2
Stress, Strain, and Constitutive Relations

2.1 Introduction

Consider the two structural members in Fig. 2.1, each acted upon by an applied weight W that is much larger than the individual weights mg, which we therefore neglect. From statics, we know that if these two members are in equilibrium, then $\Sigma \boldsymbol{F} = \boldsymbol{0}$ and $\Sigma \boldsymbol{M} = \boldsymbol{0}$. Free-body diagrams of the whole structure and the individual parts reveal that the reaction and internal forces are the same: $R_y = f_y = W$; that is, from the perspective of statics alone, these two problems are equivalent. Nevertheless, intuition tells us that the behavior of member A need not be the same as that of member B. One may fail before the other. An important question to be answered by mechanics, therefore, may be the following: Which member will likely fail first given increasing weights W? At first glance, we may be inclined to say that A will fail before B, for A is "thinner," and indeed this may well be. Yet, our information is incomplete: We have not specified what A and B are made of; A could be made of a much stronger material than B. Thinking back to statics, we realize that we never specified the properties of the materials or structures that we studied, we simply assumed that they were always rigid (i.e., infinitely stiff). In this book, however, we will see that *the individual properties of materials are central in biomechanics*. For example, we often seek to match the properties of man-made or tissue-engineered replacements to those of the native tissue or organ. Indeed, one of the continuing challenges in biomechanics is accurate characterization, or quantification, of the material behavior of both living tissues and biomaterials.

J.D. Humphrey, S.L. O'Rourke, *An Introduction to Biomechanics*,
DOI 10.1007/978-1-4939-2623-7_2

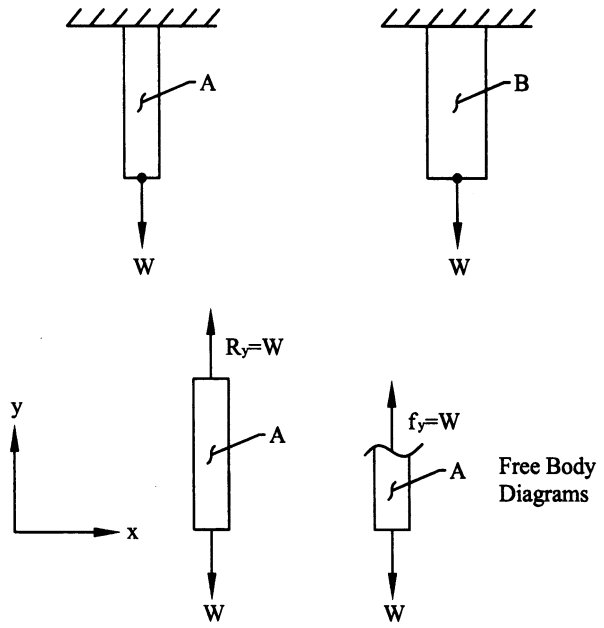


FIGURE 2.1 Contrast the potentially different responses of two simple structural members, A and B, which have the same type of fixed support at the top, the same initial length, and the same axial loading W at the otherwise free-end. Free body diagrams of the whole and a part reveal the reactions at the fixed support and the internal force.

Returning to Fig. 2.1 and the question of whether structure *A* or *B* will fail first, we first need to define what is meant by failure. In mechanics, *failure simply implies an inability to perform the intended mechanical function*. Structures *A* and *B* could thus fail by the following:

- Material failure, including fracture, tearing or rupture, as, for example, in the tearing of an anterior cruciate ligament
- Deforming excessively, which may or may not include a permanent deformation such as a severely bent (e.g., plastically deformed) surgical instrument, which does not return to its functional shape

Determination of *failure criteria* for materials is thus an important responsibility of the biomedical engineer. Recalling our intuition earlier that structure *A* may fail before *B* because *A* is thinner (Fig. 2.1) suggests that failure criteria cannot be written in terms of the applied loads alone; one must also consider the geometry. This brings us to the concept of stress.

2.2 Concept of Stress

In 1678, Robert Hooke published the anagram (in Latin) *ceiiinosssttuv,* which can be deciphered as, *ut tensio sic vis*, and translated, *as the force, so the extension*. That is, by studying the response of linear metallic springs to the application of various weights, Hooke realized that there is a one-to-one relationship for many materials between the applied load (force) and the motion (extension). Figure 2.2 shows force–extension curves for three similar but different linear springs, each described by the general formula $f = k(\ell - \ell_0)$ where ℓ is the current length, ℓ_o is the original length, and k is the so-called spring constant or stiffness. The results in Fig. 2.2 for three different springs suggest that each is characterized by an individual spring constant k_1, k_2, or k_3 (or material property). If we apply the same idea of plotting force versus extension for cylindrical specimens of various materials (e.g., aluminum or stainless steel), we quickly discover that such tests do not characterize the material. If the same loads are applied to the same material in two different labs, which use two different diameter specimens, we find different slopes in the force–extension data. Indeed, the thicker sample, albeit composed of the same material, will appear "stiffer" because it will extend less in response to the same force.[1] Hence, in contrast to Hooke's original idea, there is more to it than just "as the force, so the extension."

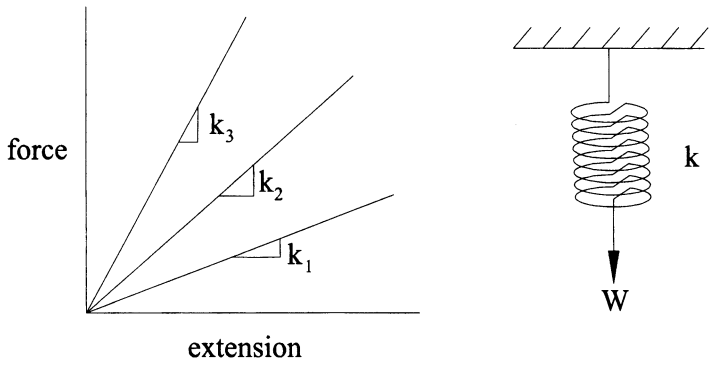


FIGURE 2.2 Force–extension behavior of three different metallic springs, which exhibit linear behaviors and thereby can be quantified by individual spring constants k (or stiffnesses). Although many springs exhibit a linear behavior, nonlinear springs exist as well.

[1] Differences between structural stiffness, which depends in part on geometry, and true material stiffness are important in clinical measurements, as discussed later.

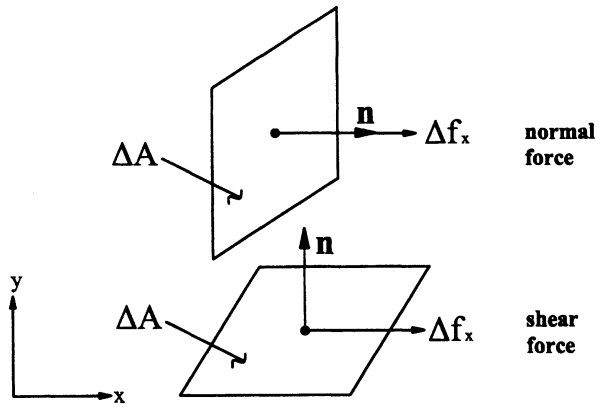


FIGURE 2.3 Schema of the x component of a differential force Δf_x (actually, the mean value of a distributed force) that acts on an area ΔA. Clearly, the effect of this single component of force on the underlying material will depend on the orientation of the area over which it acts: If the area is oriented in the same direction as the force, we expect a tension or compression, whereas if the area is oriented orthogonal to the force, we expect a shearing action. The directions of both the force **f** and the area (given by its outward unit normal vector **n**) are equally important.

In 1757, Leonard Euler realized that a better measure for analysis is a "force intensity" or *stress*. Simply put, Euler defined this intensity as a force acting normal to an area divided by the value of that area (i.e., a pressure-like quantity that we now call a normal stress). During the period 1823–1827, Augustin-Louis Cauchy formalized the concept of stress. Defined as a *force acting over an oriented area* at any point in a body, it is clear that there can be different "stresses" at the same point depending on the orientation of the applied force and the orientation of the area of interest, which implicitly says depending on the choice of a coordinate system (i.e., an origin and basis)—that is, stress is a mathematical construct; its "value" is not unique.

For example, consider a force having only an x component, say Δf_x, which acts over an area ΔA in the current (deformed) configuration of the body (Fig. 2.3). Intuitively, the effect of the same force Δf_x on the same area ΔA will have different effects depending on the orientation of ΔA, which is denoted by the outward unit normal vector $\boldsymbol{n}$ (i.e., $|\boldsymbol{n}| = 1$). For example, if $\boldsymbol{n}$ is in the direction of Δf_x, we call the force a normal force and its intensity (per unit area) a *normal stress*; if $\boldsymbol{n}$ is perpendicular to the direction of Δf_x, we call the force a shearing force and its intensity a *shear stress*. Note, therefore, that although a force could act on an area at any angle, it is generally convenient to resolve the force vector into components that are normal and parallel to the surface. Specifically, then, if we let ΔA_x denote that ΔA has an outward normal $\boldsymbol{n} = \hat{\boldsymbol{e}}_x$ and take the limit as ΔA tends to zero, then we obtain the normal stress:

$$\lim_{\Delta A \to 0} \frac{\Delta f_x}{\Delta A_x} = \frac{df_x}{dA_x} = \sigma_{xx}; \tag{2.1}$$

that is, we denote stress (i.e., a force acting over an oriented area in the current deformed configuration) with the Greek lowercase sigma, with the first and second subscripts (or indices) associated with the oriented area (i.e., face) on which the force acts and the direction of the applied force, respectively. Hence, with

$$\sigma_{(\text{face})(\text{direction})}, \tag{2.2}$$

then

$$\lim_{\Delta A \to 0} \frac{\Delta f_x}{\Delta A_y} = \frac{df_x}{dA_y} = \sigma_{yx} \tag{2.3}$$

for a shear stress in the x-y plane.

Although stresses act in the direction of that component of the force that acts at the point of interest, they are not vectors. Rather, because stress is a force acting over an oriented area, it is associated with two directions, one each for the direction of the force and the outward unit normal $\boldsymbol{n}$. Mathematically, such quantities are called tensors, but we will not exploit this character. It is useful nonetheless to represent the components of stress by arrows that act on the appropriate faces of a body in the appropriate directions. See, for example, Fig. 2.4, which shows the so-called *positive sign convention* for a 2-D state of stress relative to a Cartesian coordinate system. In particular, we shall assume that normal stresses are positive when tensile; this requires that σ_{xx} be directed in a positive direction on a positive face (i.e., one having an outward unit normal in a positive coordinate direction) and conversely that σ_{xx} be directed in a negative direction on a negative face (i.e., one having an outward unit normal in a negative coordinate direction). Indeed, for consistency, we assume the same for the shear stresses, as seen for σ_{yx} and σ_{xy} in Fig. 2.4. (Note: An easy way to remember this positive sign convention is that a positive times a positive is positive and a negative times a negative is a positive; hence, the positive sign convention requires a negative direction stress on a negative face.) As in statics, it is best to use the directions associated with the sign convention; if the computed value turns out to be negative, it simply tells us to switch the assumed direction of that component of stress.

Recall that *if a body is in equilibrium, then each of its parts must also be in equilibrium*—this holds true for any material point p. Note, therefore, that because of our sign convention (Fig. 2.4), the "two" normal stresses σ_{xx} in the figure balance and so too for the x-direction action of the "two" shear stresses σ_{yx}. If the mathematical point p represents a material particle even of infinitesimal

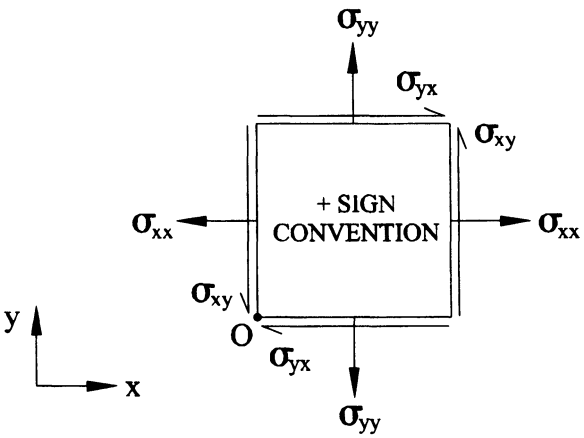


FIGURE 2.4 Positive sign convention for a 2-D state of stress, relative to Cartesian coordinates, that exists at point p but is shown over a square domain for illustrative purposes. The direction of each component is positive on a positive face (i.e., a face with an outward unit normal in the positive coordinate direction) but negative on a negative face. This convention is consistent with normal stresses being positive when tensile and it is consistent with equilibrium at a point (i.e., the balancing of equal and opposite pushes and pulls).

dimension, however, we see that equilibrium is not necessarily satisfied; that is, the two σ_{yx}'s would tend to create a couple (i.e., a pure moment, or force acting at a distance) that would tend to rotate the differential region centered at particle p. Equilibrium could be ensured by the addition of an opposing pair σ_{xy}, as seen in Fig. 2.4, wherein we have preserved both the positive sign convention (e.g., positive direction on a positive face) and the notation sigma subscript (face, direction). Consequently, $\sigma_{xy} \equiv \sigma_{yx}$ numerically at every point, which can be proven rigorously via the balance of angular momentum (i.e., $\Sigma \boldsymbol{M} = \boldsymbol{0}$ in this case) as shown below. Committing the sign convention represented in Fig. 2.4 to memory serves one well throughout mechanics.

