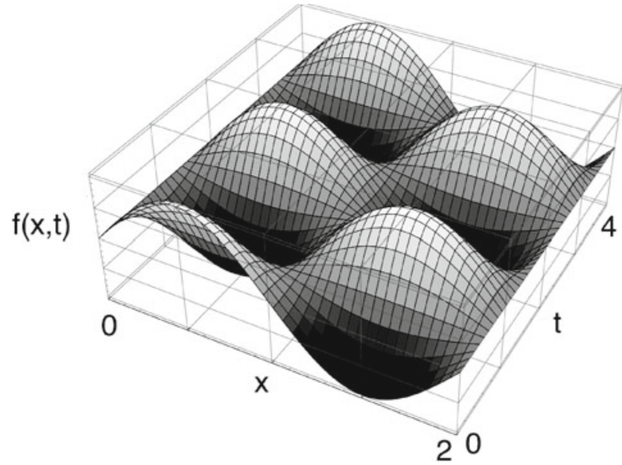

Fig. 13.5 A standing wave $f(x, t) = \sin \pi x \cos \pi t$, plotted for $0 < x < 2$ and $0 < t < 4$

A standing wave can also be written as the sum of two sinusoidal traveling waves, one to the left and one to the right. Conversely, two standing waves can be combined to give a traveling wave (Problem 8).

Since the fluid velocity v obeys the wave equation, it can also be represented as a sinusoidal wave. It is important to realize that the fluid oscillates back and forth. The fluid itself does not propagate with the wave. What propagates is the disturbance in the fluid. Sound in a fluid is a *longitudinal wave*, which means that the fluid oscillates along the same axis that the disturbance propagates (in this case, both move in the x direction). Other types of waves exist in nature, such as electromagnetic waves studied in Chap. 14. Electromagnetic waves are *transverse waves*, because the electric field oscillates in a direction perpendicular to the direction of wave propagation. Fluids cannot support significant shear stresses and only propagate longitudinal waves.

13.3 Acoustic Impedance

13.3.1 Relationships Between Pressure, Displacement and Velocity in a Plane Wave

For a plane wave traveling to the right, the pressure, displacement, and speed of the fluid have simple relationships. If the pressure change is

$$p(x, t) = p_0 \sin(kx - \omega t), \tag{13.19}$$

one can use Eqs. 13.8 and 13.12 to show that the fluid displacement is

$$\xi = \xi_0 \cos(kx - \omega t), \tag{13.20}$$

the fluid velocity is

$$v = v_0 \sin(kx - \omega t), \tag{13.21}$$

and the amplitudes are related by

$$\xi_0 = p_0 \frac{\kappa}{k} = p_0 \frac{\kappa \lambda}{2\pi} = p_0 \frac{\kappa c}{\omega}, \tag{13.22}$$

$$v_0 = \frac{p_0}{\rho_0 c} = \frac{p_0}{Z}. \tag{13.23}$$

The quantity $Z = \rho_0 c = \sqrt{\rho_0/\kappa}$ is called the *acoustic impedance* of the medium.[3] The acoustic impedance of water is about $(10^3 \text{ kg m}^{-3})(1400 \text{ m s}^{-1}) = 1.4 \times 10^6$ Pa s m^{-1}. The acoustic impedance of air is about 400 Pa s m^{-1}, so $Z_{\text{air}} \ll Z_{\text{water}}$ (Denny 1993).

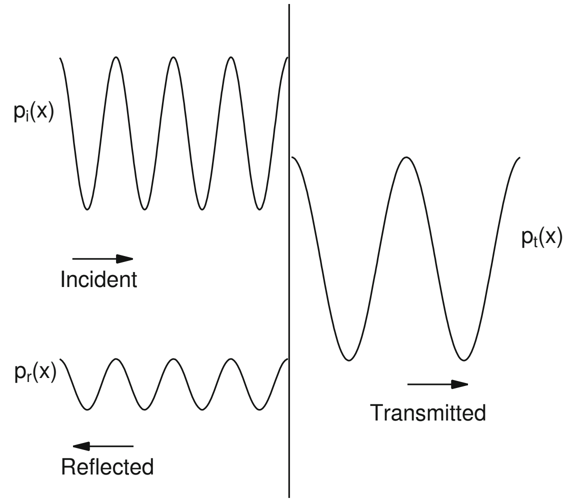

Fig. 13.6 A sound wave with pressure amplitude p_i traveling to the right is incident on a boundary separating tissue 1 on the left from tissue 2 on the right. Each tissue has a different density ρ_0 and compressibility κ. $Z_2 = 2Z_1$. Part of the wave is transmitted to the right with amplitude p_t, and part is reflected to the left with amplitude p_r. The drawing shows one instant in time

13.3.2 Reflection and Transmission of Sound at a Boundary

Consider next what happens at the boundary between two different media. Suppose a traveling wave is propagating to the right in a fluid with sound speed c_1 and acoustic impedance Z_1. At $x = 0$, it encounters a second fluid, with speed c_2 and impedance Z_2. In general, the interaction of the incoming wave with the boundary between the first and second fluids results in a reflected wave traveling to the left in fluid 1 and a transmitted (or refracted) wave traveling to the right in fluid 2 (Fig. 13.6). The acoustic impedances determine how much of the incoming wave is reflected and how much is transmitted. The waves must oscillate with the same frequency in both media. The pressure at the boundary must be the same in each medium, and the fluid velocity must also be continuous across the boundary. Let $p_i(x,t) = p_i \sin\left[\frac{\omega}{c_1}(x - c_1 t)\right]$, $p_r(x,t) = p_r \sin\left[\frac{\omega}{c_1}(x + c_1 t)\right]$, and $p_t(x,t) = p_t \sin\left[\frac{\omega}{c_2}(x - c_2 t)\right]$ be the incoming, reflected, and transmitted pressures. The velocities are related to the pressures by the acoustic impedances. At the boundary, the pressure and the velocity must be continuous. In fluid 1, the amplitude of the pressure is $p_i + p_r$, and in fluid 2, it is p_t. In fluid 1, the amplitude of the velocity is $(p_i - p_r)/Z_1$, and in fluid 2, it is p_t/Z_2. (The minus sign arises because the reflected wave is traveling to the left.) Therefore

$$p_i + p_r = p_t \tag{13.24}$$

and

$$(p_i - p_r)/Z_1 = p_t/Z_2. \tag{13.25}$$

We can solve these two equations for p_r and p_t in terms of p_i :

$$p_r = \frac{Z_2 - Z_1}{Z_2 + Z_1} p_i, \tag{13.26}$$

$$p_t = \frac{2Z_2}{Z_2 + Z_1} p_i. \tag{13.27}$$

The *intensity* I of a sound wave is a measure of the power per unit area (W m^{-2}). The instantaneous power per unit area transmitted by the wave in Eq. 13.19 at some point is

$$I(t) = p(t)v(t) = p_0 v_0 \sin^2 \omega t. \tag{13.28}$$

The average power per unit area is

$$I = \frac{1}{2} p_0 v_0 = \frac{1}{2} \frac{p_0^2}{Z}. \tag{13.29}$$

Problems 13–15 show that the *reflection* and *transmission coefficients* are

$$R = \frac{I_r}{I_i} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2, \tag{13.30}$$

[3] Strictly speaking, the acoustic impedance is the ratio $Z = p_0/v_0$, and carries information about both the amplitude ratio and the relative phase of the pressure and velocity. If the waves are in phase, Z is said to be *resistive*; if they are $\pi/2$ out of phase, Z is said to be *reactive*. The *characteristic acoustic impedance* is a property of the medium: $Z_0 = \rho_0 c$. Both have units Pa m s^{-1} or kg m^{-2} s^{-1}. For a plane wave, the impedance is resistive and $Z = Z_0$. For other waves, such as standing waves, there is a reactive component.

and

$$T = \frac{I_t}{I_i} = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2}, \tag{13.31}$$

and that $R + T = 1$.

