Chapter 1
Introduction to Solid and Fluid Mechanics

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Learning outcomes

1. Explain the difference between a solid and a fluid.
2. Describe features of stress–strain behaviour of a solid measured using a tensile testing system.
3. Explain stress–strain behaviour of biological and non-biological materials in terms of their composition.
4. Define Young's modulus.
5. Describe the measurement of Young's modulus using a tensile testing system.
6. Discuss values of Young's modulus for non-biological and biological materials.
7. Define Poisson ratio and discuss values for different materials.
8. Describe viscoelasticity, its effect on stress–strain behaviour, and models of viscoelasticity.
9. Discuss linear elastic theory and its applicability to biological tissues.
10. Define hydrostatic pressure and values in the human.
11. Define viscosity in terms of shear stress and shear rate.
12. Describe different viscous behaviours.
13. Describe measurement of viscosity.
14. Describe typical measures of viscosity for different fluids.
15. Discuss Poiseuille flow: pressure-flow relationships for flow of Newtonian fluid through a cylinder.
16. Discuss Reynolds number and flow states.
17. Discuss pressure-flow relationships in unsteady flow in cylindrical tubes.
18. Discuss energy considerations in flow including the Bernoulli equation.

An understanding of the functioning of the cardiovascular system draws heavily on principles of fluid flow and of the elastic behaviour of tissues. Indeed, much of the

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cardiovascular system consists of a fluid (blood), flowing in elastic tubes (arteries and veins). This chapter will introduce basic principles of fluid flow and of solid mechanics. This area has developed over many centuries and Appendix 1 provides details of key scientists and their contribution.

The concept of a fluid and a solid is familiar from everyday experience. However, from a physics point of view, the question arises as to what distinguishes a fluid from a solid? For a cubic volume element there are two types of forces which the volume element experiences (Fig. 1.1); a force perpendicular to a face and a force in the plane of a face. The forces perpendicular to the face cause compression of the material and this is the case whether the material is liquid or solid. The force parallel to the face is called a shear force. In a solid, the shear force is transmitted through the solid and the solid is deformed or sheared. The shear force is resisted by internal stresses within the solid and, provided the force is not too great, the solid reaches an equilibrium position. At the nano level the atoms and molecules in the solid retain contact with their neighbours. In the case of a fluid, a shear force results in continuous movement of the material. At the nano level the atoms and molecules in the fluid are not permanently connected to their neighbours and they are free to move. The key distinction between a fluid and a solid is that a solid can sustain a shear force whereas a fluid at rest does not.

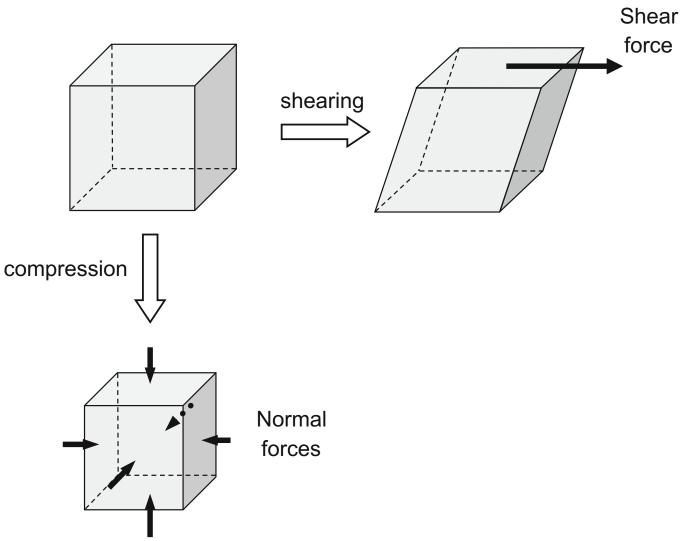


Fig. 1.1 A cube of material is subject to force parallel to a face which cause shearing and forces normal to each face which cause compression

1.1 Solid Mechanics

Solid mechanics is concerned with the relationship between the forces applied to a solid and the deformation of the solid. These relationships go by the name of the 'constitutive equations' and are important in areas such as patient-specific modelling discussed in Chap. 11. In general, these relationships are complex. For small deformations many materials deform linearly with applied force, which is fortunate as both experimental measurement and theory are relatively straightforward. This section on solid mechanics will start with 1D deformation of a material, develop linear elastic theory, then describe more complex features including those of biological materials.

1.1.1 1D Deformation

The elastic behaviour of a material is commonly investigated using a tensile testing system. A sample of the material is clamped into the system and then stretched apart. Both applied force and deformation are measured and can be plotted. Figure 1.2 shows the force-extension behaviour for steel. For many materials, such as steel and glass, the initial behaviour is linear; a doubling of applied force results in a doubling of the extension. In this region the material is elastic in that it will follow the same line on the force-extension graph during loading or unloading. The material is elastic up the point *Y*, which is called the 'yield point' but, after the yield point, the slope of the line decreases. The material is softer in that small changes in force result in large changes in extension. Beyond the yield point the material becomes plastic in that the material does not return to its original shape after removal of the force but is permanently deformed. In Fig. 1.2 further increase in force eventually leads to fracturing of the material at the point U, called the 'ultimate tensile strength' (UTS).

The force-deformation behaviour can be understood at the atomic level. The chemical bonds between atoms and molecules are deformable and small

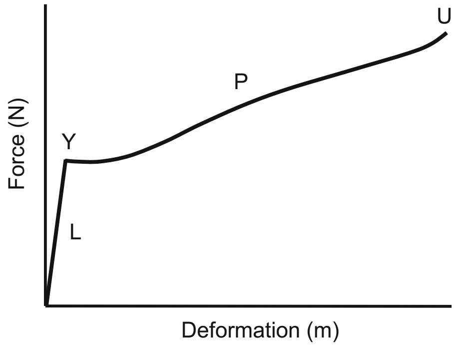


Fig. 1.2 Force-extension curve for steel. *L* linear behaviour; *Y* yield point; *P* plastic deformation; *U* (uniaxial) ultimate strength. Redrawn from Wikipedia under a GNU free documentation licence; the author of the original image is Bbanerje. https://commons.wikimedia.org/wiki/File:Hyperelastic.svg

deformations from the equilibrium position can be tolerated without change in structure. The equations governing the force-extension behaviour at the atomic level demonstrate linear behaviour and the macroscopic behaviour of a material is the composite of a multitude of interactions at the atomic and molecular level. In the plastic region there are changes in structure at the atomic and molecular level. In many materials this arises through slip processes involving the movement of dislocations or through the creation and propagation of cracks.

Biological materials are generally composite in nature. From a mechanical point of view the most important components are collagen fibres, elastin, reticulin and an amorphous, hydrophilic, material called ‘ground substance’ which contains as much as 90 % water. The elastic behaviour of the biological tissue is determined by the proportion of each component and by their physical arrangement. For example, collagen fibres in the wall of arteries are arranged in a helical pattern. Collagen is especially important in determining mechanical properties of soft biological tissues. Collagen is laid down in an un-stretched state. These unstressed fibres have a wavy, buckled shape, referred to as ‘crimp’. On application of a force, the fibres begin to straighten and the ‘crimp’ disappears and, as a result, the tissue deforms relatively easily. With increasing extension the fibres straighten fully and resist the stretch. This leads to collagen having a non-linear force-extension behaviour, which explains the non-linear force-extension behaviour of most biological soft tissues.

