Chapter 3
Portfolio Selection with Spectral Risk Measures

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Abstract In this chapter, a portfolio selection problem with spectral risk measure is considered. The spectral risk measure is a general family of coherent risk measures and is capable of reflecting investor's risk preference. A multivariate conditional heteroscedastic model with vine copulae is employed to describe the dynamics and dependence of the underlying asset returns. The technique of linear programming is used to accurately and quickly determine the optimal asset allocations. Simulation studies are conducted for investigating the impacts of the magnitude of tail dependence among the underlying assets and the degrees of risk aversion on the performance of the optimal portfolio. An empirical study is conducted by using the stock prices included in the FTSE TWSE Taiwan 100 Index. Numerical results indicate that the optimal portfolios have different reactions to different economic situations.

3.1 Introduction

In modern portfolio selection theory, the mean-variance (MV) portfolio optimization procedure introduced by Markowitz (1952; 1959) plays a crucial role in optimal asset allocations and investment diversification. In the MV procedure, investors attempt to maximize their portfolio expected return for a given level of portfolio risk, or equivalently to minimize the risk of investment with achieving a given amount of expected return, by determining the investment proportions of various securities (Markowitz 1952, 1959, 1991; Merton 1972; Kroll et al. 1984). The traditional MV portfolio problem uses standard deviation as the measure of risk and assumes that the returns of the underlying assets are independent and identically distributed (i.i.d.).

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Recently many other risk measures are more commonly used by traders in reality, for example, the value-at-risk (VaR), the expected shortfall risk (ES) and a general class of coherent risk measures, called the spectral risk measure (SRM). Thus, the optimal portfolio selection problem with risk constraints rather than standard deviation attracts more attention for practical implementation (Acerbi and Simonetti 2002; Krokhmal et al. 2002; Chabaane et al. 2006; Huang and Lin 2017). Consequently, assessing the impact regarding the selection of different risk measures on portfolio allocation is of particular importance for asset managers.

When returns are Gaussian distributed, which is parameterized through the first two moments, one could therefore well rely upon the MV framework and the choice of a risk measure is purposeless (Härdle et al. 2014). The empirical study of Adam et al. (2008) based on the monthly returns of 16 hedge funds from January 1990 to July 2001 further showed the robustness of portfolio allocation with respect to the choice of risk measures even the samples are non-Gaussian distributed. Consequently, it seems that the risk managers do not need to worry about the choice of risk measures for portfolio allocation regardless of the Gaussian assumption if the asset returns are assumed to be i.i.d.. However, many empirical studies show that hedge fund returns often exhibit autocorrelation, and have significant negative skewness and excess kurtosis (Giamouridis and Vrontos 2007; Harris and Mazibas 2010, 2013). This motivates us to consider portfolio selection problem without the i.i.d. assumption for asset returns. Furthermore, we investigate the impacts of trader's risk attitude on the performance of optimal portfolios under assuming the asset returns following a multivariate time series model.

To model the autocorrelation and conditional heteroscedasticity of each underlying asset, we consider the following model:

$$\begin{cases} X_{i,t} = f_{i,t}(\mathbf{X}_{i,t-1}, a_{i,t}), \\ a_{i,t} = \sigma_{i,t}\varepsilon_{i,t}, \\ \sigma_{i,t} = h_{i,t-1}(\sigma_{i,s}, \varepsilon_{i,s}; s = 0, \ldots, t-1), \end{cases} \tag{3.1}$$

where $X_{i,t}$ is the log return of the ith asset at time t, $f_{i,t}$ is a function of $\mathbf{X}_{i,t-1} = (X_{i,0}, X_{i,1}, \ldots, X_{i,t-1})$ and $a_{i,t}$ for $i = 1, \ldots, p$, $h_{i,t-1}$ is an $\mathcal{F}_{t-1}$ measurable function with $\mathcal{F}_{t-1}$ being the set of information from time 0 up to time $t-1$ and $\varepsilon_{i,t}$, $t = 0, 1, \ldots$, are i.i.d. innovations with zero mean and unit variance for the ith asset at time t. In addition, assets on the financial markets usually exhibit dependence. For example, the stock prices of two companies which have a complementary relationship may both increase or decrease simultaneously by public good or bad news (Zhang et al. 2015). Recent studies indicate that pair-copula decomposed models represent a more flexible way to construct multivariate distributions than standard multivariate copulae. Therefore, we model the joint distribution of $\varepsilon_{i,t}$, $i = 1, \ldots, p$, by a vine copula function. Vine copulae are able to model complex dependency patterns by using a cascade of bivariate copulae (see Aas et al. 2009; Brechmann and Schepsmeier 2013 and the references therein).

Assume that $X_{i,t}$, for $i = 1, \ldots, p$ and $t = 0, 1, \ldots$, follow model (3.1) and consider the following portfolio optimization problem:

$$\max_{\mathbf{c}_t} \ \pi(\mathbf{c}_t) = c_{1,t}\mathsf{E}_t(X_{1,t+1}) + c_{2,t}\mathsf{E}_t(X_{2,t+1}) + \cdots + c_{p,t}\mathsf{E}_t(X_{p,t+1}),$$
$$\text{subject to } \mathbf{c}_t \geq 0, \ \sum_{i=1}^{p} c_{i,t} \leq 1 \text{ and } \rho_t(\nu) \leq L, \tag{3.2}$$

where $\mathbf{c}_t = (c_{1,t}, \ldots, c_{p,t})^\top$ with $c_{i,t}$ being the holding position of $X_{i,t}$, $\mathbf{c}_t \geq 0$ is the no short-selling constraint, $\sum_{i=1}^{p} c_{i,t} \leq 1$ is the budget constraint, $\mathsf{E}_t(\cdot)$ denotes the conditional expectation given $\mathcal{F}_t$, $\rho_t(\nu)$ is the value of the time-t SRM with level ν, which reflects the degrees of risk aversion, and L is a pre-specified upper bound of risk. The main reason that we employ the SRM as the risk measure in this chapter is its link to investor's risk preference. The SRM is not only a general family of coherent risk measures (for example, the ES is a special case of the SRM), but also can reflect the degrees of risk aversion of investors since the generator of the SRM can be obtained by a trader's personal utility function. More details of the definition and properties of the SRM are introduced in Sect. 3.2.

Although model (3.1) is capable of depicting the dynamics of the underlying returns better than the traditional i.i.d. assumption, the corresponding computation of determining the optimal asset allocations in (3.2) becomes complicated. Harris and Mazibas (2013) considered a portfolio selection problem with the ES being the risk measure and employed an AR(1)-EGARCH(1,1) model to depict the marginal dynamics of the return process for each underlying asset. Moreover, they used copulae to model the dependence between the underlying assets. Since the linearization of the optimal portfolio selection problem under this realistic but complex model is difficult and not available yet in the literature, the method based on Monte Carlo simulation is proposed to obtain the optimal asset allocations. However, the simulation based method could be time consuming and the simulation biases could lead to wrong decision, especially when the optimal solution occurs on the boundary.