If the acoustic impedance of the two fluids is the same, $Z_1 = Z_2$, there is no reflected wave and the entire incoming wave is transmitted. If $Z_1 \ll Z_2$ (e.g., sound going from air to water), almost all of the sound is reflected.

13.4 Comparing Intensities: Decibels

13.4.1 The Decibel

When comparing two intensities, the range of differences is often so great that a logarithmic comparison scale is used. We first saw the decibel when discussing the frequency response of a linear system in Chap. 11. Intensity levels in dB have meaning only in terms of ratios:

$$\text{Intensity Difference (dB)} = 10\,\log_{10}\left(\frac{I_2}{I_1}\right). \tag{13.32}$$

The intensity difference can also be written in terms of pressure (or displacement or velocity) ratios:

$$\begin{aligned}\text{Intensity Difference (dB)} &= 10\log_{10}\left(\frac{I_2}{I_1}\right)\\ &= 10\log_{10}\left(\frac{p_2}{p_1}\right)^2\\ &= 20\log_{10}\left(\frac{p_2}{p_1}\right).\end{aligned} \tag{13.33}$$

This assumes that p_1 and p_2 are measured in the same medium, so the acoustic impedance does not change. If the intensity of a wave falls to 1% of its original value, the intensity difference is $10\log_{10}(0.01) = -20$ dB.

13.4.2 Measuring Hearing Response

In auditory acoustics, intensities are measured with respect to a reference intensity $I_0 = 10^{-12}$ W m^{-2}. This is the intensity of the faintest sound that a person can typically hear:

$$\text{Intensity level} = 10\log_{10}\left(\frac{I}{I_0}\right). \tag{13.34}$$

A sound that is ten times as intense as the threshold for hearing has an intensity level of 10 dB. A sound with an average intensity $I = 1$ W m^{-2} is perceived as painful, so the threshold for pain has an intensity level of about 120 dB. Table 13.1 gives the intensity in decibels for some common sounds.

Table 13.1 Approximate intensity levels of various sounds

Sound	Intensity (W m^{-2})	Level (dB, A weighting)
Rocket launch pad	10^5	170
	10^4	160
	10^3	150
	10^2	140
F-84 jet at takeoff, 25 m from the tail; Large pneumatic riveting machine (1 m); Boiler shop (maximum level); Peak sound level at a rock concert	10	130
Sound that produces pain	1	120
Woodworking shop	10^{-1}	110
Near a pneumatic drill ("jack hammer")	10^{-2}	100
Inside a motor bus	10^{-3}	90
Urban dwelling near heavy traffic	10^{-4}	80
Busy street	10^{-5}	70
Speech at 1 m	10^{-6}	60
Office	10^{-7}	50
Average dwelling	10^{-8}	40
Maximum background sound level tolerable in a broadcast studio	10^{-9}	30
Whisper; maximum background sound level tolerable in a motion picture studio	10^{-10}	20
	10^{-11}	10
Minimum perceptible sound	10^{-12}	0

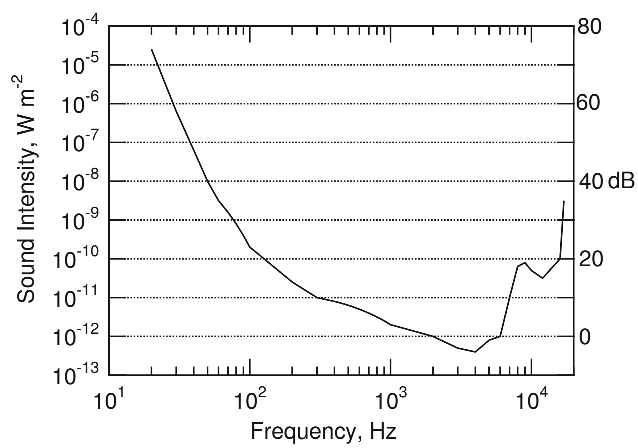

Fig. 13.7 Hearing response (MAF) curve for a young adult

The sensitivity of the ear depends on frequency. A typical hearing response curve for a young person is shown in Fig. 13.7. The *minimum auditory field* (MAF) is measured with a loudspeaker; the slightly different *minimum auditory pressure* (MAP) is measured with headphones. The ear is most sensitive to sounds between about 100 and 5000 Hz. A sound at 20 Hz will not be perceived to be as loud as one at 1000 Hz with the same intensity. Commercial sound level meters typically have two weightings. The "C" weighting has almost the same sensitivity at all frequencies. The "A" weighting more nearly mimics the response of the normal

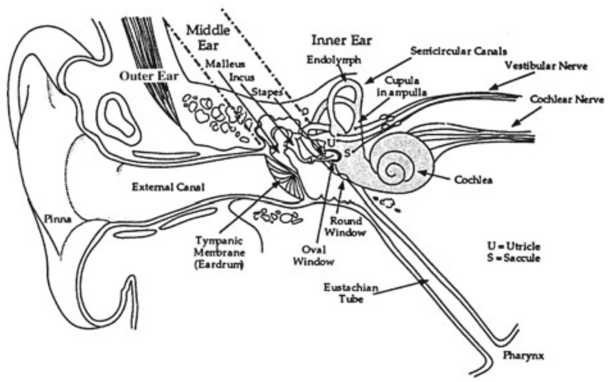

Fig. 13.8 A cross section of the ear. From Cameron et al. 1999. Used by permission

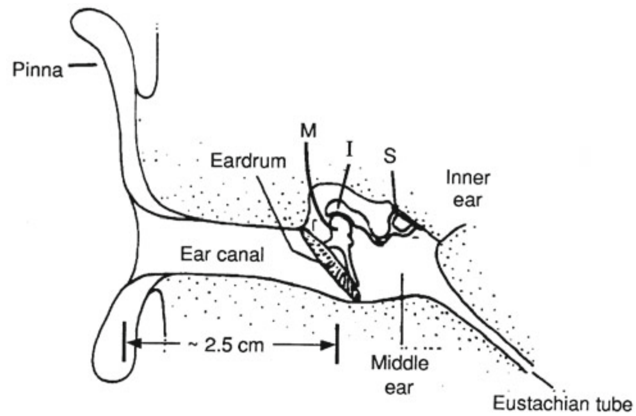

Fig. 13.9 Details of the middle ear. From Cameron et al. 1999. Used by permission

ear. Sounds with the same level when the meter is on "A" weighting will be perceived as having the same loudness.

13.5 The Ear and Hearing

A cross-section of the ear is shown in Fig. 13.8. The ear can be thought of as having three different sections, each with a unique purpose: the *external ear* gathers sound, the *middle ear* transfers energy from the air (low acoustic impedance) to the liquid of the inner ear (high acoustic impedance), and the *inner ear* transforms the signal into nerve impulses going to the brain.

The external ear consists of the *pinna*, the visible part of the ear, and an air-filled tube called the *ear canal*.