In general, however, we note that each point p could be thought of as an infinitesimal cube that is reduced in size in a limiting process. As such, each point can be thought to have six faces relative to each Cartesian coordinate system. For (x, y, z) coordinates, this implies positive and negative ΔA_x, ΔA_y, and ΔA_z faces. Moreover, given that each point can be acted upon by a force $\Delta \boldsymbol{f}$, which has a component representation relative to (x, y, z) as

$$\Delta \boldsymbol{f} = \Delta f_x \hat{\boldsymbol{i}} + \Delta f_y \hat{\boldsymbol{j}} + \Delta f_z \hat{\boldsymbol{k}} \equiv \Delta f_x \hat{\boldsymbol{e}}_x + \Delta f_y \hat{\boldsymbol{e}}_y + \Delta f_z \hat{\boldsymbol{e}}_z, \tag{2.4}$$

where $\hat{\boldsymbol{i}} \equiv \hat{\boldsymbol{e}}_x$ and so forth, there are nine possible measures (i.e., components) of stress at each point p relative to (x, y, z). They are

$$\sigma_{(\text{face})(\text{direction})} = \begin{bmatrix} \dfrac{df_x}{dA_x} & \dfrac{df_y}{dA_x} & \dfrac{df_z}{dA_x} \\ \dfrac{df_x}{dA_y} & \dfrac{df_y}{dA_y} & \dfrac{df_z}{dA_y} \\ \dfrac{df_x}{dA_z} & \dfrac{df_y}{dA_z} & \dfrac{df_z}{dA_z} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \tag{2.5}$$

which we have written in matrix form for convenience (matrices are reviewed in Appendix 6). The components σ_{xx}, σ_{yy} and σ_{zz} are *normal stresses;* they can cause extension or compression. The components σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx}, and σ_{zy} are called *shear stresses*; they can cause a body to distort, which is to say to experience changes in internal angles. Consistent with the above, this matrix is symmetric (i.e., $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, and $\sigma_{yz} = \sigma_{zy}$) for the Cauchy stress, which is a measure of actual forces acting on current oriented areas. This can be shown formally by letting the dimensions of an infinitesimal element be Δx, Δy, and Δz. Because the components of stress have units of force/area, to sum the moments about an axis such as the z axis in Fig. 2.4, we must first multiply the respective component of stress by the area over which it acts and then multiply by the associated moment arm about any point, say o (because two of the four stress components have lines of action that go through o, this point is convenient for computing the moments). Hence,

$$\sum M_z\big)_o = 0 \rightarrow -\sigma_{yx}(\Delta x \Delta z)\Delta y + \sigma_{xy}(\Delta y \Delta z)\Delta x = 0. \tag{2.6}$$

Simplifying, therefore, we have the result:

$$\sigma_{xy} = \sigma_{yx}. \tag{2.7}$$

Similarly, show that $\sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$.

Example 2.1 Referring to Fig. 2.5, what are the values of σ_{xx}, σ_{xy}, σ_{yx}, and σ_{yy} in this 2-D state of stress.

Solution: Noting that the right face is an x face (with outward unit normal $\hat{\boldsymbol{e}}_x$) and that the top face is a y face, we have $\sigma_{xx} = 120$ kPa, $\sigma_{xy} = 0$ kPa, $\sigma_{yy} = 150$ kPa, and $\sigma_{yx} = 0$ kPa. Being able to identify components of stress $\sigma_{(\text{face})(\text{direction})}$ is an important step in understanding the mechanics.

It cannot be overemphasized that stress is a mathematical concept; it is defined as a measure of a force acting over an oriented area at a point.

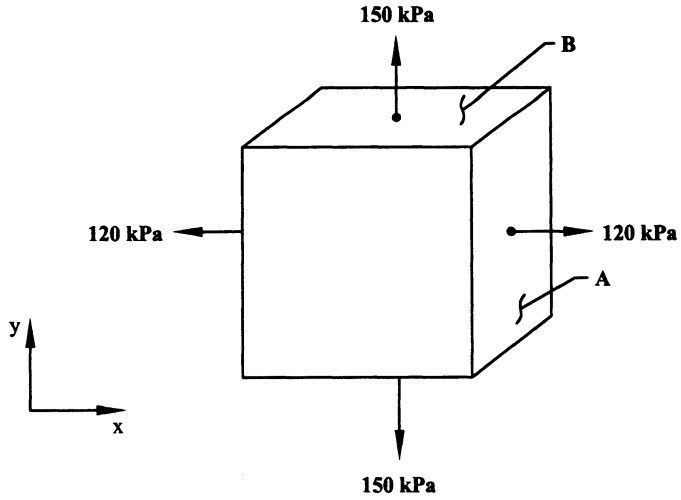


FIGURE 2.5 An illustrative 2-D state of stress acting at a point, components of which act over oriented areas that are expanded for ease of visualization. Although the magnitudes of the components can be considered arbitrary, these values are consistent with in-plane values of stress within a large artery.

Mathematically, stress is a tensor, which is defined independent of a coordinate system. Yet, *to solve practical problems, one must always compute components of stress relative to a particular coordinate system*. Because coordinate systems (which are defined by an origin and a set of base vectors) are not unique but can be defined in many different ways, many different sets of components of stress exist at the same point in a body that is subjected to a single set of applied loads. For example, for the three Cartesian coordinate systems shown in Fig. 2.6—defined by $(o; \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z)$, $(o; \hat{\boldsymbol{e}}'_x, \hat{\boldsymbol{e}}'_y, \hat{\boldsymbol{e}}'_z)$, and $(o; \hat{\boldsymbol{e}}''_x, \hat{\boldsymbol{e}}''_y, \hat{\boldsymbol{e}}''_z)$—the point p admits three different sets of components of the same stress $\sigma_{(\text{face})(\text{direction})}$:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \tag{2.8}$$

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix}, \tag{2.9}$$

and

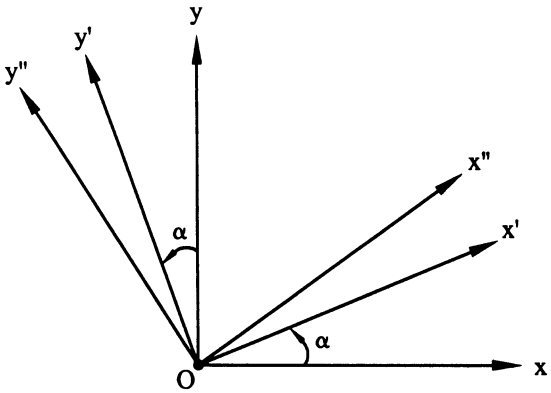


FIGURE 2.6 Interrelations, via single angles, between different Cartesian coordinate systems that share a common origin and a z-axis. The angle α is taken to increase in the direction given by the right-hand rule: counterclockwise, with the z-axis coming out of the paper.

$$\begin{bmatrix} \sigma''_{xx} & \sigma''_{xy} & \sigma''_{xz} \\ \sigma''_{yx} & \sigma''_{yy} & \sigma''_{yz} \\ \sigma''_{zx} & \sigma''_{zy} & \sigma''_{zz} \end{bmatrix}. \tag{2.10}$$

[Note: The $(\ldots)'$ and $(\ldots)''$ notation here simply denotes different coordinate systems; it does not imply differentiation as used in many courses on differential equations.]

In some cases, it may be more natural to compute one set of components, say σ_{xx}, σ_{xy},..., σ_{zz}, whereas in other cases, it may be more useful to compute another set of components, say σ'_{xx}, σ'_{xy},..., σ'_{zz}. A good example of this need is the case of a rectangular structure that consists of two members that are glued together on a 45° angle (Fig. 2.7). Because glue is stronger in shear than in extension (empirically compare removing a Postit® note by applying a normal versus a shear force), it is useful to know how much of the applied force $\boldsymbol{f}$ results in shear versus normal stresses at the glued interface; that is, we would like to know the values of σ'_{xx} and σ'_{xy}, which are computed relative to $(o; \hat{\boldsymbol{e}}'_x, \hat{\boldsymbol{e}}'_y, \hat{\boldsymbol{e}}'_z)$. Yet, from Fig. 2.7, it is clearly easier to enforce equilibrium relative to $(o; \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z)$; that is, assuming the force is applied uniformly over the surface area on which it acts, it is easy to show (see Sect. 3.3.2) that $\Sigma \boldsymbol{F} = \boldsymbol{0}$ yields $\sigma_{xx} = f/A$ and $\sigma_{xy} = 0$ on a cross section with an outward unit normal $\boldsymbol{n} = \hat{\boldsymbol{e}}_x$ that cuts through the glued region. It is clear, therefore, that multiple coordinate systems can be useful even in the same problem. Fortunately, we shall discover in Sect. 2.3 that the desired values σ'_{xx} and σ'_{xy} can be determined directly from the more easily

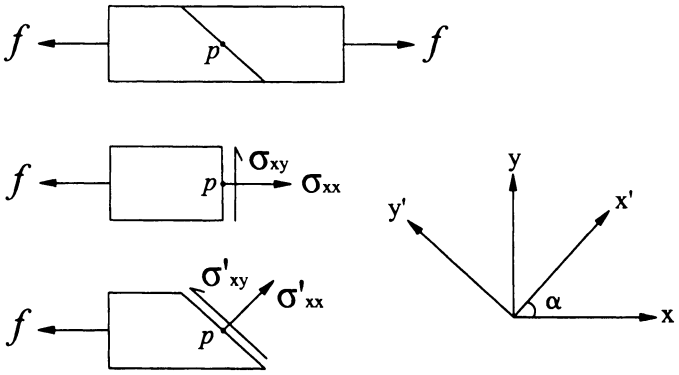


FIGURE 2.7 Free-body diagrams of the same structure cut along two cross sections: one with an outward unit normal in the direction of the applied force (which is natural for solving the equilibrium problem) and one with an outward unit normal to the glued surface (which exposes stresses that act thereon and are important with regard to possible debonding). In each case, the components are identified as $\sigma_{(\text{face})(\text{direction})}$. Note: although we could denote stresses with respect to an x′ face and x′ direction as $\sigma_{x'x'}$, we prefer to denote them as σ'_{xx} for convenience.

computed values $\sigma_{xx} = f/A$ and $\sigma_{xy} = 0$; that is, we will not need to solve the equilibrium problem for each coordinate system of interest.

Inasmuch as coordinate systems are introduced for convenience, many different coordinate systems prove useful in the wide variety of problems that fall within the domain of biomechanics. For problems in the circulatory and pulmonary systems, for example, the nearly circular nature of the arteries, capillaries, veins, and bronchioles render cylindrical-polar coordinate systems very useful. For problems involving certain cells, saccular aneurysms, the urinary bladder, and so forth, spherical coordinates are very useful. For problems in cardiac mechanics, particularly for the left ventricle, prolate spheroidal coordinates are useful. For problems in developmental cardiology, toroidal coordinates are convenient. Indeed, the list goes on and on, including more complex coordinate systems. Fortunately, regardless of the coordinate system, our notation $\sigma_{(\text{face})(\text{direction})}$ will hold; that is, we seek measures that describe the intensity of the force relative to both the oriented area on which the force acts and the direction of the applied force. In cylindrical coordinates (r, θ, z), we have (Fig. 2.8)

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}, \tag{2.11}$$

and likewise for spherical coordinates (r, θ, ϕ), we have (Fig. 2.9)

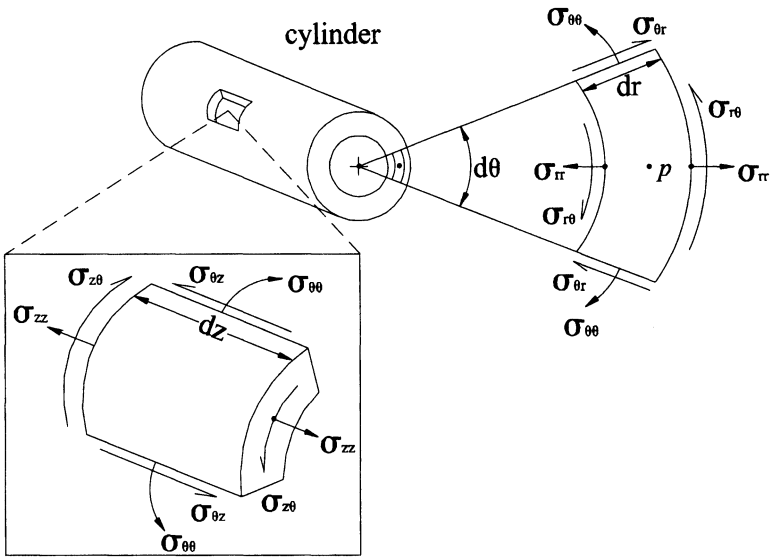


FIGURE 2.8 Components of stress relative to a cylindrical coordinate system, again using the standard notation $\sigma_{(face)(direction)}$.

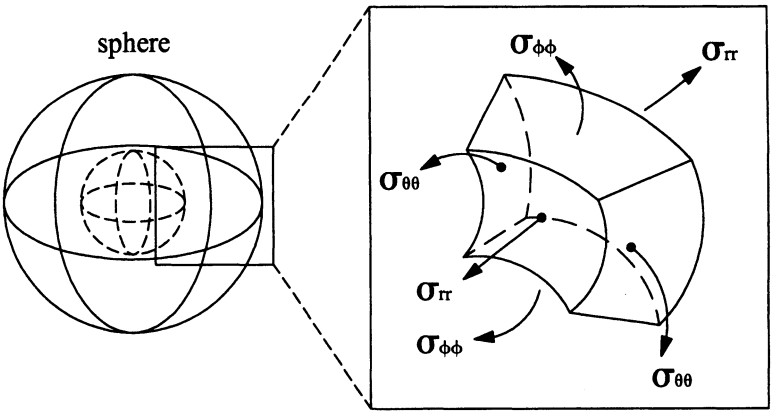


FIGURE 2.9 Normal components of stress relative to a spherical coordinate system, again denoting the components as $\sigma_{(face)(direction)}$. As an exercise, add the shearing components.