A simple 1D tensile testing system can also be used to demonstrate viscoelasticity. It was stated above that in elastic behaviour the loading and unloading curves are the same. For a viscoelastic material they are different. In elastic behaviour the application of a force results more or less immediately in deformation of the material. Viscoelastic behaviour is associated with a time- lag between the applied force and the resulting deformation. The term ‘viscoelastic’ implies that the material has a mix of elastic and viscous properties. If the tensile testing system stretches the material in a cyclic manner, then as the tissue is loaded and unloaded, the resulting force-deformation curve will be in the shape of an ellipse (Fig. 1.3). During loading the force increases but the extension increases more slowly. During unloading the

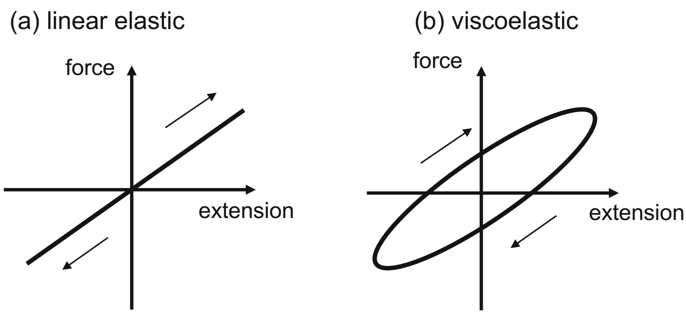


Fig. 1.3 Force-extension curves for cyclically varying force. **a** For a pure linear elastic material the loading and unloading curves are identical. **b** For a viscoelastic material the loading and unloading curves are different and are part of a loop

force decreases but the extension decreases more slowly. If the viscous component is low compared to the elastic component then the loading and unloading curves will be close together. For materials with a higher viscous component the curves are more separated and the width of the ellipse is larger.

1.1.2 Young's Modulus

In Sect. 1.1.1 the discussion of elastic behaviour was in terms of applied force and deformation. However the quantities stress and strain are more widely used in theory and experiment. The stress, σ, is the force, F, per unit area, A, and has units of pascals. The strain, ε, is the ratio of the extension, δl, divided by the original length, l, and is a dimensionless quantity.

$$\sigma = \frac{F}{A} \tag{1.1}$$

$$\varepsilon = \frac{\delta l}{l} \tag{1.2}$$

The Young's modulus, E, is a measure of the elastic behaviour of a material and is a fundamental mechanical property. Young's modulus is the ratio of stress divided by strain (Eq. 1.3). The units of E are pascals (Pa) or newtons per square metre (N m^{-2}).

$$E = \frac{\sigma}{\varepsilon} \tag{1.3}$$

Young's modulus is commonly measured using a tensile testing system. The value E is equal to the slope of the line on the stress–strain plot. For a linear elastic material the slope is constant over much of the range of stress/strain and the mechanical properties of the material may be described by a single value of E. For non-linear materials such as rubber or soft biological tissues, the value of E is dependent on the strain. For such materials the 'incremental elastic modulus' may be defined as the change in stress over the change in strain over a small section of the stress–strain curve (Eq. 1.4).

$$E_{\text{inc}} = \frac{\Delta\sigma}{\Delta\varepsilon} \tag{1.4}$$

Figure 1.4 shows the Young's modulus of a number of common materials. Note that the scale is logarithmic with a range of 9 orders of magnitude. Hard materials such as ceramics, metals and glasses have very high values of elastic modulus. These are usually quoted in gigapascals (GPa). Wood and wood products have lower values of elastic modulus, but still have a very wide range from very hard

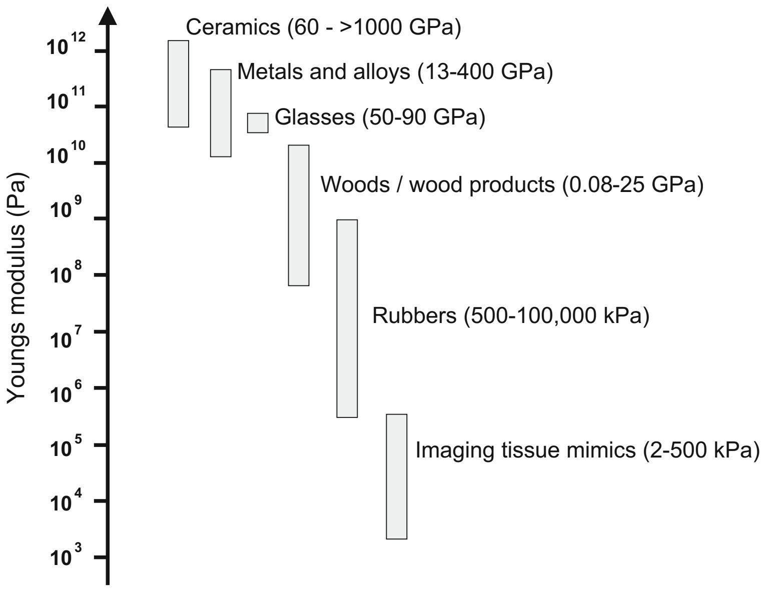


Fig. 1.4 Young's modulus *E* of common materials

woods such as oak, to very soft woods such as balsawood. Rubbers also have a very wide range from the hard vulcanised rubber used in tyres to the soft silicone rubber used in baby's dummies. The lowest elastic moduli values on the graph are for materials that mimic soft tissue used in phantoms for testing medical imaging systems. These are designed to mimic key properties of soft biological tissues, such as fat and muscle, and have low elastic modulus values in the range 2–500 kPa. Figure 1.5 shows the Young's modulus of a number of different biological tissues, taken from Sarvazyan et al. (1998). Again, there is a huge range of values. Bone and tooth enamel have the highest values of elastic modulus; liver, muscle and fat the lowest values.

The observant reader might have noted that it has been stated that the constitutive equations for soft biological tissues are complex and that the stress–strain behaviour is non-linear. How then is it justified in reporting Young's modulus, which generally applies to simple materials with linear stress–strain behaviours? This question will be addressed in Sect. 1.1.8; after more complex constitutive models have been considered.