The middle ear is a small chamber filled with air that contains three small bones, or *ossicles* shown in Fig. 13.9. It is separated from the ear canal by the *ear drum*. The bone in contact with the ear drum is called the *malleus* (it is shaped a bit like a mallet or hammer). The next bone is the *incus* (from the Latin for an anvil, which it resembles slightly). The third, in contact with the *oval window* to the inner ear, is the *stapes* (again from the Latin, for stirrup.) The *eustachian tube* leads from the middle ear to the mouth and throat (*nasopharynx*). Since the ear is sensitive to very small pressure changes, the eustachian tube serves the important function of keeping the pressure on both sides of the ear drum the same for slow changes, such as when we climb stairs or the weather changes. The walls of the eustachian tube are often collapsed together. Swallowing helps to open them up and equalize the pressure if necessary.

Sound arrives at the ear as a vibration in air. Sound energy must enter the inner ear in order to be converted into a nerve signal to the brain. Yet, the inner ear is filled with liquid. The acoustic impedance of the liquid in the inner ear is about 3500 times larger than the acoustic impedance of air. This means that without the impedance transformation by the middle ear, the intensity in the inner ear would be only about $1/1000$ of the intensity in air—a loss of about 30 dB (Problem 14).

The middle ear transforms the impedance by two mechanisms. The first is a simple area change. The ear drum vibrates in response to the pressure changes in the sound wave. If a sound wave with pressure amplitude p_{air} impinges on the ear drum of area $S_{\text{ear drum}}$, the total excess force on the ear drum is $F = p_{\text{air}} S_{\text{ear drum}}$. For this simplest model, assume that the three bones behave like a single rigid rod and there are no effects of the boundary at the circumference of the ear drum. Then, the bones transmit this force to the membrane at the oval window, which has area $S_{\text{oval window}}$. The pressure induced in the liquid in the inner ear is then $p_{\text{inner ear}} = F/S_{\text{oval window}} = p_{\text{air}} S_{\text{ear drum}}/S_{\text{oval window}}$. The area of the ear drum is about $S_{\text{ear drum}} = 64\ \text{mm}^2$, while the area of the base of the stapes is $3.2\ \text{mm}^2$ (Newman 1957). Therefore, $p_{\text{inner ear}} = 20 p_{\text{air}}$. Actually, the ear drum and the membrane at the oval window are not connected by a simple rigid rod. The malleus, incus, and stapes are pivoted in such a way that they serve as a set of levers multiplying the force at the oval window by an additional factor of 1.3. Therefore, the total pressure amplification by the middle ear is 26. This corresponds to a 28 dB change in sound intensity, which almost compensates for the 30 dB loss going from air to the liquid of the inner ear. The bones of the middle ear have muscles that change their stiffness, so they can reduce the amount of pressure amplification to protect the inner ear from very loud, low-frequency noises.

The inner ear contains three *semicircular canals*, which help control our sense of balance, and the *cochlea*, which changes the sound to nerve impulses. All are filled with liquid. The cochlea is a small spiral about the size of the tip

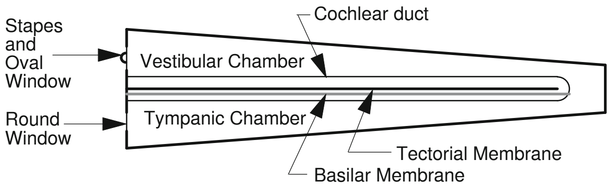

Fig. 13.10 A schematic representation of the cochlea

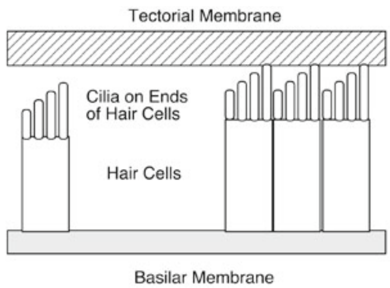

Fig. 13.11 A cross section of the cochlea. The hair cells are deformed as the basilar membrane moves relative to the tectorial membrane

of your little finger. Unwound, it is about 3 cm long. Figure 13.10 shows it schematically. There are three chambers. The *vestibular chamber* connects to the stapes in the middle ear through the oval window. At the other end of the cochlea the vestibular chamber connects to the *tympanic chamber*. The *round window* opens onto the middle ear and allows the pressure to be equalized at low frequencies. The third chamber is the *cochlear duct*.

When the stapes moves the oval window, it generates a sound wave that travels through the liquid in the cochlea. This produces a displacement of the *basilar membrane* in the third chamber, the cochlear duct. Two types of *hair cells* sit on the basilar membrane: one row of inner hair cells and three rows of outer hair cells. The hair cells in turn have very fine "hairs" on them, called *cilia*. The cilia of the outer hair cells touch another membrane, the *tectorial membrane*, but the cilia of the inner hair cells do not. A cross-section of this is shown schematically in Fig. 13.11. When the basilar and tectorial membranes are displaced by the sound wave, the cilia on the inner hair cells move in the liquid that fills the region between the membranes. It is just as if you submerged your head in a swimming pool and shook it back and forth. Your head would move in the water, but the motion of your hair would be altered as the water "dragged" it. As a result of this motion of the cilia in the liquid, the inner hair cells generate nerve impulses that then travel to the brain and provide our sensation of sound. The mechanism was discussed briefly in Sect. 9.9.

Figure 13.10 shows that the size of the cochlea changes along its length. So does its stiffness. Different locations along the cochlea therefore oscillate at different frequencies: points near the oval window (base or left side of Fig. 13.10) respond to high frequencies, while points near the apex (right side of Fig. 13.10) respond to low frequencies. The physics and neurophysiology of pitch perception and music is fascinating and complex (Sacks 2007; Hartmann 2013). For instance, some people have *absolute pitch*. They can identify a pitch (say, F-sharp) in isolation, just as normal people can identify a uniform color (say, yellow). Most people can only perceive relative pitch: one note has a higher pitch than another (for example, an E is a major third above a C).

The cochlear implant was mentioned in Chap. 7 as a way to use functional electrical stimulation to partially restore hearing. A row of electrodes is inserted along the cochlea to stimulate the nerves that are usually excited by the hair cells. Some pitch perception can be restored by performing a Fourier analysis of a sound and stimulating neurons at different places along the cochlea.

13.6 Attenuation

A plane wave of sound propagating through a medium is *attenuated*: there is a decrease in intensity because of dissipative factors such as viscosity and heat conduction, which we did not include in Sect. 13.1. The attenuation is exponential. The *amplitude attenuation coefficient*[4] is defined by

$$\alpha = -\frac{1}{p}\frac{dp}{dx}, \tag{13.35}$$

where x is the distance the wave travels in the medium. The sound pressure amplitude decays exponentially:

$$p(x) = p(0)e^{-\alpha x}. \tag{13.36}$$

Since the intensity is proportional to p^2,

$$I(x) = I(0)e^{-2\alpha x}. \tag{13.37}$$

The *intensity attenuation coefficient* is $\mu = 2\alpha$. In acoustics, the attenuation is usually expressed in decibels per meter, which is then independent of whether μ or α is used.[5]

The wave equation for acoustics is an approximation, because the basic equations of fluid dynamics are nonlinear. Therefore, effects that we have ignored, such as waveform distortion, the generation of harmonics, and increased attenuation may occur, particularly at high sound intensities.

[4] ICRU 61 (1998).

[5] Sometimes the attenuation coefficient is expressed in nepers m^{-1}, in which case the natural logarithm of the intensity or pressure ratio is used.

