8
Fundamental Balance Relations

Recall from Chap. 1 that one of the best-known equations in science is $\boldsymbol{F} = m\boldsymbol{a}$, which is called Newton's second law of motion. This equation asserts that, relative to an inertial frame of reference, the (time) rate of change of the linear momentum $m\boldsymbol{v}$ for a mass particle m must *balance* the forces $\boldsymbol{F}$ that are applied to the particle. For this reason, this "law of motion" (actually postulate) is also called the balance of linear momentum. Whereas Sir I. Newton considered only individual mass points (like the Moon or an apple), L. Euler showed that many bodies can be treated as a continuous collection of mass points (i.e., a continuum), each particle of which obeys Newton's second law. Indeed, as it turns out, three basic postulates provide the equations of motion for any continua,

Balance of mass
Balance of linear momentum
Balance of energy (i.e., first law of thermodynamics)

to which we often add the postulates of the balance of angular momentum and the entropy inequality (i.e., second law of thermodynamics), both of which can provide restrictions on the allowable constitutive relations. For example, recall from Chap. 2 that the balance of angular momentum requires that the Cauchy stress $[\sigma]$ be symmetric, which restricts possible constitutive relations that are formulated in terms of σ. Each of these five postulates can be stated as either differential equations (for systems) or integral equations (for control volumes). In this chapter, we focus on the *governing differential equations* for mass and linear momentum balance; the differential equation for energy balance is useful in bioheat transfer, which is not addressed herein. Chap. 10 addresses the control volume formulation for mass, linear momentum, and energy. Because these postulates are good for all continua, they apply equally well to biosolids and biofluids; we will see, however, that some of these equations specialize for

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individual material behaviors, which facilitates the formulation and solution of particular problems.

8.1 Balance of Mass

We shall require that the identifiable differential mass Δm be conserved for all time t; that is, in the limit as $\Delta m \to dm$,

$$\frac{d}{dt}(dm) = 0 \to \frac{d}{dt}(\rho d\forall) = 0, \tag{8.1}$$

where ρ is the mass density (having units of mass per volume) and $d\forall$ is a differential volume at any time t. Hence, we have by the product rule,

$$\frac{d}{dt}(\rho d\forall) = \frac{d\rho}{dt} d\forall + \rho \frac{d}{dt}(d\forall) = 0. \tag{8.2}$$

For simplicity, let us assume that the differential mass of interest is in the shape of a cube both at time $t = 0$ and a particular time t sometime during its history. (Of course, if the original system of interest is cuboidal, it would be expected to assume many different shapes when flowing. We simply assume that, "remarkably," it is again a cube at some time t, which is the instant on which we will focus.) Hence, let the differential volume at time t be $d\forall = dx\, dy\, dz$, which was originally a (possibly) different cube having volume $d\forall = dX\, dY\, dZ$. For a cube to deform into another cube, there can be length changes at most (i.e., no shear). Consequently, whereas the position x of a particle in the cube at time t could be a function of X, Y, and Z, in general, and thus by the chain rule

$$dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY + \frac{\partial x}{\partial Z} dZ \tag{8.3}$$

(and similarly for dy and dz). For a cube to deform into a cube, we must have at each time t, only

$$\begin{aligned} x = x(X) &\to dx = \frac{\partial x}{\partial X} dX, \\ y = y(Y) &\to dy = \frac{\partial y}{\partial Y} dY, \\ z = z(Z) &\to dz = \frac{\partial z}{\partial Z} dZ. \end{aligned} \tag{8.4}$$

Hence, $d\mathscr{V}$ at time t is

$$d\mathscr{V} = dx\,dy\,dz = \frac{\partial x}{\partial X}dX\frac{\partial y}{\partial Y}dY\frac{\partial z}{\partial Z}dZ = \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z}d\mathscr{V}, \tag{8.5}$$

where $d\mathscr{V}$ does not change in time because it is defined at $t = 0$. The rate of change of $d\mathscr{V}$ is thus

$$\frac{d}{dt}(d\mathscr{V}) = \frac{d}{dt}\left(\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z}\right)d\mathscr{V}. \tag{8.6}$$

Employing the product rule,

$$\frac{d}{dt}(d\mathscr{V}) = \left[\frac{d}{dt}\left(\frac{\partial x}{\partial X}\right)\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z} + \frac{\partial x}{\partial X}\frac{d}{dt}\left(\frac{\partial y}{\partial Y}\right)\frac{\partial z}{\partial Z} + \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}\frac{d}{dt}\left(\frac{\partial z}{\partial Z}\right)\right]d\mathscr{V}. \tag{8.7}$$

Because the original positions (X, Y, Z) are independent of time, we can interchange the order of the temporal and spatial differentiations. Using Eq. (7.7) and the chain rule, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial x}{\partial X}\right) &= \frac{\partial}{\partial X}\left(\frac{dx}{dt}\right) = \frac{\partial}{\partial X}(v_x) = \frac{\partial v_x}{\partial x}\frac{\partial x}{\partial X},\\
\frac{d}{dt}\left(\frac{\partial y}{\partial Y}\right) &= \frac{\partial}{\partial Y}\left(\frac{dy}{dt}\right) = \frac{\partial}{\partial Y}(v_y) = \frac{\partial v_y}{\partial y}\frac{\partial y}{\partial Y},\\
\frac{d}{dt}\left(\frac{\partial z}{\partial Z}\right) &= \frac{\partial}{\partial Z}\left(\frac{dz}{dt}\right) = \frac{\partial}{\partial Z}(v_z) = \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial Z}.
\end{aligned} \tag{8.8}$$

Substituting these results into Eq. (8.7) and then Eq. (8.2), we have

$$\frac{d}{dt}(\rho d\mathscr{V}) = \left[\frac{d\rho}{dt} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\right]\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z}d\mathscr{V} \tag{8.9}$$

or, by recalling Eq. (7.48),

$$\frac{d}{dt}(\rho d\mathscr{V}) = \left[\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{v})\right]d\mathscr{V} = 0. \tag{8.10}$$

Because this equation must hold for any $d\mathscr{V}$, not all zero, this implies that

$$\frac{d\rho}{dt} + \rho(\nabla \cdot v) = 0, \tag{8.11}$$

which is our (local) statement of the balance of mass. Because the mass density could differ at different points (x, y, z) or at different times, then

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t}\frac{dt}{dt} + \frac{\partial \rho}{\partial x}v_x + \frac{\partial \rho}{\partial y}v_y + \frac{\partial \rho}{\partial z}v_z, \tag{8.12}$$

similar to Eq. (7.15) (i.e., the Eulerian description of acceleration). Hence, Eq. (8.11) could also be written as $\partial\rho/\partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0$. Regardless, if ρ is a constant, $d\rho/dt = 0$, and the balance of mass requires only that

$$\nabla \cdot \boldsymbol{v} = 0. \tag{8.13}$$

This is the mass balance relation for an incompressible flow, as alluded to in Chap. 7.

Finally, it should be noted that because our final expression for mass balance can be written in vector form, it is independent of coordinate system and therefore completely general. That is, the derivation based on deforming a cube into a cube was simply used for convenience; it is not a restricted case. Using mathematics beyond that typically available to the beginning undergraduate, this derivation can be repeated exactly for an arbitrarily shaped Δm (Humphrey 2002). Herein, however, we shall simply focus on its use, not its general derivation. Given Eq. (8.13) and the definition of the del operator for various coordinate systems (Appendix 7 of Chap. 7), one can show that mass balance for an incompressible flow requires

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{8.14}$$

in Cartesians,

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{8.15}$$

in cylindricals, and

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0 \tag{8.16}$$

in sphericals. The latter two result from Exercises 7.5 and 7.6.

Observation 8.1. Note that $\nabla \cdot \boldsymbol{v} = \text{tr}[D]$; that is, the divergence of the velocity equals the sum of the diagonals of the rate of deformation when it is written in matrix form as $[D]$. Recall Eqs. (7.55)–(7.60). In Cartesians, therefore, incompressibility requires that

$$\nabla \cdot \boldsymbol{v} = D_{xx} + D_{yy} + D_{zz} = 0,$$

where D_{xx}, D_{yy}, and D_{zz} are measures of the rates at which line elements change length in the x, y, and z directions. Our intuition is thus supported by this equation: for volume to be conserved, lengthening in at least one direction must be accompanied by shortening in at least one direction.

Example 8.1 Is the following velocity field a possible incompressible flow?

$$\boldsymbol{v}(x, y, z, t) = \frac{\rho g \sin\theta}{\mu}\left(yh - \frac{y^2}{2}\right)\hat{\boldsymbol{i}}$$

where ρ is the mass density of the fluid, g (=9.81 m/s^2) is the gravitational constant, μ is the viscosity of the fluid, θ is some fixed angle (number) relative to a horizontal datum, and h is some depth of a fluid film.

Solution:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial}{\partial x}\left[\frac{\rho g \sin\theta}{\mu}\left(yh - \frac{y^2}{2}\right)\right] + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 0;$$

so yes, this is a possible incompressible flow field. This velocity field will be determined formally in Example 9.3 via the solution of the equation of motion for a particular boundary value problem.

8.2 Balance of Linear Momentum

As noted earlier, Euler showed that Newton's statement of the balance of linear momentum for a mass particle (i.e., $\boldsymbol{F} = m\boldsymbol{a}$) can be generalized for a continuum (i.e., infinite collection of particles). Hence, let us apply Newton's second law to our differential mass Δm, which we shall again take to be a differential cube having volume $\Delta x \Delta y \Delta z$ and mass density ρ (i.e., $\Delta m = \rho \Delta x \Delta y \Delta z$). Two types

of forces of importance in continuum mechanics are those that act on every particle in the continuum, called *body forces*, and those that act on the body only through its surface, the *surface forces*. Let the body force that the fluid element experiences be defined per unit mass and denoted $\boldsymbol{g} = g_x\hat{\boldsymbol{i}} + g_y\hat{\boldsymbol{j}} + g_z\hat{\boldsymbol{k}}$. The most common example of a body force is gravity. Moreover, let the forces acting on the surface of the cube be computed via the appropriate Cauchy stress ($\sigma_{(\text{face})(\text{direction})}$ relative to a prescribed coordinate system) multiplied by its respective surface area. Common surface forces are hydrostatic pressure and frictional forces between fluid particles moving relative to each other. Desiring to let the cube shrink to a point (i.e., in the limit as Δx, Δy, $\Delta z \to 0$), let the components of the stress at the center of Δm be σ_{xx}, σ_{xy}, σ_{xz}, …, σ_{zz}. Next, assume that the stress may vary from point to point[1]; thus, the stresses on each of the faces of Δm must differ from those in the center (although the stresses also vary over each face, we shall represent the stresses on a given face by their mean value, which will be appropriate as we shrink to a point). This difference from face to face is expected to be small, however, because the distance from the center, located at (x, y, z), to each face is small (e.g., $\Delta x/2$, $\Delta y/2$, $\Delta z/2$). Hence, we consider a Taylor's series expansion about the center. For example, for the normal stress on the positive x face, we have

$$\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right) + \text{H.O.T.}, \tag{8.17}$$

(where *H.O.T.* stands for higher-order terms such as Δx^2 and so forth, which are negligible with respect to Δx, as shown in Sect. 3.1 of Chap. 3) and similarly for each component and each face. Remembering that we are summing forces (i.e., stresses acting over oriented areas), we have (Fig. 8.1)

$$\begin{aligned}\sum F_x = ma_x \to &\left\{\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right)\right] - \left[\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{\Delta x}{2}\right)\right]\right\}\Delta y \Delta z \\ &+ \left\{\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right)\right] - \left[\sigma_{yx} - \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\right\}\Delta x \Delta z \\ &+ \left\{\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(\frac{\Delta z}{2}\right)\right] - \left[\sigma_{zx} - \frac{\partial \sigma_{zx}}{\partial z}\left(\frac{\Delta z}{2}\right)\right]\right\}\Delta x \Delta y \\ &+ \rho g_x \Delta x \Delta y \Delta z = \rho \Delta x \Delta y \Delta z a_x.\end{aligned} \tag{8.18}$$

[1] This is similar to that done in solids as, for example, letting the moment in the beam element $M(x)$ be $M(x) + \Delta M(x)$ at $x + \Delta x$ and so on. See also Sect. 3.1 of Chap. 3.

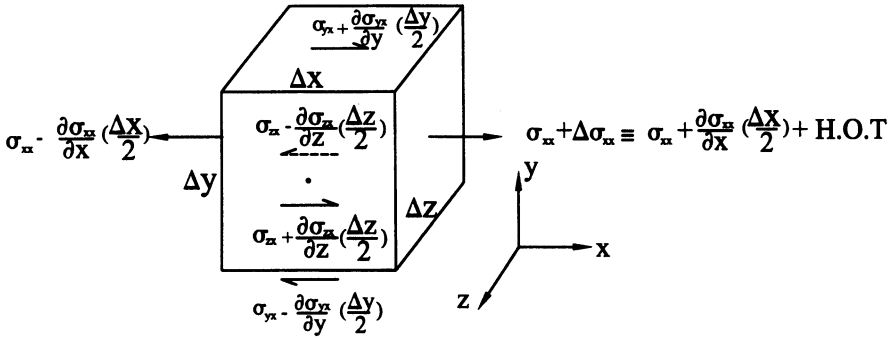


FIGURE 8.1 Force balance for a fluid element of cuboidal shape that is accelerating and subjected to body forces. For simplicity, x-direction contributions only are given.