In the literature, linear programming (LP) is widely used in portfolio selection under the i.i.d. assumption. LP is a fast algorithm to obtain accurate estimates of the optimal asset allocations, especially when the optimal solution occurs on the boundary. Due to the principal that potential return rises with an increase in risk, the optimal solution of the portfolio selection problem usually occurs on the boundary and thus LP is a suitable technique for solving it. For example, Markowitz (1952) used LP to solve the MV portfolio selection problem. Rockafellar and Uryasev (2000) considered portfolio selection problem with ES and proposed a linearization to select the optimal portfolio by LP. Recently, Huang and Lin (2017) proposed a linearization scheme to approximate the original portfolio selection problem and then obtain the optimal asset allocations by LP when the SRM is used as the risk measure.

In the simulation study, we conduct several scenarios to investigate the accuracy of the proposed LP for obtaining the optimal allocations, the effects of the magnitude of tail dependence and the degrees of risk aversion on the performance of the optimal portfolio. We also conduct empirical studies by using the underlying stock prices included in FTSE TWSE Taiwan 100 Index. Our empirical results indicate that the optimal portfolios have different reactions to different economic situations.

The remainder of this chapter is organized as follows. Section 3.2 reviews some backgrounds including coherent measures of risk, utility functions, SRM and vine copulae. The LP of Huang and Lin (2017) for solving (3.2) with model (3.1) is introduced in Sect. 3.3. Simulation studies are presented in Sect. 3.4. Section 3.5 demonstrates empirical results by using the stock prices included in the FTSE TWSE Taiwan 100 Index. Concluding remarks are given in Sect. 3.6. Computational details are presented in the Appendix.

3.2 Backgrounds

3.2.1 Coherent Measures of Risk

Let $\mathcal{G}$ be the set of random portfolio returns, ρ be a risk measure, which is a mapping from $\mathcal{G}$ into $\mathbb{R}$, and X denote the return of an asset.

(A1) Translation invariance: If A is a deterministic portfolio with guaranteed return α, then for all $X \in \mathcal{G}$ we have $\rho(X + A) = \rho(X) - \alpha$.
(A2) Subadditivity: For all X and $Y \in \mathcal{G}$, $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$.
(A3) Positive homogeneity: For all $\lambda \geq 0$ and all $X \in \mathcal{G}$, $\rho(\lambda X) = \lambda\rho(X)$
(A4) Monotonicity: For all X and $Y \in \mathcal{G}$ with $X \leq Y$, we have $\rho(Y) \leq \rho(X)$.
(A5) Law invariance: For any portfolio returns X and Y with distribution function F_X and F_Y, respectively, if $F_X = F_Y$, then $\rho(X) = \rho(Y)$.
(A6) Comonotonic additivity: For any comonotonic random variables X and Y, $\rho(X + Y) = \rho(X) + \rho(Y)$.

A risk measure satisfying (A1)–(A4) is called coherent (Artzner et al. 1999). Unfortunately, the popular risk measure, VaR, is not coherent since VaR fails to comply with the subadditivity property and thus does not provide good incentives with respect to portfolio diversification. In addition, it is not in general continuous with respect to the confidence level α. Consequently VaR is sensitive to small changes in α when it is applied to discontinuous distributions (Acerbi and Tasche 2002). On the other hand, Dhaene et al. (2004) showed that the ES is a coherent, law invariant (A5) and comonotonic additive (A6) risk measure. Thus, the ES can be treated as a coherent extension of the VaR.

3.2.2 Utility Function

When a consumer or an investor exposed to uncertainty, a risk-averse investor might choose to accept with a low but guaranteed payment, rather than choosing an investment with high expected returns but also with high risk of losing money. Let $U(x)$ be the utility function of a risk-averse investor, where x denotes the wealth. The aversion to risk implied by a utility function $U(\cdot)$ is to be assumed as a form of concavity (Pratt 1964). The more the curvature of a concave function $U(x)$, the more the risk aversion is there. Hence, a more risk-averse investor prefers a more conservative investment. In the following, three popular utility functions are briefly introduced through the absolute risk-aversion, denoted by $A(x) = -U''(x)/U'(x)$, and the relative risk-aversion, abbreviated as $R(x) = -xU''(x)/U'(x)$, (Leroy and Werner 2001):

1. Constant Absolute Risk-Aversion (CARA): If $A(x)$ is a positive constant which is independent of wealth x, then we call the corresponding utility function being CARA. For example, the negative exponential utility function defined by $U(x) = -e^{-\nu x}$ is a CARA utility.
2. Constant Relative Risk-Aversion (CRRA): If $R(x)$ is a positive constant R which is independent of wealth x, then we call the corresponding utility function being CRRA. If $R = 1$, then the utility function of CRRA can be written as $U(x) = \ln x$, for $x > 0$, which is called log utility. If $R \neq 1$, then $U(x) = \frac{x^{1-R}}{1-R}$, for $x > 0$, which is called power utility.
3. Hyperbolic Absolute Risk-Aversion (HARA): If a utility function satisfies $A(x) = -U''(x)/U'(x) = 1/(ax + b)$, which is a hyperbolic function of x, then it is called HARA. In particular, the HARA encompasses the CARA and CRRA cases since it reduces to the CARA if $a = 0$ and reduces to the CRRA if $b = 0$. In general, if $ab \neq 0$, the utility function of the HARA can be written as

$$U(x) = \begin{cases} \log(x - x_s), & \text{if } a = 1, \\ \dfrac{(x - x_s)^{1-R^*}}{1 - R^*}, & \text{otherwise,} \end{cases}$$

for $x > x_s$, and $U(x) = -\infty$, for $x \leq x_s$, where $R^* = 1/a$ and $x_s = -b/a$.

3.2.3 *Spectral Measures of Risk*

A general class of coherent risk measures, called spectral risk measure (SRM), is defined by

$$M_\phi(X) = -\int_0^1 \phi(p) F_X^{\leftarrow}(p) dp, \tag{3.3}$$

where $F_X^{\leftarrow}(p) = \inf\{x | F_X(x) \geq p\}$ and $\phi \in \mathscr{L}^1([0,1])$ is called the risk aversion function of the risk measure $M_\phi(X)$. In addition, ϕ is said to be an "admissible" risk spectrum if it is non-negative, non-increasing and $\int_0^1 \phi(p)dp = 1$. SRM is a coherent measure of risk if ϕ is an admissible risk spectrum (Acerbi 2002). In the realm of spectral measures, an investor can optimize a portfolio in a more articulated way by expressing her subjective risk aversion via the function ϕ (Acerbi and Simonetti 2002).