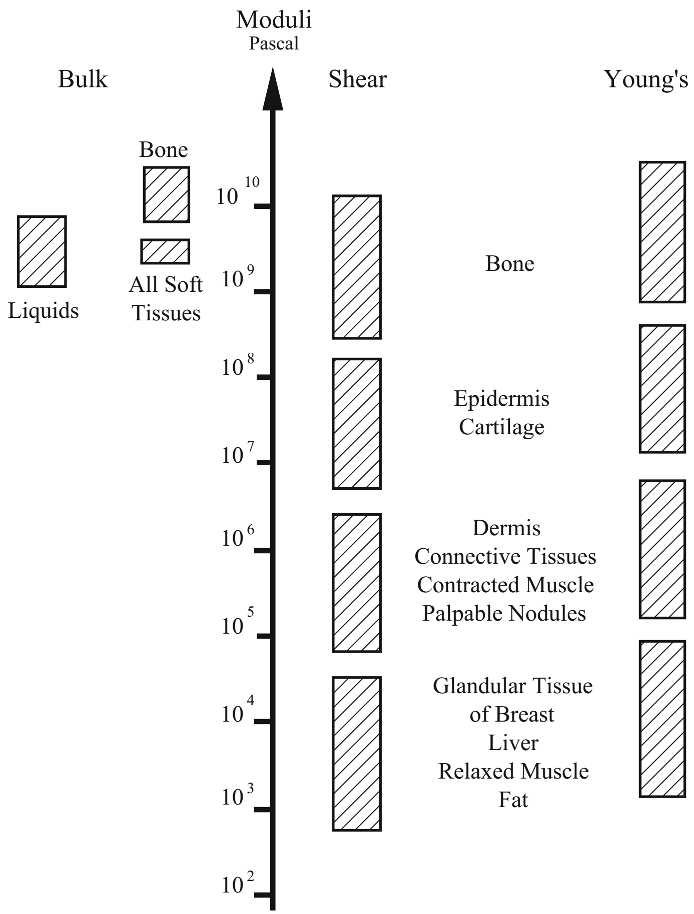


Fig. 1.5 A summary of data from the literature concerning the variation of the shear modulus, Young's modulus and bulk modulus for various materials and body tissues. Reproduced from Sarvazyan et al. (1995), with permission of Springer

1.1.3 Poisson's Ratio

When a material is stretched in the z direction there is usually compression in the x and y directions, and when a material is compressed there is usually expansion in the other two directions. This is called the Poisson effect. The Poisson ratio v is given by the fractional change in length in the x direction divided by the fractional change in length in the z direction (Eq. 1.5).

$$\upsilon = \frac{\delta x / x}{\delta z / z} \tag{1.5}$$

For incompressible materials, that is, materials where the volume does not change when loaded, the Poisson ratio has a value of 0.5. Soft biological tissues contain large amounts of water and have Poisson values close to 0.5. For many materials such as metals, glasses and concrete, the Poisson value is in the range 0.2–0.4.

1.1.4 Models of Viscoelastic Behaviour

Elasticity and viscosity can be represented by a spring and a dashpot. The spring responds immediately to being stretched, which represents the purely elastic behaviour of a material. For a dashpot, there is a delay between the stretching force and the extension, which represents the viscous behaviour of a material. A viscoelastic material can be represented as a combination of a spring and a dashpot and there are various configurations, three of which are shown in Fig. 1.6. These are; the Maxwell model where the spring and the dashpot are in series, the Voigt model where the spring and the dashpot are in parallel and a model consisting

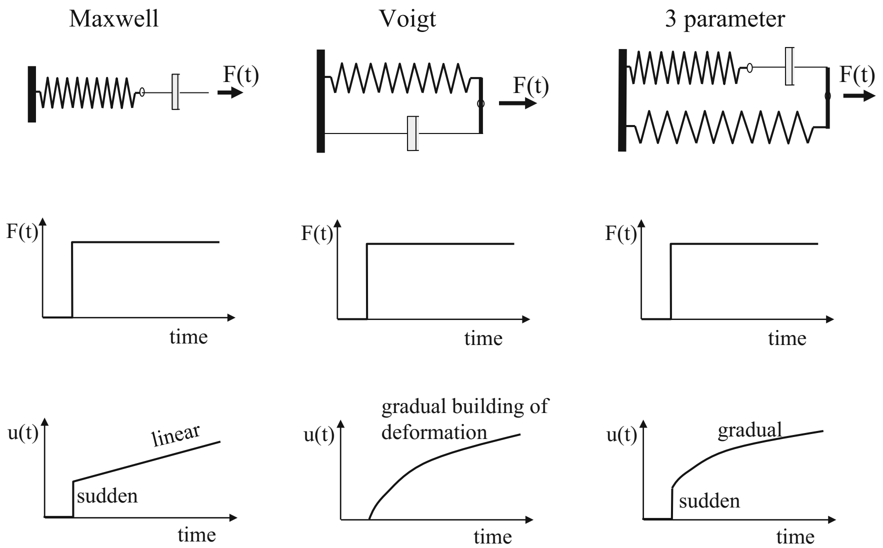


Fig. 1.6 *Top row* models of viscoelastic behaviour using combinations of a spring (elasticity) and dashpot (viscosity); Maxwell model, Voigt model, 3-parameter model. *Middle row* a sudden force is applied to the tissues. *Bottom row* The distension *u* is shown as a function of time for each model

of two springs and a dashpot. Figure 1.6 also shows the behaviour of each of these models. For the Maxwell model there is a sudden extension which corresponds to the spring stretching immediately followed by a linear increase caused by the dashpot slowly extending. For the Voigt model there is a gradual increase in the deformation. For the 3-parameter model, there is a sudden extension followed by a gradual extension. In practice, for a particular material, the model which best describes the experimentally determined behaviour of the material is chosen.

1.1.5 Linear Elastic Theory (Isotropic)

The Young's modulus and the Poisson ratio are two examples of parameters which describe the elastic behaviour of materials. In this section other parameters will be defined. In order to simplify things, the approach is taken of assuming that the material is linear elastic with no viscous components. It will also be assumed that the material is isotropic (having the same behaviour in all directions).

The bulk modulus, *B,* describes the ability of the material to resist a change in volume. A cube of material of volume, *V,* is subjected to a pressure, *P*, (Fig. 1.7a). The pressure increases by a small amount, δP. This leads to a reduction in volume by a small amount, δV. The bulk modulus is the change in pressure divided by the change in volume.

$$B = -\frac{\delta P}{\delta V} \tag{1.6}$$

The shear modulus, *G,* describes the ability of a material to withstand a shearing force (Fig. 1.7b). The shear modulus is the shear stress divided by the shear strain.

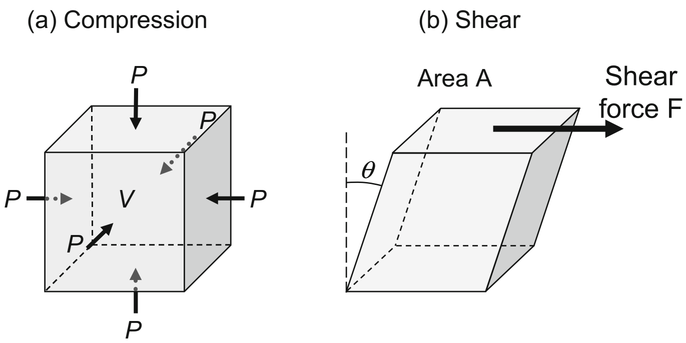


Fig. 1.7 **a** Compression—the cube of volume *V* is subject to a pressure *P* on all faces. **b** Shear—the cube is subject to a force *F* on one face which causes the cube to shear by an angle θ

$$G = \frac{F/A}{\tan(\theta)} \tag{1.7}$$

For the simple elastic material described in this section, G and E are related by Eq. (1.8) which involves both E and the Poisson ratio v. If the material is also incompressible, then Poisson's ratio v is 0.5 and the equation simplifies (Eq. 1.9).

$$G = \frac{E}{2(1 + \upsilon)} \tag{1.8}$$

$$G = \frac{E}{3} \tag{1.9}$$

Figure 1.5 shows values of G and B, along with E, for biological tissues. The range of values for bulk modulus is about one order of magnitude, whereas those for shear modulus and Young's modulus range over 7 orders of magnitude.