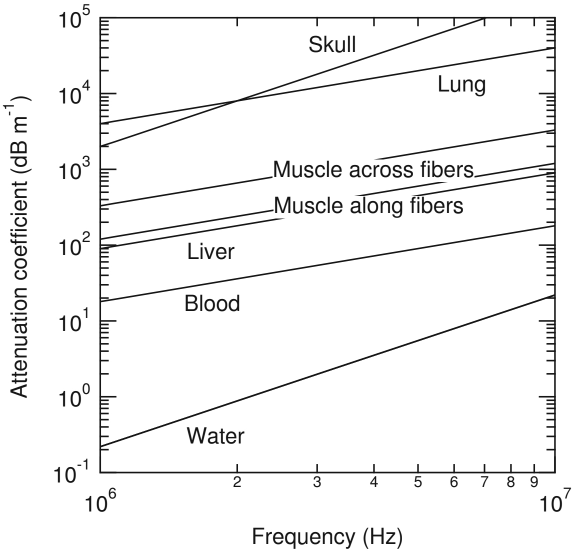

Fig. 13.12 Representative values of the attenuation coefficient for ultrasound

In air, the attenuation depends on the frequency of the sound and the temperature and humidity of the air (Lindsay and Beyer 1989; Denny 1993). Sound that we can hear (in the frequency range of 20 Hz to 20 kHz) is attenuated by about 0.1–10 dB km^{-1}. Water transmits sound better than air, but its attenuation is an even stronger function of frequency. It also depends on the salt content. At 1000 Hz, sound attenuates in fresh water by about 4×10^{-4} dB km^{-1}. The attenuation in sea water is about a factor of ten times higher (Lindsay and Beyer 1989). The low attenuation of sound in water (especially at low frequencies) allows aquatic animals to communicate over large distances (Denny 1993).

The attenuation of sound depends strongly on frequency. Figure 13.12 shows some representative values. As a rule of thumb, at ultrasonic frequencies the attenuation is proportional to frequency, with the constant of proportionality being 100 dB m^{-1} MHz^{-1}. There are large variations in attenuation in tissue, depending on the age of the subject and other factors. Values can be found in Appendix A of ICRU 61 (1998).

There can also be scattering of the sound from some object, just as there is for light. The total scattering cross-section for the object is defined by

$$\sigma_s = \frac{W_s}{I_0}, \qquad (13.38)$$

where W_s is the total power scattered and I_0 is the incident intensity. As in Chap. 14, the differential scattering cross-section can also be defined. Scattering can be increased by using an ultrasound contrast agent (Faez et al. 2013).

13.7 Diagnostic Uses of Ultrasound

Ultrasound has several uses in medicine. The most common is to provide diagnostic images that complement those made with x-rays, nuclear medicine, and magnetic resonance.[6] Ultrasound does not provide the image quality of these other methods, and it is susceptible to artifacts (see Problems 28–31), but it can be performed in real time, at low cost, with a small instrument at the patient's bedside. In general, the different medical imaging techniques compete with one another. Each has its own advantages and disadvantages (Glide-Hurst et al. 2010).

The highest frequency sounds that we can hear ($\approx$ 15 kHz) have a wavelength in water of 0.1 m. One property of waves is that diffraction limits our ability to produce an image. Only objects larger than or approximately equal to the wavelength can be imaged effectively. This property is what limits light microscopes to resolutions equal to about the wavelength of visible light, 500 nm. If we used audible sound to form images, our resolution would be limited to about 0.07 m, which would be a poor image indeed. To overcome this difficulty, higher frequencies (ultrasound) are used. Typically, diagnostic ultrasound uses frequencies on the order of 1 to 15 MHz, corresponding to wavelengths of 1.4 to 0.1 mm in tissue. Higher frequencies would result in even shorter wavelengths, but higher frequency sound has increased attenuation, which ultimately sets an upper bound to the useful frequency.

13.7.1 Ultrasound Transducers

Ultrasound is typically produced using a *piezoelectric transducer*. A piezoelectric material converts a stress (or pressure) into an electric field, and vice versa. A high-frequency oscillating voltage applied across a piezoelectric material creates a sound wave at the same frequency. Conversely, an oscillating pressure applied to a piezoelectric material creates an oscillating voltage across it. Measurement of this voltage provides a way to record ultrasonic waves. Thus, the same piezoelectric material can serve as both source and detector. One piezoelectric material often used in medical transducers is lead zirconate titanate (PZT). Its density is 7.5×10^3 kg m^{-3}, the speed of sound in the material is 4065 m s^{-1}, and the acoustic impedance is 30×10^6 Pa s m^{-1}. About half of the electrical energy is converted to sound energy, and vice versa.

[6] See Kremkau (2006); Carson and Fenster (2008) or Wells (2006).

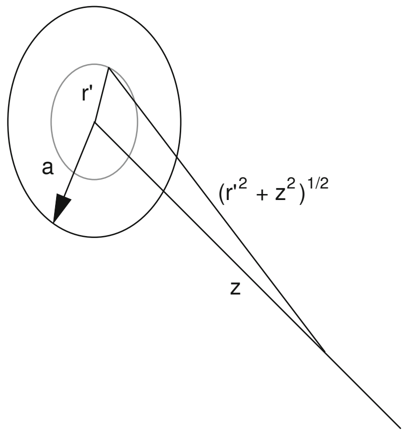

Fig. 13.13 Coordinate system for calculating the intensity of sound radiated from a transducer of radius a. The z axis passes through the center of the transducer and is perpendicular to it

There are some important features of the radiation pattern from a transducer that we review next. Consider a circular transducer or piston, the surface of which is oscillating back and forth in a fluid. Both faces set up disturbances in the fluid; however, we consider the radiation from only one face, since the transducer is placed in a holder which prevents radiation from the rear surface. We can easily calculate the intensity along the z-axis, which we set up perpendicular to the piston and passing through its center, as shown in Fig. 13.13.

The displacement of the face of the transducer, ξ, is the same as the displacement of the fluid in contact with it. The entire face of the piston, and therefore the fluid immediately in front of it, vibrates with a fluid velocity $d\xi/dt = v_0 \cos\omega t$.[7] Each small element of the vibrating fluid creates a wave that travels radially outward, the points of constant phase being expanding hemispheres. The amplitude of each spherical wave decreases as $1/r$, the intensity falling as $1/r^2$. We want the pressure at a point z on the axis of the transducer. It is obtained by summing up the effect of all the spherical waves emanating from the face of the transducer. At time t the phase of the wave is the same as the phase of the wave leaving the annular ring $r'dr'$ at the earlier time $t - r/c$:

$$p \propto \frac{d\xi(z,t)}{dt} \propto \int_0^a 2\pi r' dr' \frac{\cos[\omega(t - r/c)]}{r}.$$

This is easily evaluated by changing variables. Since $r^2 = r'^2 + z^2$, $2rdr = 2r'dr'$:

$$p \propto 2\pi \int_{r=z}^{r=\sqrt{a^2+z^2}} rdr \frac{\cos[\omega(t - r/c)]}{r}$$

$$= \frac{2\pi}{k}\left[\sin[\omega(t - \frac{1}{c}\sqrt{a^2+z^2})] - \sin[\omega(t - z/c)]\right].$$

To find the average intensity, we square and average over one period. The result is

$$I \propto \sin^2\left[\frac{\omega}{2c}\left(z - \sqrt{a^2+z^2}\right)\right]. \tag{13.39}$$

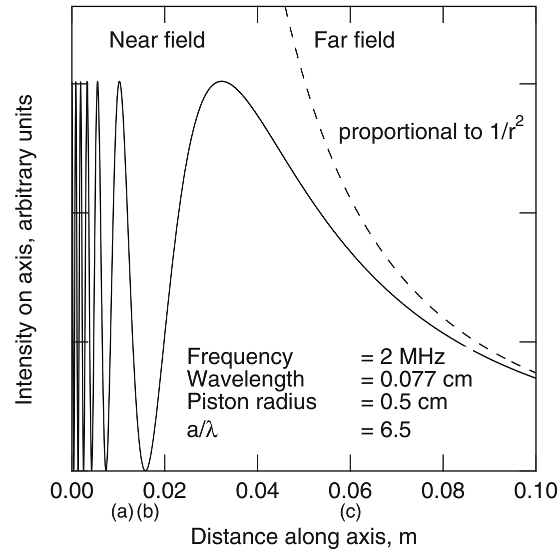

Fig. 13.14 The sound intensity on the axis of a circular transducer. The sound frequency is 2 MHz, and the transducer radius is 0.5 cm. Both near- and far-field regions are shown. Labels (**a**), (**b**), and (**c**) show the positions of the transverse radial scans in Fig. 13.15

The result is plotted in Fig. 13.14 for a fairly typical but small transducer ($a = 0.5$ cm, $f = 2$ MHz).