Acerbi (2002) further mapped any rational investor's subjective risk aversion (or utility preference) onto a SRM. For example, if we consider the exponential utility function defined over random outcomes x by $U(x) = -e^{-\nu x}$, where $\nu > 0$, then the risk aversion function $\phi(\cdot)$ is defined by setting $\phi(p) \propto e^{-\nu p}$. To satisfy the constraint $\int_0^1 \phi(p)dp = 1$, we have $\phi(p) = \frac{\nu e^{-\nu p}}{1-e^{-\nu}}$, where $0 < p < 1$.
Additionally, since the ES can be expressed as

$$\mathrm{ES}(X) = -\frac{1}{\alpha}\int_0^\alpha F_X^{\leftarrow}(p)dp = -\int_0^1 \phi_{ES_\alpha}(p)F_X^{\leftarrow}(p)dp, \text{ for } 0 \le \alpha \le 1,$$

where $\phi_{ES_\alpha}(p) = \frac{1}{\alpha}\mathrm{I}_{\{p\le\alpha\}}$ with $\mathrm{I}_{\{\cdot\}}$ being an indicator function, thus the SRM defined in (3.3) can be expressed as a weighted average of expected shortfalls (Acerbi 2004).

3.2.4 Vine Copulae: C- and D-Vines

Traditionally, traders evaluate the performance and risk of a portfolio under the multivariate Gaussian assumption. However, many empirical studies found that this assumption is not adequate for financial data (Danielsson et al. 2006; Morton et al. 2006; Giamouridis and Vrontos 2007). Copulae help to release the Gaussian assumption and offer a general class of joint distributions. It uses a copula function to link the marginal distributions of individual asset returns to depict the dependence structure.

Copula has recently become increasingly popular in many fields of applications for constructing multivariate distributions (Choros et al. 2013, 2014). It establishes the link between the univariate margins and the multivariate distribution functions. The main concern in practical implementation is how to identify an adequate family of copulae. A rich variety of bivariate copula families is well-investigated in the literature (Joe 1997; Nelsen 2006). However, the choice of adequate families for higher dimensions is more challenging. Standard multivariate copulae such as the multivariate Gaussian, Student-t and Archimedean copulae lack the flexibility of accurately modeling the dependence among larger numbers of variables. In stead of generalizing the standard multivariate copulae by increasing the complexity of their structures, vine copulae propose to model multivariate dependency by using and benefiting from the rich variety of bivariate copulae as building blocks (Joe 1996; Bedford and Cooke 2001, 2002; Kurowicka and Cooke 2006).

Vine copulae are flexible graphical models for describing multivariate distributions by decomposing a multivariate density into a series of bivariate copulae, or called pair-copulae, where each pair-copula can be chosen independently from each others (Aas et al. 2009; Brechmann and Schepsmeier 2013). This decomposition allows for an enormous flexibility in modeling asymmetries and tail dependence of a large number of variables. Aas et al. (2009) proposed a method for statistical inference of pair-copula decomposed models. Brechmann and Schepsmeier (2013) established an R package, called CDVine, which provides functions and tools for statistical inference of canonical vine (C-vine) and D-vine copulae, where the C- and D-vines are two successful and popular vine copula families in many applications (see Brechmann and Schepsmeier 2013, and the references therein). In the following, we employ the multivariate distribution with 4 variables as an example to briefly illustrate the 4-dimensional C- and D-vines.

There are 12 different 4-dimensional C-vine forms and 12 different 4-dimensional D-vine forms, and none of them are the same. The 4-dimensional C-vine structure is generally represented as

$$\begin{aligned} f_{1234}(\mathbf{x}) = & f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot f_4(x_4) \cdot \\ & c_{12}\{F_1(x_1), F_2(x_2)\} c_{13}\{F_1(x_1), F_3(x_3)\} c_{14}\{F_1(x_1), F_4(x_4)\} \cdot \\ & c_{23|1}\{F(x_2 \mid x_1), F(x_3 \mid x_1)\} c_{24|1}\{F(x_2 \mid x_1), F(x_4 \mid x_1)\} \cdot \\ & c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\} \end{aligned} \tag{3.4}$$

and the 4-dimensional D-vine structure is represented as

$$\begin{aligned} f_{1234}(\mathbf{x}) = & f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot f_4(x_4) \cdot \\ & c_{12}\{F_1(x_1), F_2(x_2)\} c_{23}\{F_2(x_2), F_3(x_3)\} c_{34}\{F_3(x_3), F_4(x_4)\} \cdot \\ & c_{13|2}\{F(x_1 \mid x_2), F(x_3 \mid x_2)\} c_{24|3}\{F(x_2 \mid x_3), F(x_4 \mid x_3)\} \cdot \\ & c_{14|23}\{F(x_1 \mid x_2, x_3), F(x_4 \mid x_2, x_3)\}, \end{aligned} \tag{3.5}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $f_{1234}(\mathbf{x})$ is the joint density of (X_1, X_2, X_3, X_4), $f_i(x_i)$ is the marginal density of X_i, $F_i(x_i)$ is the distribution function of X_i for $i = 1, 2, 3, 4$, $F(x_2 \mid x_1)$ is the conditional distribution function of X_2 given X_1, $c_{12}\{F_1(x_1), F_2(x_2)\}$ is a pair copula density of X_1 and X_2, $c_{23|1}\{F(x_2 \mid x_1), F(x_3 \mid x_1)\}$ is the conditional pair copula density of X_2 and X_3 given X_1 and so on. The details of the deviation of (3.4) and (3.5) are given in the Appendix.

The C- and D-vine trees help us to easily memorize the decompositions of (3.4) and (3.5). For example, the corresponding structure of a 4-dimensional C-vine including 3 trees is shown in Fig. 3.1a. In the first tree, the dependencies of the first and second variables, of the first and third, of the first and fourth, and so on, are modeled by pair copulae. That is, if we assign the orders 1,…,4 to the four random variables, then the pairs of $(1, 2)$, $(1, 3)$, $(1, 4)$, ... are modeled by bivariate copulae. In the second tree, $(2, j \mid 1)$ denotes the conditional dependence of the second and the jth variables given the first variable, for $j = 3, 4$, and a bivariate copula is employed to model each conditional distribution. In the third tree, we denote the conditional dependence of $(2, 3 \mid 1)$ and $(2, 4 \mid 1)$ by $(3, 4 \mid 1, 2)$ and again model the conditional joint distribution of $(3, 4 \mid 1, 2)$ by a bivariate copula. By comparing the C-vine trees with the decomposition given in (3.4), the pairs shown in the C-vine trees are exactly the same with the components of the pair copulae in (3.4). Similarly, Fig. 3.1b presents the corresponding 4-dimensional D-vine trees to (3.5).