For the material described in this section, linear elastic isotropic, there are four parameters which describe elastic behaviour; Young's modulus E, Poisson ratio v, bulk modulus B and shear modulus G. However, only 2 of these are independent. In other words knowledge of 2 of these quantities (for example Poisson ratio and bulk modulus) allows estimation of the other 2 (Appendix 2).

1.1.6 Linear Elastic Theory (Generalised)

The theory described in the previous section will be developed further, but without the constraint of isotropy. So far the theory has been developed by considering mainly stress and deformation in 1D. However in most cases, the behaviour applies in 3D and the solid will deform in all three directions. Figure 1.8 shows the forces in 1 dimension for stretching of a column of material, in 2D for a square of material and in 3D for a cube. For the column of material there is a single extension force. For the square, each face has a force perpendicular to the face and also a shear force. For the cube, each face has a perpendicular force and two shear forces. The types of deformation which result are shown in the lower images. The number of stress and strain components needed to describe behaviour increases from 1D to 3D. For 1D it is just one stress and one strain. For 2D four components are needed and for 3D nine components are needed. Much more data is needed to describe behaviour in 3D than in 2D or 1D.

Table 1.1 shows the number of elastic constants required for the constitutive model as a function of the number of dimensions and of the degree of anisotropy. The most general case is a material which is fully anisotropic; behaviour in x, y and z is different. An orthotropic material has at least two planes of symmetry. An example of an orthotropic material might be a section of tree showing concentric rings. The third case is an isotropic material. The table shows the number of elastic

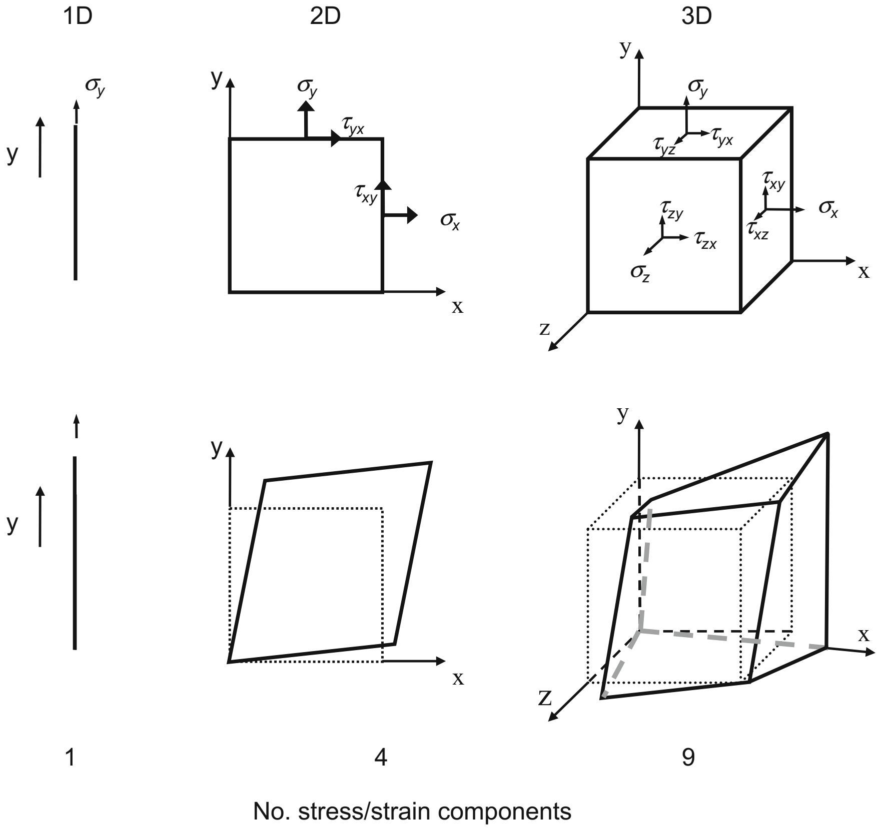


Fig. 1.8 Forces on a solid in 1D, 2D and 3D, and typical deformations. The number of stress and strain components needed to fully describe behaviour is 1 for 1D, 4 for 2D and 9 for 3D

Table 1.1 Number of elastic constants required as a function of the number of physical dimensions and the degree of isotropy

	1D	2D	3D
Anisotropic	1	9	21
Orthotropic	1	5	9
Isotropic	1	2	2

constants that are needed for 1D, 2D and 3D constitutive models. For 1D deformation only one constant is needed, the Young's modulus E. The number of constants needed increases as the model becomes more complex and as the number of dimensions increases so that for a 3D anisotropic constitutive model, 21 elastic constants are needed. Complex constitutive models require measurements of many elastic constants, which is challenging.

1.1.7 Constitutive Models for Non-linear Elasticity

It has been stated above that many biological materials are not described by linear stress–strain behaviour and instead are non-linear. A number of constitutive models have been developed which give non-linear behaviour. Models based on hyperelastic behaviour are widely used (Fig. 1.9); discussed further in e.g. Humphrey (2002). The number of constants varies from one model to the next, one parameter for the Neo-Hookean model, 2 for the Mooney-Rivlin model, three for the Yeoh model and so on. Generally, the constitutive model is developed which fits the experimentally determined data.

1.1.8 Materials in Practice

The linear elastic constitutive models described in the last two sections are actually relatively simple. Many materials are complex. They may be non-homogeneous and contain internal structures, be anisotropic, or have non-linear elastic, viscoelastic and plastic behaviour depending on the stress and strain. Many biological materials exhibit all of the above characteristics. In general constitutive equations are complex. Constitutive models such as these require advanced mathematics and are not