Simplifying and taking the limit, we have

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \frac{1}{\Delta x\Delta y\Delta z}\left(\left[\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z+\rho g_x\Delta x\Delta y\Delta z-\rho a_x\Delta x\Delta y\Delta z\right)=0, \tag{8.19}$$

or, as our final result in the x direction,

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}+\rho g_x=\rho a_x. \tag{8.20}$$

Note that the first subscript on the stress denotes the face on which the force acts, whereas the second subscript denotes the direction of the force—each σ in this equation appropriately has x for the second subscript because this is an x-direction force balance. Balance in the y-direction (Fig. 8.2) similarly yields

$$\begin{aligned}\sum F_y = ma_y \rightarrow &\left\{\left[\sigma_{yy}+\frac{\partial \sigma_{yy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right]-\left[\sigma_{yy}-\frac{\partial \sigma_{yy}}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\right\}\Delta x\Delta z\\ &+\left\{\left[\sigma_{xy}+\frac{\partial \sigma_{xy}}{\partial x}\left(\frac{\Delta x}{2}\right)\right]-\left[\sigma_{xy}-\frac{\partial \sigma_{xy}}{\partial y}\left(\frac{\Delta x}{2}\right)\right]\right\}\Delta y\Delta z\\ &+\left\{\left[\sigma_{zy}+\frac{\partial \sigma_{zy}}{\partial z}\left(\frac{\Delta z}{2}\right)\right]-\left[\sigma_{zy}-\frac{\partial \sigma_{zy}}{\partial z}\left(\frac{\Delta z}{2}\right)\right]\right\}\Delta x\Delta y\\ &+\rho g_y\Delta x\Delta y\Delta z=\rho\Delta x\Delta y\Delta z a_y.\end{aligned} \tag{8.21}$$

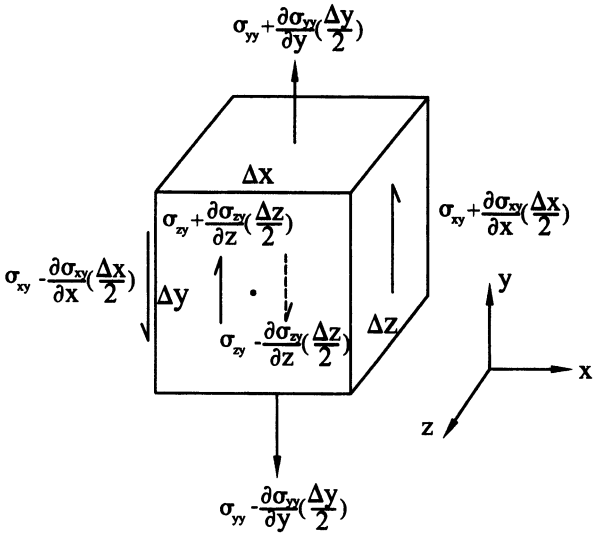


FIGURE 8.2 Similar to Fig. 8.1 except for the y-direction.

Again, simplifying and taking the limit, we have

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}+\rho g_y=\rho a_y. \tag{8.22}$$

Similarly, in the z direction, we find (do it)

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho g_z=\rho a_z. \tag{8.23}$$

Equations (8.20), (8.22), and (8.23) are the general equations of motion relative to a Cartesian coordinate system. Because we did not specify any particular material behavior (i.e., constitutive relation) in this derivation, these equations are true for *all* materials that can be regarded as continua. Indeed, in the case of statics (i.e., no accelerations), we recover Eqs. (3.8)–(3.10), which were derived for solids but likewise are good for all continua.

Similar equations can be found for other coordinate systems. For example, in cylindricals, we have

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{\partial \sigma_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\rho g_r=\rho a_r, \tag{8.24}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{z\theta}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+\rho g_\theta=\rho a_\theta, \tag{8.25}$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}+\rho g_z=\rho a_z, \tag{8.26}$$

and in sphericals

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi}+\frac{1}{r}\left(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi}+\sigma_{\theta r}\cot\theta\right)+\rho g_r=\rho a_r, \tag{8.27}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi}+\frac{1}{r}\left[2\sigma_{r\theta}+\sigma_{\theta r}+\left(\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)\cot\theta\right]+\rho g_\theta=\rho a_\theta, \tag{8.28}$$

$$\frac{\partial \sigma_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}+\frac{1}{r}\left[2\sigma_{r\phi}+\sigma_{\phi r}+\left(\sigma_{\phi\theta}+\sigma_{\theta\phi}\right)\cot\theta\right]+\rho g_\phi=\rho a_\phi. \tag{8.29}$$

8.3 Navier–Stokes Equations

To specialize the equations of motion for an incompressible Newtonian behavior, the incompressible $(\nabla\cdot\boldsymbol{v}=0)$ Navier-Poisson equation [Eq. (7.66)] can be substituted into the equations of motion [Eqs. (8.20), (8.22), and (8.23)]. For example, for Cartesian coordinates, the x-direction equation

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}+\rho g_x=\rho a_x \tag{8.30}$$

becomes

$$\begin{aligned}\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial y}\left[2\mu\left(\frac{1}{2}\right)\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\right]\\ +\frac{\partial}{\partial z}\left[2\mu\left(\frac{1}{2}\right)\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\right]+\rho g_x=\rho a_x,\end{aligned} \tag{8.31}$$

or

$$-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 v_x}{\partial x^2}+\mu\frac{\partial^2 v_x}{\partial y^2}+\mu\frac{\partial^2 v_y}{\partial y\partial x}+\mu\frac{\partial^2 v_x}{\partial z^2}+\mu\frac{\partial^2 v_z}{\partial z\partial x}+\rho g_x=\rho a_x. \tag{8.32}$$

Now, if we let

$$2\mu\frac{\partial^2 v_x}{\partial x^2} = \mu\frac{\partial^2 v_x}{\partial x^2} + \mu\frac{\partial^2 v_x}{\partial x^2} \tag{8.33}$$

and if we interchange the order of mixed derivatives $\partial^2/\partial y\partial x$ to $\partial^2/\partial x\partial y$ and so forth, then Eq. (8.32) can be written as

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x + \mu\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = \rho a_x, \tag{8.34}$$

or

$$-\frac{\partial p}{\partial x} + \mu\nabla^2 v_x + \rho g_x + \mu\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{v}) = \rho a_x. \tag{8.35}$$

Note: The Laplacian $\nabla^2 \equiv \nabla\cdot\nabla$, which is computed easily. Consistent with the above incompressibility assumption, $\nabla\cdot\boldsymbol{v} = 0$; thus, our final relation in the x direction is

$$-\frac{\partial p}{\partial x} + \mu\nabla^2 v_x + \rho g_x = \rho a_x. \tag{8.36}$$

Similarly, the y-direction equation

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z} + \rho g_y = \rho a_y \tag{8.37}$$

becomes

$$\begin{aligned}\frac{\partial}{\partial x}\left[2\mu\left(\frac{1}{2}\right)\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(-p + 2\mu\frac{\partial v_y}{\partial y}\right)\\ + \frac{\partial}{\partial z}\left[2\mu\left(\frac{1}{2}\right)\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] + \rho g_y = \rho a_y,\end{aligned} \tag{8.38}$$

or

$$-\frac{\partial p}{\partial y} + \mu\nabla^2 v_y + \rho g_y + \mu\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{v}) = \rho a_y. \tag{8.39}$$

Incompressibility thus yields

$$-\frac{\partial p}{\partial y}+\mu\nabla^2 v_y+\rho g_y=\rho a_y. \tag{8.40}$$

Finally,

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho g_z=\rho a_z \tag{8.41}$$

can be shown (do it) to reduce to:

$$-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z+\rho g_z=\rho a_z. \tag{8.42}$$

Considering the three component equations, we see that the incompressible Navier–Stokes equations (due to Navier (1785–1836) and Stokes (1819–1903)) can be written more generally in vector notation as

$$-\nabla p+\mu\nabla^2\boldsymbol{v}+\rho\boldsymbol{g}=\rho\boldsymbol{a}, \tag{8.43}$$

which is good for any coordinate system. Finally, for an Eulerian approach, recall from Eq. (7.21) that the acceleration has two contributions: local and convective. Writing these explicitly yields our final form for the *incompressible Navier–Stokes equation*:

$$-\nabla p+\mu\nabla^2\boldsymbol{v}+\rho\boldsymbol{g}=\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right). \tag{8.44}$$

Hence, this system of equations consisting of the equation of motion (8.44) and the incompressible mass balance equation [see Eq. (8.13)],

$$\nabla\cdot\boldsymbol{v}=0, \tag{8.45}$$

represent our four governing differential equations (three scalar momentum equations and one scalar mass equation) for an incompressible Newtonian fluid in terms of our four unknowns (pressure and three components of velocity). Because of the convective acceleration terms, these are nonlinear coupled partial differential equations, which are difficult to solve in general; one must often resort to numerical methods. We shall see in Chaps. 9 and 11, however, that a number of useful solutions can be found analytically in Cartesian, cylindrical, and spherical coordinates. In cylindrical coordinates, for example, the incompressible Navier–Stokes equations are

$$-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r = \rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right), \tag{8.46}$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta = \rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right), \tag{8.47}$$

$$-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z = \rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right), \tag{8.48}$$

which clearly appear as formidable coupled equations (each contains all four unknowns). Because blood vessels, airways, ureters, medical tubing, and so forth are cylindrical in cross section, these equations (combined with mass balance) are perhaps the most important in biofluid mechanics; they are the focus of much of Chap. 9, in which we will find exact solutions for a few important classes of problems.

Here, however, let us note that in certain problems, the Navier–Stokes equations simplify considerably. For example, G. Stokes suggested that it would be useful to consider flows in which the viscous effects are much greater than the inertial (i.e., convective acceleration) effects; that is, in slow (or *creeping*) flows, the Navier–Stokes equation reduces to

$$-\nabla p+\mu\nabla^2\boldsymbol{v}+\rho\boldsymbol{g}=\rho\frac{\partial\boldsymbol{v}}{\partial t}, \tag{8.49}$$

which is a linear second-order differential equation.

Conversely, another simplification can be made if we assume that the viscous effects are small. Although all fluids resist deformation to some degree, as noted earlier there are problems wherein the viscosity of the fluid is negligible. In this case, the fluid is called *inviscid* and the Navier–Stokes equation reduces to the so-called *Euler equation*:

$$-\nabla p+\rho\boldsymbol{g}=\rho\boldsymbol{a}, \tag{8.50}$$

where the acceleration includes both local and convective parts in general. The Euler equation is thus a nonlinear first-order differential equation. A fluid that experiences only incompressible and inviscid flows is called an *ideal fluid*.

Finally, if the fluid is truly static, then $\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{a} = \mathbf{0}$, and the Navier–Stokes equation becomes

$$-\nabla p + \rho \boldsymbol{g} = \mathbf{0}, \tag{8.51}$$

which is a linear first-order differential equation. Although this equation is often derived in courses on Engineering Statics, the derivation is typically much different. Regardless, let us examine the following simple example.

Example 8.2 Consider a container of fluid at rest with a Cartesian coordinate defined as positive downward and the origin located at the surface of the fluid (Fig. 8.3). Find the hydrostatic pressure p at the depth h.

Solution: From Eqs. (8.36), (8.40), and (8.42), with $\boldsymbol{g} = +\rho g \hat{\boldsymbol{j}}$, given the downward oriented coordinate direction, we have

$$-\frac{\partial p}{\partial x} + 0 = 0, \quad -\frac{\partial p}{\partial y} + \rho g = 0, \quad -\frac{\partial p}{\partial z} + 0 = 0.$$

From the first and third equations, $p = p(y)$ at most, and the partial derivative becomes an ordinary derivative. Solving by integration,

$$\frac{dp}{dy} = \rho g \rightarrow \int \frac{d}{dy}(p) dy = \int \rho g \, dy$$

and, consequently, we have

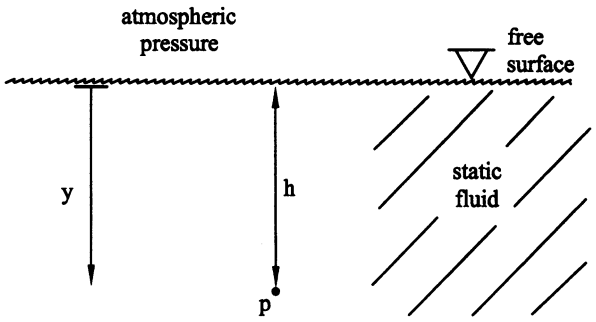


FIGURE 8.3 Determination of the pressure as a function of depth in a static fluid.

$$p(y) = \rho g y + c.$$

The integration constant c is found from the boundary conditions. Here, note that the so-called *gauge pressure* is defined as the *absolute pressure* minus atmospheric pressure. If we assume an atmospheric pressure at the surface, then $p(y=0)=0$(gauge) and $c=0$. Thus, $p(y)=\rho g y$. At $y=h$, therefore, we obtain the well-known result that $p=\rho g h$ at depth h (that pressure increases with depth is easily appreciated as we swim deeper in a pool). In a sense, then, this is a solution of the Navier–Stokes equation. Because of the importance and utility of the Navier–Stokes equation, much of Chap. 9 is devoted to its solution.