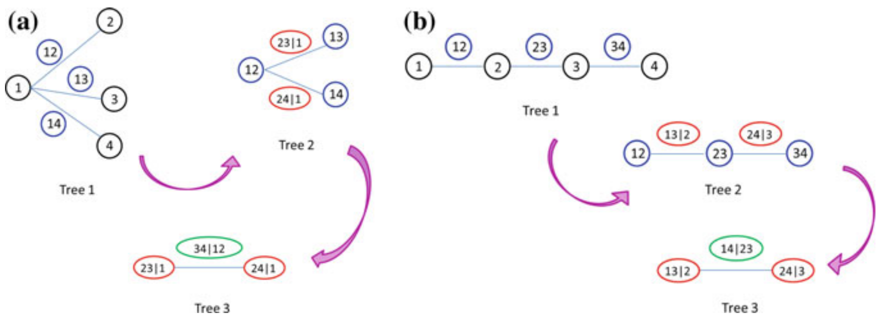

Fig. 3.1 **a** A 4-dimensional C-vine tree. **b** A 4-dimensional D-vine tree

3.3 Methodology

Rockafellar andUryasev (2000; 2002) proposed a scheme of linearization of the optimization problem (3.2) with the ES under the assumption of i.i.d. returns. In the following, we present their technique with the ES in our notation. First, rewrite the ES:

$$ES_{\alpha,t} = -\mathsf{E}_t(Y_{t+1} \mid -Y_{t+1} > \xi_{\alpha,t}) = \xi_{\alpha,t} + \frac{1}{\alpha}\mathsf{E}_t(-Y_{t+1} - \xi_{\alpha,t})^+,$$

where $Y_{t+1} = \sum_{m=1}^{p} c_{m,t} X_{m,t+1}$ is the portfolio return at time $t+1$ and $\xi_{\alpha,t}$ is the corresponding VaR of Y_{t+1} with respect to α level at time $t+1$ conditional on $\mathcal{F}_t$. Then, the optimization problem (3.2) with the ES can be rewritten as

$$\max_{\mathbf{c}_t, \xi_{\alpha,t}, z_1, \ldots, z_t} \mathsf{E}_t(Y_{t+1}) \text{ subject to } \mathbf{c_t} \geq 0,\ \sum_{m=1}^{p} c_{m,t} \leq 1,\ \text{and}$$

$$\begin{cases} \xi_{\alpha,t} + \frac{1}{t\alpha}\sum_{i=1}^{t} z_i & \leq L, \\ z_i & \geq 0, \\ z_i + \xi_{\alpha,t} & \geq -Y_i,\ \text{for } i = 1, \ldots, t, \end{cases} \tag{3.6}$$

by incorporating z_i's to extend the set of unknown parameters. In (3.6), the objective function and the constraints are now linear functions of the unknown parameters $\{\mathbf{c}_t, \xi_{\alpha,t}, z_1, \ldots, z_t\}$ and thereby a LP technique can be used to obtain $\mathbf{c}_t$.

However, many empirical studies show that the return processes of the underlying assets in financial markets usually exhibit autocorrelation, negative skewness, kurtosis, conditional heteroscedasticity and tail dependence (Giamouridis and Vrontos 2007; Choros et al. 2013, 2014). It is of particular importance for asset managers to incorporate these features of the financial time series data when creating an investment or hedging portfolio. In order to model the autocorrelation and conditional heteroscedasticity, we assume that the mth underlying return process $X_{m,t}$, $m = 1, \ldots, p$, follows (3.1) and the joint distribution of $(\varepsilon_{1,t}, \ldots, \varepsilon_{p,t})$ is modeled by a C- or D-vine for depicting the multidimensional dependence among the underlying assets. Model (3.1) includes various financial time series models which are widely used in the market. For example, the ARMA-GARCH and ARMA-EGARCH models are two particular cases being commonly discussed in the economic, statistical, and financial literatures (see Bollerslev 1986; Nelson 1990; Duan 1995; Brandt and Jones 2006; Harveya and Sucarrat 2014).

Huang and Lin (2017) extended the i.i.d. scenario of Rockafellar and Uryasev (2000; 2002) to a more realistic situation as illustrated in (3.1) and linearize the nonlinear optimization problem in (3.2) with SRM. In particular, if we employ the ES, which is a special case of the SRM, as the risk measure, then the optimization problem (3.2) can be rewritten as

$$\max_{\mathbf{c}_t, \xi^*_{\alpha,t}, z_1, \ldots, z_t} \mathsf{E}_t(Y_{t+1}), \text{ subject to } \mathbf{c_t} \geq 0, \ \sum_{m=1}^{p} c_{m,t} \leq 1, \text{ and}$$

$$\begin{cases} L \geq -\sum_{m=1}^{p} c_{m,t}\mu_{m,t} + \xi^*_{\alpha,t} + \frac{1}{t\alpha}\sum_{i=1}^{t} z_i, \\ z_i \geq 0 \\ z_i \geq -\xi^*_{\alpha,t} - \kappa_i, \ \text{for } i = 1, \ldots, t, \end{cases} \tag{3.7}$$

where $\mu_{m,t} = \mathsf{E}_t(X_{m,t+1})$, $\kappa_i = \sum_{m=1}^{p} c_{m,t}\sigma_{m,t+1}\varepsilon_{m,i}$, $\xi^*_{\alpha,t}$ is the corresponding VaR of κ_{t+1} with respect to α level at time $t+1$ conditional on $\mathcal{F}_t$. From comparing the expressions of (3.6) and (3.7), one can find the following three major changes:

1. The 1st term on the right-hand-side of the 1st inequality in (3.7) stands for the autocorrelated part.
2. On the right-hand-side of the 3rd inequality in (3.7) since the m-th summand of κ_i includes the conditional volatility $\sigma_{m,t+1}$, thus κ_i reflects the effect of conditional heteroscedasticity.
3. The role of the i.i.d. returns $X_{m,i}$ in (3.6) for each fixed m is replaced by the i.i.d. innovations $\varepsilon_{m,i}$ contained in κ_i in (3.7).

3.4 Simulation Study

In this section, we conduct several simulation scenarios to investigate the accuracy of the LP, the effects of the magnitude of tail dependence and the degrees of risk aversion on the performance of the optimal portfolio.

3.4.1 A 2-Dimensional Case

First, for the purpose of demonstration we concentrate on $p = 2$.