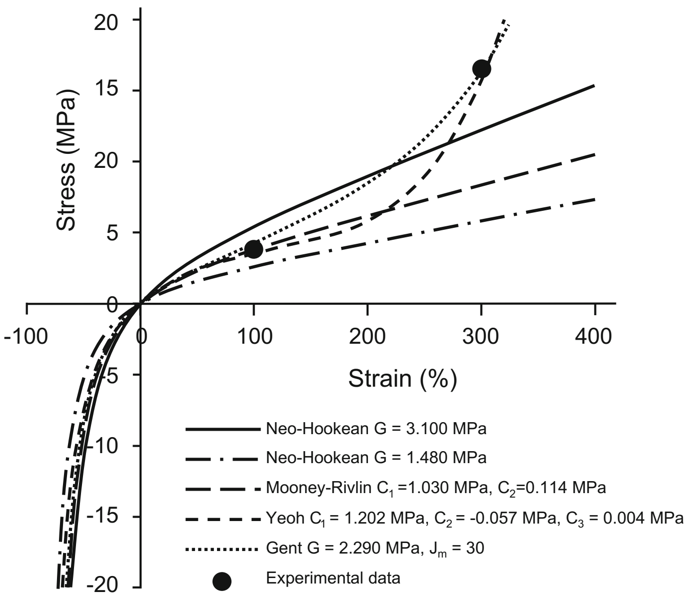


Fig. 1.9 Stress–strain curves for different hyperelastic constitutive models; redrawn from Wikipedia, author Bbanerje, under the GNU free documentation licence

covered in this book. The interested reader is referred to texts by Humphrey (2002), Cowin and Humphrey (2001), Cowin and Doty (2007) and Holzapfel and Ogden (2006). In practice, simplified constitutive models are often used, especially in medical use. There are various reasons for this; most importantly being the reduced number of necessary parameters, which in turn are easier to measure experimentally and make the models computationally less expensive and easier to understand.

An example of this simplification is the reporting of Young's modulus values for biological tissues (as shown in Fig. 1.5) even when the linear elastic constitutive model is recognised as being inappropriate compared to more complex models, such as, for example, a hyperelastic model. There may be several reasons for this simplification. Linear elastic theory is easy to understand whereas hyperelastic theory is more difficult. There are many different hyperelastic models, whereas there is only one linear elastic theory. There is only one constant, Young's modulus, which allows comparison of results between different papers. Different viscoelastic models use different constants so it is harder to compare data from different studies. Young's modulus is easy to measure experimentally, whereas viscoelastic models with several parameters require more complicated measurements.

An important consideration, which will be discussed in Chaps. 10 and 11, is the complexity of constitutive models required for estimation of data from computational modelling. When idealised tissue geometries are used, it is possible to specify in microscopic detail the microstructure of the arterial wall, including collagen fibre orientation and the associated mechanical properties. In this case, it is possible to use more complex constitutive models. However when data from the individual patient is used, it is not possible to obtain detailed information on geometry at the microscopic level from medical images, and it is not possible to measure, in the individual patient, the many different elastic constants. In this case, it is common practice to use quite simple constitutive models in which only 1–2 elastic constants are used.

1.2 Fluid Mechanics

Fluid mechanics is the study of fluids and the forces acting on them. It is divided into fluid statics, which is the study of fluids at rest and fluid kinematics, which is the study of fluids in motion. The principle fluid of interest in the cardiovascular system is blood, and later chapters will deal specifically with blood and blood flow. This section will describe the general principles of fluid mechanics, using illustrations from blood flow where relevant.

1.2.1 Hydrostatic Pressure

The pressure in a fluid is dependent on depth within the fluid

$$P_{\mathrm{d}} = P_{\mathrm{o}} + g\rho d \tag{1.10}$$

where, P_{d} is the pressure at depth d, P_{o} is the pressure at the surface, g is the gravitational constant and ρ is the density of the fluid. The excess pressure, $g\rho d$, is called the hydrostatic pressure. Hydrostatic pressure arises because of the effect of gravity; the weight of the fluid produces a force which operates in a downward direction. This produces a pressure gradient, which operates in an upward direction. In the case of a static fluid (e.g. a puddle or water in a bath) these two forces balance each other.

The average person is some 1.8 m tall and, when standing, there will be a variation in pressure in the cardiovascular system arising from the hydrostatic pressure. The overall pressure difference from the head to the toes is 136 mmHg; from the heart to the toes is around 100 mmHg. This is the minimum pressure (for a person of 1.8 m height) which the heart must provide in order to lift blood from the toes to the heart. It will be seen in later chapters that the maximum pumping pressure of the heart (systolic pressure) in a healthy individual is typically just slightly more than this minimum value at around 120 mmHg (but increasing with age). This pressure gradient arising from gravity is considerably reduced when the person lies down, where the head, heart and toes are all in the same vertical plane.

1.2.2 Shear Force and Strain Rate

A fluid is unable to retain an unsupported shape. A fluid will flow, taking up the shape of the container and coming to rest. Deformation of the fluid occurs as a result of shear forces. It was noted above that a solid can sustain a shear force whereas a fluid deforms under the action of a shear force. In a fluid at rest, there is no movement and hence no shear forces are present.

For a solid, a shear force will cause a strain (represented by the angle θ in Fig. 1.7); the larger the shear force then the larger the strain. The constitutive model for a solid concerns the relationship between shear force and strain. In contrast, for a liquid, application of a shear force results in an increase in the strain with time (larger angle θ–Fig. 1.10). The constitutive equations for a liquid concern the relationship between shear force and rate of change of strain (strain rate).

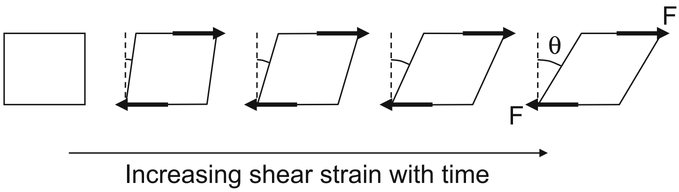


Fig. 1.10 In a liquid a shear force results in increasing strain with time; i.e. increase in the angle θ

1.2.3 Viscosity

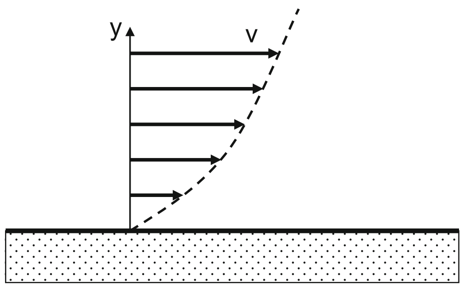
Usually fluid motion is of interest with respect to a boundary or wall (Fig. 1.11). The velocity v of the fluid alters with distance y from the wall due to frictional forces within the fluid and between the fluid and the boundary. At the wall, the fluid velocity is zero and the velocity increases with distance from the wall. The variation in velocity with distance is called the shear rate (dv/dy). The viscosity μ is defined as the ratio of the shear stress τ to the shear rate (Eq. 1.11).

$$\mu = \frac{\tau}{dv/dy} \tag{1.11}$$

Viscosity is a fundamental property of a fluid. It describes the ability of the fluid to resist deformation by a shear force, and the viscosity has a major role in defining the velocity values near to surfaces.