Observation 8.2. In approximately 220 B.C., the Greek mathematician Archimedes derived a very important relation in fluid statics that relates the amount of fluid displaced by an immersed solid to the force exerted on that solid by the fluid (the so-called buoyant force). Although we could derive this result by considering an arbitrarily shaped solid, for convenience let us consider a solid cylinder, as shown in Fig. 8.4. The weight of the cylinder is $W = \rho_s g(\pi a^2)(h_2 - h_1)$, where ρ_s is the mass density of the solid and a is its radius. Whereas this force tends to cause the solid to "sink," the difference in pressures on the bottom and top surfaces tends to push upward on the solid. This *buoyant force* $F_B = (p_2 - p_1)\pi a^2 = \left(\rho_f g h_2 - \rho_f g h_1\right)\pi a^2$; see Example 8.2. If

$W > F_B$, then the solid will sink;
$W = F_B$, then the solid is neutrally buoyant;
$W < F_B$, then the solid will float.

In particular, any body that remains fully submerged at a fixed depth (i.e., where it is placed) is said to be neutrally buoyant. In this case,

$$\rho_s g\left(\pi a^2\right)(h_2 - h_1) = \rho_f g\left(\pi a^2\right)(h_2 - h_1),$$

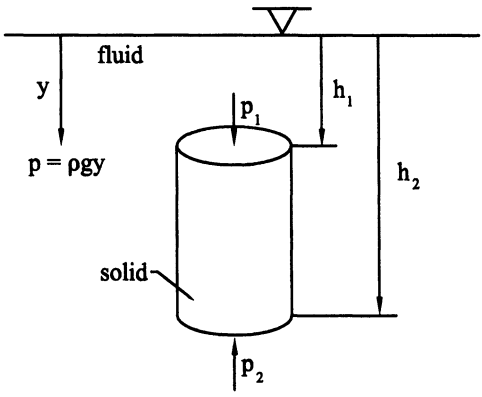


FIGURE 8.4 Schema illustrating Archimedes' principle. Assuming that the submerged solid is a cylinder merely simplifies the analysis, the consequence of which is very general.

or $\rho_s = \rho_f$. Regardless, we see that the buoyant force

$$F_B = \rho_f g\left(\pi a^2\right)(h_2 - h_1) = \rho_f g \forbidden{V_s},$$

where $\forbidden{V_s}$ is the volume of the solid that is in the fluid, which is equal to the volume of the displaced fluid. Archimedes' principle states, therefore, that the net buoyant force exerted on a solid by a fluid equals the force of gravity on the liquid that is displaced by the solid.

Archimedes' principle is often used in mechanical tests on soft tissues. Because soft tissues tend to have a slightly higher mass density ($\rho \sim 1{,}050$ kg/m^3) than the physiologic solution in which they are placed, they tend to sink, especially when mounting fixtures are affixed to them. To render the tissue neutrally buoyant, therefore, a volume-occupying low-density material (e.g., Styrofoam) can be attached to the fixtures so that the weight of the total volume of fluid displaced by the specimen and fixture equals the tissue-fixture weight. Hence, the only loads on the tissue will be those imposed by the materials testing unit.

[Note to student/instructor: It may be advisable to proceed to Chap. 9 at this time and return to the following sections on inviscid fluids and methods of measurements only if desired.]

8.4 The Euler Equation

Comparison of the incompressible Navier–Stokes equation [Eq. (8.44)] to the Euler equation [Eq. (8.50)] reveals that the former is a system of coupled second-order partial differential equations (PDEs), whereas the latter is a system of coupled first-order PDEs. First-order equations tend to be easier to solve, but because of the convective part of the acceleration [i.e., $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$], both equations are nonlinear, and it is often this nonlinearity that poses the greatest difficulty in solution. For this reason, it can be shown that a judicious choice of a coordinate system can be helpful in trying to solve even the Euler equation.

Toward this end, let us define two new terms. Let a *pathline* be defined as the locus of points through which a material particle passes in a flow field. An example would be the path taken by a leaf as it flows down a river. Let a *streamline* be defined as a locus of points where the velocity is everywhere tangent. This mathematical definition is less intuitive than that for the pathline. In cases of steady flows, however, the two lines coincide. Hindsight reveals that it can be convenient, particularly in steady flows, to define a coordinate system such that one of the coordinate axes coincides locally with a streamline. Hence, let us consider the following.

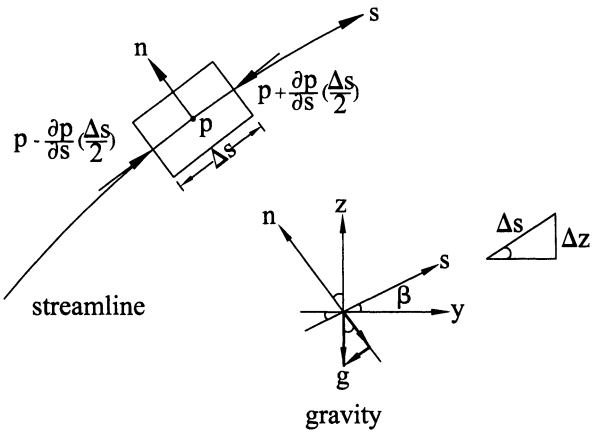


FIGURE 8.5 Differential fluid element in two dimensions relative to streamline coordinates. Remember, therefore, that a streamline is drawn tangent to the velocity vector at every point in a flow field. Note, too, the relation of the streamline coordinates with respect to the Cartesian coordinates, where z is now taken to be vertical, similar to most derivations of this equation in the literature; x is out of the paper.

Question: How do we write Euler's equation in terms of locally orthogonal streamline coordinates (s, n, x)? We could use coordinate transformations to get from equations in terms of (x, y, z) to those in terms of (s, n, x), or we could directly rederive Euler's equation in terms of s, n, and x. Let us adopt the second approach, which is straightforward and will reinforce our earlier derivation.

First, recall, that the constitutive equations for an incompressible, Newtonian fluid are $\sigma_{xx} = -p + 2\mu D_{xx}$, $\sigma_{yy} = -p + 2\mu D_{yy}$, $\sigma_{zz} = -p + 2\mu D_{zz}$, $\sigma_{xy} = 2\mu D_{xy}$, and so on. For inviscid fluids, therefore, $\mu = 0$ and $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, which is a hydrostatic state of stress (i.e., an inviscid fluid cannot support a shear stress). Second, recall from Eqs. (7.3) and (7.4) that a hydrostatic state of stress at a point relative to one coordinate system is also hydrostatic relative to any coordinate system at that point. Now, to rederive Euler's equation in terms of s, n, and x, consider the differential fluid element taken along a streamline in Fig. 8.5. Let us assume further that the normal stresses are $\sigma_{ss} = \sigma_{nn} = \sigma_{xx} = -p$ at the center of this element and that the pressure can vary from point to point and possibly with time [i.e., $p = p(s, n, x, t]$. Nonetheless, we shall focus only on possible changes in the streamline direction. Thus, at the positive and negative s faces (i.e., at $s \pm \Delta s/2$), we allow the pressure to be slightly larger or smaller than the value p at the center: On these faces, we let the pressure be $p \pm \Delta p$. Moreover, for the flow to be in the positive s direction, the pressure must be higher at the negative s face. Hence, using a typical Taylor series expansion, at any fixed time t,

$$p\left(s+\frac{\Delta s}{2}\right)=p(s)+\frac{\partial p}{\partial s}\left(\frac{\Delta s}{2}\right),$$
$$p\left(s-\frac{\Delta s}{2}\right)=p(s)-\frac{\partial p}{\partial s}\left(\frac{\Delta s}{2}\right), \tag{8.52}$$

and we see that the pressure gradient $\partial p/\partial s<0$ for flow in the positive s direction. Again, higher-order terms in the Taylor's series have been neglected for, in hindsight, they would be negligible. Linear momentum balance in s thus requires that

$$\sum F_s = ma_s \rightarrow \left(p-\frac{\partial p}{\partial s}\frac{\Delta s}{2}\right)\Delta n\Delta x-\left(p+\frac{\partial p}{\partial s}\frac{\Delta s}{2}\right)\Delta n\Delta x - \rho(g\sin\beta)\Delta s\Delta n\Delta x=\rho a_s\Delta s\Delta n\Delta x. \tag{8.53}$$

Simplifying, we have

$$-\frac{\partial p}{\partial s}\Delta s\Delta n\Delta x-\rho g(\sin\beta)\Delta s\Delta n\Delta x=\rho a_s\Delta s\Delta n\Delta x. \tag{8.54}$$

Dividing this equation by the differential volume and taking the limit, we obtain

$$\lim_{\substack{\Delta s\to 0\\ \Delta n\to 0\\ \Delta x\to 0}}\frac{1}{\Delta s\Delta n\Delta x}\left(-\frac{\partial p}{\partial s}\Delta s\Delta n\Delta x-\rho g(\sin\beta)\Delta s\Delta n\Delta x-\rho a_s\Delta s\Delta n\Delta x\right)=0, \tag{8.55}$$

or

$$-\frac{\partial p}{\partial s}-\rho g\sin\beta=\rho a_s. \tag{8.56}$$

Because $\sin\beta=\partial z/\partial s$, we have

$$-\frac{\partial p}{\partial s}-\rho g\frac{\partial z}{\partial s}=\rho a_s=\rho\left(\frac{\partial v_s}{\partial t}+v_s\frac{\partial v_s}{\partial s}+v_n\frac{\partial v_s}{\partial n}+v_x\frac{\partial v_s}{\partial x}\right). \tag{8.57}$$

Let us now exploit our choice of a streamline coordinate system. Because the velocity vector is everywhere tangent to a streamline, the only component of the velocity is v_s; that is, $\boldsymbol{v}=v_s(s,\ n,\ x,\ t)\hat{\boldsymbol{e}}_s$ relative to streamline coordinates, whereas $\boldsymbol{v}=v_x(x,\ y,\ z,\ t)\hat{\boldsymbol{e}}_x+v_y(x,\ y,\ z,\ t)\hat{\boldsymbol{e}}_y+v_z(x,\ y,\ z,\ t)\hat{\boldsymbol{e}}_z$ relative to a usual Cartesian system. Both represent possible unsteady 3-D flows, but the

simplification is clear for the streamline system. Hence, with $v_n = v_x = 0$, the s-direction Euler equation becomes

$$-\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s} = \rho\left(\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s}\right). \tag{8.58}$$

Enforcing linear momentum balance via a summation of the forces in the n direction similarly yields

$$\begin{aligned}\sum F_n = ma_n \rightarrow \left(p - \frac{\partial p}{\partial n}\frac{\Delta n}{2}\right)\Delta s \Delta x - \left(p + \frac{\partial p}{\partial n}\frac{\Delta n}{2}\right)\Delta s \Delta x \\ - \rho(g\cos\beta)\Delta s \Delta n \Delta x = \rho a_n \Delta s \Delta n \Delta x.\end{aligned} \tag{8.59}$$

Simplifying,

$$-\frac{\partial p}{\partial n}\Delta s \Delta n \Delta x - \rho g(\cos\beta)\Delta s \Delta n \Delta x = \rho a_n \Delta s \Delta n \Delta x. \tag{8.60}$$

Dividing this by $\Delta\forall$ and taking the limit, we obtain

$$\lim_{\substack{\Delta s \to 0 \\ \Delta n \to 0 \\ \Delta x \to 0}} \frac{1}{\Delta s \Delta n \Delta x}\left(-\frac{\partial p}{\partial n}\Delta s \Delta n \Delta x - \rho g(\cos\beta)\Delta s \Delta n \Delta x - \rho a_n \Delta s \Delta n \Delta x\right) = 0, \tag{8.61}$$

or

$$-\frac{\partial p}{\partial n} - \rho g \cos\beta = \rho a_n. \tag{8.62}$$

Because $\cos\beta = \partial z/\partial n$, we get

$$-\frac{\partial p}{\partial n} - \rho g \frac{\partial z}{\partial n} = \rho a_n. \tag{8.63}$$

For a centripetal acceleration, $a_n = \partial v_n/\partial t - v_s^2/R$, where $v_n \equiv 0$ and R is the radius of curvature for the streamline. Thus,

$$-\frac{\partial p}{\partial n} - \rho g \frac{\partial z}{\partial n} = \rho\left(-\frac{v_s^2}{R}\right). \tag{8.64}$$

In summary, for flow in the *s*-*n* plane, the Euler equation relative to streamline coordinates reduces to two equations [Eqs. (8.58) and (8.64)] in terms of two unknowns: v_s and p. Clearly, these equations should be easier to solve than the more general equations in terms of four unknowns (pressure and three components of the velocity).

Observation 8.3. It can be shown that the Laplacian of the velocity

$$\nabla^2 \boldsymbol{v} = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla \times (\nabla \times \boldsymbol{v}).$$

Hence, for an incompressible flow,

$$\nabla^2 \boldsymbol{v} = -\nabla \times \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta}$ is the vorticity vector. In this case, the incompressible Navier–Stokes equations can be written as

$$-\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} = \rho \boldsymbol{a} \rightarrow -\nabla p - \mu(\nabla \times \boldsymbol{\zeta}) + \rho \boldsymbol{g} = \rho \boldsymbol{a}.$$

Note, therefore, that the Navier–Stokes equation reduces to the Euler equation ($\mu = 0$) when the flow is irrotational ($\boldsymbol{\zeta} = \boldsymbol{0}$) regardless of the viscosity. In other words, any incompressible, irrotational flow that satisfies the Euler equation will likewise satisfy the full Navier–Stokes equations, as we will see in Chap. 11. One must be careful, however, because Euler solutions will not satisfy viscous boundary conditions, such as those on shear stress.