There are several important features of Fig. 13.14. Close to the transducer there are large oscillations in intensity along the axis; there are corresponding oscillations perpendicular to the axis, as shown in Fig. 13.15. The maxima and minima form circular rings. This is called the *near field* or *Fresnel zone*. Further away the intensity falls as $1/r^2$, in the *far field* or *Fraunhoffer zone*. The depth of the Fresnel zone is approximately a^2/λ. For the example shown (2 MHz, transducer diameter 1 cm), the depth is about 3 cm; for a larger transducer or higher-frequency ultrasound, it would be greater.

In the far field, approximations can be made to simplify the calculation. The intensity is then given by

$$I \propto \frac{1}{r^2}\left(\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right)^2. \tag{13.40}$$

Function $J_1(x)$ is the *Bessel function of order 1*. It is found in math tables and is available in many spreadsheets. The angular dependence of the far-field intensity is plotted in Fig. 13.16. If you want the ultrasound to be transmitted

[7] We use $d\xi/dt$ because it is in phase with the excess pressure.

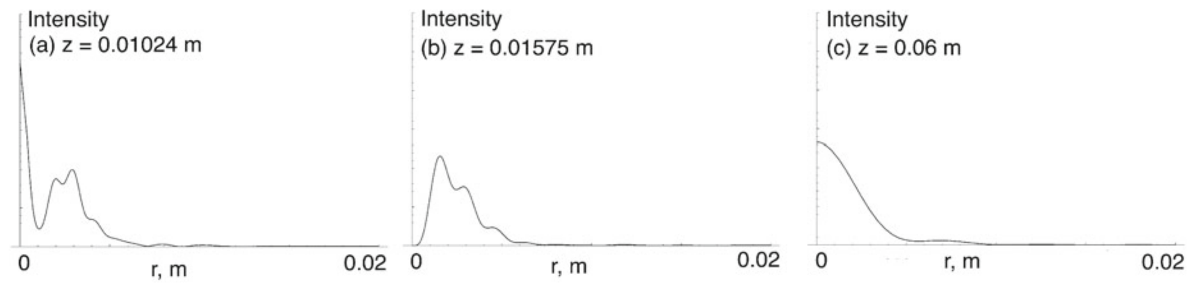

Fig. 13.15 Scans across the beam from the transducer shown in Fig. 13.14. **a** In the near field at an on-axis maximum 0.01024 m from the transducer. **b** In the near field at an on-axis minimum 0.01575 m from the transducer. **c** In the far field 0.060 m from the transducer

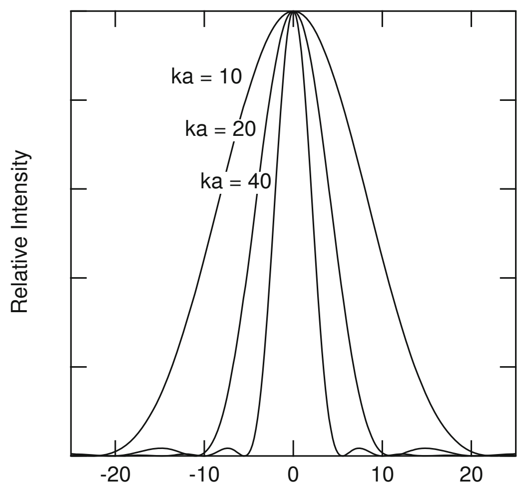

Fig. 13.16 The far-field intensity as a function of angle, calculated from Eq. 13.40. The value $ka = 10$ corresponds to 1 MHz and transducer radius $a = 0.25$ cm. The value $ka = 20$ corresponds to $f = 2$ MHz and $a = 0.25$ cm or $f = 1$ MHz and $a = 0.5$ cm. Value $ka = 40$ corresponds to 4 MHz and $a = 0.25$ cm or $f = 2$ MHz and $a = 0.5$ cm, the case examined in Fig. 13.14

mainly in the direction normal to the face of the transducer, select a transducer with a larger radius. If the transducer is used as a detector, a larger transducer is more selective in the direction perpendicular to its face. By shaping the face of the transducer, it is possible to bring the beam to a focus at some particular depth. This improves the spatial resolution and increases the strength of the returning echo. Ultrasound imaging may be done in the near field, the far field, or the transition region. Modern transducers typically consist of an array of transducers which may lie on a straight or curved line. They can be driven in such a way as to produce waves that come to a focus, or that travel in an off-axis direction (see Hendee and Ritenour 2002 or Fig. 13.18).

The impedance of a typical transducer is about 30×10^6 Pa s m^{-1}, so it is necessary to have an impedance-matching material between the transducer and the patient's skin (see Problem 16).

13.7.2 Pulse Echo Imaging

Most ultrasonic imaging is based on a pulse-echo technique. A short pulse (typically 0.5 µs in duration with a central frequency of about 5 MHz) is applied to the tissue by a piezoelectric transducer. The pulse travels with a speed of about $c = 1540$ m s^{-1} (or 1.54 mm µs^{-1}). Whenever it approaches a boundary between two tissues having different acoustic impedances, part of the incident pulse is reflected as an echo, which can be detected by the same piezoelectric transducer. The longer the time Δt between the generation and detection of the pulse, the farther away the reflecting boundary. In general, the distance from the source to the boundary is $\Delta x = c\Delta t/2$. Multiple boundaries produce multiple echoes, with each echo corresponding to a different distance from the source to boundary. A plot of echo intensity versus time is called an *A scan*. An A scan of the eye is shown in Fig. 13.17. As the attenuation is high, it is customary to increase the gain of the receiving amplifier as the echo time increases.