The viscosity of a fluid may be measured using a variety of techniques with instruments called ‘viscometers’. Commonly, a cone plate viscometer is used which consists of a flat circular plate and a moving circular plate in the shape of a cone positioned above the flat plate. The fluid is introduced into the space between the plates and the upper plate is spun at increasingly high speeds. From the rotational speed the shear rate is calculated. Figure 1.12 shows plots of shear stress versus shear rate for a variety of different fluids along with the corresponding plots of viscosity versus shear rate. The different types of fluid behaviours are listed below

Fig. 1.11 Effect of a surface (wall) on fluid velocity. The velocity at the wall is zero and there is a change in velocity with distance from the wall



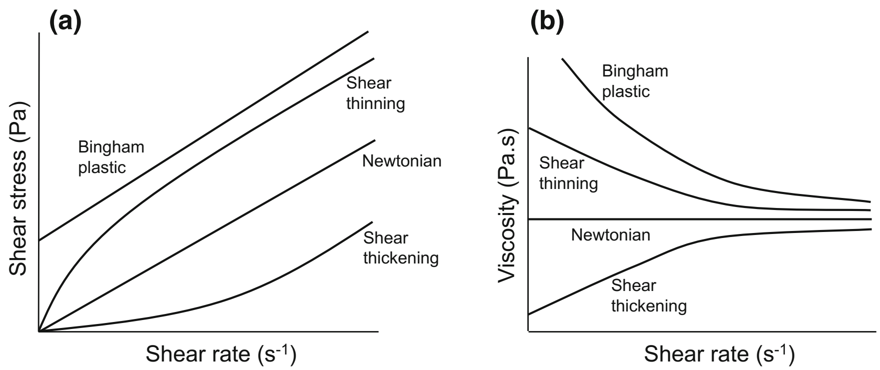


Fig. 1.12 Behaviour of different types of fluid with shear rate; **a** shear stress–shear rate plots, **b** viscosity shear rate plots

- Newtonian. The shear stress increases linearly with shear rate, and consequently the viscosity is a constant and independent of shear rate. Examples are water and motor oil.
- Non-Newtonian
  - Bingham plastic. A yield stress must be exceeded before the fluid will flow. Examples are toothpaste and mayonnaise.
  - Shear thinning (also called ‘dilatant’). The viscosity decreases with increasing shear rate. Examples are tomato ketchup, blood and latex paint.
  - Shear thickening (also called ‘pseudoplastic’). The viscosity increases with increasing shear rate. Examples are silly putty and cornstarch solution.

Any fluid which does not have a linear stress–shear rate behaviour is a non-Newtonian fluid. Bingham plastic fluids, shear thinning fluids and shear thickening fluids are all examples of non-Newtonian fluids. Blood is a shear thinning fluid and the viscous behaviour of blood will be discussed in detail in Chap. 3. For many materials the viscosity reaches a constant value at higher shear rates and one way of comparing different liquids is to quote the high-shear rate viscosity.

A simple and inexpensive way to measure fluid viscosity is to fill a funnel with the fluid and time how long it takes for the fluid to drain out. This method is commonly used in drilling, for example, to check the quality of drilling mud. The viscosity is approximately proportional to the emptying time from the funnel. Table 1.2 gives rough values of emptying time for different fluids, though it is noted that the calculated values are for illustration only and are not claimed to be accurate. Fluids such as water, blood and oil take 2 min or less to drain out. Thicker fluids such as glycerol and honey take minutes to hours whereas peanut butter can take days. The calculated time for window putty is 3 years, and the longest calculated time is for pitch at 1000s of years. Pitch is an example of a fluid with a viscosity so high that initial inspection would suggest it to be a solid; a block of

Table 1.2 High-shear viscosity of different liquids

Fluid	High-shear viscosity (Pa s)	Time to drain
Water	0.001	26 s
Blood	0.004	29 s
Olive oil	0.08	2 min
Glycerol	1	17 min
Honey	5	2 h
Peanut butter	250	3 days
Window putty	100,000	3 years
Pitch	230,000,000	1000s years

Calculated time for different fluids to empty from a funnel (for illustrative purposes only)

pitch shatters like glass when hit with a hammer. However, it is a liquid and it will flow given sufficient time. The longest running viscosity measurement is on a funnel filled with pitch, which was set up in 1927 (Edgeworth et al. 1984). The funnel was filled and allowed to settle for 3 years after which the end of the funnel was cut. After 80 years only 8 drops of pitch had fallen from the funnel.

1.2.4 *Steady Flow in a Tube*

The flow of fluids in tubes was investigated by the French physicist Poiseuille. Figure 1.13 shows the basic parameters needed to describe flow in a cylinder. The pressure at the right side of the tube is P and at the left $P + \Delta P$. The pressure drop across the tube, ΔP, gives rise to a flow, Q. The formula derived by Poiseuille relating these variables is shown in Eq. (1.12).

$$Q = \frac{\Delta P \pi D^4}{128 L \mu} \tag{1.12}$$

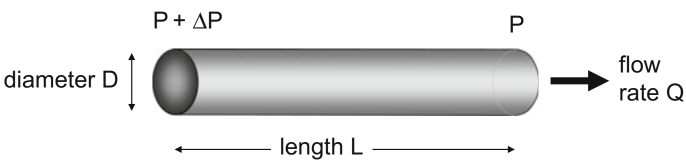


Fig. 1.13 Quantities relevant to flow in a cylindrical tube

where, L is the tube length, D is the diameter and μ is the viscosity. For any given tube of constant cross-section the flow rate is proportional to the pressure difference. The formula also shows that the pressure difference required to sustain a certain flow rate Q, is inversely proportional to the 4th power of the diameter. In other words a decrease in diameter by a factor of 2 requires a 16-fold increase in pressure to maintain the same flow rate.

For flow in a cylinder the resistance to flow R is defined as the pressure difference divided by the flow rate (Eq. 1.13). For Poiseuille flow this gives Eq. (1.14). The resistance is proportional to the length of the tube and to viscosity and is inversely proportional to diameter to the fourth power.

$$R = \frac{\Delta P}{Q} \tag{1.13}$$

$$R = \frac{128 L \mu}{\pi D^4} \tag{1.14}$$

The velocity as a function of position across the diameter is called the 'velocity profile'. For steady flow of a Newtonian fluid in a long straight tube it can also be shown that the velocity profile has the shape of a parabola (Fig. 1.14). For a parabolic velocity profile the velocity is maximum in the middle of the tube and is zero at the tube wall. The shape is not dependent on the viscosity.