8.5 The Bernoulli Equation

The so-called Bernoulli equation is one of the most used, yet probably *most misused*, equations in fluid mechanics. As we shall see, Bernoulli's equation is an algebraic equation that is much easier to solve than the differential equations of Navier–Stokes or Euler. This simplification does not come without a price, however, for there are five important restrictions that must be respected for the Bernoulli equation to apply. The best way to appreciate restrictions is to derive carefully the equation of interest—let us so begin.

8.5.1 Bernoulli Equation for Flow Along a Streamline

Let us first derive Bernoulli's equation from the Euler equation, relative to a streamline coordinate system (s, n, x). Hence, the first two restrictions are those for Euler's equation: incompressible flow and negligible viscosity. Recall that

such a fluid is said to be ideal. Next, let us restrict our attention to a steady flow wherein $\partial \boldsymbol{v}/\partial t = \mathbf{0}$. This provides our third restriction. Hence, Eq. (8.58) reduces to

$$\frac{\partial p}{\partial s} + \rho g \frac{\partial z}{\partial s} + \rho v_s \frac{\partial v_s}{\partial s} = 0. \tag{8.65}$$

Next, note that, in general, the pressure and velocity can each vary from point to point: that is, $p = p(s, n, x)$ and $v_s = (s, n, x)$. Consequently,

$$\begin{aligned} dp &= \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn + \frac{\partial p}{\partial x} dx, \\ dv_s &= \frac{\partial v_s}{\partial s} ds + \frac{\partial v_s}{\partial n} dn + \frac{\partial v_s}{\partial x} dx. \end{aligned} \tag{8.66}$$

Yet, if we restrict our attention to flow *along a streamline* s, whereby $dn = dx = 0$, then

$$dp = \frac{\partial p}{\partial s} ds, \quad dv_s = \frac{\partial v_s}{\partial s} ds, \quad dz = \frac{\partial z}{\partial s} ds. \tag{8.67}$$

This suggests that if we integrate Eq. (8.65) *along* a streamline, we obtain

$$\int \frac{\partial p}{\partial s} ds + \int \rho g \frac{\partial z}{\partial s} ds + \int \rho v_s \frac{\partial v_s}{\partial s} ds = \int 0 \, ds, \tag{8.68}$$

or

$$\int dp + \int \rho g \, dz + \int \rho v_s \, dv_s = c. \tag{8.69}$$

Assuming further that the mass density and gravitational constant do not vary with position in the z direction, our final relation is

$$p + \rho g z + \frac{1}{2} \rho v_s^2 = c, \tag{8.70}$$

or, as it is most often written,

$$\frac{p}{\rho} + g z + \frac{v_s^2}{2} = C, \tag{8.71}$$

where $C = c/\rho$. Again, however, we emphasize that this—Bernoulli's—equation can be used only if all of the following restrictions are met:

1. Incompressible flow
2. Inviscid fluid
3. Steady flow
4. Flow along a streamline
5. Constant gravitational forces

Before illustrating some solutions to the Bernoulli equation, let us consider a few additional interesting findings.

8.5.2 *Bernoulli Equation for Irrotational Flow*

In this subsection, we show that Bernoulli's equation holds at all points in a flow field, not just along a streamline, if the flow is irrotational and the other four restrictions are still satisfied. Hence, recall that an irrotational flow is one in which fluid elements moving in the flow field do not undergo any rigid rotation. Moreover, the vorticity vanishes if the flow is irrotational (i.e., $\boldsymbol{\zeta} = \mathbf{0} = \nabla \times \boldsymbol{v}$). Recall, too, that for an incompressible fluid, mass balance requires that $\nabla \cdot \boldsymbol{v} = 0$, and for an inviscid fluid, $\mu = 0$; thus, the linear momentum equation for an ideal fluid reduces to Euler's equation, $-\nabla p + \rho \boldsymbol{g} = \rho \mathbf{a}$, where

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \tag{8.72}$$

in an Eulerian formulation. Substituting this equation into Euler's equation for a steady flow, we obtain

$$-\frac{1}{\rho}\nabla p + \boldsymbol{g} = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}. \tag{8.73}$$

Now, from vector calculus, it can be shown that (see Exercise 7.14)

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla(\boldsymbol{v} \cdot \boldsymbol{v}) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}). \tag{8.74}$$

For an irrotational flow, however, $\nabla \times \boldsymbol{v} = \mathbf{0}$; thus, Euler's equation for steady, irrotational flow can be written

$$-\frac{1}{\rho}\nabla p + \boldsymbol{g} = \frac{1}{2}\nabla(\boldsymbol{v} \cdot \boldsymbol{v}), \tag{8.75}$$

where $\boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}||\boldsymbol{v}|\cos 0 = v^2$, with v^2 a scalar. Hence, Euler's equation becomes

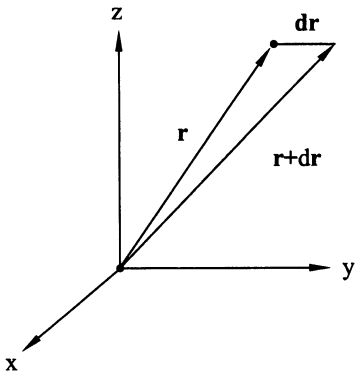


FIGURE 8.6 Position vector $\boldsymbol{r}$ and a small change therefrom.

$$-\frac{1}{\rho}\nabla p + \boldsymbol{g} = \frac{1}{2}\nabla v^2. \tag{8.76}$$

At this point, it is important to note that we have not yet specified a coordinate system and, in particular, we have not specified streamline coordinates. Thus, consider a generic displacement of a particle in the flow field from position $\boldsymbol{r}$ to position $\boldsymbol{r} + d\boldsymbol{r}$ (Fig. 8.6). The displacement vector $d\boldsymbol{r}$ is an arbitrary infinitesimal displacement in any direction. If the only body force is the force due to gravity, then $\boldsymbol{g} = -g\hat{\boldsymbol{k}}$, with z a vertical direction, as in most applications of Bernoulli's equation. Taking the dot product of $d\boldsymbol{r} = dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}$ with each of the terms in Eq. (8.76), we have

$$\begin{aligned}-\frac{1}{\rho}\left(\hat{\boldsymbol{i}}\frac{\partial p}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial p}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial p}{\partial z}\right)\cdot\left(dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}\right) + \left(-g\hat{\boldsymbol{k}}\right)\cdot\left(dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}\right)\\ = \frac{1}{2}\left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x}(v^2) + \hat{\boldsymbol{j}}\frac{\partial}{\partial y}(v^2) + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}(v^2)\right)\cdot\left(dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}\right),\end{aligned} \tag{8.77}$$

or

$$\begin{aligned}-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) - g\,dz\\ = \frac{1}{2}\left(\frac{\partial(v^2)}{\partial x}dx + \frac{\partial(v^2)}{\partial y}dy + \frac{\partial(v^2)}{\partial z}dz\right),\end{aligned} \tag{8.78}$$

which can be written more compactly as

$$-\frac{1}{\rho}dp - g\,dz = \frac{1}{2}d\left(v^2\right). \tag{8.79}$$

Integration yields the final result, namely

$$\int\frac{1}{\rho}dp + \int g dz + \int\frac{1}{2}d\left(v^2\right) = C \rightarrow \frac{p}{\rho} + gz + \frac{v^2}{2} = C, \tag{8.80}$$

which is the same equation that we obtained in Eq. (8.71) by focusing our attention along a streamline. We see, therefore, that Bernoulli's equation is valid between *any* two points in the field if the flow is irrotational; if the flow is not irrotational, the Bernoulli equation is still valid at any two points along a streamline. Summarizing then, our five basic restrictions for using the Bernoulli equation are (1) incompressible, (2) inviscid, (3) steady, (4) along a streamline *or* in an irrotational flow, and (5) constant gravitational forces.

For any two appropriate points, say 1 and 2, the Bernoulli equation thus becomes

$$\frac{p_1}{\rho} + gz_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + gz_2 + \frac{1}{2}v_2^2, \tag{8.81}$$

which reveals its simple algebraic character and, consequently, why many are tempted to (mis)use it. We will consider a few simple examples later to illustrate how we might use this simple equation.

First, however, note the following. Because it came from Euler's equation, Bernoulli's equation is also a statement of the balance of linear momentum in an inertial reference frame. Being a single algebraic equation, it can be solved for only one unknown. Of course, regardless of the formulation—Navier–Stokes, Euler, or Bernoulli—one must always simultaneously satisfy both the balance of mass and the balance of linear momentum, with mass balance providing one additional equation and thus the ability to solve one additional unknown. Although we have derived a differential equation for mass balance, let us consider a special case here. For flow into and out of a rigid, impermeable pipe or nozzle, the net *volumetric flow* in, Q_{in}, must equal the net volumetric flow out, Q_{out}. These flows are defined by

$$Q_{in} = \int \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dA_{in} = \int \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dA_{\text{out}} = Q_{\text{out}}, \tag{8.82}$$

where $\hat{\boldsymbol{n}}$ is an outward unit vector normal to the cross-sectional area A of interest. If $\boldsymbol{v}$ is taken to be uniform across the differential area of interest and in the $\pm$ direction of the outward unit normal vector, then

$$Q_{\text{in}} = Q_{\text{out}} \rightarrow \overline{v}_1 A_1 = \overline{v}_2 A_2, \tag{8.83}$$

where 1 and 2 denote the inlet and outlet, respectively, and the overbar denotes a mean value. This simple form of mass balance is often used in problems using the Bernoulli equation, as we will now see. In combination with Bernoulli, it allows us to solve for two unknowns between two appropriate points 1 and 2.

Example 8.3 It can be shown experimentally that Bernoulli's equation can be used in computations for flows through constrictions but *not* for flows through expansions. The reason for this is that in the latter case, adverse pressure gradients can disturb the flow such that there is a reversal and thus significant viscous losses. Bernoulli assumes no viscous effects and therefore does not apply. We shall see in Chap. 10 that the flow in an expansion can be handled easily using the energy equation. For a constriction, such as a nozzle or needle, find the injection pressure needed to achieve an exit flow of v_o if the flow exits into a fluid of pressure $P_o = P_{\text{atm}}$.

Solution: Assuming that we know the cross-sectional area within the inlet to the needle A_i and its exit area A_o, mass balance requires that $v_i A_i, = v_o A_o$, which allows us to compute v_i, given the value of v_o. If we assume that the needle is short, we would expect negligible viscous losses. Indeed, if we further select a centerline streamline, where the velocity gradient $\partial v/\partial r$ should be zero due to the symmetry of v, viscous losses should be small and, thus, we can use Bernoulli. Assuming a horizontal situation (Fig. 8.7),

$$\frac{P_i}{\rho} + \frac{1}{2}\left(\frac{v_o A_o}{A_i}\right)^2 = \frac{P_o}{\rho} + \frac{1}{2}(v_o)^2 \rightarrow P_i = P_o + \frac{1}{2}\rho v_o^2 \left[1 - \left(\frac{A_o}{A_i}\right)^2\right].$$

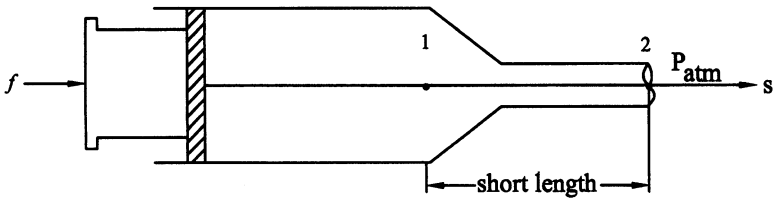


FIGURE 8.7 Flow through a nozzle (e.g., syringe and needle) over a short length. The importance of length on the viscous effects will be demonstrated in Chap. 10.

Example 8.4 Water flows steadily up a short, vertical, 2.54-cm-diameter pipe and discharges to atmospheric pressure (Fig. 8.8). If a pressure of 16 kPa drives the fluid at a volumetric flow rate Q of 5 L/min, what height does the fluid reach?

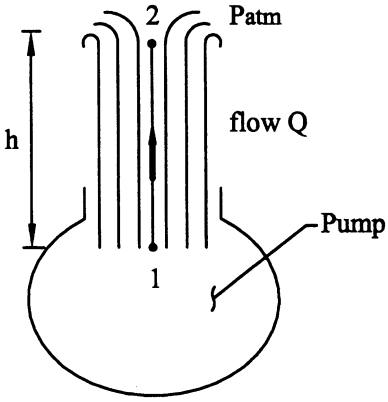


FIGURE 8.8 Flow from a vertical tube/pump that discharges to atmosphere. Because of the influence of gravity, fluid particles will rise to a particular height and then fall.

Solution:
Given:

$A_1 = \pi(1.27)^2 = 5.07\ \text{cm}^2$	$z_1 = 0$
$Q = 5\dfrac{\text{L}}{\text{min}}$	$z_2 = h\ \text{m}$
$p_1 = 16\ \text{kPa}$	$g = 9.81\dfrac{\text{m}}{\text{s}^2}$
$p_2 = 0$ (gauge)	$\rho = 1000\dfrac{\text{kg}}{\text{m}^3}$.