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}, \tag{2.12}$$

each at every point p. It is important to review and understand that which is represented in these figures.

Independent of the specific coordinate system, a *1-D state of stress* is one in which only one component of stress (e.g., σ_{xx}) is nonzero relative to the prescribed coordinate system; a *2-D state of stress* is one in which four components of stress (e.g., σ_{xx}, σ_{yy}, σ_{xy}, and σ_{yx}, three of which are independent because $\sigma_{xy} = \sigma_{yx}$) may be nonzero relative to the chosen coordinate system; a *3-D state of stress* is one in which all nine components (six of which are independent) may be nonzero in general.

In summary, the concept of stress is a mathematical one. Stress may be computed at each point in a continuum body; when resolved with respect to a coordinate system, there are nine components at each point, although only six components of the Cauchy stress are independent relative to each 3-D coordinate system—three normals and three shears. Because coordinate systems can be related via transformation relations, the various components of stress can be related through transformation relations. Let us now derive these useful relations for Cartesian components.

2.3 Stress Transformations

Consider a 2-D state of stress relative to either $(o; \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y)$ or $(o; \hat{\boldsymbol{e}}'_x, \hat{\boldsymbol{e}}'_y)$ as shown in Fig. 2.10. Because these figures merely represent the stresses that act at point p, we can cut either square part in order to represent components relative to both coordinate systems in a single figure. Anticipating the need to sum forces and moments to enforce equilibrium (of the parts), let the diagonally cut part be of uniform width Δz and length Δy along the vertical cut edge. Hence, the three exposed areas of interest are computed easily, as shown in Fig. 2.11.

From geometry, we have $\sin\alpha = \text{opp/hyp}$ and $\cos\alpha = \text{adj/hyp}$, where $\text{opp} \equiv \Delta x$ and $\text{adj} \equiv \Delta y$. Hence, $\text{hyp} = \Delta y/\cos\alpha = \Delta y \sec\alpha$. Now, if we multiply through by the width Δz, then we have the result that the area that σ'_{xx} and σ'_{xy} act over is given by $\Delta z\,(\text{hyp}) = \Delta A \sec\alpha$, with $\Delta y \Delta z = \Delta A$ being the area over which σ_{xx} and σ_{xy} act. Similarly, σ_{yy} and σ_{yx} act over $\Delta z(\text{opp}) = \Delta z(\text{hyp}) \sin\alpha = \Delta A \sec\alpha \sin\alpha = \Delta A \tan\alpha$. Now, we are ready to sum forces. Balancing forces (i.e., stresses multiplied by the areas over which they act) in the x' direction requires that we find the components in the x' direction. Clearly, the x-directed forces must be multiplied by $\cos\alpha$, whereas the y-directed forces must be multiplied by $\sin\alpha$ to get the x' direction components. Hence, equilibrium yields

$$\sum F_{x'} = 0 = \sigma'_{xx}\Delta A \sec\alpha - (\sigma_{xx}\Delta A)\cos\alpha - (\sigma_{yx}\Delta A \tan\alpha)\cos\alpha - (\sigma_{yy}\Delta A \tan\alpha)\sin\alpha - (\sigma_{xy}\Delta A)\sin\alpha$$

or

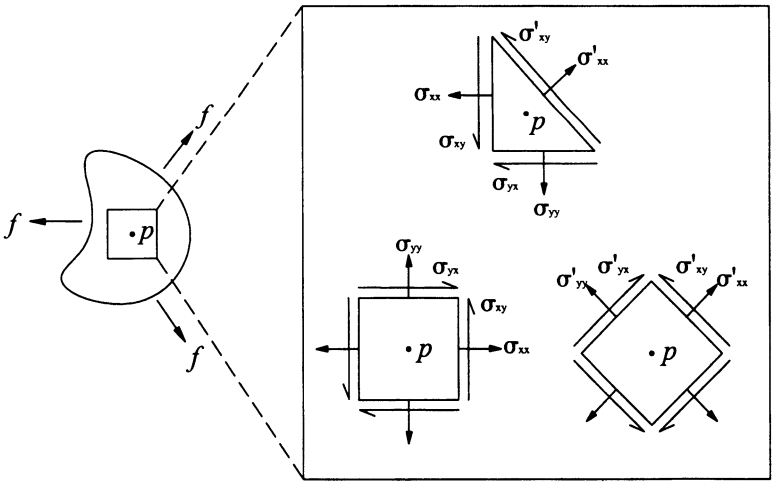


FIGURE 2.10 General 2-D state of stress at the point p emphasizing again that the components are defined with respect to the orientation of the area over which they act (i.e., the face) and the direction of the applied force (i.e., the direction). Hence, different sets of components coexist at the same point. This allows us to make fictitious cuts that expose, on the same element, components relative to different coordinate systems.

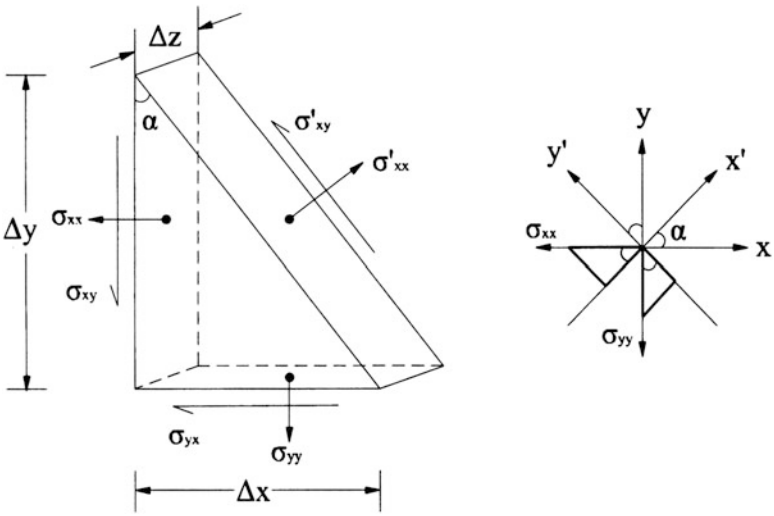


FIGURE 2.11 Detailed diagram of the fictitious element from Fig. 2.10 with the 2-D components of stress isolated relative to two different Cartesian coordinate systems. Remembering that if a body is in equilibrium, then each of its parts are in equilibrium, we can therefore use a force balance to relate the components of stress for the two coordinate systems. Alpha is an arbitrary cutting angle.

$$\sigma'_{xx} = \sigma_{xx}\cos^2\alpha + 2\sigma_{xy}\sin\alpha\cos\alpha + \sigma_{yy}\sin^2\alpha, \tag{2.13}$$

wherein we let $\sigma_{xy} = \sigma_{yx}$ from above, and we see that the ΔA cancels throughout, thereby rendering the equation valid for arbitrarily chosen (small) dimensions about point p. It is important to realize, therefore, that the continuum concept of stress actually represents an average force intensity within a small region (neighborhood) centered about the point of interest.

Recalling the trigonometric identities

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}, \quad \sin^2\alpha = \frac{1 - \cos 2\alpha}{2}, \quad \sin 2\alpha = 2\sin\alpha\cos\alpha \tag{2.14}$$

Equation (2.13) can be rewritten as

$$\sigma'_{xx} = \sigma_{xx}\frac{1 + \cos 2\alpha}{2} + \sigma_{xy}\sin 2\alpha + \sigma_{yy}\frac{1 - \cos 2\alpha}{2}, \tag{2.15}$$

or

$$\sigma'_{xx} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\alpha + \sigma_{xy}\sin 2\alpha. \tag{2.16}$$

Given that Eq. (2.13) is a perfectly acceptable way to compute σ'_{xx} from values of stress relative to (o; $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$) for any α, one might ask: Why use the trigonometric identities to obtain the alternate form [Eq. (2.16)]? This is actually a good question, the answer to which comes from *hindsight*. Throughout this text, we must remember that even what may appear to be simple or obvious may have taken great thinkers many years to realize (e.g., nearly 150 years passed between Hooke's ideas on force to Cauchy's on stress). We will see below that Eq. (2.16) is extremely convenient in one particular application. It is also important to remember that we, as students, benefit from the many hours, days, indeed weeks or even years of thought by many which resulted in simplifications we have today.

Forces in the y' direction (Fig. 2.11) can similarly be balanced, namely

$$\begin{aligned}\sum F_{y'} = 0 = \sigma'_{xy}\Delta A\sec\alpha + (\sigma_{xx}\Delta A)\sin\alpha - \left(\sigma_{xy}\Delta A\right)\cos\alpha \\ - \left(\sigma_{yy}\Delta A\tan\alpha\right)\cos\alpha + \left(\sigma_{yx}\Delta A\tan\alpha\right)\sin\alpha,\end{aligned}$$

or

$$\sigma'_{xy} = 2\sin\alpha\cos\alpha\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right) + \left(\cos^2\alpha - \sin^2\alpha\right)\sigma_{xy}. \tag{2.17}$$

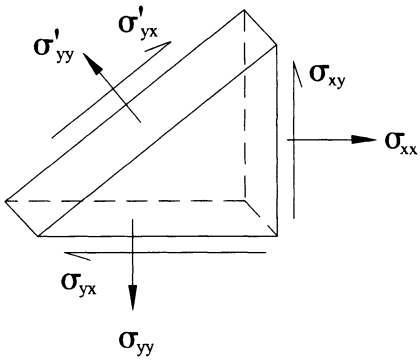


FIGURE 2.12 Alternate fictitious cut (cf. Fig. 2.11) to expose y′-face components of stress σ.

Again, using trigonometric identities, we can rewrite this equation as

$$\sigma'_{xy} = \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\alpha + \sigma_{xy} \cos 2\alpha. \tag{2.18}$$

Remembering that if a body is in equilibrium, then all of its parts are in equilibrium, we often fictitiously cut a body into multiple different parts to expose, on cut surfaces, specific components of stress of interest. Because the selection of the oblique cutting plane in Fig. 2.10 did not isolate a y' surface (i.e., an area with outward unit normal $\hat{\boldsymbol{e}}'_y$), we must consider another free-body diagram that isolates σ'_{yy} and σ'_{yx} (Fig. 2.12). Doing so, we can again balance forces in x' and y'. This is left as an exercise; thus, show that given such a cut,

$$\sum F_{x'} = 0 \rightarrow \sigma'_{yx} \Delta A \sec\alpha = -(\sigma_{xx} \Delta A \tan\alpha) \cos\alpha - \left(\sigma_{xy} \Delta A \tan\alpha\right) \sin\alpha + \left(\sigma_{yy} \Delta A\right) \sin\alpha + \left(\sigma_{yx} \Delta A\right) \cos\alpha,$$

or

$$\sigma'_{yx} = 2 \sin\alpha \cos\alpha \left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right) + \left(\cos^2\alpha - \sin^2\alpha\right)\sigma_{xy}. \tag{2.19}$$

Again, using trigonometric identities, we have the alternate form

$$\sigma'_{yx} = \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\alpha + \sigma_{xy} \cos 2\alpha, \tag{2.20}$$

which is the same as Eq. (2.18), as it should be (i.e., σ'_{xy} must equal σ'_{yx} to satisfy the balance of angular momentum for a rectangular body cut parallel to x' and y'). Finally, show that

$$\sum F_{y'} = 0 \rightarrow \sigma'_{yy} \Delta A \sec\alpha = (\sigma_{xx} \Delta A \tan\alpha) \sin\alpha - \left(\sigma_{xy} \Delta A \tan\alpha\right) \cos\alpha + \left(\sigma_{yy} \Delta A\right) \cos\alpha - \left(\sigma_{yx} \Delta A\right) \sin\alpha,$$

or

$$\sigma'_{yy} = \sigma_{xx} \sin^2\alpha - 2\sigma_{xy} \sin\alpha \cos\alpha + \sigma_{yy} \cos^2\alpha. \tag{2.21}$$

This equation can then be written as

$$\sigma'_{yy} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\alpha - \sigma_{xy} \sin 2\alpha. \tag{2.22}$$

Together, Eqs. (2.16), (2.18), (2.20), and (2.22) show that the components of a 2-D state of stress relative to one Cartesian coordinate system can be related to those of any other Cartesian system sharing a common origin. All that is needed is the angle α that relates the two coordinate systems; indeed, as a check, we see that at $\alpha = 0$, the $(o; x, y)$ and $(o; x', y')$ coordinate systems coincide, and our transformations yield $\sigma'_{xx} = \sigma_{xx}$ at $\alpha = 0$, and so on, as they should. Although it can be shown that similar transformation relations hold for 3-D states of stress and also for other coordinate systems, we will not go into the details here. Rather, the most important things to realize are that the concept of stress is defined at every point in a continuum body and that the components of the stress (tensor) are not unique; they are determined by the coordinate system of interest. Fortunately, one does not have to solve the equations of equilibrium to determine the value of each component of stress relative to each coordinate system. Rather, *one only needs to solve equilibrium once (in terms of the coordinate system that is most convenient) and then to compute any related component of interest through the transformation relations.* Because these derivations did not require us to specify the material, these relations are good for any solid or fluid as long as the continuum assumption is valid. We will thus use these transformations throughout this book.

Example 2.2 Consider the 2-D state of stress in Fig. 2.5. Find the values of stress σ'_{xx}, σ'_{yy}, and σ'_{xy} for $\alpha = 45°$.