When the tube is not long and straight the velocity profile will vary, depending on the shape and length of the vessel. At the entrance to a long pipe the velocity profile is initially flattened but after a certain length (called the 'inlet length') the velocity profile becomes parabolic, and it remains parabolic after this point

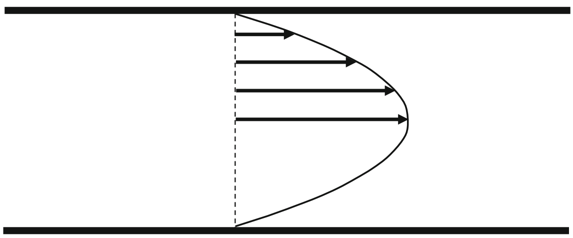


Fig. 1.14 Parabolic velocity profile for flow of a Newtonian fluid in a long straight tube

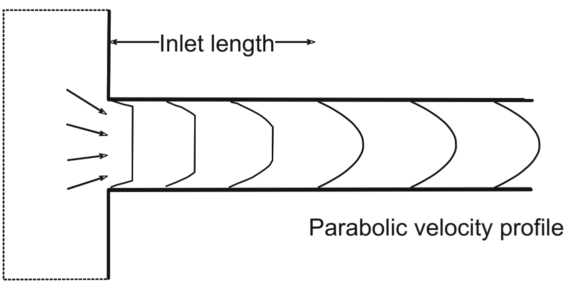


Fig. 1.15 Velocity profiles at the entrance to a long straight tube. The length after which the velocity profile is stable is called the inlet length

(Fig. 1.15). This stable region of flow is called ‘fully developed flow’. Equation (1.15) shows a formula for calculating inlet length IL (from MacDonald 1974). In this case the velocity profile at a specific point in the vessel is dependent on the viscosity.

$$IL = 0.04dRe \tag{1.15}$$

where, d is diameter and Re is Reynolds number.

1.2.5 Reynolds Number and Flow States

We will continue with consideration of flow at the entrance to a straight pipe. The flow can be divided into two regions (Fig. 1.16). Flow near the entrance is dominated by inertial forces and the velocity profile in the region is flat. Flow further along the tube is dominated by viscous forces and the velocity profile changes with distance from the vessel wall. The interface between these two regions is called the boundary layer.

The Reynolds number Re of a fluid is defined as the ratio of inertial to viscous forces. For flow in a tube the Reynolds number is given by Eq. (1.16). The Reynolds number is dimensionless.

$$Re = \frac{\rho VD}{\mu} \tag{1.16}$$

where V is velocity, ρ is density, D is diameter and μ is the viscosity. As the velocity of a fluid increases its behaviour will change. There are three flow states which are related to the Reynolds number, and also the velocity. At low Reynolds numbers and low fluid velocities the flow is laminar. Fluid elements follow clearly defined paths. This behaviour is associated with Reynolds numbers of less than about 2300. In turbulent flow fluid elements follow erratic paths which randomly vary with time.

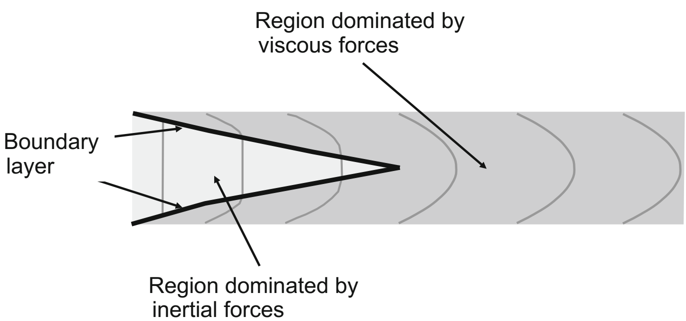


Fig. 1.16 Regions of flow at the entrance to a long straight tube and the boundary layer

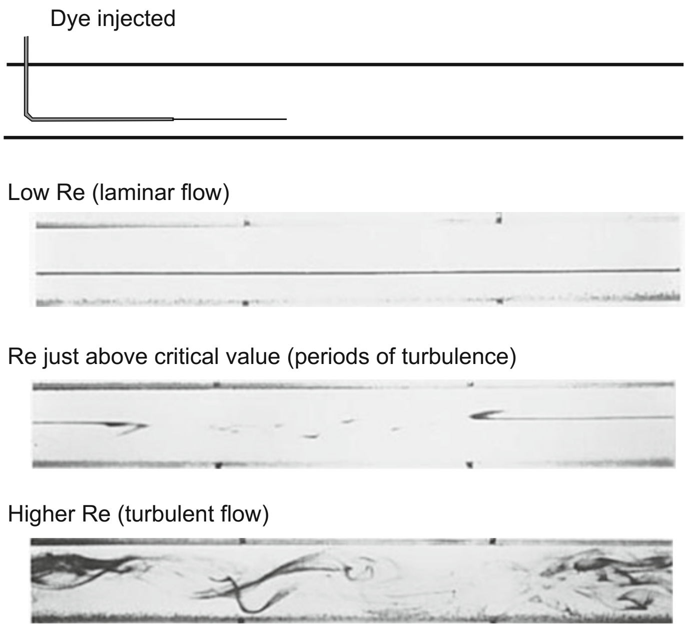


Fig. 1.17 Flow states demonstrated by injection of dye. At low *Re* the flow is laminar and the dye moves along a straight path. At values of *Re* just above a critical value the flow oscillates between laminar and turbulent. At higher *Re* the flow is turbulent and the dye follows multiple erratic paths. Reprinted with permission from: Caro CG , Pedley TJ, Schroter RC, Seed WA, assisted Parker KH; Mechanics of the circulation 2nd edition, 2011, Cambridge University Press

This occurs at high Reynolds numbers greater than about 2500. Between these two states the flow is termed transitional. The fluid is considered to be ‘disturbed’ with intermittent periods of turbulence and laminar flow.

Figure 1.17 shows the different flow types. Ink is injected into a tube to visualise features within the flow. At low Reynolds numbers there is laminar flow and the dye moves along a single path. At Reynolds numbers just above the critical value there are intermittent periods of laminar flow and turbulence. The dye has periods where it follows a clearly defined straight path and periods where it has an erratic path. For higher Reynolds numbers there is turbulent flow and the dye follows multiple erratic paths.

1.2.6 Unsteady Flow in Tubes

When the pressure varies in a tube, so will the flow rate. A time-varying pressure gradient results in a time-varying flow (Fig. 1.18). The inertia of the fluid gives rise to a time lag between the flow and pressure.

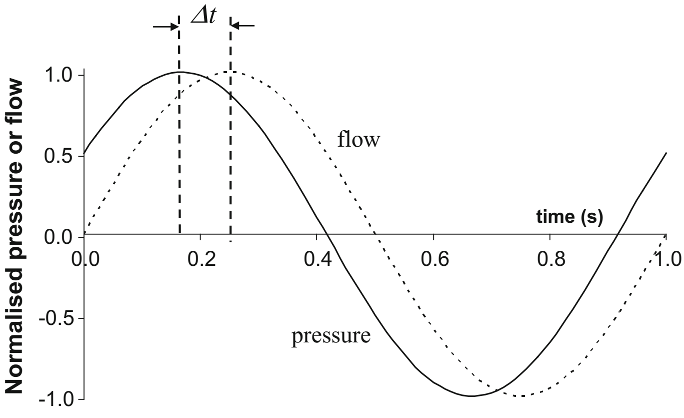


Fig. 1.18 Time-varying sinusoidal pressure in a tube will result in time varying sinusoidal flow. There is a time lag between flow and pressure as a result of the inertia of the fluid