Assume:

1. Incompressible
2. Inviscid
3. Steady flow (given)
4. Along a streamline (given)
5. Constant gravitational forces

Moreover, let us assume that the velocity $v_2 = 0$ at the maximum height of the fluid column. Hence, from mass balance

$$Q = v_1 A_1 \rightarrow v_1 = \frac{Q}{A_1} = \frac{5\,\text{L/min}}{5.07\,\text{cm}^2}\left(\frac{1000\,\text{cm}^3}{1\,\text{L}}\right) = 986\frac{\text{cm}}{\text{min}}$$

or $v_1 = 0.164$ m/s. Hence, from Bernoulli,

$$\frac{p_1}{\rho} + gz_1 + \frac{v_1^2}{2} = \frac{p_2}{\rho} + gz_2 + \frac{v_2^2}{2} \rightarrow \frac{p_1}{\rho} + \frac{1}{2}v_1^2 = gh,$$

or

$$h = \frac{p_1}{\rho g} + \frac{1}{2g}v_1^2 = \frac{16000\,\mathrm{N/m^2}}{(1000\,\mathrm{kg/m^3})(9.81\,\mathrm{m/s^2})} + \frac{(0.164\,\mathrm{m/s^2})}{2(9.81\,\mathrm{m/s^2})} = 1.64\,\mathrm{m}.$$

Note that ρg is sometimes called the *specific weight* and denoted by γ, not to be confused with the *specific gravity* $\mathrm{SG} = \rho/\rho_{\mathrm{H_2O}}$ at 4 °C. Given that 1 kPa = 7.5 mmHg, what might this suggest with regard to how far blood might travel if an open needle (having a different diameter) were placed in the heart?

Example 8.5

Note that Bernoulli and mass balance provide two equations:

$$\frac{p_1}{\rho} + gz_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + gz_2 + \frac{1}{2}v_2^2, \quad v_1A_1 = v_2A_2,$$

which can be used to solve for the two velocities, v_1 and v_2, along a straight horizontal streamline s in a steady, converging, ideal flow, with A_1 and A_2 known. To do so, however, we must independently compute or measure the pressures p_1 and p_2. Assuming a negligible gravitational field, determine if the pressure gauges in Fig. 8.9 can be used to determine the pressures along the center streamline.

Solution: Because we do not know $\boldsymbol{v}$ as a function of (x, y, z) or (r, θ, z), we cannot determine if Bernoulli holds across the streamline (i.e., if $\nabla \times \boldsymbol{v} = \boldsymbol{0}$, then Bernoulli may hold for any two points). Hence, let us recall the original Euler equations for a steady ideal flow:

$$-\frac{\partial p}{\partial s} - \rho g\frac{\partial z}{\partial s} = \rho v_s\frac{\partial v_s}{\partial s}, \quad -\frac{\partial p}{\partial n} - \rho g\frac{\partial z}{\partial n} = -\frac{\rho v_s^2}{R}$$

from Eqs. (8.58) and (8.64). In particular, from the n-direction equation with $g \sim 0$,

$$\frac{\partial p}{\partial n} = \frac{\rho v_s^2}{R},$$

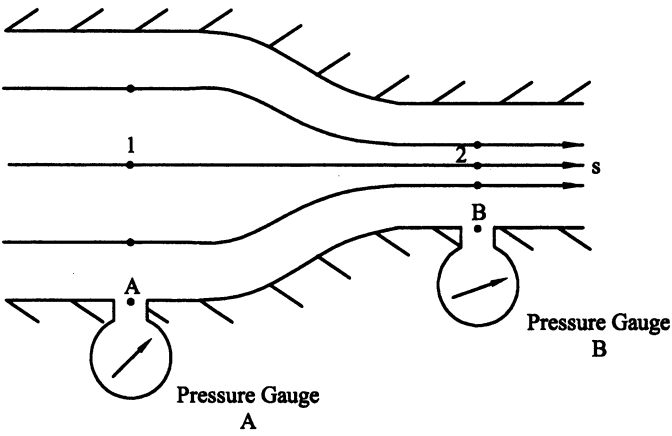


FIGURE 8.9 A simple internal flow that converges from a larger to a smaller diameter tube. Assume that the pressure gauges are connected flush to the wall of the tubing and that they are filled with an incompressible fluid.

where R is the radius of curvature of the streamline. Noting that $R \to \infty$ for the locally parallel horizontal streamlines within the regions associated with gauges A and B, then at each gauge, $\partial p/\partial n = 0$, which states that p does not vary in the normal direction when the streamlines are locally parallel. Hence, the pressure measured by these gauges, at the wall, equals the pressures at 1 and 2, and Bernoulli and mass balance can determine v_1 and v_2 in terms of measured p_1, p_2, A_1 and A_2.

8.5.3 Further Restrictions for the Bernoulli Equation

We have suggested that Bernoulli's equation is perhaps the most used and misused equation in fluid mechanics. The latter observation should cause us to respect the noted restrictions: the flow must be incompressible, inviscid, steady, irrotational or along a streamline, and within a constant gravitational field. The last restriction is seldom a concern in the research laboratory or clinical environment; hence, let us focus on the first four restrictions. If we know the velocity field, it is obviously easy to check the incompressible ($\nabla \cdot \boldsymbol{v} = 0$), steady ($\partial \boldsymbol{v}/\partial t = \mathbf{0}$), and irrotational ($\nabla \times \boldsymbol{v} = \mathbf{0}$) restrictions. This would be the case wherein we *measure* $\boldsymbol{v}(x, y, z, t)$ and seek to use Bernoulli to *calculate* the pressure field. In many cases, however, we may only know the velocity at a few select points, not everywhere; hence, rigorously checking these restrictions is not always so easy. With regard to the inviscid ($\mu = 0$) restriction, we know that all fluids resist flowing to some degree and, thus, have

a nonzero μ. The key question then is whether the viscous effects (losses) are negligible with respect to other factors in the problem. This can often be answered only via experience or by comparing solutions of the problem with and without viscosity, which defeats the purpose of seeking an easier approach. Hence, let us record some well-established observations based on others' experiences.

It is well known that viscous (frictional) effects become more and more important over longer lengths of tubes. The Bernoulli equation should thus be restricted to short lengths (e.g., in a needle). In cases of long lengths (e.g., IV tubing from the bag to the patient), one must solve the full differential equations of motion or employ the semi-empirical methods of Chap. 10. Note, too, that flow from a syringe into a needle is an example of a converging flow. Experience reveals that Bernoulli holds in many converging flows for which the flow field is not *turbulent* (i.e., fluctuating randomly). In contrast, Bernoulli should not be used to compute flows in diverging geometries or sudden expansions. Adverse pressure gradients can disturb the flow within such geometries, resulting in separation of the flow from the wall and the formation of recirculation zones (e.g., eddies; Fig. 8.10). Note, therefore, that stenoses in the vasculature can be considered as a converging geometry upstream (proximal) but a diverging geometry downstream (distal). Hence, there is a possibility of complex flows, particularly *flow separation* and recirculation zones just distal to the stenosis (Fig. 8.11). Therefore, Bernoulli's equation should not be used across a severe stenosis, although it may be used to estimate the maximum velocity in the stenosis, given proximal data. Bernoulli may sometimes be used in cases of gentle bends, although complex *secondary flows* can develop in curved tubes, which disallow the use of Bernoulli (Fig. 8.12). Likewise, Bernoulli may be used for internal flows entering a rounded entrance

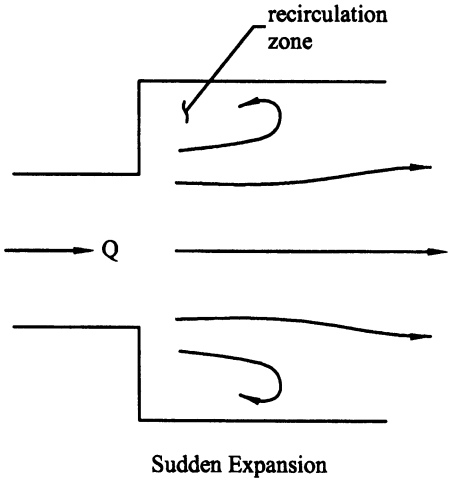


FIGURE 8.10 Formation of recirculation zones (sometimes referred to as eddies) downstream (i.e., distal) of a sudden expansion. Such eddies can dissipate considerable energy.

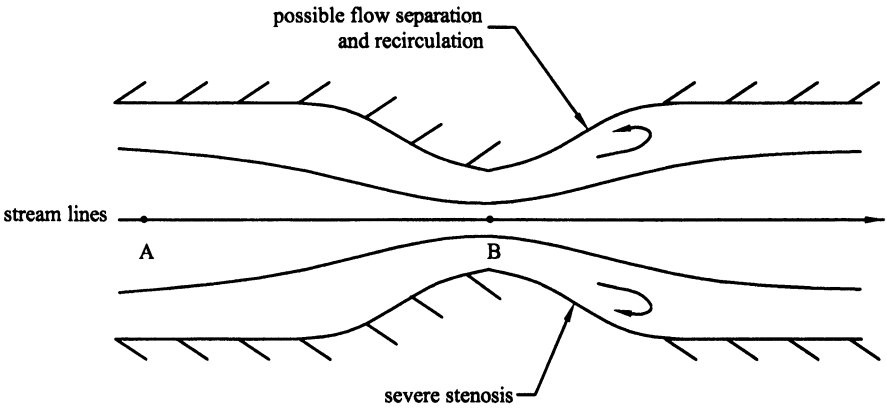


FIGURE 8.11 Similar to that in Fig. 8.10 except for flow through a stenosis.

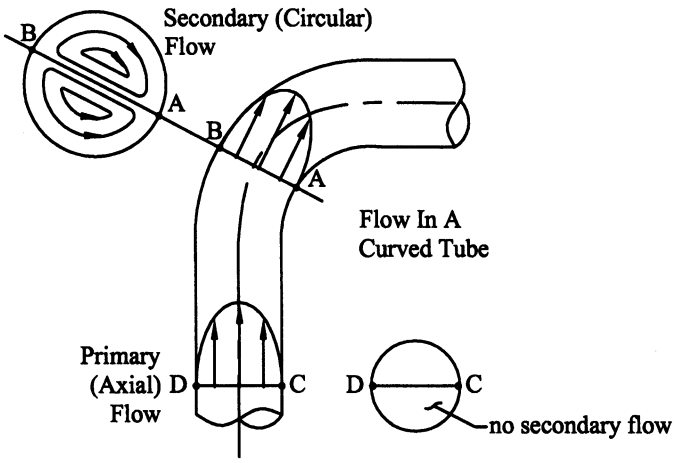


FIGURE 8.12 Secondary flows develop in curved tubes and are characterized by components of the velocity in the circumferential as well as the axial direction.

(converging flow), but a sharp entrance may disturb the flow and disallow its use. Finally, Bernoulli cannot be applied across a pump or propeller. Hence, Bernoulli could not be used to compare inlet and outlet velocities for an intravascular ventricular-assist device (IVAD). In this case, the semi-empirical methods of Chap. 10 would be an appropriate first approximation.

Example 8.6 Determine the time t_f it takes for a cylindrical container with a small central hole to drain.

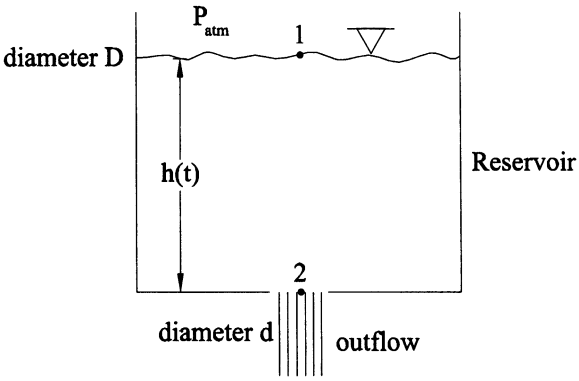


FIGURE 8.13 Fluid draining from a reservoir through a centrally located bottom hole.

Solution: Referring to Fig. 8.13, let us consider a streamline from the free surface, at 1, to the drain, at 2. Assuming an atmospheric pressure at 1 and 2, Bernoulli's equation reduces to

$$gh + \frac{1}{2}v_1^2 = \frac{1}{2}v_2^2,$$

or

$$v_2^2 - v_1^2 = 2gh(t),$$

where we emphasize that h varies with time t. Mass balance gives $v_1A_1 = v_2A_2$; thus,

$$v_2 = v_1\frac{\pi D^2/4}{\pi d^2/4} = v_1\frac{D^2}{d^2}$$

and, therefore,

$$v_1^2\left(\frac{D^4}{d^4} - 1\right) = 2gh(t) \rightarrow v_1 = \sqrt{\frac{d^4 2gh(t)}{D^4 - d^4}}.$$

Now, we recognize that $v_1, = -dh/dt$ and, therefore,

$$\frac{1}{\sqrt{h(t)}}\left(-\frac{dh}{dt}\right) = \frac{d^2\sqrt{2g}}{\sqrt{D^4 - d^4}}.$$

Integrating with respect to time,

$$\int_H^0 \frac{1}{\sqrt{h}} \frac{dh}{dt} dt = \int_0^{t_f} \frac{-d^2\sqrt{2g}}{\sqrt{D^4 - d^4}} dt,$$

or

$$-2\sqrt{H} = \frac{-d^2\sqrt{2g}}{\sqrt{D^4 - d^4}} t_f;$$

thus,

$$t_f = 2\sqrt{\frac{H\left(D^4 - d^4\right)}{2gd^4}} = \sqrt{\frac{2H\left(D^4 - d^4\right)}{gd^4}}.$$

Example 8.7

Evaluate the pressure difference between points A and B in Fig. 8.11. Assume aortic values such that $\overline{v}$ at A is 0.15 m/s, that the diameter at A is 0.03 m, and that the diameter at B is 0.01 m. Assume that $\rho = 1{,}060$ kg/m^3.