To form a two-dimensional image, it is necessary to scan in many different directions. In a *B scan* the brightness of the screen corresponds to the intensity of the echo, plotted versus position in the body in the plane of the scan. The B-scan transducer sends a narrow beam into the body. The direction of the beam is rapidly changed to cover a fan-shaped region of the body. This can be done with an oscillating or rotating transducer head (often containing three transducers), with an array of transducers that are pulsed sequentially, or with a *phased array* of transducers that are pulsed together. The operation of sequential pulsing or a phased array can be understood by referring to Fig. 13.18. The basic principle of using

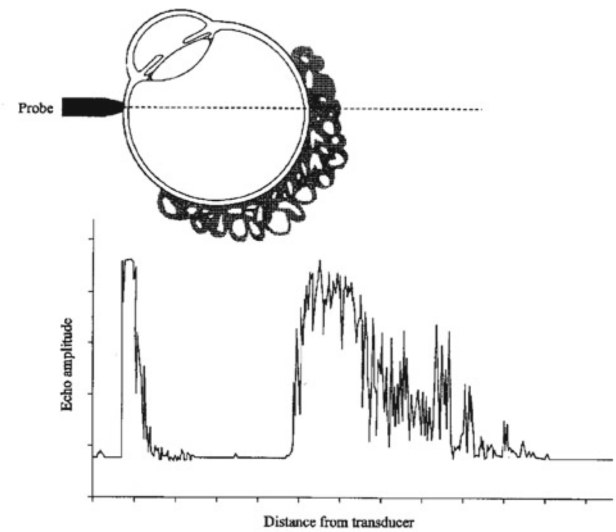

Fig. 13.17 An A scan of the eye. From ICRU 61, p. 2. Used by permission

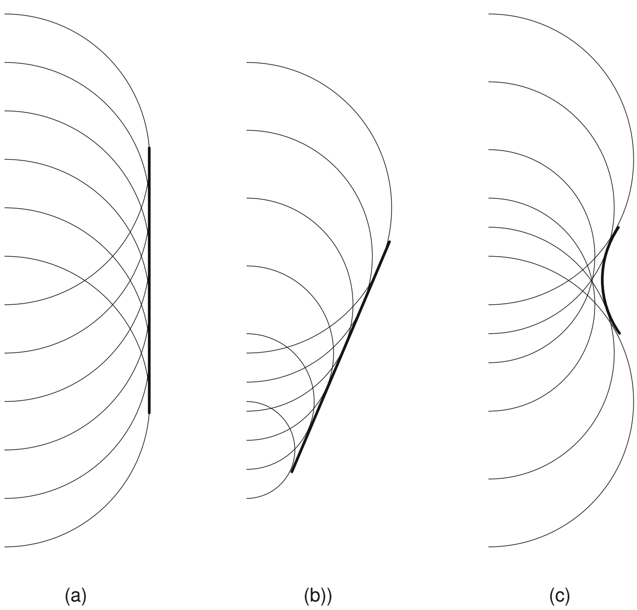

Fig. 13.18 How a phased array or delayed-pulse array works. Five transducers have been pulsed; the semi-circles show the propagating lines of constant phase from each one. The thick lines show the advancing wave front. In (**a**) all five transducers have been pulsed at the same time. The signals from each transducer add along the plane wave front traveling to the right. In (**b**) the top transducer was pulsed first. Each lower transducer was pulsed at successively later times, so the pulses have not traveled as far. This steers the beam downward. In (**c**) the outer transducers were pulsed first. As one goes inward, each transducer was pulsed later than the one before. This focuses the beam. The same technique can be used to steer or focus the sensitivity to the scattered wave during detection

Fig. 13.19 A B scan of a 16-week fetus

multiple one-dimensional (x) echo scans along different lines through the body is explored in Problems 24 and 25.

Two-dimensional ultrasound is widely used in diagnostic medicine; for instance in monitoring the fetus during pregnancy. Figure 13.19 shows a typical ultrasound image of a fetus.

Other imaging methods include motion or *M mode* to observe the beating heart as a function of time, and detecting sound backscattered from structures in an organ that are smaller than a wavelength.

As the tissue response to high intensity ultrasound is nonlinear, harmonics of the original ultrasound pulse are generated in the tissue; the second-harmonic signals are used to form *harmonic images*.

13.7.3 The Doppler Effect

When the source of an ultrasound wave is moving, the frequency of the wave observed by a stationary receiver is different than the frequency of the source. This phenomenon is called the *Doppler effect*. When the source is moving toward the receiver, the frequency is higher, and when the source moves away from the receiver, the frequency is lower.

To see why this happens, consider the source moving to the right with speed v_s in a fluid for which the speed of sound is c. At $t = 0$, the source emits the crest of a wave with period T (frequency $f = 1/T$). The wave travels to the right. This crest takes a time $t = L/c$ to reach a stationary receiver a distance L away. At $t = T$, one period later, another crest is emitted by the source. This crest takes less time to reach the receiver because the source has moved closer to the receiver. Specifically, the distance from source to receiver is now $L - v_s T$, so the crest reaches the receiver at $t = T + (L - v_s T)/c$. The time T' between crests reaching the receiver is $T' = T + (L - v_s T)/c - L/c = T(1 - v_s/c)$. The frequency observed by the receiver is

$$f' = 1/T' = \frac{f}{1 - v_s/c}. \tag{13.41}$$

If the source is moving toward the receiver with a speed equal to 10% of the speed of sound, then f' is about 11% higher than f. When the source is moving away from the receiver, $f' = f/(1 + v_s/c)$ (see Problem 33). It is not difficult to include the effect of motion of the reflecting surface at an angle with the ultrasound beam.

In medical ultrasound applications, the detected wave is often a reflection from moving tissue, such as red blood cells. In this case, the relationship between the frequency f produced by a stationary source and the frequency f' received by the stationary receiver after reflection from an object moving away from it at speed v_o is (see Problem 34)

$$f' = f\frac{1 - v_o/c}{1 + v_o/c}. \tag{13.42}$$

The difference in frequency between f and f' contains information about the speed of the object (Problem 36). Doppler ultrasound is used in medicine to measure speed, such as the speed of moving blood cells. Often the Doppler shift is measured for a pulse of ultrasound, so that one can be sure of the depth at which the Doppler shift occurred. A distribution of red cell velocities can be measured by looking at the Doppler shift frequency spectrum. In *color flow imaging* the velocity information from Doppler imaging is superimposed on a B-scan ultrasound image.

13.7.4 Elastography

Several techniques are being developed to measure the elastic properties of tissue. For example, an A-mode signal is measured with and without a static force on the tissue; the slight changes in signal reflect changes in tissue density.

In *shear wave elastography*, the shear wave propagates so slowly that images of the displacement $\eta(x, t)$ can be measured using traditional ultrasound techniques. The distribution of wave speeds can be determined from these images, and then the distribution of the shear modulus can be calculated. This method has been used to analyze breast cancer tumors, which tend to have a higher shear modulus than the surrounding healthy breast tissue (Berg et al. 2012).

13.7.5 Safety

The skin intensities used in diagnostic ultrasound range from 0.1 W m^{-2} for an obstetric examination to 25,000 W m^{-2} for some procedures that image the heart or blood vessels. These intensities occur over a small area of the body and for a limited period of time. Many studies have been done to see if any harm results from these sound intensities. No harmful effects have been found.

13.8 Therapeutic Uses of Ultrasound

The primary potential causes of harm from ultrasound are also used for therapy. They are *diathermy,* the heating of the tissue because of the energy deposited, and *cavitation*, a process in which very high intensity sound waves cause tiny bubbles of steam to form and then collapse violently. Cavitation requires intensities of 3.5×10^7 W m^{-2} or more.

High-intensity-focused ultrasound is made possible by phased arrays. It is possible to accurately locate the focal spot by applying lower intensity pulses that only heat the tissue a few degrees. The temperature in the tissue is mapped using magnetic resonance imaging, described in Chap. 18. Once the focal region is in the desired target, a longer pulse is applied to heat the tissue to the desired temperature. The technique is called magnetic resonance guided focused ultrasound (MRgFUS) or magnetic resonance guided high intensity focused ultrasound (MRgHIFUS). This technique is used for fat reduction (Saedi and Kaminer 2013), breast (Merckel et al. 2013), and prostate cancer, relief of pain from metastatic cancer, and neurosurgery.