Solution: From Eqs. (2.13), (2.21), and (2.17), we have

$$\begin{aligned}
\sigma'_{xx} &= 120(\cos 45°)^2 + 2(0)\cos 45° \sin 45° + 150(\sin 45°)^2 \\
&= 120\left(\frac{\sqrt{2}}{2}\right)^2 + 150\left(\frac{\sqrt{2}}{2}\right)^2 = 135\text{kPa}, \\
\sigma'_{yy} &= 120\left(\frac{\sqrt{2}}{2}\right)^2 + 150\left(\frac{\sqrt{2}}{2}\right)^2 = 135\text{kPa}, \\
\sigma'_{xy} &= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{150 - 120}{2}\right) + 0 = 15\text{kPa}.
\end{aligned}$$

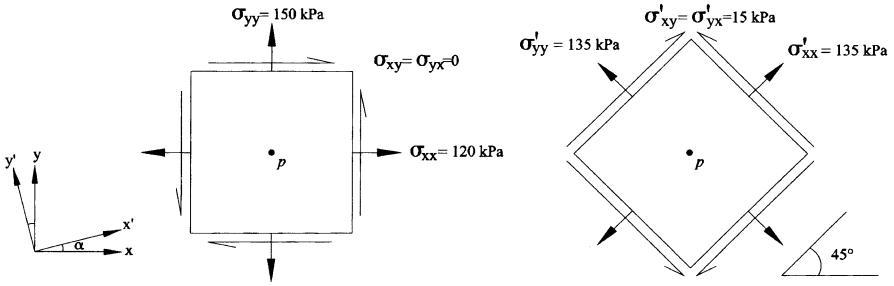


FIGURE 2.13 Two-dimensional state of stress from Fig. 2.5 with components computed relative to both the original x-y coordinate system and an x′-y′ coordinate system with $\alpha = 45°$.

Hence, the state of stress at point p can also be represented as in Fig. 2.13. We see, therefore, that a "shearless" state of stress with respect to one coordinate system need not be shearless in general. Indeed, it can be seen from Eq. (2.17) that if $\sigma_{xy} = 0$, then $\sigma'_{xy} = 0$ only if $\sigma_{xx} = \sigma_{yy}$ or $\alpha = 0$ or $\alpha = 90°$.

2.4 Principal Stresses and Maximum Shear

Given that different values of normal stresses and shear stresses can be computed at the same point in a body depending on the choice of coordinate system (e.g., different Cartesian coordinate systems related via the arbitrary angle α), it is natural to ask if a particular coordinate system exists relative to which the normal or shear stresses are maximum or minimum. The answer, of course, is yes, which will prove very important. For example, if we plot σ'_{xx} as a function of α according to Eq. (2.13) (e.g., for values from Example 2.1 of $\sigma'_{xx} = 120$ kPa, $\sigma_{yy} = 150$ kPa, $\sigma_{xy} = 0$ kPa), we obtain the result shown in Fig. 2.14, with σ'_{xx} minimum at $\alpha = 0°$ and maximum at $\alpha = 90°$ in this case. Recall from calculus, therefore, that general max/min problems require us to compute a first derivative with respect to the quantity of interest. Hence, to find a maximum or minimum normal stress in two dimensions, relative to Cartesian coordinates, differentiate Eq. (2.16) with respect to α and set the result equal to zero; that is,

$$\frac{d\sigma'_{xx}}{d\alpha} = \frac{\sigma_{xx} - \sigma_{yy}}{2}(-\sin 2\alpha)(2) + \sigma_{xy}(\cos 2\alpha)(2) = 0, \tag{2.23}$$

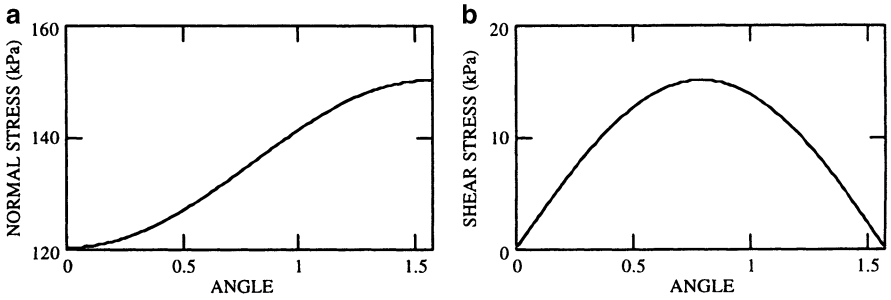


FIGURE 2.14 Plot of the normal and shear stresses σ'_{xx} and σ'_{xy} as a function of $\alpha \in [0, \pi/2]$ radians (i.e., 90°). Note that the local extrema for the normal stress occur at $\alpha = 0$ and 90°, whereas the local maximum for the shear stress is at $\alpha = 45°$.

or

$$\frac{\sin 2\alpha_p}{\cos 2\alpha_p} = \frac{\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})/2} = \tan 2\alpha_p. \tag{2.24}$$

Hence, the maximum or minimum normal stresses σ'_{xx} occur when α is given by

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})}\right). \tag{2.25}$$

We denote this value of α as α_p because *the maximum/minimum normal stresses are called principal values*. Note that whenever $\sigma_{xy} = 0$, then $\alpha_p = 0$, which is to say, σ_{xx} and σ_{yy} are the max/min values of the normal stress. This was the case in Example 2.2 and thus Fig. 2.14. Conversely, if $\sigma_{xx} = \sigma_{yy}$, then $\tan 2\alpha_p = \infty$. Recall that the tangent function goes to infinity at $\pi/2$ radians; hence in this case, $2\alpha_p = \pi/2$ radians, which is to say, $\alpha_p = \pi/4$ radians whenever $\sigma_{xx} = \sigma_{yy}$ regardless of the value of σ_{xy}. All other values of α_p are computed easily.

Now, if we substitute the value of $\alpha = \alpha_p$ into Eqs. (2.16), (2.18), and (2.22) for σ'_{xx}, σ'_{yy}, and σ'_{xy}, we will find $\sigma_1 \equiv \sigma'_{xx})_{\max/\min}$, $\sigma_2 \equiv \sigma'_{yy})_{\max/\min}$, and the value of shear associated with these so-called *principal values of stress* σ_1 and σ_2. This is easily done numerically, but it proves useful to note the following. The tangent of an angle equals the opposite over the adjacent. Hence, we can think of a triangle with an angle $2\alpha_p$ and sides as shown in Fig. 2.15 (this is hindsight for which we introduced the above trigonometric identities). Hence, we have

$$\sin 2\alpha_p = \frac{\sigma_{xy}}{H}, \tag{2.26}$$

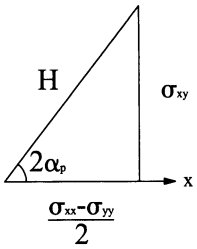


FIGURE 2.15 Trigonometric interpretation of the angle α_p, which is associated with the principal values of stress in two dimensions.

where

$$H = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{2.27}$$

and, similarly,

$$\cos 2\alpha_p = \frac{(\sigma_{xx} - \sigma_{yy})/2}{H}. \tag{2.28}$$

Using these relations for $\cos 2\alpha_p$ and $\sin 2\alpha_p$ in Eq. (2.16), we have

$$\begin{aligned}\sigma'_{xx})_{\max/\min} = \sigma'_{xx}(\alpha = \alpha_p) \equiv & \frac{\sigma_{xx} + \sigma_{yy}}{2} \\ & + \frac{\sigma_{xx} - \sigma_{yy}}{2}\left(\frac{(\sigma_{xx} - \sigma_{yy})/2}{H}\right) + \sigma_{xy}\left(\frac{\sigma_{xy}}{H}\right),\end{aligned} \tag{2.29}$$

wherein the second and third terms have a common denominator and can be combined. Multiplying this combined term by unity (i.e., H/H), we have

$$\sigma'_{xx})_{\max/\min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\left((\sigma_{xx} - \sigma_{yy})/2\right)^2 + \sigma_{xy}^2}{H}\frac{H}{H}, \tag{2.30}$$

or, finally,

$$\sigma_1 \equiv \sigma'_{xx})_{\max/\min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}. \tag{2.31}$$

Hence, we see that it is easy to compute one of the principal stresses. Show that $\sigma'_{yy})_{\max/\min}$ yields the same result; that is, the two principal values of stress are given by the same equation with the plus/minus signs preceding the radical delineating the two.

Next, note that if we compute σ'_{xy} at $\alpha = \alpha_p$, we obtain from Eq. (2.18)

$$\sigma'_{xy}(\alpha = \alpha_p) = \left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)\left(\frac{\sigma_{xy}}{H}\right) + \sigma_{xy}\left(\frac{(\sigma_{xx} - \sigma_{yy})/2}{H}\right) = 0; \tag{2.32}$$

that is, the shear stress associated with the max/min normal (or principal) stresses is always zero. In other words, *a principal state of stress simply imposes extension or compression, not shear*, relative to the principal directions (defined by α_p).

Finally, one can ask similarly: At what value of α is the shear maximum or minimum? In this case, we differentiate Eq. (2.18) with respect to α and set the result equal to zero. Doing so, we obtain

$$\frac{d\sigma'_{xy}}{d\alpha} = \frac{\sigma_{yy} - \sigma_{xx}}{2}(\cos 2\alpha)(2) + \sigma_{xy}(-\sin 2\alpha)(2) = 0, \tag{2.33}$$

or

$$\frac{\sin 2\alpha_s}{\cos 2\alpha_s} = \frac{(\sigma_{yy} - \sigma_{xx})/2}{\sigma_{xy}} = \tan 2\alpha_s. \tag{2.34}$$

Denoting the value of α at which the shear is max/min as α_s, we thus have

$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{\sigma_{yy} - \sigma_{xx}}{2\sigma_{xy}}\right). \tag{2.35}$$

Here, we see that if $\sigma_{yy} = \sigma_{xx}$, then $\alpha_s = 0$ and the associated σ_{xy} is an extremum; conversely, if $\sigma_{xy} = 0$, then $2\alpha_s = \pi/2$ or $\alpha_s = \pi/4$. Recalling that the shear stress is zero when the state of stress is principal, this reveals that α_s and α_p differ by $\pi/4$ or 45°. Substituting the value of α_s into Eq. (2.18) and using ideas similar to those in Fig. 2.15, we find that

$$\sigma'_{xy})_{\max/\min} = \sigma'_{xy}(\alpha = \alpha_s) = \frac{\sigma_{yy} - \sigma_{xx}}{2}\left(\frac{\sigma_{yy} - \sigma_{xx}}{2H}\right) + \sigma_{xy}\left(\frac{\sigma_{xy}}{H}\right), \tag{2.36}$$

which can be written as

$$\sigma'_{xy})_{\max/\min} = \pm\sqrt{\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2} \tag{2.37}$$

or because of the squared term, it is often written (which is the same as H above)

$$\tau_m \equiv \sigma'_{xy})_{\max/\min} = \pm\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}. \tag{2.38}$$

Here, note two things. First, the normal stresses at $\alpha = \alpha_s$ are nonzero in general (this is different from the vanishing shear at $\alpha = \alpha_p$), but computed easily. Second, if the principal stresses occur at $\alpha = 0$, then σ_{xx} and σ_{yy} are principal, whereas $\sigma_{xy} = 0$. In this case, $\sigma'_{xy})_{\max/\min}$ is simply one-half the difference between the principal values [cf. Eq. (2.38)]. Indeed, it can be shown (do it) that this is the case in general:

$$\tau_m = \pm\frac{\sigma_1 - \sigma_2}{2}, \tag{2.39}$$

where σ_1 and σ_2 are the principal values, usually ordered $\sigma_1 > \sigma_2$.

Example 2.3 For the state of stress in Example 2.1 ($\sigma_{xx} = 120$ kPa, $\sigma_{yy} = 150$ kPa, $\sigma_{xy} = 0$ kPa), find α_p and α_s and discuss.

Solution: From Eq. (2.25), we have

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{\sigma_{xy}}{(\sigma_{xx}-\sigma_{yy})/2}\right) = \frac{1}{2}\tan^{-1}\left(-\frac{0}{15}\right) = 0$$

and, therefore, the $(o; \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y)$ coordinate system is principal; that is, the values of σ'_{xx} and σ'_{yy} are max/min at $\alpha = 0$, which is consistent with Fig. 2.14 and the finding in Example 2.2 that $\sigma'_{xx}\,(\alpha = 45°) = \sigma'_{yy}\,(\alpha = 45°) = 135$ kPa, which is an intermediate value between 120 and 150 kPa.

From Eq. (2.35), we have

$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{150-120}{2(0)}\right) = \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

or 45°. Hence, the value of $\sigma'_{xy})_{\max/\min} = 15$ kPa, as computed in Example 2.2.

Example 2.4 Given the 2-D state of stress $\sigma_{xx} = -p$, $\sigma_{yy} = -p$, and $\sigma_{xy} = 0$, show that such a "hydrostatic state of stress" exists relative to all coordinate systems.

Solution: Recall from Eq. (2.13) that

$$\sigma'_{xx} = \sigma_{xx}\cos^2\alpha + 2\sigma_{xy}\sin\alpha\cos\alpha + \sigma_{yy}\sin^2\alpha;$$

hence for our state of stress,

$$\sigma'_{xx} = -p\left(\cos^2\alpha + \sin^2\alpha\right) = -p \quad \forall\alpha$$

and similarly for σ'_{yy}. Likewise, recall from Eq. (2.17) that

$$\sigma'_{xy} = \sin\alpha\cos\alpha\left(\sigma_{yy} - \sigma_{xx}\right) + \left(\cos^2\alpha - \sin^2\alpha\right)\sigma_{xy},$$

and thus for our state of stress,

$$\sigma'_{xy} = \sin\alpha\cos\alpha(-p + p) + 0 = 0 \quad \forall\alpha.$$

A similar finding can be shown in three dimensions. Thus, *a hydrostatic state of stress* (in two dimensions, $\sigma_{xx} = \sigma_{yy} = -p$ and $\sigma_{xy} = 0$, or in three dimensions, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ and $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$) *is principal relative to all coordinate systems*. This is a very special case.