Solution: Although Bernoulli should not be used across a sudden expansion, it can be used along a central streamline between sections at A and B. Bernoulli becomes

$$\frac{p_1}{\rho} + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2,$$

where $v_1 A_1 = v_2 A_2$. Hence, the pressure difference is

$$p_1 - p_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) = \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] = \frac{1}{2}\rho v_1^2\left[\left(\frac{\pi d_1^2}{\pi d_2^2}\right)^2 - 1\right],$$

or

$$\begin{aligned} p_1 - p_2 &= \frac{1}{2}\left(1060\frac{\text{kg}}{\text{m}^3}\right)\left(0.15\frac{\text{m}}{\text{s}}\right)^2\left[\left(\frac{0.03\,\text{m}}{0.01\,\text{m}}\right)^4 - 1\right] \\ &= 954\frac{\text{kg}}{\text{ms}^2} = 954\left(\text{kg}\frac{\text{m}}{\text{s}^2}\right)/\text{m}^2 = 954\,\text{Pa}, \end{aligned}$$

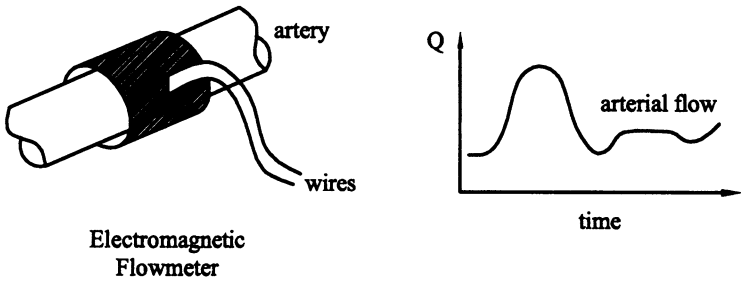


FIGURE 8.14 Schema of the time-varying volumetric flow rate Q measured in vivo using an electromagnetic flowmeter.

where 7.5 mmHg = 1 kPa; hence, $p_1 - p_2 = 7.155$ mmHg. Note that pressures can be measured chronically in animals using indwelling catheters whereas flows are often measured with implanted flowmeters (e.g., Fig. 8.14).

Example 8.8

Under what conditions can you compute the pressure in the system in Fig. 8.15? Recall that streamlines must be parallel and straight, where the radius of curvature is infinity, in order for $\partial p/\partial n = 0$. Consider multiple possibilities.

Solution 1: For flow along a streamline between points 2 and 4,

$$\frac{p_2}{\rho} + gz_2 + \frac{1}{2}v_2^2 = \frac{p_4}{\rho} + gz_4 + \frac{1}{2}v_4^2.$$

From overall mass balance, $v_2 A_2 = v_4 A_4$. For the pipe from point 2 to point 4, $A_2 = A_4$; therefore, $v_2 = v_4$. With $v_2 = v_4$ and $z_2 = z_4$, Bernoulli's equation

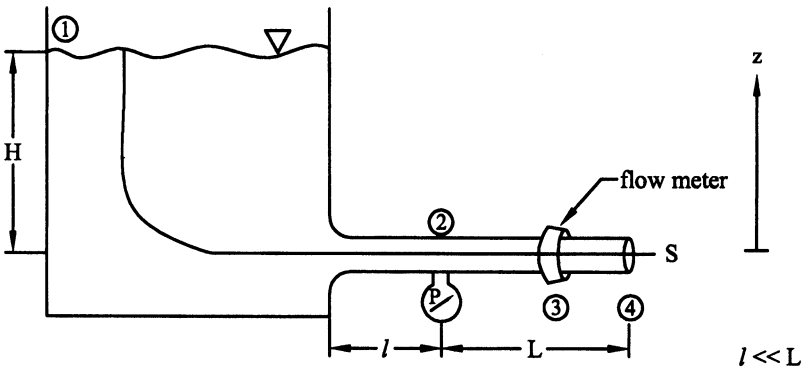


FIGURE 8.15 Flow from a reservoir through a segment of rigid tubing.

suggests that the pressures p_2 and p_4 are equal. Because we discharge to atmospheric pressure at an assumed subsonic velocity, $p_4 = 0$ (gauge) and the pressure at point 2 is also predicted to be zero. Would we expect this to be the case particularly given that we are driving the flow only via a pressure gradient? Recall that Bernoulli should not be used over long distances.

Solution 2: For flow along a streamline between points 1 and 2, assuming a rounded entrance at the chamber-tube interface,

$$\frac{p_1}{\rho} + gz_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + gz_2 + \frac{1}{2}v_2^2,$$

where the pressure at point 1 is zero (gauge) and $v_1 \ll v_2$ if $A_1 \gg A_2$; thus,

$$gz_1 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2 \rightarrow p_2 = \rho gH - \frac{1}{2}\rho v_2^2,$$

where v_2 is nonzero and equal to v_3 (which is measured via the flowmeter) by mass balance. We observe, therefore, that one application of Bernoulli suggests that $p_2 = 0$, whereas another yields $p_2 = \rho gH - \frac{1}{2}\rho v_2^2$. Bernoulli is applicable across contractions and over short distances.

Example 8.9

A forced vortex flow is given by $\boldsymbol{v} = r\omega_o \hat{\boldsymbol{e}}_\theta$, where ω_o is constant (Fig. 8.16). Determine if Bernoulli's equation can be used to determine the pressure difference between two radial locations. Ignore gravity.

Solution: For $\boldsymbol{v} = v_r\hat{\boldsymbol{e}}_r + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{e}}_z$, we have

$$v_r = 0, \quad v_\theta = r\omega_o, \quad v_z = 0.$$

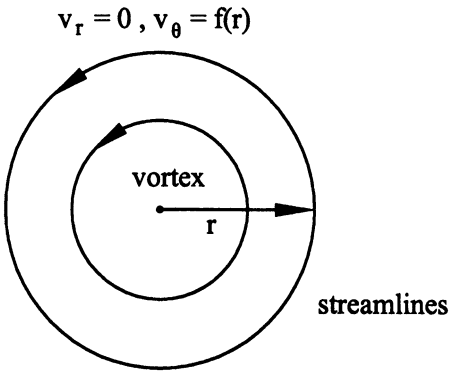


FIGURE 8.16 Schema of a vortex flow. In a forced vortex, $v_\theta = r\omega$, whereas in a free vortex, $v_\theta = c/r$, where c is a constant. What is the vorticity for each?.

To use the Bernoulli equation, several assumptions must be met:

1. The flow must be steady. Checking in each direction, we get

$$\frac{\partial}{\partial t}(v_r) = 0, \quad \frac{\partial}{\partial t}(r\omega_o) = 0, \quad \frac{\partial}{\partial t}(v_z) = 0;$$

   thus, this restriction is satisfied.

2. The fluid must be incompressible. For cylindrical coordinates,

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z).$$

   Checking $\nabla \cdot \boldsymbol{v} = 0$, we get

$$\frac{1}{r}\frac{\partial}{\partial r}(r(0)) + \frac{1}{r}\frac{\partial}{\partial \theta}(v\omega_o) + \frac{\partial}{\partial z}(0) = 0;$$

   thus, this restriction is satisfied.

3. Under certain situations, we can assume that the fluid is inviscid. The validity of this assumption must be established via experience.

4. The flow must be along a streamline or it must be irrotational, where $\nabla \times \boldsymbol{v} = \mathbf{0}$. For cylindrical coordinates, $\nabla \times \boldsymbol{v} = \mathbf{0}$ is given by

$$\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \left(\frac{1}{r}\frac{\partial (rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\boldsymbol{e}}_z = \mathbf{0}.$$

   With $v_r = v_z = 0$, we have

$$\nabla \times \boldsymbol{v} = \left(-\frac{\partial v_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{1}{r}\frac{\partial (rv_\theta)}{\partial r}\right)\hat{\boldsymbol{e}}_z = \frac{1}{r}\frac{\partial (r^2\omega_o)}{\partial r}\hat{\boldsymbol{e}}_z$$

or, in the z direction,

$$\frac{1}{r}(2r\omega_o) = 2\omega_0.$$

Therefore, the flow is not irrotational, and Bernoulli's equation cannot be used unless along a streamline.

8.6 Measurement of Pressure and Flow

One of the most important advances in the development of the modern method of scientific investigation was the realization (by Galileo and others) that theory and experiment must go hand in hand. Theory is needed to design and interpret experiments, which, in turn, are needed to test theories. Experimentation often involves the identification of specific functional relationships between the dependent and independent variables that theory establishes to be important, as well as the calculation of the numerical values of the associated material parameters. Recall from Fig. 1.9 of Chap. 1 that theories, like hypotheses, are motivated by basic observations. Both observation and experimentation require measurements.

Measurement implies that we assign a numerical value to a quantity, often via a comparison to some standard. For example, if we desire to measure the length of an object, we may choose to quantify the length in terms of meters, where 1 m = 1,650,763.73 wavelengths of the orange-red radiation of krypton-86 in a vacuum. Standard weights and measures are kept by governmental agencies such as the National Institute for Standards and Technology (NIST) in the United States and the International Bureau of Weights and Measures in Sèvres, France.

In the modern laboratory, most systems for measurement consist of three components: a transducer, a signal conditioner, and a recorder. A *transducer* is simply any device that converts a physical quantity of interest into another quantity that is more easily measured. Perhaps the simplest transducer is the mercury thermometer, which "converts" temperature (thermal energy) into the displacement of a column of mercury that is easily measured against a ruled background. Most modern transducers convert physical quantities into electrical outputs, either a voltage or a current. A signal conditioner often consists of a combination of amplifiers and filters. Amplifiers modify the range of a signal, whereas filters remove unwanted portions of a signal. A recorder may be any device that archives the measurement; it may take various forms, including a still camera, an analog video recorder plus an analog-to-digital (A/D) converter, or a digital camera and digital memory. Amplifiers, conditioners, and recorders are discussed in detail in courses on instrumentation. Here, let us focus on a few basic transducers.

8.6.1 Pressure

A pressure is a net force per unit area that acts normal to and into a surface area. Stephen Hales was apparently the first, in 1733, to measure the pressure in an artery under "normal" conditions. Specifically, Hales inserted a small-diameter vertical tube into the carotid artery of a horse and recorded the height to which

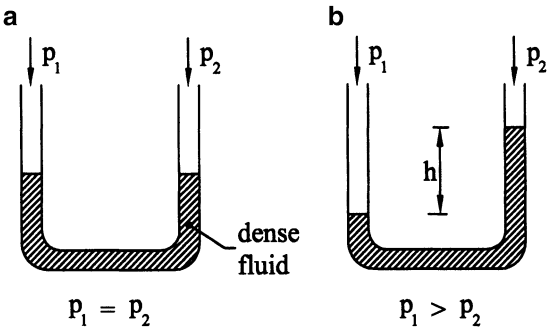


FIGURE 8.17 A U-tube manometer. Relative differences in height between the fluid in the two tubes indicates a difference in pressure acting on each column of fluid.

the blood rose in the tube. A simple free-body diagram of such an experiment reveals that the blood was acted upon by atmospheric pressure p_0 from above and arterial pressure p_a from below. With $p_a > p_0$, the net vertical force due to these pressures, $(p_a - p_0)\pi a^2$, where a is the inner radius of the tube, balanced the weight of the column of blood, $W = \rho g(\pi a^2 h)$ where h is the height. Hence, this simple transducer (tube) allowed the gauge pressure $p = p_a - p_0$ to be inferred simply in terms of the height that the blood rose ($p = \rho g h$), as we know from fluid statics. Albeit the first method of measurement, this clearly is not the easiest. (Note: The blood could easily reach a value of 2 m in an excited animal.)

A major advancement in the measurement of blood pressure, therefore, was the use of a U-shaped mercury manometer by Poiseuille in 1828. The principle of operation of a U-tube manometer is very simple (Fig. 8.17). Mercury orignally proved convenient because of its high density: $SG_{Hg} = \rho_{Hg}/\rho_{H_2O}(4°C) = 13.55$, where $\rho_{H_2O}(4°C) = 1000\,kg/m^3$. Why? Of course, blood pressure (e.g., 120/80) continues to be measured by physicians using the units mmHg (where 7.5 mmHg = 1 kPa).

Electrical-based resistance strain gauges were first used in the physiologic measurement of blood pressure in 1947. Briefly, the fluid pressure elastically deformed a thin metal diaphragm within the transducer, the deformation of which was measured by a strain gauge and calibrated. Hence, an analysis similar to that of LEHI beam bending (Chap. 5) allowed the design of such transducers. Although strain gauge transducers are still used, the ability to use miniature piezoelectric crystals or fiber optics in catheters has revolutionized in vivo measurements (see, e.g., the website for Millar Instruments in Houston, TX). A piezoelectric material is one that generates an electrical output in direct response to an applied load. For more on physiologic measurements and, in particular, the need for adequate frequency responses, see Chap. 11 of Milnor (1989).