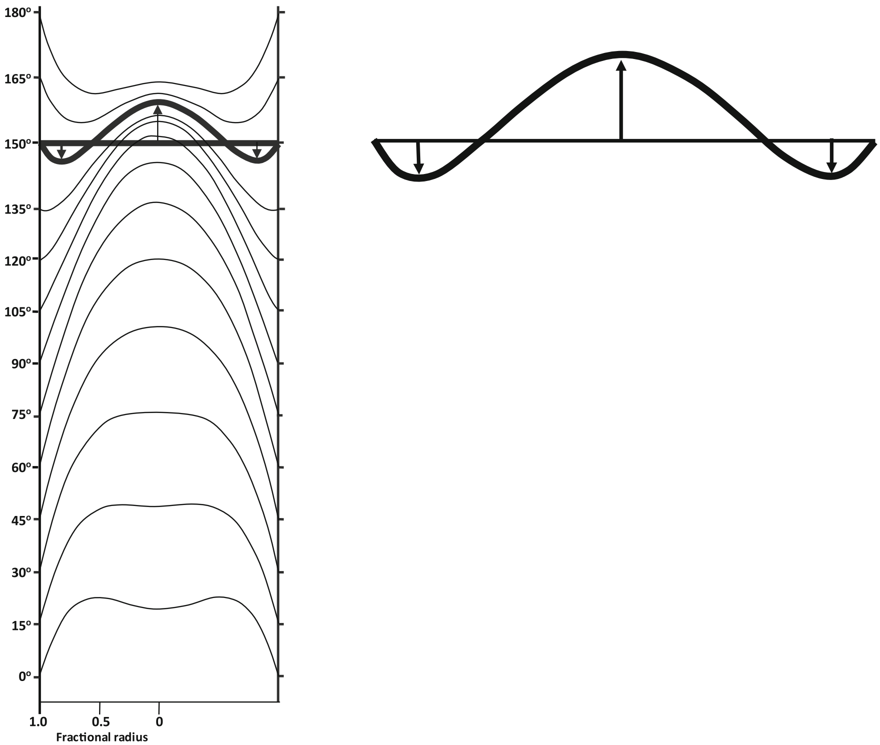


Fig. 1.19 Calculated velocity profiles as a function of phase for a sinusoidal pressure in a straight tube containing a Newtonian fluid

When flow in a tube varies with time, velocity profiles are not parabolic (Fig. 1.19); velocity profiles for a sinusoidal pressure gradient are shown. The inertia of flow in the central core prevents the flow following the pressure gradient, whereas lower velocities at the wall can follow the pressure gradient. This results in complex velocity profiles. In some cases there is both forward flow and reverse flow present at the same time.

1.2.7 Energy Considerations and the Bernoulli Principle

The Bernoulli principle is that an increase in fluid velocity is associated with a decrease in pressure. In general, for flow along a streamline in which there are no energy losses (i.e. no viscosity), the energy is constant at all points along the streamline and is distributed between gravitational potential energy, pressure energy and kinetic energy. Bernoulli's Eq. (1.17) is simply a statement of conservation of energy.

$$P + h\rho g + \frac{\rho v^2}{2} = \text{constant} \tag{1.17}$$

where P is pressure, h is height, ρ is density, g is the gravitational constant and v is velocity.

Assuming that the effects of gravity can be ignored, Bernoulli's equation can be simplified as shown in Eq. (1.18).

$$P + \frac{\rho v^2}{2} = \text{constant} \tag{1.18}$$

Bernoulli's equation is considered further in Chap. 15 with respect to the pressure of blood in a diseased artery in which there is lumen reduction. In practice viscosity leads to energy losses and this is also considered in Chap. 15.

Appendix 1: Scientists Involved in the Development of Fluid and Solid Mechanics

Many of the scientists below have made contributions in many scientific areas. Only the areas relevant to fluid mechanics and solid mechanics as mentioned in this chapter are described below.

Bernoulli, Daniel 1700–1782. Proposed an inverse relationship between fluid velocity and pressure which came to be known as the Bernoulli principle and an equation which described this, known as the Bernoulli equation.

Bingham, Eugene 1878–1945. Proposed a mathematical form for the behaviour of fluids which have a yield stress before flowing. Subsequently the fluids were referred to as 'Bingham plastics'.

Hagen, Gotthilf 1797–1884. Formulated a law relating pressure drop and flow rate for flow of a fluid in pipes. This was independently discovered by Jean Poiseuille. Subsequently called the Hagen–Poiseuille law, or just the Poiseuille law.

Hooke, Robert 1635–1703. Discovered the law of elasticity (extension proportional to force). Subsequently called Hooke's law.

Maxwell, James 1831–1879. Proposed a model which described the stress–strain behaviour of materials with viscous and elastic behaviour. Subsequently called the Maxwell model.

Newton, Issac 1642–1727. Developed a theory of mechanics for which the second law states that a force acting on a mass will produce an acceleration. The unit of force, the newton (N), is named after him. Newton also derived the equation for fluids relating shear stress to shear strain rate. Fluids which have a linear stress–strain rate behaviour are said to be Newtonian.

Pascal, Blaise 1623–1662. Undertook studies of pressure, demonstrating that hydrostatic pressure is governed by elevation difference, not on the weight of the fluid. The unit of pressure, the pascal (Pa), is named after him.

Poiseuille, Jean 1797–1869. Formulated a law relating pressure drop and flow rate for flow of a fluid in pipes. This was independently discovered by Gotthilf Hagen. Subsequently called the Hagen–Poiseuille law, or just the Poiseuille law.

Poisson, Simeon 1781–1840. Defined a ratio to express the contraction of a material in the direction transverse to the direction of stretching. Subsequently called Poisson's ratio.

Reynolds, Osborne 1842–1912. Investigated flow in pipes, deriving a dimensionless quantity to describe the transition from laminar to turbulent flow. Subsequently called Reynold's number.

Voigt, Woldemar 1850–1919. Studied the elastic behaviour of materials formulating a model which could be used to describe the stress–strain behaviour. Subsequently called the Voigt model.

Young, Thomas 1773–1829. Performed experiments on the stretching of materials. Originated the concept of the elastic modulus as a fundamental property of a material. The elastic modulus was subsequently called the Young's modulus.

Appendix 2: Relationships Between the 4 Elastic Constants for a Linear Isotropic Material

Input constants	Output equations			
	$E =$	$v =$	$G =$	$B =$
E, v	–	–	$\frac{E}{2(1+\upsilon)}$	$\frac{E}{3(1-2\upsilon)}$
$\boldsymbol{E}$, $\boldsymbol{G}$	–	$\frac{E-2G}{2G}$	–	$\frac{EG}{3(3G-E)}$
$\boldsymbol{E}$, $\boldsymbol{B}$	–	$\frac{3B-E}{6B}$	$\frac{3BE}{9B-E}$	–
v, $\boldsymbol{G}$	$2G(1+\upsilon)$	–	–	$\frac{2G(1+\upsilon)}{3(1-2\upsilon)}$
v, $\boldsymbol{B}$	$3B(1-2\upsilon)$	–	$\frac{3B(1-2\upsilon)}{2(1+\upsilon)}$	–
$\boldsymbol{G}$, $\boldsymbol{B}$	$\frac{9BG}{3B+G}$	$\frac{3B-2G}{6B+2G}$	–	–

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