1. Generate observations of the mth underlying return process from the following AR(1)-EGARCH(1, 1) model, $m = 1, 2$,

$$\begin{cases} X_{m,t} &= \phi_{m,0} + \phi_{m,1} X_{m,t-1} + a_{m,t}, \\ a_{m,t} &= \sigma_{m,t} \varepsilon_{m,t}, \\ \log \sigma^2_{m,t} &= k_m + G_m \log \sigma^2_{m,t-1} + A_m[|\varepsilon_{m,t-1}| - \mathsf{E}(|\varepsilon_{m,t-1}|)] + L_m \varepsilon_{m,t-1}, \end{cases} \tag{3.8}$$

   where $(\varepsilon_{1,t}, \varepsilon_{2,t})$ are i.i.d. samples from a bivariate t distribution with zero means, unit variances, correlation ρ, and $\nu_1 = \nu_2 = \nu$. In particular,

$$\mathsf{E}(|\varepsilon_{m,t-1}|) = \frac{2\sqrt{\nu-2}\,\Gamma[(\nu+1)/2]}{(\nu-1)\Gamma(\nu/2)\sqrt{\pi}}.$$

2. Solve the optimization problem defined in (3.1) and (3.2) for ES and SRM cases, where $\alpha = 0.05$ for the ES and the generating function $\phi(\cdot)$ for the SRM is set to be $\phi(p) = 10e^{-10p}/(1 - e^{-10})$ for $0 \le p \le 1$.

The expected returns (on the upper panel) and the values of risks (on the lower panel) of portfolios with different holding weights, c_1, of the 1st underlying asset under the model (3.8) are presented in Figs. 3.2 and 3.3 with ES and SRM, respectively.

The parameters in (3.8) are set to be $\rho = 0.5$, $\nu = 10$, $\phi_{1,0} = 0.01$, $\phi_{2,0} = 0.0105$, $\phi_{1,1} = 0.02$, $\phi_{2,1} = 0.0199$, $k_1 = k_2 = -0.3$, $A_1 = A_2 = 0.1776$, $G_1 = G_2 = 0.95$ and $L_1 = L_2 = -0.05$, and the upper bound L of the ES (or SRM) is set up to be the value of the ES (or SRM) of the portfolio with $c_1 = c_2 = 0.5$. Figure 3.2 plots the results of the ES case and Fig. 3.3 presents the results of the SRM with $T = 250$ on the left panel and $T = 500$ on the right panel. The red dashed lines in the lower panel denote the predetermined upper bound of the risk. If the value of the risk of a specified c_1 is below the red dashed line, then we plot the corresponding point in green, otherwise we mark the point in blue. The red circles on the upper panel denote the optimal solution calculated from the LP, which are close to the optimal selection of c_1 shown in Figs. 3.2 and 3.3, especially we increase the number of observations T to 500. This phenomenon confirms the accuracy of the proposed method in this 2-dimensional case.

3.4.2 The Impacts of Tail-Dependence

In this section, we investigate the impacts of tail-dependence under bear or bull markets. Consider the case of 10 assets, where assets 1–5 are independent and assets 6–10 have nonlinear tail dependency. We employ a 5-dimensional D-vine to model

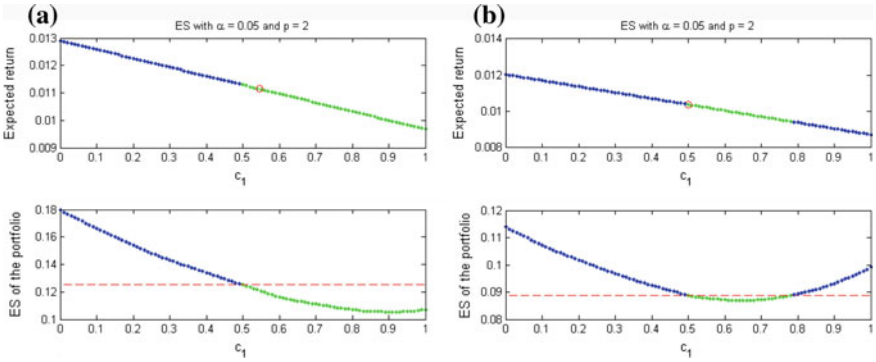

Fig. 3.2 The expected returns and the values of the ES of portfolios with different holding weights c_1 of the 1st underlying asset under model (3.8), where the numbers of observations are **a** $T = 250$ and **b** $T = 500$. XFGexp_rtn_ES_2d

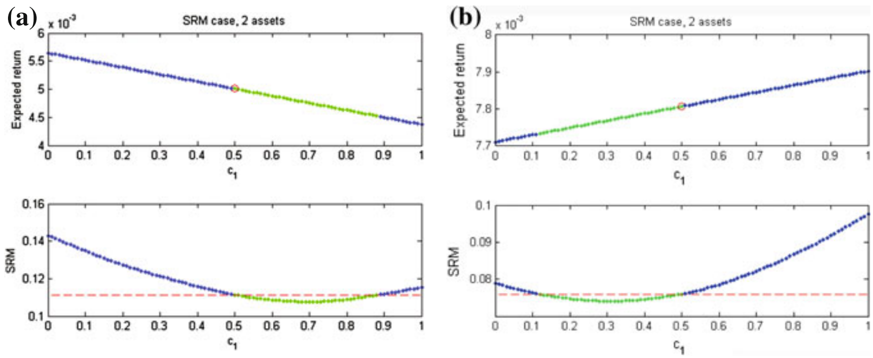

Fig. 3.3 The expected returns and the values of the SRM of portfolios with different holding weights c_1 of the 1st underlying asset under model (3.8), where the numbers of observations are **a** $T = 250$ and **b** $T = 500$. XFGexp_rtn_SRM_2d

the joint distribution of the dependent assets 6–10. In particular, we employ bivariate Clayton and Gumbel copulae to describe the nonlinear tail dependency between assets 6–10 in the first tree of the D-vine for bear and bull markets, respectively, where the copula parameters are randomly chosen from a $U(3,5)$ random variable. By using the same settings as in Sect. 3.4.1, except for setting $\phi_{i,0} = 0.1$, for $i = 1, \ldots, 5$, and $(\phi_{i,0}, \phi_{i,1}, k_i) = (0.11, 0.02, -0.28)$, for $i = 6, \ldots, 10$, to enlarge the expected returns of the assets in the bull market case, the optimal allocations are solved by the proposed LP method with ES under the bear and bull markets, separately.

We compute the sums of the weights of the assets 6–10 under bear and bull markets separately. The average of the holding proportions of assets 6–10 in the optimal portfolio based on 100 random replications is around 37% for the bear market and is around 90% for the bull market. These values reveal interesting and reasonable phenomenon. In a bear market, since the lower tail dependence of the assets 6–10 are modeled by a D-vine with Clayton copulae, the prices of the assets 6–10 tend to decrease simultaneously. In practice, diversification strategies are employed by investors in tough economic times. Hence, the independent assets 1–5 are more attractive to investors than the lower tail-dependent assets 6–10 in bear markets. On the contrary, the upper tail dependent assets have higher chance to be selected in the optimal portfolio than the independent assets in bull markets since the assets with upper tail dependencies tend to increase simultaneously.