Finally, the student should be aware that Otto Möhr showed in 1895 that the simple trigonometric structure of these relations [Eqs. (2.26)–(2.28)] for max/min components of stress can be represented easily in a 2-D diagram called Mohr's circle. The interested reader is encouraged to explore this representation via any standard textbook entitled *Strength of Materials* or *Mechanics of Materials*. We shall not discuss Mohr's circles herein because the computer (or calculator) has rendered these computations so easy (compared to the slide rule) that Mohr's circle is no longer needed even though some still use it because of its visual appeal. Rather, we refer the reader to Fig. 2.16, which reviews the methods discussed herein.

2.5 Concept of Strain

Mechanics is, of course, the study of forces and the associated motions. In dynamics, we tend to study the motion (i.e., kinematics) in terms of quantities like the velocity vector $\boldsymbol{v}$ or the acceleration vector $\boldsymbol{a}$. These will likewise prove central to our discussion of biofluid mechanics in Chaps. 7–10. In biosolid mechanics, however, our primary interest is usually the displacement vector $\boldsymbol{u}$.

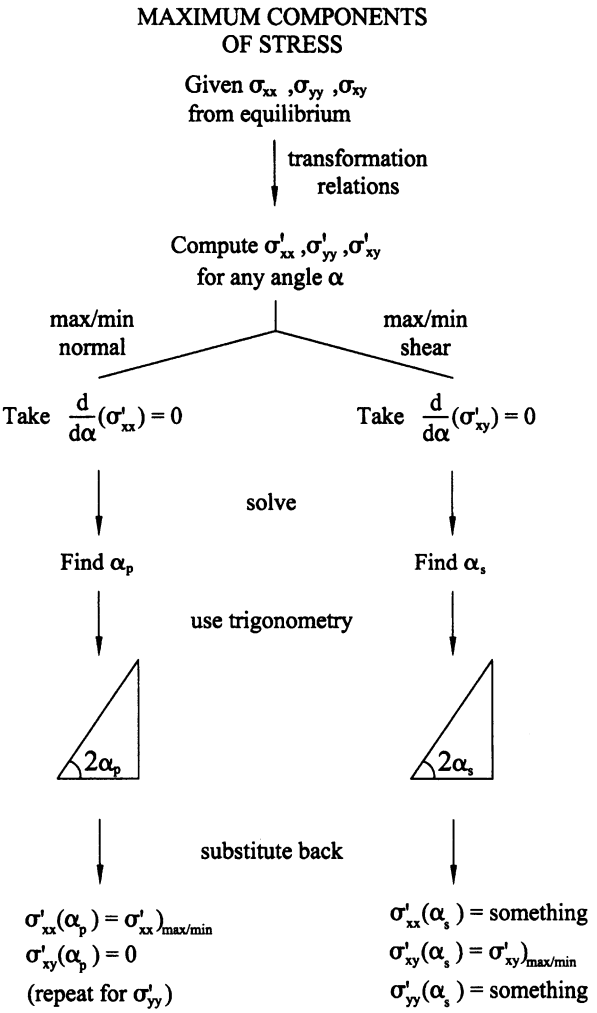


FIGURE 2.16 Flowchart showing our approach for determining max/min values of stress (or strain) relative to preferred coordinate axes.

Basically, a *displacement vector* quantifies the difference between where we (a point) are, denoted by a position vector $\boldsymbol{x}$, and where we were originally, denoted by a position vector $\boldsymbol{X}$. Thus, $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ (Fig. 2.17). Because each point in a body can displace separately (provided certain compatibilities are maintained between neighboring points, except in cases of fracture, of course), the displacement vector can vary with position and time, namely

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}, \tag{2.40}$$

where the position vector $\boldsymbol{x}$ also depends on which point (i.e., originally located by $\boldsymbol{X}$) is being tracked. Because $\boldsymbol{u}$ is a vector, it has components relative to the selected coordinate system. With respect to Cartesian coordinates, we may write

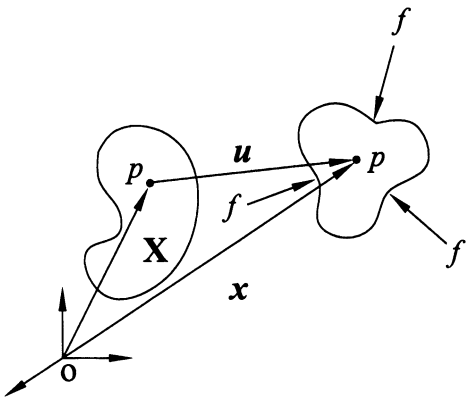


FIGURE 2.17 Schema of the displacement vector $\boldsymbol{u}$ of a generic point $\boldsymbol{p}$ from its location $\boldsymbol{X}$ in an undeformed reference configuration to its location $\boldsymbol{x}$ in a deformed configuration.

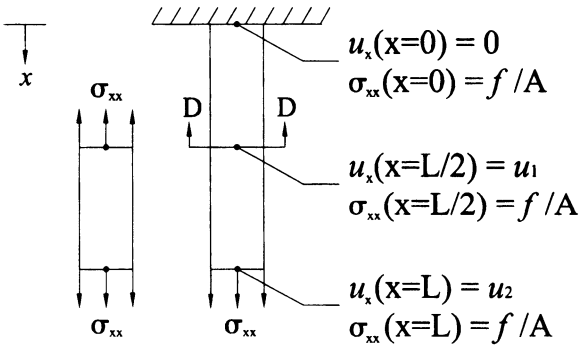


FIGURE 2.18 Displacements at various locations in a uniformly loaded, vertically suspended structural member. Note, in particular, that the value of the displacement varies from point to point (i.e., it is nonuniform), whereas the value of the xx component of stress does not vary.

$$\boldsymbol{u}=u_x\hat{\boldsymbol{i}}+u_y\hat{\boldsymbol{j}}+u_z\hat{\boldsymbol{k}}\equiv u_x\hat{\boldsymbol{e}}_x+u_y\hat{\boldsymbol{e}}_y+u_z\hat{\boldsymbol{e}}_z, \tag{2.41}$$

where u_x, u_y, and u_z are the components relative to the chosen Cartesian coordinate system. As a simple example, consider a slender structural member that is fixed at its upper end and loaded by a uniformly distributed force at its other end (Fig. 2.18). From Sect. 2.2, we can show that each cross section (e.g., that obtained via the cutting plane D-D) has a stress $\sigma_{xx}=f/A$, where f is the total axial force and A is the cross-sectional area. If the member is of homogeneous composition, we would expect this same stress at each point to cause the same response. An obvious question then is whether the displacement $\boldsymbol{u}$ can serve as a good measure of this response: Do we expect a one-to-one relation between the stress and the displacement at a given point? A quick examination of the problem reveals that the answer is no. Whereas the value of σ_{xx} is the same at

all points in this member, the displacement clearly differs from point to point. Because of the fixed support at $X=0$, the u_x displacement there is zero. Conversely, the u_x displacement at the end of the member is a maximum.[2] As it turns out, the displacement gradient $\partial u_x/\partial X$, like stress, is the same at each point in this simple problem. Without going into details, Cauchy showed during the period 1827–1841 that certain combinations of displacement gradients (called strains) are convenient for relating to the stress. Indeed, because stress and strain are both mathematical concepts, or definitions, various nineteenth and early twentieth-century investigators (Almansi, Green, Kirchhoff, and others) showed that different definitions of stress and strain can be equally useful in different situations. One of the commonly used definitions of strain in biomechanics is that due to George Green in 1841. In terms of Cartesian components, it can be computed via

$$
\begin{aligned}
E_{XX} &= \frac{\partial u_X}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u_X}{\partial X}\right)^2 + \left(\frac{\partial u_Y}{\partial X}\right)^2 + \left(\frac{\partial u_Z}{\partial X}\right)^2\right],\\
E_{YY} &= \frac{\partial u_Y}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u_X}{\partial Y}\right)^2 + \left(\frac{\partial u_Y}{\partial Y}\right)^2 + \left(\frac{\partial u_Z}{\partial Y}\right)^2\right],\\
E_{ZZ} &= \frac{\partial u_Z}{\partial Z} + \frac{1}{2}\left[\left(\frac{\partial u_X}{\partial Z}\right)^2 + \left(\frac{\partial u_Y}{\partial Z}\right)^2 + \left(\frac{\partial u_Z}{\partial Z}\right)^2\right],\\
E_{XY} &= \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X} + \frac{\partial u_X}{\partial X}\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\frac{\partial u_Y}{\partial Y} + \frac{\partial u_Z}{\partial X}\frac{\partial u_Z}{\partial Y}\right) = E_{YX},\\
E_{YZ} &= \frac{1}{2}\left(\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y} + \frac{\partial u_X}{\partial Y}\frac{\partial u_X}{\partial Z} + \frac{\partial u_Y}{\partial Y}\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\frac{\partial u_Z}{\partial Z}\right) = E_{ZY},\\
E_{ZX} &= \frac{1}{2}\left(\frac{\partial u_Z}{\partial X} + \frac{\partial u_X}{\partial Z} + \frac{\partial u_X}{\partial Z}\frac{\partial u_X}{\partial X} + \frac{\partial u_Y}{\partial Z}\frac{\partial u_Y}{\partial X} + \frac{\partial u_Z}{\partial Z}\frac{\partial u_Z}{\partial X}\right) = E_{XZ}.
\end{aligned}
\tag{2.42}
$$

Relations are similar, but more complex, for other coordinate systems such as cylindrical and spherical. For a complete derivation and interpretation of these relations, see Humphrey (2002). Suffice it to say, however, that one of the reasons that these relations are so useful is that they are insensitive to rigid-body translations or rotations; that is, the components of the *Green strain* measure only the deformation part of a total motion, where we note that it is the deformation (changes in length or internal angle due to applied loads) that we

[2] If the overall deformation is homogeneous, careful experimental measurements show that $u_x = \Lambda X - X = (\Lambda - 1)X$, where Λ is just a number, a so-called stretch ratio. Stretch ratios are used extensively in Chap. 6.

wish to relate to the stress. Clearly, however, Green's definition of strain is nonlinear (quadratic) in terms of the displacement gradients. Even for the simple (idealized) example in Fig. 2.18, which consists only of an axial extension and associated lateral thinning, we have

$$E_{XX} = \frac{\partial u_X}{\partial X} + \frac{1}{2}\left(\frac{\partial u_X}{\partial X}\right)^2, \qquad E_{YY} = \frac{\partial u_Y}{\partial Y} + \frac{1}{2}\left(\frac{\partial u_Y}{\partial Y}\right)^2,$$
$$E_{ZZ} = \frac{\partial u_Z}{\partial Z} + \frac{1}{2}\left(\frac{\partial u_Z}{\partial Z}\right)^2. \tag{2.43}$$

As it turns out, the nonlinear terms can introduce considerable complexity into the solution of the full boundary value problem. We will consider such problems in Chaps. 6 and 11.

Here, let us consider a tremendous simplification. IF the displacement is small, then $x \sim X$ from $u_X = x - X$ and similarly for $y \sim Y$ and $z \sim Z$; IF the displacement gradients are small, then the nonlinear terms can be neglected in comparison to the linear terms (e.g., if $\partial u_X/\partial X \sim 0.001$, then $\frac{1}{2}(\partial u_X/\partial X)^2 \sim 0.0000005$ is small in comparison); and IF the rigid-body rotations are small (see below), then the Green strains can be *approximated* as

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}, \tag{2.44}$$

where

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \varepsilon_{yx},$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) = \varepsilon_{zy}, \tag{2.45}$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = \varepsilon_{zx},$$

where ε_{xx}, ε_{yy}, and ε_{zz} are the extensional components and ε_{xy}, ε_{yz}, and ε_{xz} are the shear components of the linearized strain.