8.6.2 Flow

The history of measuring physiological flows dates back to at least 1628 and Harvey, but recent advances in technology have revolutionized the field. Nevertheless, let us consider a simple, theoretically motivated method. Noting from the previous subsection that static pressures are easy to measure, let us exploit Bernoulli's equation (i.e., a theory) to design a device to measure (i.e., perform an experiment or make an observation) the velocity of a flowing fluid. If we consider a horizontal streamline, then Bernoulli's equation (8.71) becomes

$$\frac{p_1}{\rho} + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2. \tag{8.84}$$

Now, let us define the so-called stagnation pressure p_0. *A stagnation pressure* is that value of pressure at a point in a flow field where the fluid is decelerated to zero velocity due to nonviscous effects. Hence, from Bernoulli, we see that if point 2 is a stagnation point, then

$$\frac{p_1}{\rho} + \frac{1}{2}v_1^2 = \frac{p_0}{\rho} \rightarrow v_1 = \sqrt{\frac{2(p_0 - p_1)}{\rho}}. \tag{8.85}$$

In other words, we can infer the velocity by measuring a pressure difference along a streamline for a fluid of known, constant mass density ρ. A possible experimental setup to exploit this theoretical result is shown in Fig. 8.18a. Recall that $\partial p/\partial n = 0$ if the streamlines are parallel [Eq. (8.64)] in the absence of gravity and with $R \rightarrow \infty$; hence, a wall tap can measure the pressure at point 1, whereas a tube filled with an incompressible fluid/gel will stop the flow at point 2 and thus create a stagnation point 0. The difference in pressures $p_0 - p_1$ can thus be measured simply by a U-shaped manometer and we see again that theory guides the design of many transducers. Shown in Fig. 8.18b is a pitot-static tube (pronounced pea-toe), which is designed based on a similar idea and assuming that the thin tube (~0.0625 in. diameter) does not disturb the flow significantly. Indeed, another method of measuring a flow velocity is to use a heated wire. The rate of cooling of the wire can be related to the velocity of the flow; actually, one measures the current supplied to the wire to maintain it at a constant temperature. Such *hot-wire* anemometers are commercially available as small as 0.02 mm in diameter and 0.1 mm long, with a 50-kHz frequency response.

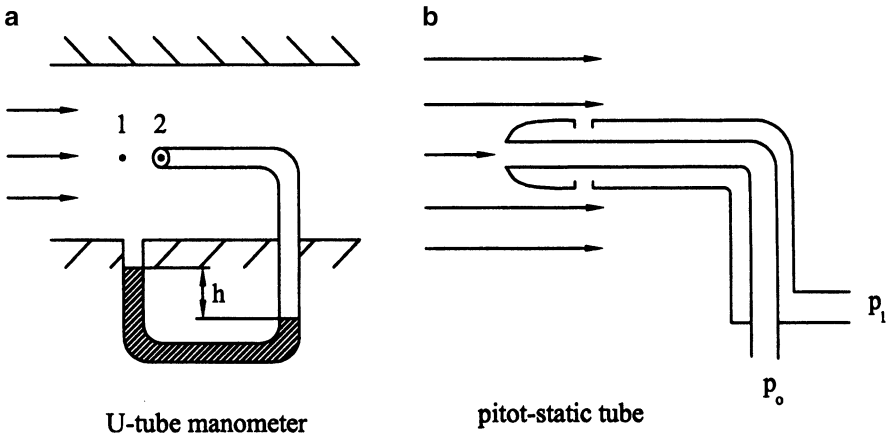


FIGURE 8.18 A pitot-static tube, which allows velocity to be inferred via the simpler measurement of pressure. The motivation for the simple device lies in the theory (Bernoulli equation) and reminds us that theory should always guide the design and interpretation of experiments.

Whereas the pitot-static tube and hot-wire devices measure the velocity at a point, many physiological and clinical situations necessitate that one simply measure the volumetric flow rate Q. An important advance in this regard was the *electromagnetic flowmeter*, developed between 1968 and 1974. Briefly, these devices are based on the fundamental discoveries of Michael Faraday (around 1832) that the motion of an electrically conductive material within a magnetic field generates an electromotive force [see Milnor (1989) for further details]. These flowmeters must be surgically placed around the vessel directly (Fig. 8.14), and with calibration, the output signal is related to the mean flow. Blood is electrically conductive, of course, because of the many ions within.

In some cases, of course, one may wish to simply know qualitative characteristics about the flow field rather than quantitative information. Visualizing flows can be as simple as placing floats on the surface and watching their motion or similarly seeding a flow field with neutrally buoyant fluorescent markers and imaging their motions. Another method is to inject a dye into the flow field (e.g., Fig. 8.19); indeed the common diagnostic tool of angiography uses X-rays to image the motion of a radio-opaque contrast agent that is

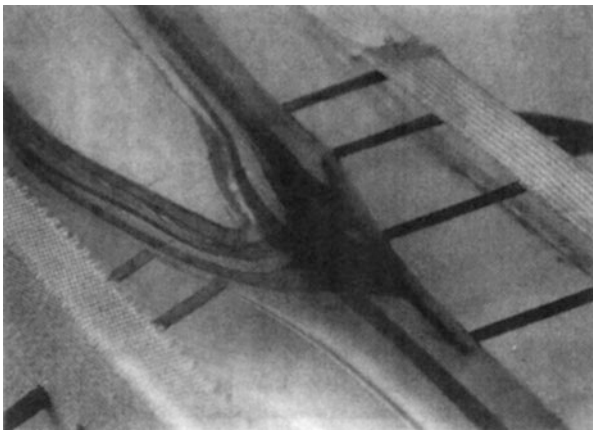


FIGURE 8.19 Visualization of the flow through a model carotid bifurcation. Colored dye is introduced into the flow stream, which allows pathlines to be visualized. In steady flows, pathlines and streamlines coincide. Albeit not quantitative, flow visualization can provide important clues into important aspects of a flow, which, in turn, allow us to focus theoretically or computationally on that which is important. With permission from Lippincott Williams & Wilkins.

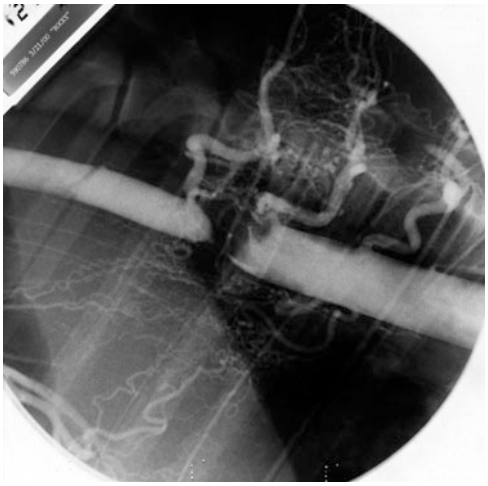


FIGURE 8.20 An aortogram (X-ray) from an experimental model of aortic coarctation, which is induced by inflating an occluding balloon around the aorta. Coarctation models are used to induce and then study hypertension proximal to the occlusion. Note the extensive development of collateral vessels to shunt blood around the obstruction and thereby respond to the insult. The development of new blood vessels is called angiogenesis, which is an important area of current research, as it relates to cancer (tumors develop vessels to supply nutrients and oxygen), tissue engineering (tissues which need to be fed in vivo if they are to survive post-implantation), and recovery from severe injuries such as a myocardial infarction. Angiogenesis research requires input from both biosolid and biofluid mechanics. (Courtesy Dr. M. Miller, Texas A&M University).

injected into the bloodstream. Angiography remains the primary method for diagnosing coarctations, aneurysms, and obstructive atherosclerotic lesions (Fig. 8.20).

Advances in technology have led to many additional, sophisticated methods for quantifying velocities and flows. Ultrasonic (1–8 MHz) transducers and laser Doppler anemometers (LDA) both rely on the Doppler shift (i.e., the frequency shift experienced by waves when the distance between the generator and receiver changes). In the LDA, for example, one focuses a laser beam on a point (i.e., small volume) in the flow field, which scatters when it hits indigenous or seeded particles in the flow. A frequency shift between the scattered and reference light is proportional to the velocity of the scatterer. LDA is widely used in the laboratory to study the complex flow fields within tapering, branching models of the vasculature or airways. Clinically, Doppler ultrasound and magnetic resonance angiography are powerful tools for noninvasively measuring local flows. The interested student is encouraged to research these modalities further.

8.7 Navier–Stokes Worksheets

One quickly discovers that the solution of the Navier–Stokes and Euler equations for many different problems and different coordinate systems follow the same steps. Consequently, we have found it useful to use "work-sheets" to formulate such problems in a consistent way. These worksheets guide us through the identification of the physical problem (e.g., via free-body diagrams) and the identification of appropriate assumptions such as steady flow ($\partial \boldsymbol{v}/\partial t = \boldsymbol{0}$), axisymmetric flow ($\partial \boldsymbol{v}/\partial \theta = \boldsymbol{0}$), no body forces ($\boldsymbol{g} = \boldsymbol{0}$), and so forth. Listing such assumptions and then identifying the terms within the mass balance and Navier–Stokes (or Euler) equations that drop out accordingly allow us to find the reduced differential equations that require solution, subject to appropriate initial-boundary conditions. We encourage the student to make multiple photocopies of these worksheets, which can then be used to solve the problems of interest.

Navier–Stokes Worksheet (Cartesians)

Problem Statement:

Assumptions:

1. Newtonian	3.	5.	7.	9.
2. Incompressible	4.	6.	8.	10.

Mass Balance: $\nabla \cdot \boldsymbol{v} = 0$:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Linear Momentum: $-\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} = \rho \boldsymbol{a}$:

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x = \rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right),$$

$$-\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y = \rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right),$$

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z = \rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right)$$

Reduced Governing Differential Equations:

Boundary/Initial Conditions:

Navier–Stokes Worksheet (Cylindrical)

Problem Statement:

Assumptions:				
1. Newtonian	3.	5.	7.	9.
2. Incompressible	4	6.	8.	10

Mass Balance: $\nabla \cdot \boldsymbol{v} = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Linear Momentum: $-\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} = \rho \boldsymbol{a}$:

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r$$
$$= \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right),$$
$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta$$
$$= \rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$
$$= \rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

Reduced Governing Differential Equations:

Boundary/Initial Conditions:

Chapter Summary

The goal of this chapter was to derive and present general *equations of motion* for incompressible fluids, both ideal (i.e., inviscid) and Newtonian (i.e., described by the Navier-Poisson constitutive relation). We began by deriving mass balance in spatial form (i.e., in the current configuration, as is typically the case in biofluid mechanics) for any fluid, which when reduced for incompressible behavior revealed that the divergence of the velocity field must then be zero (Eq. 8.13). Although often called the continuity equation, we prefer the more descriptive terminology of *mass balance*.

We then showed that linear momentum balance for an ideal fluid results in the famous *Euler equations* whereas linear momentum balance for a Newtonian fluid results in the famous *Navier-Stokes equations*. Because the former is actually a special case of the latter, one can focus on the derivation of the Navier-Stokes equation (Sect. 8.3). It is particularly important to note that, in both cases, mass balance and linear momentum balance together provide the requisite equations to determine the unknowns of interest: pressure plus 1, 2, or 3 components of velocity in 1-D, 2-D, and 3-D, respectively. Moreover, just as we noted in Chap. 2 for biosolids, one should first determine components of velocity (a vector) relative to the coordinate system that renders easiest the mathematical solution and only then via appropriate transformation relations determine the velocities, accelerations, rates of deformation, or stresses that are most meaningful physically or biologically. Illustrative solutions of the Navier-Stokes equations are found in Chap. 9.

It was also shown that integration of the Euler equations for certain flow fields or along particular directions can yield the so-called *Bernoulli equation*, one of the most used and misused equations in fluid mechanics. As noted in Sect. 8.6.2, a general solution of the Bernoulli equation can reveal a clever experimental approach and thus motivate the design of both a novel transducer and an associated experiment. In contrast, the temptation to misuse this equation (i.e., to not respect limitations imposed by the assumptions that led first to the Euler equation and then to the Bernoulli equation) stems primarily from the ease of computation—in contrast to the coupled nonlinear differential equations known as the Euler equations, the Bernoulli equation is algebraic and thus solved more readily. One infamous example of the misuse of Bernoulli is the computation of a pressure drop across a stenosis in an artery or vein. Bernoulli does not apply to this situation because of the adverse pressure gradient that develops just distal to the stenosis. Hence, one should solve either the Navier-Stokes equation, which can only be done numerically, or a control volume based energy equation, the latter of which is derived in Chap. 10. Hence, as we have emphasized throughout this text, it is essential to remember the limitations associated with each derived result.

Appendix 8: Differential Equations

Albeit not without controversy and debate (Boyer 1949; Bell 1986), it is generally agreed that Sir I. Newton invented the basic ideas of calculus and that he was so motivated largely by problems of mechanics. The two basic areas of this subject are, of course, the differential and the integral calculus.

Differential equations allow us to determine how quantities of interest (dependent variables) vary in space and time (independent variables). Such equations can depend on but one independent variable (yielding an ordinary differential equation) or they can simultaneously depend on multiple independent variables (thus yielding partial differential equations); they can appear singly or as systems of equations that must be solved simultaneously; and they can be linear or nonlinear. There is, therefore, great motivation for the biomechanicist to be well versed in methods of solving differential equations and the student is well advised to complete multiple courses in this important area.

Although we see that the Navier–Stokes and Euler equations of motion, in combination with mass balance, represent coupled nonlinear partial differential equations (PDEs), we will consider only simple cases herein and thereby focus primarily on linear ordinary differential equations (ODEs). For example, consider a simple ODE of the form

$$\frac{d^n}{dx^n}(f(x)) = g(x). \tag{A8.1}$$

Such equations arise frequently, as in Chaps. 5 and 9, particularly when $n = 2$ or 4. The best way to solve such equations is directly via integration.