3.4.3 The Impact of the Degrees of Risk Aversion

In this section, we investigate the performance of the optimal portfolios with different degrees of risk aversion, where each asset return process is assumed to follow an AR(1)-EGARCH(1,1) process. Consider that an investor plans to construct a portfolio by solving (3.2) with 30 assets subject to his personal risk attitude with a HARA utility function $U(x) = \log(x + b)$, where $b \in (-1, 0)$. Let ε be a positive constant satisfying $\max(0, b) < \varepsilon < 1 + b$ and set the generating function $\phi(p)$ of the SRM to be

$$\phi(p) = \begin{cases} \dfrac{-\log \varepsilon}{\eta}, & 0 \le p < \varepsilon - b, \\ \dfrac{-\log(p+b)}{\eta}, & \varepsilon - b \le p \le 1, \end{cases} \tag{3.9}$$

where $\eta = b \log \varepsilon - (1 + b) \log(1 + b) + (1 + b) - \varepsilon > 0$ and b reflects the degrees of risk aversion of an investor.

Figure 3.4a presents boxplots of the optimal expected returns obtained by solving (3.2) with a generating function of the SRM defined in (3.9), where $b = -0.2, -0.3, -0.5$, $\varepsilon = 10^{-4}$ and the number of replications is 100. Figure 3.4b presents the corresponding utility functions, where other parameters in model (3.8) are set to be the same as in Sect. 3.4.1. In Fig. 3.4b, the solid lines are the tangents at $x = 0.7$ for the 3 utility functions. Since the slope of the tangent line in the case of $b = -0.5$ is larger than the others, thus investors having the utility function with $b = -0.5$ are more aggressive than those with $b = -0.2$ and -0.3. Figure 3.4a indicates that less risk-averse or more aggressive investors have larger expected returns than conservative investors.

3.5 Empirical Studies

We carry out our empirical investigation by using underlying assets stock price data included in the FTSE TWSE Taiwan 100 Index. We selected 79 stocks from 100 underlying assets included in the Taiwan 100 Index, where the daily returns from 1,

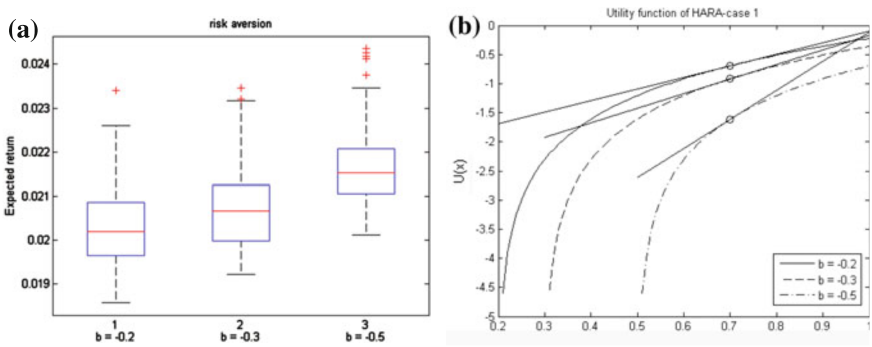

Fig. 3.4 **a** Boxplots of the optimal expected returns obtained by solving (3.2) with a generating function of the SRM defined in (3.9), where $b = -0.2$, –0.3 and –0.5. **b** The corresponding utility functions for $b = -0.2$, -0.3 and -0.5. XFGexp_rtn_SRM

December 2004 through 3, July 2014 (2365 observations) are used for investigation. This period includes a number of financial crises, for example, the subprime lending, stagflation, the Lehman crisis, the Greek government-debt crisis as well as the U.S. monetary policy-QE2. These events caused financial markets to have large volatility variation. In the following, we divide the time period into three sub-periods for the investigation: December 2004 to November 2007 (denoted by P1), representing relatively favorable market conditions (737 observations), December 2007 to December 2010 (denoted by P2), representing more extreme market conditions (764 observations) and January 2011 to 3, July 2014 (denoted by P3), representing improved market conditions (864 observations). We construct a self-financing trading strategy by using the proposed LP method to daily rebalance the portfolio with the 79 stocks for each of the three sub-periods. In particular, the FTSE TWSE Taiwan 100 Index is used as our benchmark for comparison. In the following, we use P1 as an example to illustrate the details of the investigation:

1. Let $P_{m,t}$ and $FTSE_t$ be the price of the mth asset and FTSE TWSE Taiwan 100 Index at time t, where $t = 0$ stands for the date of 1, December 2004.
2. Let V_t denote the value of our portfolio at time t and V_{250} be the same with the value of FTSE TWSE Taiwan 100 Index on 5, December 2005. That is,

$$V_{250} = FTSE_{250} = b^{(250)} \sum_{m=1}^{p} c_{m,250} P_{m,250} + Cash_{250},$$

where $Cash_{250} = FTSE_{250}(1 - \sum_{m=1}^{p} c_{m,250})$ is the amount invested in the bank, $b^{(250)} = FTSE_{250} \sum_{m=1}^{p} c_{m,250} / \sum_{m=1}^{p} c_{m,250} P_{m,250}$ is a scalar such that $V_{250} = FTSE_{250}$, $c_{m,250}$ are obtained by solving (3.2) with ES of level $\alpha = 0.05$ by the proposed LP method, and each underlying return process is modeled by an AR(1)-EGARCH(1,1) based on $X_{m,t} = \ln P_{m,t} - \ln P_{m,t-1}$ for $t = 1, \ldots, 250$ and $m = 1, \ldots, 79$.

3. At time $t = 251$, the value of our portfolio is $V_{251^-} = b^{(250)} \sum_{m=1}^{p} c_{m,250} P_{m,251} + e^{r_{\text{day}}} Cash_{250}$ prior to adjusting the allocations, where r_{day} is the daily riskfree interest rate and is set up as 0.01/250 in our investigation. By using the data $P_{m,t}$, $t = 1, \ldots, 251$, we reestimate the dynamic models of each return process and compute the updated optimal allocations, which are proportional to $c_{m,251}$ obtained from solving (3.2) by LP, where the value of the updated portfolio, denoted by V_{251^+}, is the same as V_{251^-}. That is,

$$V_{251^+} = b^{(251)} \sum_{m=1}^{p} c_{m,251} P_{m,251} + Cash_{251}, \tag{3.10}$$

where $b^{(251)} = V_{251^-} \sum_{m=1}^{p} c_{m,251} / \sum_{m=1}^{p} c_{m,251} P_{m,251}$ is a scalar such that $V_{251^-} = V_{251^+}$ for satisfying self-financing, and $Cash_{251} = V_{251^-} (1 - \sum_{m=1}^{p} c_{m,251})$ is the amount invested in the bank after the reallocation.
4. Repeat Step 3 until the end of P1.