Similarly for cylindricals, $\boldsymbol{u} = u_r\hat{\boldsymbol{e}}_r + u_\theta\hat{\boldsymbol{e}}_\theta + u_z\hat{\boldsymbol{e}}_z$ and the linearized (often called small) strains are

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{zr} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{bmatrix}, \tag{2.46}$$

where

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \varepsilon_{\theta r}, \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, & \varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) = \varepsilon_{z\theta}, \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \varepsilon_{zr}.
\end{aligned} \tag{2.47}$$

Finally, for sphericals, the linearized strains are

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\theta} & \varepsilon_{\phi\phi} \end{bmatrix}, \tag{2.48}$$

where

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \varepsilon_{\theta r}, \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, & \varepsilon_{\theta\phi} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r}\cot\phi\right) = \varepsilon_{\phi\theta}, \\
\varepsilon_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}, & \varepsilon_{r\phi} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) = \varepsilon_{\phi r}.
\end{aligned} \tag{2.49}$$

Of course, the exact (nonlinear) components can likewise be represented as 3×3 matrices because they too consist of nine components (six independent) relative to a particular coordinate system. It is also very important to note that we have not derived the exact (nonlinear) or the approximate (linear) relations for strain; we have merely listed the results. In many introductory textbooks, the linearized relations are often derived poorly, primarily in an attempt to make the derivation "accessible" to the beginning reader. We prefer to adhere to the adage stated in the preface: To tell the truth, nothing but the truth, but not the whole truth until the student is ready to appreciate the whole truth. Hence, rather than derive these relations poorly, let us merely consider a few 1-D or 2-D examples to illustrate their meaning and usage. First, consider a motion described by the displacement vector $\boldsymbol{u} = u_x\hat{\boldsymbol{e}}_x + u_y\hat{\boldsymbol{e}}_y$, with components

$$u_x = (\Lambda - 1)X, \qquad u_y = 0, \tag{2.50}$$

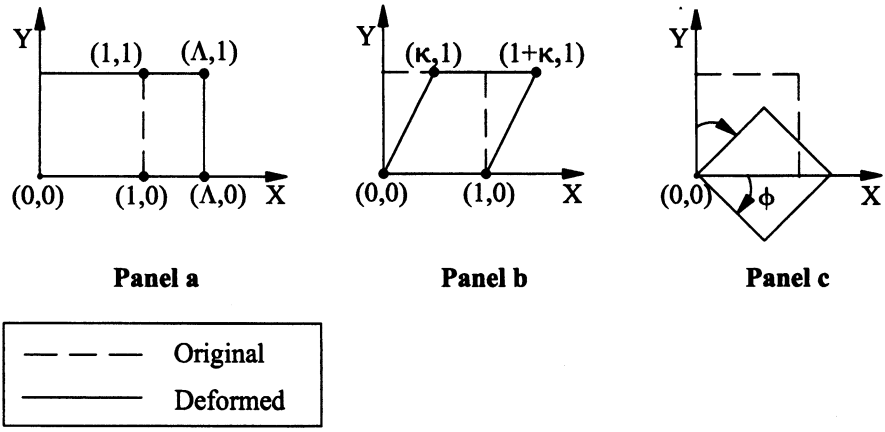


FIGURE 2.19 Schema of three simple motions: (**a**) a 1-D extension by the amount Λ, (**b**) a simple shear via the measure κ, and (**c**) a rigid-body rotation given by the angle ϕ. In each case, note that we assume that the motion is homogeneous; that is, although the displacements differ from point to point, their spatial gradients do not. In other words, each point experiences the same strain in a homogeneous motion, although each point need not experience the same displacement.

where Λ is a number close to unity. That is, as the body deforms, none of its material particles displace vertically, whereas particles may displace in the X direction differently: at $X = 0$, there is no displacement, which is to say that the left edge is fixed, whereas the right edge displaces the most. Hence, current positions of points originally at (X, Y) are given by $x = \Lambda X$ and $y = Y$, which allows us to map material points from original to current places (cf. Fig. 2.17). For example, point $(X, Y) = (0, 0)$ stays put, whereas point $(X, Y) = (1, 1)$ goes to $(x, y) = (\Lambda, 1)$ for any value of Λ. Finally, for Λ near unity (which satisfies the above requirement that the displacement and displacement gradients are both small for this problem with zero rigid-body rotation), the linearized strain is

$$\varepsilon_{xx} = \Lambda - 1, \qquad \varepsilon_{yy} = 0, \qquad \varepsilon_{xy} = 0. \tag{2.51}$$

As can be seen in Fig. 2.19a, this motion represents a 1-D extension only. That the linearized values of strain differ from the exact (nonlinear) values is seen easily given that

$$E_{xx} = (\Lambda - 1) + \frac{1}{2}(\Lambda - 1)^2 = \frac{1}{2}(\Lambda^2 - 1) \equiv \frac{1}{2}(\Lambda + 1)(\Lambda - 1),$$
$$E_{xy} = 0, \qquad E_{yy} = 0.$$

Second, consider a motion described by

$$u_x = \kappa Y, \qquad u_y = 0, \tag{2.52}$$

where κ is a number close to zero. Hence, $x = X + \kappa Y$ and $y = Y$ allows us to map points from the undeformed to the deformed configuration. For example, point $(X, Y) = (0, 0)$ stays put again, whereas point $(X, Y) = (0, 1)$ goes to $(x, y) = (\kappa, 1)$. Moreover,

$$\varepsilon_{xx} = 0, \qquad \varepsilon_{yy} = 0, \qquad \varepsilon_{xy} = \frac{1}{2}\kappa, \tag{2.53}$$

and as can be seen from Fig. 2.19b, this motion is one of *simple shear* (if $x = X + \kappa Y$ and $y = Y + \kappa X$, then we would have a *pure shear*). Third, consider a motion given by

$$\begin{aligned} x &= \cos\phi X + \sin\phi Y, \\ y &= -\sin\phi X + \cos\phi Y, \end{aligned} \tag{2.54}$$

where ϕ is some fixed angle. Hence, the displacements [differences between where we (a point) are (x, y) and where we were (X, Y)] are

$$\begin{aligned} u_x &= (\cos\phi - 1)X + \sin\phi Y, \\ u_y &= (-\sin\phi)X + (\cos\phi - 1)Y. \end{aligned} \tag{2.55}$$

Consequently, the linearized strains are

$$\varepsilon_{xx} = \cos\phi - 1, \qquad \varepsilon_{yy} = \cos\phi - 1, \qquad \varepsilon_{xy} = \frac{1}{2}(\sin\phi - \sin\phi) = 0. \tag{2.56}$$

Clearly, ε_{xx} and ε_{yy} equal zero *if and only if* $\phi = 0$. If we use these displacements to map the motions of points demarcating a unit square, we find that this case represents a rigid-body rotation about the z axis (Fig. 2.19c). Although we do not expect strains to arise due to rigid-body motions, increasingly larger values of ϕ wrongly suggest increasing extensional strains. As we stated earlier, therefore, *the approximate (linearized) relations for strain are only good for small deformations and small rotations*. This is extremely important to remember in biomechanics, especially in soft tissue biomechanics wherein the deformations and rigid-body motions are often large (finite). This is the case for the heart, for example, which twists, shortens, shears, and becomes much smaller in diameter upon contraction. There are, nonetheless, many articles in the literature that use the small strain measure to study the heart—this is wrong and the reader must beware.

Example 2.5 Show that, in contrast to the linearized measure ε_{xx}, the exact measure E_{XX} is insensitive to the rigid-body motion in Eq. (2.54).

Solution: From Eq. (2.42), the 2-D strain E_{XX} is

$$E_{XX} = \frac{\partial u_X}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u_X}{\partial X}\right)^2 + \left(\frac{\partial u_Y}{\partial X}\right)^2\right],$$

where for the rigid-body rotation,

$$\frac{\partial u_X}{\partial X} = \cos\phi - 1, \qquad \frac{\partial u_Y}{\partial X} = -\sin\phi.$$

Thus,

$$\begin{aligned} E_{XX} &= \cos\phi - 1 + \frac{1}{2}\left[(\cos\phi - 1)^2 + (-\sin\phi)^2\right] \\ &= \cos\phi - 1 + \frac{1}{2}\left(\cos^2\phi - 2\cos\phi + 1 + \sin^2\phi\right) \\ &= \cos\phi - 1 + \frac{1}{2}(-2\cos\phi + 2) = 0 \quad \forall\phi. \end{aligned}$$

As an exercise, the reader should confirm that $E_{YY} = 0$ and $E_{XY} = 0$ for this rigid-body motion as well.

Observation 2.1. Although we illustrated a few simple states of strain using examples based on the displacements of four points that define a 2-D rectangular domain, we must realize that strains cannot be computed, in general, by simply knowing the displacements at a few points. Strains are computed from displacement gradients, which requires that we know the displacement *field* {i.e., the displacement as a continuous function of position [e.g., $\boldsymbol{u} = \boldsymbol{u}(x, y, z)$]}. Experimentally, however, we cannot measure the displacement at all points; we can only measure the displacements at a finite, often small, number of points. In practice, therefore, one often introduces *interpolation functions*, which allow one to estimate displacements between measurement points. Because the mathematics of interpolation is well established, knowledge of these functions aids the experimentalist in designing the number and placement of markers for measuring displacements. For example, the minimum number of points to estimate the mean 3-D strain in the wall of the heart is four, which forms a

tetrahedron.[3] Conversely, the minimum number of points needed to estimate the mean 2-D strain on the surface of the heart is three, which form a triangle. By using multiple sets of four or three markers, one can begin to map region-to-region differences in strain using interpolation functions. The biomechanicist should thus be familiar with interpolation, even though we leave such study for intermediate and advanced courses. See, for example, Humphrey (2002).

In Chaps. 3–6, we will seek to *relate the deformations (strains) to the applied loads (stresses) that act on the body*. To do this, we will see that we must use equilibrium equations to determine the stresses that exist at each point, which, in turn, will be related to the strains at the same point through functions that quantify the material behavior (i.e., through constitutive relations). Consequently, it is very important to note the following. Like stress, strain can have different components at each point (given the same deformation) depending on the coordinate system to which it is referred. Fortunately, similar to Eqs. (2.13)–(2.22) for stress, it can be shown that strain transforms in like fashion:

$$\begin{aligned}
\varepsilon'_{xx} &= \varepsilon_{xx}\cos^2\alpha + 2\varepsilon_{xy}\sin\alpha\cos\alpha + \varepsilon_{yy}\sin^2\alpha,\\
\varepsilon'_{yy} &= \varepsilon_{xx}\sin^2\alpha - 2\varepsilon_{xy}\sin\alpha\cos\alpha + \varepsilon_{yy}\cos^2\alpha,\\
\varepsilon'_{xy} &= 2\sin\alpha\cos\alpha\left(\frac{\varepsilon_{yy}-\varepsilon_{xx}}{2}\right) + (\cos^2\alpha - \sin^2\alpha)\varepsilon_{xy},
\end{aligned} \tag{2.57}$$

where α is again the angle that relates the $(o; x, y, z)$ and $(o; x', y', z')$ Cartesian coordinate systems. Note: If $\alpha = 0$, then the components relative to the two systems are equal, as they should be. Similarly, principal values for strain are determined at $\alpha = \alpha_p$ [cf. Eqs. (2.25)–(2.39)], namely

$$\varepsilon_{1,2} = \varepsilon'_{xx}\big)_{\max/\min} = \varepsilon'_{yy}\big)_{\max/\min} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \varepsilon_{xy}^2}, \tag{2.58}$$

with

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{\varepsilon_{xy}}{(\varepsilon_{xx}-\varepsilon_{yy})/2}\right). \tag{2.59}$$

---

[3] Note that the discussion in Chap. 5 in Humphrey (2002) contains an error. It correctly notes the need for a minimum of three line segments, but wrongly suggests that they can be obtained from three coplanar points.

Similarly, the maximum value of the shearing strain is determined at $\alpha = \alpha_s$; that is,

$$\varepsilon'_{xy})_{\max/\min} = \varepsilon'_{xy}(\alpha_s) = \pm\sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \varepsilon_{xy}^2}, \tag{2.60}$$

where

$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{\varepsilon_{yy} - \varepsilon_{xx}}{2\varepsilon_{xy}}\right). \tag{2.61}$$

Whereas the stress transformation equations were derived via equilibrium considerations and thus force balances, here we must take a different approach. Consider, for example, the linearized extensional strain $\varepsilon'_{xx} = \partial u'_x/\partial x'$, just as $\varepsilon_{xx} = \partial u_x/\partial x$. Recall from calculus that two coordinate systems can be related via a coordinate transformation (Fig. 2.6), specifically

$$x' = x\cos\alpha + y\sin\alpha, \qquad y' = -x\sin\alpha + y\cos\alpha \tag{2.62}$$

whereby $x' = x$ and $y' = y$ if $\alpha = 0$ (i.e., if the coordinate systems coincide). Because displacement is just a vector (i.e., difference between position vectors; Fig. 2.17), we have similar relations for each component,

$$u'_x = u_x\cos\alpha + u_y\sin\alpha, \qquad u'_y = -u_x\sin\alpha + u_y\cos\alpha, \tag{2.63}$$

where, of course, the displacement components can each vary from point to point in the body: $u_x = u_x(x, y)$ and $u_y = u_y(x, y)$, and likewise $u'_x = u'_x(x', y')$ and $u'_y = u'_y(x', y')$. Yet, from Eq. (2.62), the primed coordinates are a function of the unprimed coordinates, namely $x' = x'(x, y)$ and $y' = y'(x, y)$, and, consequently,

$$u'_x = u'_x\Big(x'(x, y), y'(x, y)\Big), \qquad u'_y = u'_y\Big(x'(x, y), y'(x, y)\Big), \tag{2.64}$$

which is to say, u'_x and u'_y also depend on position (x, y).

Hence, we can compute strains relative to (x', y') using the chain rule:

$$\varepsilon'_{xx} = \frac{\partial u'_x}{\partial x'} = \frac{\partial u'_x}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'_x}{\partial y}\frac{\partial y}{\partial x'} \tag{2.65}$$

and so forth. Toward this end, let us first solve for x and y in terms of x' and y' [from Eq. (2.62), which represents two equations and two unknowns]:

$$x = x' \cos\alpha - y' \sin\alpha, \qquad y = x' \sin\alpha + y' \cos\alpha. \tag{2.66}$$

Now, we have from Eq. (2.65), using Eqs. (2.63) and (2.66),

$$\begin{aligned}\varepsilon'_{xx} &= \left(\frac{\partial u_x}{\partial x}\cos\alpha + \frac{\partial u_y}{\partial x}\sin\alpha\right)(\cos\alpha) + \left(\frac{\partial u_x}{\partial y}\cos\alpha + \frac{\partial u_y}{\partial y}\sin\alpha\right)(\sin\alpha) \\ &= \frac{\partial u_x}{\partial x}\cos^2\alpha + \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\sin\alpha\cos\alpha + \frac{\partial u_y}{\partial y}\sin^2\alpha,\end{aligned} \tag{2.67}$$

whereby, from Eq. (2.45), we have the desired result,

$$\varepsilon'_{xx} = \varepsilon_{xx}\cos^2\alpha + 2\varepsilon_{xy}\sin\alpha\cos\alpha + \varepsilon_{yy}\sin^2\alpha, \tag{2.68}$$

which is similar in form to the relation for σ'_{xx}. It is left as an exercise for the reader to find the transformation equations for ε'_{yy} and ε'_{xy}.