The neurosurgical use of focused ultrasound has an interesting history. It was first proposed in the 1940s, but the large impedance difference between skull and soft tissue meant that a portion of the skull had to be removed in order for the ultrasound to reach the brain. Focus was achieved with a plastic lens in front of the transducer. A special water-filled container coupled the transducer to the surface of the brain. The development of phased transducer arrays made it possible to focus without a special lens and also to make corrections for the ultrasound waves passing through the skull (Clement and Hynynen 2002). There is high energy absorption at the skull, so transducer arrays covering a large part of the skull are used, and the water that couples the array to the scalp is cooled. Some patients have been treated for chronic neuropathic pain (Jeanmonod et al. 2012). Other uses are reviewed by Monteith et al. (2013) and Ellis et al. (2013).

Another use of ultrasound is *lithotripsy,*[8] the destruction of kidney stones using sharply focused ultrasound. Lithotripsy uses extremely intense, pulsed ultrasound waves. The peak intensity is about 3.8×10^8 W m^{-2}. The sound is intense enough so that bubbles of steam form and then collapse. When they collapse near the surface of the stone they

[8] *Litho-* means stone.

"hammer" on the stone. With repeated blows, the stone shatters. These smaller pieces may pass in the urine, avoiding surgery.

Symbols Used in Chap. 13

Symbol	Use	Units	First used page
a	Transducer radius	m	372
c	Speed of sound	m s^{-1}	364
c_{shear}	Speed of shear wave	m s^{-1}	365
f, g, h	Arbitrary functions		366
f, f'	Frequency	Hz	366
k	Wave number	m^{-1}	366
l	Length	m	364
p	Excess pressure	Pa	364
s_n	Normal stress	Pa	364
s_s	Shear stress	Pa	365
r, r'	Position	m	372
t	Time	s	363
v	Fluid or particle velocity	m s^{-1}	365
v_s, v_o	Velocity of source, observer	m s^{-1}	375
x, y, z	Position	m	363
E	Young's modulus	Pa	364
F	Force	N	364
G	Shear modulus	Pa	365
I	Intensity	W m^{-2}	367
J_1	Bessel function of order 1		372
L	Distance	m	374
M	Mass	kg	365
P	Pressure	Pa	364
R	Reflection coefficient		367
S	Area	m^2	364
T	Transmission coefficient		367
T	Period	s	375
V	Volume	m^3	364
W_s	Power scattered	W	371
Z	Acoustic impedance	Pa s m^{-1}or kg m^{-2} s^{-1}	367
α	Amplitude attenuation coefficient	m^{-1}	370
κ	Compressibility	Pa^{-1}	364
ϵ_n	Normal strain		364
ϵ_s	Shear strain		365
λ	Wavelength	m	366
η	Displacement from equilibrium in a shear wave	m	365
μ	Intensity attenuation coefficient	m^{-1}	370
ρ	Density	kg m^{-3}	364
σ	Scattering cross section	m^2	371
θ	Angle		372
ξ	Displacement from equilibrium	m	363
ω	Angular frequency	s^{-1}	366

Problems

Section 13.1

Problem 1. Show that $1/\sqrt{\rho_0 \kappa}$ has units of speed.

Problem 2. Show that the pressure p satisfies the wave equation. Hint: Use Eqs. 13.13 and 13.14. Differentiate to obtain $\partial^2 p/\partial x^2$ and $\partial^2 p/\partial t^2$. Also use the fact that when multiple partial derivatives are taken, the order of differentiation can be interchanged (Appendix N).

Problem 3. Show that v and ρ also satisfy the wave equation.

Problem 4. Derive Eq. 13.15.

Section 13.2

Problem 5. Use the chain rule, with $u = x - ct$, to show that $f(x - ct)$ obeys the wave equation for any function f. Show that $g(x + ct)$ also obeys the wave equation.

Problem 6. Calculate the wavelength in air for the lowest audible frequency (20 Hz for most people) and the highest audible frequency (20 kHz for most young people).

Problem 7. The ear canal is about 2.5 cm long. It is open to the air at one end and closed by the ear drum at the other. This can cause a standing wave to form, which has a pressure node (zero amplitude) at the opening and pressure maximum at the ear drum. What is the longest wavelength of a standing wave that is set up? What frequency does this correspond to? Compare this to the most sensitive frequency of the ear (Fig. 13.7).

Problem 8. Use the trigonometric identity $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ to show that a traveling wave can be written as the sum of two out-of phase standing waves, and that a standing wave can be written as the sum of two oppositely-propagating traveling waves.

Section 13.3

Problem 9. Derive the relationships between p_0, ξ_0, and v_0 (Eqs. 13.22 and 13.23), where p_0, ξ_0, and v_0 are the amplitudes of a sinusoidally varying plane wave.

Problem 10. For the following five tissues, calculate the density and compressibility (data are from Hendee and Ritenour 2002).

Tissue	Z (Pa s m^{-1})	c (m s^{-1})
Fat	1.38×10^6	1475
Brain	1.55×10^6	1560
Blood	1.61×10^6	1570
Muscle	1.65×10^6	1580
Bone	6.10×10^6	3360

Problem 11. Show that the intensity of a sound wave (Eq. 13.29) can be written as $\frac{1}{2}ZV^2$, as $\frac{1}{2}PV$, or as $\frac{1}{2}\frac{P^2}{Z}$.

Problem 12. The threshold for audible sound is 10^{-12} W m^{-2}. Use Eq. 13.29 to convert this to the amplitude of the pressure oscillation in air, using $Z_{air} = 400$ Pa s m^{-1}. Compare this to 10^5 Pa (atmospheric pressure), and to 5×10^{-6} Pa (which is on the order of the amplitude of random pressure variations in the air due to thermal motion). Are the pressure oscillations small? Perform the same analysis for the threshold for pain, $I = 1$ W m^{-2}.

Problem 13. When an incident sound wave in fluid 1 encounters the boundary with fluid 2, the reflection coefficient, R, is defined as the fraction of the incident intensity that is reflected. Derive an expression for R in terms of Z_1 and Z_2. Use the data in Problem 10 to calculate what fraction of the incident intensity is reflected at the boundary going from muscle to fat. Do the same for the boundary going from fat to muscle.

Problem 14. When an incident sound wave in fluid 1 encounters the boundary with fluid 2, the transmission coefficient, T, is defined as the fraction of the incident intensity that is transmitted. Derive an expression for T in terms of Z_1 and Z_2. Hint: recall that fluids 1 and 2 are different, so that the value of Z in Eq. 13.29 is different for the incident and transmitted waves.

Problem 15. Use the results of Problems 13 and 14 to show that $R + T = 1$.

Problem 16. (a) Show that when sound goes from a transducer with $Z_{transducer} = 30 \times 10^6$ Pa s m^{-1} to tissue with $Z_{tissue} = 1.5 \times 10^6$ Pa s m^{-1}, the transmission coefficient is $T = 0.18$.

(b) Show that a coupling medium between the transducer and tissue will maximize the overall transmission if $Z_{coupling} = \sqrt{Z_{transducer} Z_{tissue}}$. Show that in that case the transmission is $T = 0.36$. Ignore interference effects ($\lambda \gg$ the thickness of the coupling medium).

Section 13.4

Problem 17. If the intensity of a sound wave falls to half its original value, what is the change in dB?

Section 13.5

Problem 18. A sound wave with intensity of 10^{-12} W m^{-2} is the threshold for hearing. Convert that to a pressure amplitude P. Convert the pressure amplitude to a displacement amplitude using Eq. 13.22, with $f = 1$ kHz, $\kappa_{air} = 10^{-5}$ Pa^{-1}, and $c_{air} = 344$ m s^{-1}. Compare your result with the size of an atom, which is on the order of 0.1 nm. Surprised?