Figure 3.5a–c plot the values of our trading strategy and the FTSE TWSE Taiwan 100 Index for P1, P2 and P3, respectively, where the black line is the values of the Taiwan 100 Index and the upper bound L of the risk is set to be 0.02, 0.03 or 0.05. In Fig. 3.5a–c, the values of the self-financing portfolio with $L = 0.05$ (green line) fluctuate more than those of $L = 0.02$ (red lines) and 0.03 (blue lines) no matter which economic situation is since a more aggressive trading strategy (with larger L) could gain more profits by taking more risks. In particular, the optimal portfolio tends to be more aggressive (with larger L) in bull markets and be more conservative (with smaller L) in bear markets. For example, during the financial crisis from December 2007 to June 2009 in Fig. 3.5b, the optimal portfolios with smaller L perform better than those with larger L.

In practice, investors would not use a fixed L for selecting their optimal portfolio, but rely on constructing the efficient frontier with various L instead. The discussion of how to construct the optimal portfolio through the efficient frontier framework is beyond the scope of this chapter. The objective of this chapter is to demonstrate that the proposed LP is useful to obtain the optimal allocations under conditional heteroscedastic models with more general risk measures than standard deviation. The

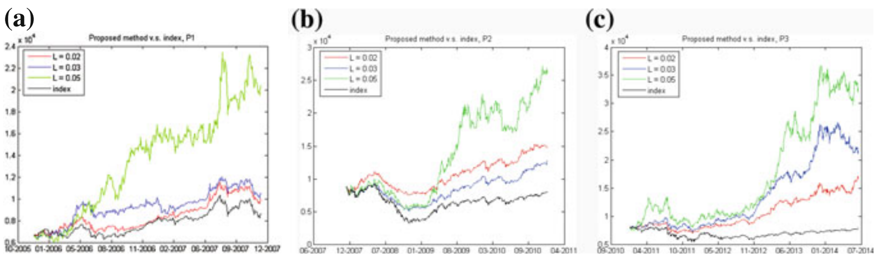

Fig. 3.5 The values of the self-financing trading strategy and the FTSE TWSE Taiwan 100 Index for **a** P1 **b** P2 **c** P3 with different fixed upper bounds of risk. XFGTWSE100_strategy_fixedESlevel

empirical study is designed to investigate whether the optimal portfolio would react to different economic situations if we consider a more complex but more realistic model. Please note though that we did not consider transaction costs in the daily reallocation and also allow to hold fractional numbers of shares of assets. What we have done is to provide an accurate and fast computational method for the investors who use model (3.1) to depict the dynamics of the underlying assets and obtain their optimal allocations of the assets by solving (3.2).

3.6 Concluding Remarks

In this chapter, we considered a portfolio optimization problem with the SRM, where the dynamics of the underlying return processes are depicted by autoregressive and conditional heteroscedastic models. The tail-dependence of the underlying assets is modeled by a CD-vine copula. A linearization of the optimal portfolio selection problem is used to compute the optimal asset allocations accurately and quickly. Simulation studies are conducted to investigate several interesting economic phenomena. First, we demonstrate the accuracy of the LP method for solving the optimal portfolio problem by using the case of two underlying assets. Second, we reveal that the optimal portfolio tends to diversify the investing risk by selecting the independent assets in bear market. Third, the less risk-averse investors achieve larger expected returns than conservative investors. The empirical study indicates that the optimal portfolio tends to be aggressive in bull markets and be conservative in bear markets.

Appendix

Derivation of (3.4) and (3.5)
To show the 4-dimensional C-vine, first note that

$$f_{1234}(\mathbf{x}) = f_1(x_1) f(x_2 \mid x_1) f(x_3 \mid x_1, x_2) f(x_4 \mid x_1, x_2, x_3), \tag{3.11}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $f_{1234}(\mathbf{x})$ is the joint density of (X_1, X_2, X_3, X_4), $f_i(x_i)$ is the marginal density of X_i for $i = 1, 2, 3, 4$, $f(x_2 \mid x_1)$ is the conditional density of X_2 given X_1 and so on. In addition, we have the following identities

$$\begin{aligned}
f(x_2 \mid x_1) &= c_{12}\{F_1(x_1), F_2(x_2)\} f_2(x_2), \\
f(x_3 \mid x_1, x_2) &= \frac{f(x_2, x_3 \mid x_1)}{f(x_2 \mid x_1)} \\
&= c_{23|1}\{F(x_2 \mid x_1), F(x_3 \mid x_1)\} f(x_3 \mid x_1) \\
&= c_{23|1}\{F(x_2 \mid x_1), F(x_3 \mid x_1)\} c_{13}\{F_1(x_1), F_3(x_3)\} f_3(x_3),
\end{aligned}$$

and

$$f(x_4 \mid x_1, x_2, x_3) = \frac{f(x_3, x_4 \mid x_1, x_2)}{f(x_3 \mid x_1, x_2)}$$
$$= c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\} f(x_4 \mid x_1, x_2)$$
$$= c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\} \frac{f(x_2, x_4 \mid x_1)}{f(x_2 \mid x_1)}$$
$$= c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\} c_{24|1}\{F(x_2 \mid x_1), F(x_4 \mid x_1)\} f(x_4 \mid x_1)$$
$$= c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\} c_{24|1}\{F(x_2 \mid x_1), F(x_4 \mid x_1)\}$$
$$c_{14}\{F_1(x_1), F_4(x_4)\} f_4(x_4).$$

By substituting the above identities into (3.11), we have

$$\begin{aligned} f_{1234}(\mathbf{x}) = & f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4) \\ & c_{12}\{F_1(x_1), F_2(x_2)\} c_{13}\{F_1(x_1), F_3(x_3)\} c_{14}\{F_1(x_1), F_4(x_4)\} \\ & c_{23|1}\{F(x_2 \mid x_1), F(x_3 \mid x_1)\} c_{24|1}\{F(x_2 \mid x_1), F(x_4 \mid x_1)\} \\ & c_{34|12}\{F(x_3 \mid x_1, x_2), F(x_4 \mid x_1, x_2)\}. \end{aligned}$$

Therefore, (3.4) holds.