As we saw earlier, one way to infer components of strain based on experimental measurements is to place multiple markers (points) on the specimen and to follow their motions. From these motions, we then construct displacement vectors at each point to identify the displacement field, as, for example, $\boldsymbol{u} = \boldsymbol{u}(X, Y, Z)$, from which one can compute the appropriate displacement gradients and thus strains. Indeed, using noncontacting methods [e.g., video, X-ray, magnetic resonance imaging (MRI), laser Doppler] to track the motions of multiple surface or embedded markers is a common way to "measure" strains in soft tissues and even cells. Such approaches are used in applications ranging from gait analysis to quantifying cardiac motion in health and disease. Figure 2.20 shows, for example, that all six components of the finite Green strain are nonzero and changing throughout the cardiac cycle; likewise, they vary from point to point. For more on cardiac motions, see Humphrey (2002).

Here, however, let us consider devices called *strain gauges*, which are useful for inferring surface strains on many engineering structures, from bridges to components on airplanes, as well as hard biological tissues. Briefly, in 1856, Lord Kelvin (William Thompson) reported three important observations: The electrical resistance of metallic wires increases with increasing mechanical loads applied along their long axis, different materials have different sensitivities, and the Wheatstone bridge can be used to measure well the changes in resistance. These observations led to the invention of the electrical-resistance strain gauge (Fig. 2.21). These gauges are glued onto the surface of the specimen, which allows them to deform with the underlying specimen. By deforming with the specimen, the electrical resistance changes in the wires, which, in turn, provides (via calibration) the value of the associated extensional strain (i.e., in the predominant direction of the wires). These gauges are very

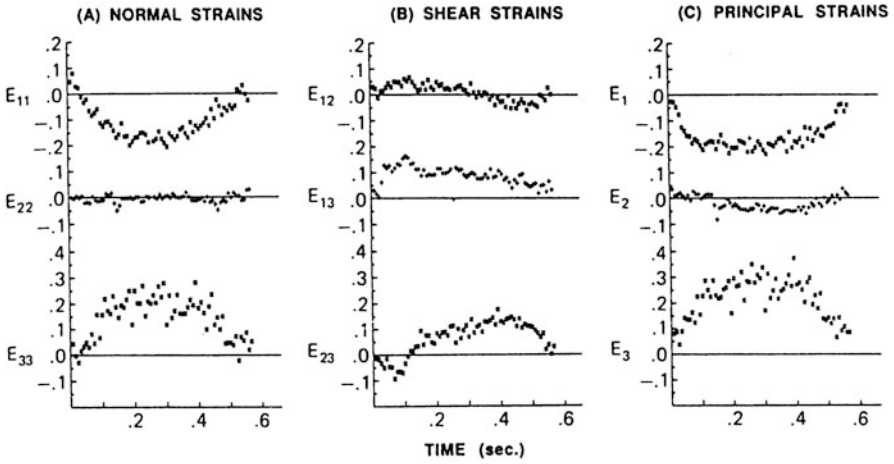


FIGURE 2.20 All six components of the Green strain (extensional and shear) calculated from the motions of small metallic markers that were implanted within the wall of an animal heart. Note that 11, 22, and 33 denote circumferential, axial, and radial components of strain, respectively, with 12, 23, and so forth denoting the associated shears. The principal values, are E_1, E_2, and E_3. Clearly, all six components are nonzero, finite in magnitude, and time varying over the cardiac cycle. [From Waldman et al. (1985), with permission from Lippincot Williams & Wilkins.].

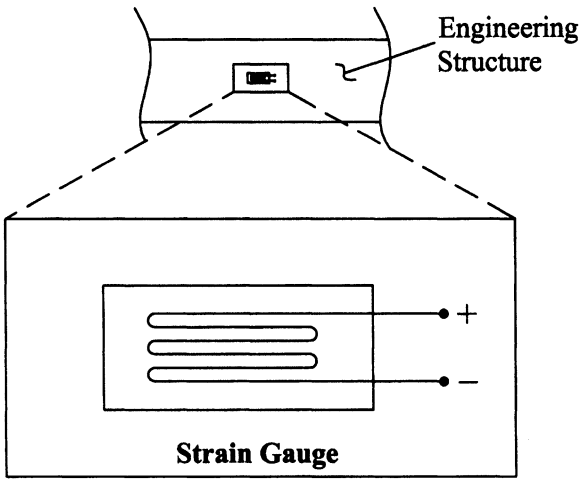


FIGURE 2.21 Schema of an electrical-resistance strain gauge that is glued onto a structure of interest. Such gauges are commonly used in aerospace, civil, and mechanical engineering to measure strains in materials that experience small strains. They are likewise useful in biomechanics for measuring strains in transducers, select biomaterials, and hard tissues such as bone and teeth.

useful when the strains are small, as, for example, in bones, teeth, and biomaterial implants such as an artificial hip. One limitation, however, is that these gauges can only measure extensional components, not shears. To know completely the strain at a point, however, we must know both extensional and shearing strains in general. Fortunately, theory supports experiment, thus allowing us to make the necessary measurements as shown next.

Example 2.6 Design an experimental set-up using strain gauges whereby one can measure a complete 2-D strain in a small region (i.e., averaged over a small region even though strain is, strictly speaking, defined at a point).

Solution: From Eq. (2.57), we see that an extensional strain relative to a primed coordinate system is related to the 2-D components of strain relative to an original coordinate system. If an extensional strain ε'_{xx} is measurable by a strain gauge, then measuring three extensional strains would provide three equations for the three components ε_{xx}, ε_{yy}, and ε_{xy}; that is, as illustrated in Fig. 2.22,

$$\begin{aligned}
\varepsilon'_{xx} &= \varepsilon_{xx}\cos^2\alpha_1 + 2\varepsilon_{xy}\cos\alpha_1\sin\alpha_1 + \varepsilon_{yy}\sin^2\alpha_1,\\
\varepsilon''_{xx} &= \varepsilon_{xx}\cos^2\alpha_2 + 2\varepsilon_{xy}\cos\alpha_2\sin\alpha_2 + \varepsilon_{yy}\sin^2\alpha_2,\\
\varepsilon'''_{xx} &= \varepsilon_{xx}\cos^2\alpha_3 + 2\varepsilon_{xy}\cos\alpha_3\sin\alpha_3 + \varepsilon_{yy}\sin^2\alpha_3,
\end{aligned}$$

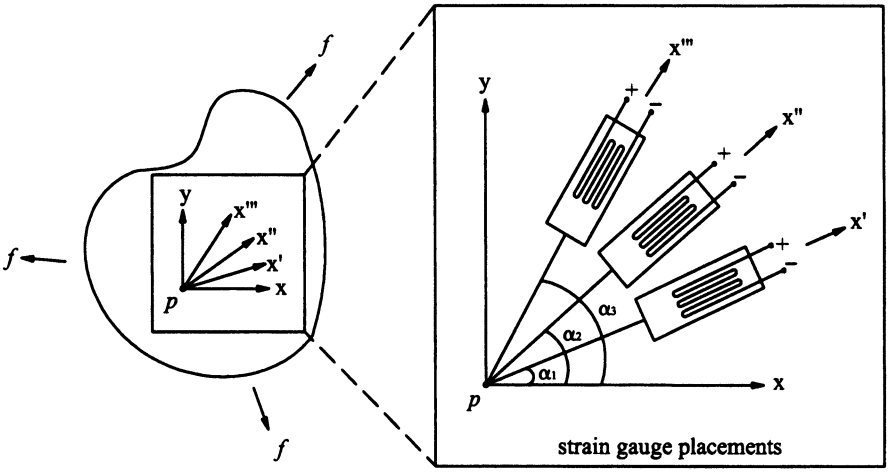


FIGURE 2.22 Placement of three strain gauges to form a so-called strain rosette. It is assumed that each gauge is affixed to the surface at a known angle. Although small, strain gauges are obviously of finite, not infinitesimal, size, thus information from rosettes necessarily represent mean values of strain within the region of measurement.

where the angles α_1, α_2, and α_3 relate the coordinate systems to a baseline x direction. Clearly then, these equations represent three equations in terms of three unknowns (ε_{xx}, ε_{yy}, ε_{xy}) provided that ε'_{xx}, ε''_{xx}, ε'''_{xx}, α_1, α_2, and α_3 are measured. (Note: Whereas the strains come from the resistance changes, the angles are known because we are the ones who glue the gauges onto the surface). Although any values of α_1, α_2, and α_3 are fine, certain values are preferred. For example, $\alpha_1 = 0$, $\alpha_2 = \pi/4$, and $\alpha_3 = \pi/2$ radians or $\alpha_1 = 0$, $\alpha_2 = \pi/3$, and $\alpha_3 = 2\pi/3$ radians are common. For example, let us consider the former case:

$$\varepsilon_{0^\circ} = \varepsilon'_{xx}(\alpha = 0) = \varepsilon_{xx},$$

$$\varepsilon_{45^\circ} = \varepsilon''_{xx}\left(\alpha = \frac{\pi}{4}\right) = \varepsilon_{xx}\left(\frac{\sqrt{2}}{2}\right)^2 + 2\varepsilon_{xy}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \varepsilon_{yy}\left(\frac{\sqrt{2}}{2}\right)^2,$$

$$\varepsilon_{90^\circ} = \varepsilon'''_{xx}\left(\alpha = \frac{\pi}{2}\right) = \varepsilon_{yy}.$$

Hence, if ε'_x, ε''_{xx}, ε'''_{xx} are known from the gauges, $\varepsilon_{xx} = \varepsilon_{0^\circ}$, $\varepsilon_{yy} = \varepsilon_{90^\circ}$, and $\varepsilon_{xy} = \varepsilon_{45^\circ} - \varepsilon_{0^\circ}/2 - \varepsilon_{90^\circ}/2$ are thereby measurable.

We emphasize, therefore, that *theory is indispensable in the design of experiments; it tells us what to measure, why, and to what accuracy*. Moreover, theory reveals the inherent limitations and restrictions. Given that the strain gauge provides information that is averaged along its length and that clusters (rosettes) of gauges further average information over the enclosed region, strain gauges should not be used in areas where large gradients (i.e., point-to-point differences) are expected in the strain field. Again, theory will often reveal the domain of applicability.

In summary, although Hooke's suggestion in the late seventeenth century, "as the force, so the extension," was profound, we now see that Euler, Cauchy, Green, and others in the eighteenth, nineteenth, and early twentieth centuries showed that the mathematical concepts of stress and strain are often much more useful in continuum mechanics than the physical quantities of force and extension. Being mathematical concepts, however, stress and strain are merely definitions, not physical realities or experimental measurables. Stress and strain can thus be defined in different ways, to suit the particular need, and, fortunately, they can be inferred from experimental "measurables" such as forces, dimensions, and displacements. Because stress and strain are but mathematical concepts, having different components depending on the coordinate system to which they are referred, they cannot be sensed directly by a cell and thus cannot be the stimulus for mechanotransduction (Humphrey 2001) even though many have suggested otherwise. These quantities can nevertheless be conveniently

correlated with mechanosensitive responses by cells (e.g., altered gene expression due to a microgravity environment) and thus they can serve as important metrics in phenomenological theories that have predictive capability. More on this later. First, however, let us explore mathematical relationships between stress and strain that serve to quantify the behavior of particular materials.

2.6 Constitutive Behavior

Mathematical relations that describe the response of a material to applied loads under conditions of interest are called *constitutive relations* because this response depends on the internal makeup, or constitution, of the material. That is, given the same overall dimensions, a piece of rubber will respond differently than a piece of metal to the same forces because of the marked differences in their internal makeup—long-chain molecules that are held together via covalent and van der Waals bonds versus collections of atoms that are held together by metallic bonds. Indeed, even different metals and metal alloys respond differently because of differences in their internal makeup and so too for collagenous tissues such as tendons and the cornea, each of which consist largely of type I collagen, albeit with very different microstructural arrangements. Likewise, the conditions of interest must be specified. Rubber, for example, behaves very differently below its glass transition temperature than it does at room temperature or above its melting point. Quantifying, via constitutive relations, the different (solidlike and fluidlike) behaviors of molecules, cells, tissues, organs, biomaterials, and other materials under conditions of importance in biomedical engineering are critically important to both analysis and design.

As noted in Sect. 1.7, there are five general steps in a constitutive formulation, which can be easily remembered via the acrostic DEICE. First, we must *delineate* general characteristics of the behavior. For example, we must determine if the behavior is solidlike or fluidlike. The former is said to admit a shear stress in equilibrium, no matter the value of shear; the latter is said to be incapable of supporting a shear stress in equilibrium, which is to say that it will flow as long as the shear is applied. We emphasize that although one generally thinks of solids and fluids as phases of matter, in continuum mechanics we really seek to delineate solidlike versus fluidlike *behaviors*. For example, most people would classify glass to be a solid at room temperature, and indeed it exhibits solidlike behaviors at these temperatures. Yet, over many years to centuries, one also finds that glass flows at room temperature, as evidenced by the vertical variations in the thickness of glass window panes in Gothic churches in Europe. Hence, it is really the behavior under the condition of interest, including timescales, that is most important, and we may equally well model glass as a solid or a fluid at room temperature depending on the problem

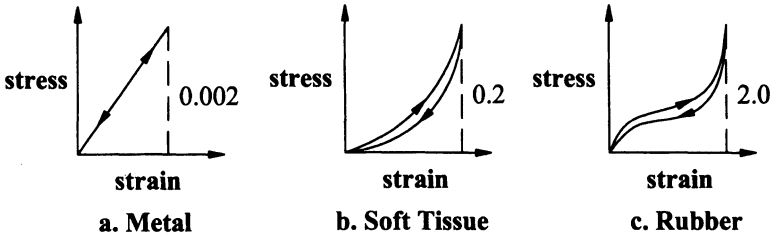


FIGURE 2.23 Qualitative comparison of the stress–strain behavior of three classes of materials: metals, soft tissues, and elastomers. Note the different order of magnitudes of the associated strains (from 0.002 to 2.0) and that the soft tissues and elastomers not only exhibit nonlinear behaviors, but they also reveal a slight hysteresis (i.e., noncoincident loading and unloading curves). The values of stress would obviously be very different as well, but we simply emphasize the general character of the curves here.

at hand. Inasmuch as this is clear, we can loosely talk about solids versus fluids, as most do; we will discuss particular constitutive behaviors and relationships for biofluids in Chap. 7.