Problem 19. The ear can just hear sound at about 1000 Hz at a level that corresponds to a pressure change of 2×10^{-5} Pa. Atmospheric pressure is 10^5 Pa. Since atmospheric pressure is due to collisions of molecules with the eardrum, there are pressure fluctuations because of fluctuations in the number of collisions in time Δt. We can expect that $\Delta p/p$ is about 1/(number of collisions)$^{1/2}$. Suppose that the eardrum has area S and that when detecting a signal at 1000 Hz it averages over a time interval of 0.5 ms. The number of collisions per unit area per unit time is given by $nv/4$, where n is the number of air molecules per unit volume and v is an average velocity of 482 m s^{-1}. The radius of the eardrum is 4.5 mm. Find $\Delta p/p$.

Problem 20. People use many cues to estimate the direction a sound came from. One is the time delay between sound arriving at the left and right ears. Estimate the maximum time delay. Ignore any diffraction effects caused by the head.

Section 13.6

Problem 21. Find the conversion between α in dB m^{-1} and m^{-1} (as in $I = I_0 e^{-\alpha x}$).

Section 13.7

Problem 22. An ultrasound pulse used in medical imaging has a frequency of 5 MHz and a pulse width of 0.5 µs. Approximately how many oscillations of the sound wave occur in the pulse? The number of oscillations is sometimes called the quality, Q, of the pulse. A pulse with little damping has $Q \gg 1$, whereas a heavily damped pulse has $Q \approx 1$. Is the ultrasound pulse heavily damped?

Problem 23. A heavily damped pulse does not represent a single frequency. Consider a pulse $p(t)$ having the shape

$$p(t) = e^{-(t/\tau)^2} \cos \omega_0 t.$$

Using the techniques developed in Sect. 11.9, calculate the Fourier transform of this pulse. Determine the shape of the power spectrum. How is the parameter τ related to the width of the power spectrum? What is the central frequency of the power spectrum?

Problem 24. Suppose you send a short ultrasound pulse into the body at $t = 0$, and observe echoes at $t = 31$, 79, and 95 µs. How far from the source are the three tissue boundaries? Assume $c = 1540$ m s^{-1} in each tissue, and ignore attenuation. Draw a line corresponding to the x-axis ($x = 0$ is the source location), and draw a dot at the position corresponding to each boundary. You have just created an A scan, where each dot represents a boundary.

Problem 25. Suppose you emit an ultrasound pulse in the x direction from a source at each of eight different positions y. Each pulse receives a series of echoes, as shown in the table below (Echo times are in µs.):

y (mm):	0	10	20	30	40	50	60	70
Echo 1	35	37	39	40	45	47	48	49
Echo 2	97	98	58	56	57	96	91	90
Echo 3			71	73	71			
Echo 4			99	99	98			

Draw an x–y coordinate system ($x = 0$ is location of the source) and put a bright spot corresponding to each echo. Assume $c = 1540$ m s^{-1} in each tissue, and ignore attenuation. You have just created a two-dimensional ultrasound image.

Problem 26. Assume the attenuation is proportional to frequency, and is given by 100 dB m^{-1} MHz^{-1}. If you use a 5-MHz ultrasound wave to image a surface 30 mm below the surface of the skin, the measured echo is what fraction of the original intensity? Ignore impedance differences at the surface of the skin and assume that 100 % of the wave is reflected by the surface, so that the reduction of the echo intensity is caused entirely by attenuation. Remember that you must consider the round-trip distance traveled by the wave. Express you answer in dB.

Problem 27. The intensity of echoes depends on not only the nature of the boundary they reflect from, but also the distance to the boundary. Consider a boundary that reflects 50 % of the incident wave intensity. Compare the intensity of the echoes recorded by the detector for boundaries 10, 20, and 30 mm from the source. Assume an attenuation coefficient of 500 dB m^{-1}. Ignore any inverse-square fall-off. Clinical ultrasound imaging devices often use a technique called *time gain compensation* to selectively amplify later echoes, thereby correcting for the effect of attenuation that you just calculated.

Problem 28. The depth resolution of an ultrasound image depends on the speed of sound and the duration of the ultrasound pulse. A pulse having a duration of 0.5 µs has what spatial width (assume $c = 1540$ m s^{-1})? Structures smaller than the spatial pulse width are difficult to resolve using ultrasound imaging.

Problem 29. Ultrasound images are often generated using a series of ultrasound pulses, with echoes detected from each pulse. Images are obtained more quickly if the time between pulses is short. However, if this time is too short, echoes from consecutive pulses overlap, making the ultrasound signal difficult to interpret. Assume the deepest structure you wish to image is 80 mm from the source, and the speed of sound is 1540 m s^{-1}. What is the minimum time between pulses you can use without overlapping echoes? How many pulses per second does this correspond to? If you need to use 256 pulses in order to build up a two-dimensional image, how many images can you generate per second? Can you generate images at the video rate (30 frames per second)?

Problem 30. Suppose that an ultrasound wave is traveling to the right in muscle, toward a 3-mm thick layer of fat. (Use the data in Problem 10 for the acoustic properties of these tissues.) Part of the wave reflects off the left surface of the fat (echo 1), but part is transmitted and then reflects off the right surface, producing a wave traveling to the left. Part of this is detected as echo 2, but part of this left-traveling wave undergoes two additional reflections, traveling back and forth through the fat before being detected as echo 3. Echo 3 is called a *reverberation echo* and is one source of artifact in an ultrasound image. You can have more than one, since the wave can reflect back and forth between the left and right surfaces multiple times. Calculate the time between the first three echoes, and the relative intensities of each one (ignore attenuation).

Problem 31. Assume a fat-muscle boundary is 50 mm below the tissue surface. Calculate the intensity of the reflected wave, ignoring attenuation, using the data in Problem 10. Now, assume there is a bone that lies in the region from 20 to 30 mm below the surface, with the fat-muscle boundary still 50 mm below the surface. Calculate the intensity of the wave reflected from the fat-muscle boundary, accounting for the front and back bone surfaces, ignoring attenuation. If the minimum measurable intensity is -25 dB, will the fat-muscle boundary be observable in each case? In general, surfaces behind a bone do not appear in ultrasound images. The bone casts an *acoustic shadow*.

Problem 32. Verify Eq. 13.39. Show that for $z \gg a$, the intensity falls off as z^{-2}.

Section 13.7.3

Problem 33. Show that when a source of sound waves is moving away from the receiver, the frequency of the source, f, and the frequency measured by the receiver, f', are related by $f' = f/(1 + v_s/c)$.

Problem 34. Suppose a stationary source sends ultrasound waves to the right. They are reflected from an object moving to the right with speed v_o, and then are recorded by the stationary receiver (the receiver and source are at the same location). Derive the relationship in Eq. 13.42 between the frequency of the source, f, and the frequency recorded by the receiver, f', using the following steps.

(a) Find the time t_1 when the receiver records a signal that was emitted by the source at $t = 0$, traveled a distance L, was reflected, and then returned to the receiver.

(b) Find the time t_2 when the receiver records a signal that was emitted by the source at $t = T$, traveled a distance $L + \Delta L$, was reflected, and then returned to the receiver.

(c) Relate the distance ΔL to the speed of the object.
(d) Solve for $T' = t_2 - t_1$.
(e) Determine f' in terms of f.

Problem 35. Show that if $v_o \ll c$, Eq. 13.42 reduces to $f' = f(1 - 2v_o/c)$.

Problem 36. Solve Eq. 13.42 for v_o as a function of f'/f. This allows you to measure the emitted and received frequencies and determine the speed of the object.

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