On the other hand, the 4-dimensional D-vine is obtained through the following representation:

$$f_{1234}(\mathbf{x}) = f_2(x_2) f(x_3 \mid x_2) f(x_1 \mid x_2, x_3) f(x_4 \mid x_1, x_2, x_3). \tag{3.12}$$

By using a similar argument to the derivation of the C-vine, we have the following identities:

$$\begin{aligned} f(x_3 \mid x_2) &= c_{23}\{F_2(x_2), F_3(x_3)\} f(x_3), \\ f(x_1 \mid x_2, x_3) &= c_{13|2}\{F(x_1 \mid x_2), F(x_3 \mid x_2)\} f(x_1 \mid x_2) c_{12}\{F_1(x_1), F_2(x_2)\} f_1(x_1), \\ f(x_4 \mid x_1, x_2, x_3) &= c_{14|23}\{F(x_1 \mid x_2, x_3), F(x_4 \mid x_2, x_3)\} c_{24|3}\{F(x_2 \mid x_3), F(x_4 \mid x_3)\}, \\ & \quad c_{34}\{F_3(x_3), F_4(x_4)\} f_4(x_4). \end{aligned}$$

Therefore, (3.12) can be rewritten as

$$\begin{aligned} f_{1234}(\mathbf{x}) = & f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4) \\ & c_{12}\{F_1(x_1), F_2(x_2)\} c_{23}\{F_2(x_2), F_3(x_3)\} c_{34}\{F_3(x_3), F_4(x_4)\} \\ & c_{13|2}\{F(x_1 \mid x_2), F(x_3 \mid x_2)\} c_{24|3}\{F(x_2 \mid x_3), F(x_4 \mid x_3)\} \\ & c_{14|23}\{F(x_1 \mid x_2, x_3), F(x_4 \mid x_2, x_3)\} \end{aligned}$$

and (3.5) holds.

References

Aas, K., Czado, C., Frigessi, A., & Bakken, H. (2009). Pair-copula constructions of multiple dependence. *Insurance: Mathematics and Economics*, *44*, 182–198.

Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking and Finance*, *26*, 1505–1518.

Acerbi, C. (2004). Coherent representation of subjective risk aversion. In G. Szego (Ed.), *Risk measures for the 21st Centry*. New York: Wiley.

Acerbi, C., & Simonetti, P. (2002). Portfolio optimization with spectral measures of risk, Working paper, Abaxbank, Italy.

Acerbi, C., & Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, *26*, 1487–1503.

Adam, A., Houkari, M., & Laurent, J.-P. (2008). Spectral risk measures and portfolio selection. *Journal of Banking and Finance*, *32*, 1870–1882.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, *9*, 203–208.

Bedford, T., & Cooke, R. M. (2001). Probability density decomposition for conditionally dependent random variables modeled by vines. *Annals of Mathematics and Artificial Intelligence*, *32*, 245–268.

Bedford, T., & Cooke, R. M. (2002). Vines-a new graphical model for dependent random variables. *Annals of Statistics*, *30*, 1031–1068.

Brandt, M. W., & Jones, C. S. (2006). Volatility forecasting with range-based EGARCH models, *Journal of Business and Economic Statistics, 24*, 470–486.

Brechmann, E. C., & Schepsmeier, U. (2013). Modeling dependence with C- and D-vine copulas: The R package CDVine. *Journal of Statistical Software*, *52*, 1–27.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*, 307–327.

Chabaane, A., Laurent, J.-P., Malevergne, Y., & Turpin, F. (2006). Alternative risk measures of alternative investments. *Journal of Risk*, *8*, 1–32.

Choros, B., Härdle, W., & Okhrin, O. (2013). Valuation of collaterablized debt obligations with hierarchical Archimedean copulae. *Empirical Finance*, *24*, 42–62. doi:10.1016/j.jempfin.2013.08.001.

Choros, B., Härdle, W., & Overbeck, L. (2014). Copula dynamics in CDOs. *Quantitative Finance*, *14*, 1573–1585. doi:10.1080/14697688.2013.847280.

Danielsson, J., Jorgensen, B. N., Sarma, M., & De Vries, C. G. (2006). Comparing downside risk measures for heavy tailed distributions. *Economics Letters, 92*, 202–208.

Dhaene, J., Goovaerts, M. J., Kaas, R., Vanduffel, S., & Vyncke, D. (2004). Capital requirements, risk measures and comonotonicity. *Belgian Actuarial Bulletin*, *4*, 53–61.

Duan, J. C. (1995). The GARCH option pricing model. *Mathematical Finance, 5*, 13–32.

Giamouridis, D., & Vrontos, I. D. (2007). Hedge fund portfolio construction: A comparison of static and dynamic approaches. *Journal of Banking and Finance*, *31*, 199–217.

Härdle, W., Nasekin, S., Lee, D., & Phoon, K. F. (2014). TEDAS - Tail event driven asset allocation. *Journal of Financial Econometrics*, submitted, SFB 649 DP 2014-32.

Harris, R. D. F., & Mazibas, M. (2010). Dynamic hedge fund portfolio construction. *International Review of Financial Analysis*, *19*, 351–357.

Harris, R. D. F., & Mazibas, M. (2013). Dynamic hedge fund portfolio construction: A semi-parametric approach. *Journal of Banking and Finance*, *37*, 139–149.

Harveya, A., & Sucarrat, G. (2014). EGARCH models with fat tails, skewness and leverage. *Computational Statistics & Data Analysis, 76,* 320–338.

Huang, S. F., & Lin, T. Y. (2017). A linearization of portfolio optimization problem with general risk measures under multivariate conditional heteroscedastic models. *Journal of Forecasting*, submitted.

Joe, H. (1996). Families of m-variate distributions with given margins and $m(m-1)/2$ bivariate dependence parameters. In L. Rüschendorf, B. Schweizer & M.D. Taylor (Eds.), *Distributions with fixed marginals and related topics*. California: Institute of Mathematical Statistic.

Joe, H. (1997). *Multivariate models and dependence concepts*. London: Chapman & Hall.

Krokhmal, P., Palmquist, J., & Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of Risk*, *4*.

Kroll, Y., Levy, H., & Markowitz, H. M. (1984). Mean-variance versus direct utility maximization. *Journal of Finance*, *39*, 47–61.

Kurowicka, D., & Cooke, R. M. (2006). *Uncertainty analysis with high dimensional dependence modelling*. New York: Wiley.

Leroy, S. F., & Werner, J. (2001). *Principles of financial economics*. New York: Cambridge University Press.

Markowitz, H. M. (1952). Portfolio selection. *Journal of Finance*, *7*, 77–91.

Markowitz, H. M. (1959). *Portfolio selection*. New York: Wiley.

Markowitz, H. M. (1991). *Portfolio selection: Efficient diversification of investment*. Cambridge: Blackwell.

Merton, R. C. (1972). An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis*, *7*, 1851–1872.

Morton, D., Popova, E., & Popova, I. (2006). Efficient fund of hedge funds construction under downside risk measures. *Journal of Banking & Finance, 30*, 503–518.

Nelson, D. B. (1990). Stationarity and persistence in the GARCH(l, l) model. *Econometric Theory*, *6*, 318–334.

Nelsen, R. B. (2006). *An introduction to copulas* (2nd ed.). Berlin: Springer.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica, 32*, 122–136.

Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk, 2*, 21–41.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance*, *26*, 1443–1471.

Zhang, J. Z., Härdle, W., Chen, Y. C., & Bommes, E. (2015). Distillation of news flow into analysis of stock reactions. *Journal of Business & Economic Statistics*, *34*, 547–563.