Note, therefore, that Eq. (A8.1) can be written as

$$\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}(f(x))\right) = g(x) \rightarrow \frac{d}{dx}(\text{something}) = g(x), \tag{A8.2}$$

whereby we can integrate with respect to x to obtain

$$\int \frac{d}{dx}(\text{something})dx = \int g(x)dx \rightarrow \text{something} = \int g(x)dx + c. \tag{A8.3}$$

The constant of integration c requires additional information for solution. If the integration is with respect to a spatial variable, we say that we need a *boundary condition* to find c; if the integration is with respect to time, we say that we need an *initial condition* (i.e., a condition at the time of initiation of the process, usually at time $t = 0$ or perhaps $t = -\infty$).

Note from Eq. (A8.3) that the form $d/dx(something)dx$ permits a simple integration; hence, we should try to put ODEs in this form whenever possible. For example, note that

$$\frac{df}{dx}+\frac{1}{x}f(x) \equiv \frac{1}{x}\frac{d}{dx}(x\,f(x)); \tag{A8.4}$$

hence, if we have

$$\frac{df}{dx}+\frac{1}{x}f(x)=g(x)\rightarrow\frac{1}{x}\frac{d}{dx}(x\,f(x))=g(x) \tag{A8.5}$$

and multiplication by x permits a simple solution for $f(x)$,

$$\int\frac{d}{dx}(xf(x))dx \equiv xf(x)=\int xg(x)dx+c. \tag{A8.6}$$

This form [Eq. (A8.4)] occurs frequently in cylindrical coordinates, with x replaced by the radial coordinate r. A similar situation arises if

$$\frac{df}{dx}+\frac{2}{x}f(x) \equiv \frac{1}{x^2}\frac{d}{dx}\left(x^2f(x)\right). \tag{A8.7}$$

Multiplication by x^2 thus yields $d/dx(something)$, which is easily integrated. Regardless of the form, the direct integration of an ODE reduces the problem to one of integral calculus, and integration tables for $\int g(x)dx$, $\int xg(x)dx$, and so forth become very useful, as do methods such as integration by parts:

$$\int_a^b udv=uv\Big|_a^b-\int_a^b vdu. \tag{A8.8}$$

Of course, not all ODEs can be put in a simple form to allow direct integration. Another commonly encountered form in mathematics is the linear, second-order ODE with constant coefficients:

$$\frac{d^2f}{dx^2}+a_1\frac{df}{dx}+a^2f=0. \tag{A8.9}$$

Experience reveals that such equations admit exponential solutions of the form $f(x)\propto e^{\lambda x}$. Because the equation is linear, we know that its solution is unique. Hence, if we can find any solution (e.g., by trial and error or by guessing), then we will have found THE solution. If we guess that $f(x)\propto e^{\lambda x}$, then Eq. (A8.9) becomes

$$\lambda^2e^{\lambda x}+a_1\lambda e^{\lambda x}+a_2e^{\lambda x}=\left(\lambda^2+a_1\lambda+a_2\right)e^{\lambda x}=0\ \ \forall x. \tag{A8.10}$$

Clearly, this equation is satisfied for all x by ensuring that $\lambda^2 + a_1\lambda + a_2 = 0$, which is a (simple) quadratic equation in λ. Hence, whereas the method of Eq. (A8.3) reduces the differential equation to a problem of integral calculus, here we have reduced it to one of algebra, noting that the solution of our quadratic equation is

$$\lambda^2 + a_1\lambda + a_2 = 0 \rightarrow \lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}. \tag{A8.11}$$

Clearly, there are three possible types of solution. (1) The values of λ_1 and λ_2 can be real and distinct, whereby

$$f(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \tag{A8.12}$$

where c_1 and c_2 are arbitrary constants (like the constant of integration in Eq. A8.6) that must be determined via boundary or initial conditions. (2) The values of λ_1 and λ_2 can be real and equal, whereby

$$f(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} \quad (\lambda \equiv \lambda_1 = \lambda_2). \tag{A8.13}$$

Finally, (3) the values of λ_1 and λ_2 can be complex conjugates (with a_1 and a_2 real), whereby

$$f(x) = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x}. \tag{A8.14}$$

Using Euler's relations, however,

$$e^{ibx} = \cos bx + i \sin bx, \qquad e^{-ibx} = \cos bx - i \sin bx, \tag{A8.15}$$

we can alternatively write the solution as

$$f(x) = d_1(\cos bx + i \sin bx)e^{ax} + d_2(\cos bx - i \sin bx)e^{ax} \tag{A8.16}$$

or by defining $c_1 = d_1 + d_2$ and $c_2 = (d_1 - d_2)i$, we have

$$f(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{A8.17}$$

In summary, it cannot be overemphasized that the student of biomechanics must be well versed in the methods of applied mathematics, including differential equations, which requires undergraduate and graduate courses beyond the basic 2-year sequence required of all students in engineering. This appendix merely addressed two of the simpler cases encountered in introductory problems.

Exercises

8.1 Derive the equation of motion [Eq. (8.23)] for the z direction.

8.2 Derive the incompressible Navier–Stokes equation (8.42) for the z direction taking into account the contributions from the local and convective acceleration.

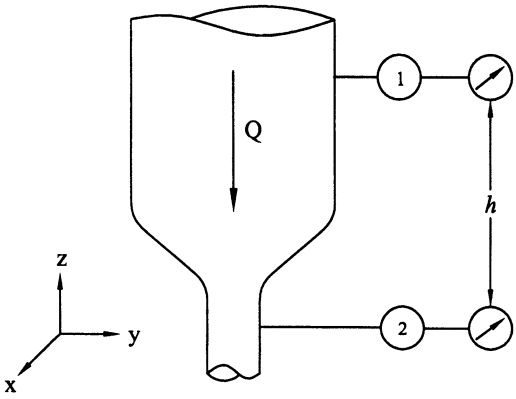
8.3 The general incompressible Navier–Stokes equation can be written as $-\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g} = \rho \boldsymbol{a}$. (a) Write the equation that governs fluid statics and (b) use it to find the relationship among the pressure p, the density ρ, and gravity $\boldsymbol{g}$ in a beaker of water.

8.4 Given the vectorial form of the incompressible Navier–Stokes equation, find the r-, θ-, and z-direction equations in cylindrical coordinates. Hint: Remember that the base vectors change with direction in cylindrical coordinates (see Exercise 7.5).

8.5 Repeat Exercise 8.4 for spherical coordinates (see Exercise 7.6). This is nontrivial.

8.6 An incompressible fluid flows through the device shown in Fig. 8.21. If the pressure at the two gauges is the same, then find the value of the diameter at Section 2 given that the diameter at Section 1 is 2 cm, the velocity at Section 1 is 10 cm/s, and the height h is 10 cm.

FIGURE 8.21



8.7 Derive the streamline direction Euler equation for the case where the streamlines are horizontal and all parallel; assume an arbitrary body force vector $\boldsymbol{g}$, which may include gravity and electromagnetic effects (see Fig. 8.22).

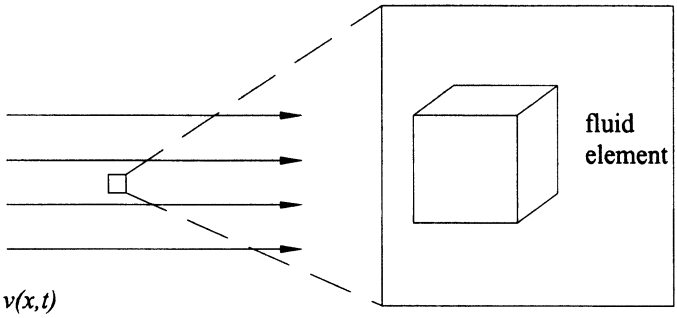


FIGURE 8.22

8.8 Explain why the pressure is constant across straight-parallel streamlines if the fluid is ideal and the flow is steady. Assume no gravity.

8.9 Specific gravity SG is defined as

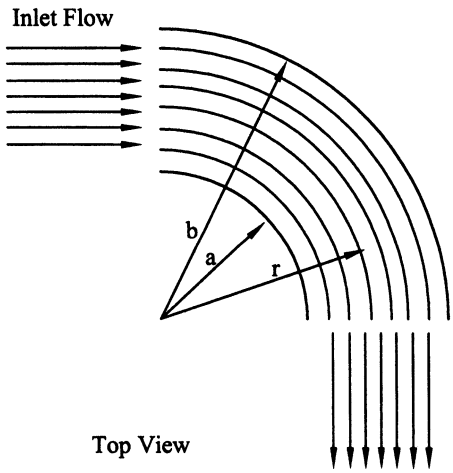
$$\text{SG} = \frac{\rho}{\rho_{\text{H}_2\text{O}}\,\text{at}\,4\,^\circ\text{C}},$$

where $\rho_{\text{H}_2\text{O}}$ at 4 °C is 1,000 kg/m^3. Compute the mass density for the following fluids at room temperature (20 °C) if SG = 1.26 (glycerin), SG = 13.55 (mercury), SG = 1.025 (seawater), and SG = 0.998 (water).

8.10 Glycerin exits a pipe at a mean velocity of $v_o = 1$ m/s and rises to a height $h = 2$ m as in Fig. 8.8. Find the value of the exit pressure assuming gravity $\boldsymbol{g} = 9.81$ m/s^2 $(-\hat{\boldsymbol{j}})$ acts down and the pipe is vertical.

8.11 Water flows through a 90° elbow of a water slide that is open on the top to the atmosphere (Fig. 8.23). Assume steady, ideal, irrotational flow and that $v_s = c/r$ in the bend, where c is constant. For $b > a$, and including gravity, determine whether the flowing fluid will be deeper on the inside (i.e., at $r = a$) or outside (at $r = b$) in the bend.

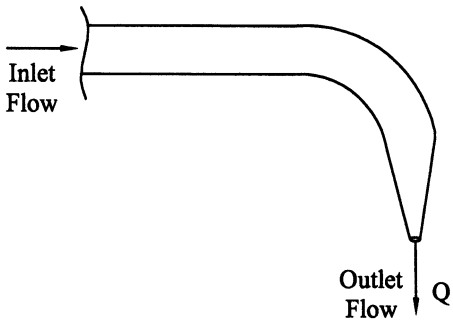
FIGURE 8.23



8.12 Can Bernoulli's equation be used to relate the pressures and velocities at any two points in a flow field given $v_x = x + y$, $v_y = x - y$, $v_z = 0$ plus negligible viscosity and negligible body forces? Why?

8.13 A dentist uses a device similar to that in Fig. 8.24. If the supply volumetric flow rate is Q, and the nozzle tip is d in diameter, what is the exit velocity.

FIGURE 8.24



8.14 Rederive Archimedes' principle if the immersed solid is a cube of length a.

8.15 Rederive Archimedes' principle if the immersed solid is of arbitrary shape.

8.16 Compare the buoyant force on a ring of artery versus a ring of silicone rubber. Assume that each is 4 mm in internal diameter, 0.8 mm in thickness, and 3 mm in length. Compute the volume of Styrofoam that would need to be glued to each ring to render it neutrally buoyant. Assume that the density of an artery is 1,050 kg/m^3, that of silicone is 1,500 kg/m^3, and that of Styrofoam is 40 kg/m^3.

8.17 If the density of blood is ~1,060 kg/m^3 and the blood rose 2 m in Hales' experiment on blood pressure in a horse, what was the arterial blood pressure (gauge) in kPa and mmHg, where 7.5 mmHg = 1 kPa and 1 Pa = 1 N/m^2 (1 N = 1 kg m/s^2).

8.18 If blood pressure is 120 mmHg as measured using a mercury manometer, what would the value be using a water manometer (i.e., $\times$ mmH_2O)? Recall that $SG_{Hg} = 13.55$.

8.19 The *venturi meter* is a commonly used device in engineering to measure internal flows. Research this device and rederive the requisite equations that guide its design and use.

8.20 Show that

$$\nabla^2 \boldsymbol{v} = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla \times (\nabla \times \boldsymbol{v}).$$

8.21 Given that

$$\nabla \cdot \boldsymbol{v} = \mathrm{tr}[D]$$

is true regardless of coordinates, show that it is true for sphericals see [Eq. (7.60)].

8.22 Take two sheets of paper and hold them closely together while blowing air between them. What happens? Does the forced flow between the paper cause the sheets to move farther apart or closer together? Use Bernoulli's equation to explain your observation.

8.23 Bernoulli's equation is often used to explain *lift* of an airfoil (i.e., an airplane wing). In particular, airfoils are designed such that the distance from the leading edge to the trailing edge is longer on the top than on the bottom surface. Consequently, the air must move faster over the top surface (to reach the trailing edge at the same time as the air traveling along the bottom surface). Hence, from Bernoulli, we predict a lift (i.e., a pressure greater on the bottom than on the top surface).

Some have similarly argued that Bernoulli can be used to "explain" why red blood cells tend to move to the central region of an artery. Note, therefore, that it will be shown in Chap. 9 that the velocity field for a steady, incompressible flow in a circular tube is

$$\boldsymbol{v} = c\left(1 - \frac{r^2}{a^2}\right)\hat{\boldsymbol{e}}_z,$$

where c is constant, a is the inner radius of the tube, and r is a cylindrical-polar coordinate $r \in [o, a]$. This velocity field suggests that, similar to the airfoil, the velocity is higher on the side of the cell closest to the centerline. Is it reasonable to use Bernoulli's equation in this case and why? See Fig. 8.25.

FIGURE 8.25