It is also very important to determine if a material's response to an applied load is *linear* or *nonlinear*. For example, if we apply increasingly greater loads (stresses), do we observe proportionate or disproportionate increases in extension (strain). Metals and bone tend to exhibit a linear stress–strain response under small strains (i.e., strains that do not cause permanent changes in the microstructure and, thus, properties). In contrast, elastomers and soft tissues tend to exhibit nonlinear stress–strain responses under large strains without a permanent change in structure, as seen in Fig. 2.23. Nonlinear behavior is much harder to quantify. Indeed, note that elastomers and soft tissues exhibit behaviors very different from those of traditional engineering materials (e.g., metals) because of their long-chain polymeric structure. In particular, much of the behavior of such polymers depends on changes in the underlying conformations of the molecules (i.e., their inherent order or disorder). Their mechanical behaviors are thus said to be governed by *entropic* mechanisms in contrast to *energetic* mechanisms that govern the lattice atomic structure in metals. It is, of course, the biopolymers (proteins) elastin and collagen that dominate soft tissue behavior—entropic changes in which complicate the associated quantification.

Another important characteristic exhibited by some solids under certain conditions is a so-called *elastic* behavior. By elastic, it is meant that the material does not dissipate any energy as it deforms. In other words, the path followed by the material in a stress–strain plot is the same during loading and unloading and the material will recover its original size and shape when all loads are removed. Moreover, an elastic behavior suggests that a material responds instantaneously to an applied load (again, the importance of timescale). Whereas metals exhibit an elastic response under small strains, tissues and rubber only exhibit a

"nearly" elastic behavior under many normal conditions. That the behavior is not purely elastic is evidenced, in part, by the small differences between the loading and unloading curves (hysteresis) in Fig. 2.23, the dissipation being due, in part, to moving the structural proteins within the viscous, proteoglycan dominated ground substance matrix. Fung calls the nearly elastic behavior of soft tissues *pseudoelastic* and offers some ideas to simplify the quantification (Fung 1990). Constitutive relations for such behavior are discussed in Chap. 6.

If the behavior of a material is independent of the position within the body/structure from which it was taken, we say that the material is *homogeneous*. Obviously, a fiber-reinforced composite like steel-reinforced concrete would not be homogeneous because the steel and surrounding matrix exhibit very different behaviors. In contrast, many metals and rubberlike materials are often homogeneous or at least nearly so, notwithstanding impurities. Although soft tissues are also composites, consisting of elastin, various collagens, proteoglycans, water, and so forth, there are cases in which it is reasonable to consider an associated homogenized behavior. Examples may include describing the behavior of skin, lung parenchyma, myocardium, bone, or even brain tissue under certain circumstances. In other cases, however, accounting for the heterogeneity due to layering (e.g., intima, media, and adventitia in blood vessels or even cortical versus cancellous bone) is essential.

Finally, if the behavior of a material is independent of its orientation within the body/structure, we say that its response is *isotropic*. Whereas many metals exhibit isotropy under small strains and rubber exhibits isotropy under large strains, tendons (with axially oriented type I collagen) and the stalks of plants clearly would not exhibit an isotropic response. Indeed, most tissues exhibit

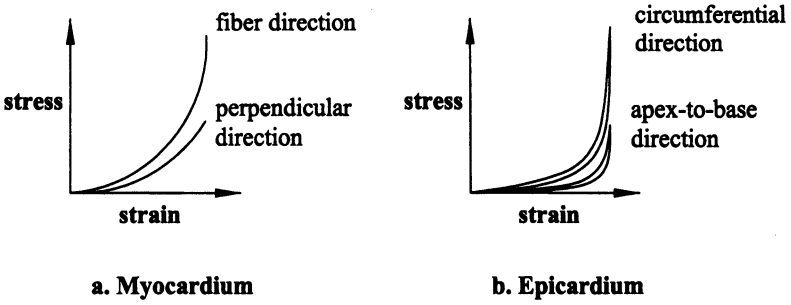


FIGURE 2.24 Schema of typical stress–strain data from a thin slab of noncontracting myocardium and associated epicardium. Both exhibit nonlinear anisotropic behaviors over finite strains, but the epicardium is more strongly nonlinear because of the initially very compliant behavior that is thought to arise due to the highly undulated collagen (cf. Fig. 1.8) in the unloaded state. Also shown is the slight hysteresis exhibited by the primarily collagenous epicardium; muscle tends to exhibit greater hysteresis (not shown).

anisotropic responses (see Fig. 2.24), which again are more difficult to quantify in general. Later in this section, we will discuss two different anisotropics and compare their quantification to that of isotropy for the case of small strains.

Whereas we seek to characterize the responses of materials in terms of concepts such as linearity, elasticity, homogeneity, and isotropy, we emphasize again that these are but descriptors of behavior; no material is linear, elastic, homogeneous, or isotropic. Rather, material behaviors and the constitutive relations that describe them depend on the conditions of interest. Water, for example, behaves differently depending on the temperature; it can behave as a gas (steam), liquid (fluid), or solid (ice), each of which requires a different constitutive descriptor. Common metals also exhibit markedly different behaviors under different conditions. Under the action of a shear stress, the atoms comprising the lattice structure of a metal move relative to one another. If the shear is small (remember, even if the shear is zero relative to one coordinate system, shears will exist relative to other coordinate systems except in the very special case of a hydrostatic pressure as discussed in Example 2.4), the atoms maintain their bonds with their original neighbors, and upon the release of the loads, they return to their original positions (i.e., deform elastically). Under larger strains, however, the atoms cannot maintain bonds with their original neighbors and they slip relative to one another and form new bonds with new neighbors (this process is called yielding). Thus, when the load is released, they remain "permanently" displaced rather than going back to their original positions (Fig. 2.25). This is called plastic set, and this inelastic behavior is called *plasticity*. For constitutive relations in plasticity, see Khan and Huang (1995).

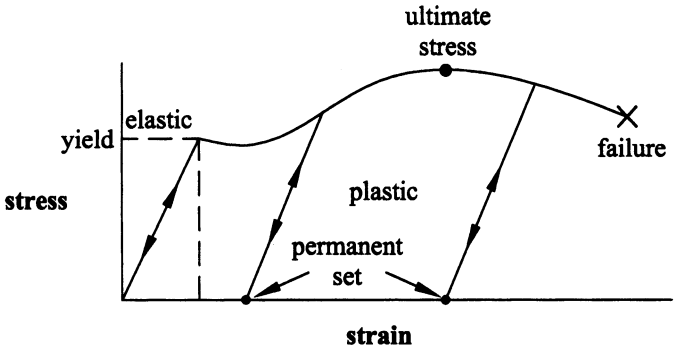


FIGURE 2.25 Schema of the stress–strain behavior of a metal that exhibits a linearly elastic response over small strains but a plastic (i.e., nonrecoverable) response thereafter. In particular, the loading and unloading curves in the plastic domain have a similar character as those in the elastic domain except that the subsequent "yield point" increases with increased plastic deformation (a so-called hardening) up to a point called the ultimate stress. Yield and failure occur due to excessive shear stresses in such ductile materials.

We see, therefore, that the strain level of interest can also dictate the constitutive behavior of a material. Although it is important to analyze plastic deformations in many fields of engineering (e.g., in metal forming), we seldom design implant biomaterials to exceed their yield point under the action of in vivo loads. Hence, in this book, we will focus on elastic behavior. Finally, note that soft tissues behave differently depending on whether they are hydrated, heated excessively, or exposed to certain medications. *Because constitutive relations describe material behavior, not the material itself*, the bioengineer must always be mindful of the specific conditions under which the material will perform, knowing that multiple constitutive relations may be necessary to describe the behaviors of the same material under different conditions.

Observation 2.2. One of the most important, and challenging, areas within biomechanics remains the formulation of constitutive relations to describe material responses to applied loads under biological conditions of interest. Generally, such formulations require measurement of the geometry, applied loads, and resulting deformations, or strains. A basic tenet of experimental biomechanics is that one should design tests that represent simple initial or boundary value problems for this facilitates both the measurement and the interpretation of the data. Yet, many biological tissues and organs, particularly in disease, have inherent geometric complexities or material heterogeneities that render the experimentation more challenging.

A method for inferring surface strains for complex geometries that is gaining increasing usage in biomechanics is referred to as *Digital Image Correlation*, or DIC. Briefly, using non-contacting imaging, one seeks to identify surface characteristics within sequential configurations of the body as it is loaded. Local correlation of these characteristics from configuration to configuration over large portions of the surface allows one to estimate point-wise displacements, which via the use of interpolation (see Observation 2.1) or similar methods can be used to compute "full-field" strains. Having information on geometry, applied loads, and strains is typically sufficient to estimate the associated material properties, provided the functional form of the constitutive relation is known. Such estimations are known as inverse methods. A recent innovation in soft tissue mechanics is a panoramic digital image correlation (p-DIC) method, which allows one to simultaneously monitor displacements along the axial length and around the entire circumference of a cylindrical specimen such as an arterial aneurysm (Genovese et al. 2013). Because DIC allows one to quantify components of the deformation gradient, one can easily compute both rigid body rotations and strains, the latter of which can be computed relative to a convenient coordinate system or via transformation relations (Chap. 2) relative to principal directions. Innovations in experimental

methods, such as p-DIC, promise to provide increasingly better information on soft tissues that exhibit regionally varying, anisotropic, nonlinear responses under finite strains, particularly in injury repair and disease processes.

2.6.1 *Illustrative Characteristic Behaviors*

Figure 2.26 shows illustrative data from a uniaxial test on a bovine chordae tendineae. This tissue connects the heart valve to the papillary muscle within the ventricular cavity of the heart; it consists primarily of uniaxially oriented type I collagen having only a slight undulation when unloaded. As seen in the figure, chordae (similar to tendons and ligaments of the joints) exhibit a nonlinear stress–stretch response over finite (not infinitesimal) but moderate strains. Because of the highly oriented collagen fibers, chordae are strongly anisotropic; because of the slight hysteresis upon cyclic loading/unloading, there is slight energy dissipation; because of the presence of a thin membranous covering (sheath), the tissue is not homogeneous. *Nonlinearity, inelasticity, anisotropy, and heterogeneity are common characteristics of soft tissues.* Figure 2.24 shows similar responses by excised noncontracting myocardium and epicardium. The latter is a thin collagenous membrane that covers the outer surface of the heart. Whereas the myocardium consists primarily of locally parallel muscle fibers embedded in a 3-D plexus of collagen and a ground substance matrix, the epicardium consists primarily of a 2-D plexus of collagen and elastin embedded in its ground substance matrix (proteoglycans and bound water). The collagen fibers tend to be highly undulated in both tissues in an unloaded configuration, hence the initially very compliant, perhaps isotropic response by the epicardium that is followed by a rapid stiffening (due to the straightening of the fibers). The initially greater stiffness of the myocardium is due to the presence of myofibers. Although the chordae and epicardium consist of very similar constituents, their behaviors are very different because of the different microarchitectures. Histology, the study of the fine

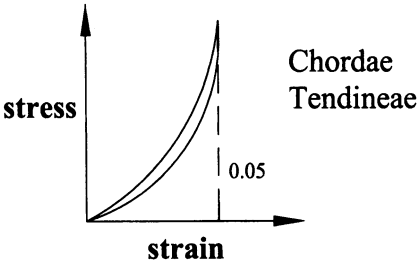


FIGURE 2.26 Schema of stress–strain data from a uniaxial test on an excised chordae tendineae, the thin stringlike tissue that connects the heart valve to the papillary muscle. Note the small, but not infinitesimal, strain. Many ligaments and tendons exhibit similar behavior.

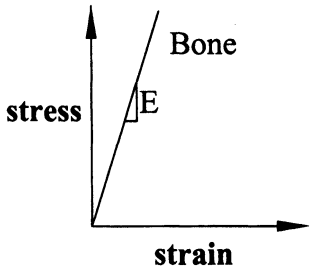


FIGURE 2.27 Schema of stress–strain data from bone prior to yield, which reveals an initially linear, nearly elastic response. Note that bone (type I collagen impregnated with hydroxyapatite) is much stiffer and less extensible than the chordae (Fig. 2.26; primarily type I collagen).

structure of tissues, thus plays an important role in constitutive formulations. We will consider soft tissue constitutive relations in Chap. 6.

In contrast, Fig. 2.27 shows results from a uniaxial test on bone. Note the much smaller range of strain and the near linear behavior. Although not shown, bone exhibits anisotropy and it is heterogeneous—cortical and cancellous bone being very different, as discussed in Chap. 4. Quantification of the stress–strain behavior of bone is discussed in Sect. 2.7. Although we could discuss much more about the characteristic behaviors of these and other solids, we refer the student to texts on material science and biomaterials (e.g., Askeland 1994; Ratner 2003), which emphasize the need for biomechanics and material science to go hand-in-hand. Here, we simply note that we will focus in Chaps. 2–5 primarily on a class of material behaviors that we refer to as LEHI:

Linear: linear stress-strain behavior and linearized kinematics
Elastic: no dissipation and the loading/unloading curve coincide
Homogeneous: same material behavior everywhere in the material/body
Isotropic: same material response in all directions at a point

2.6.2 Hookean LEHI Behavior

Due largely to A.L. Cauchy, S.D. Poisson, G. Lamé, L.M.H. Navier, and G. Green in the early to mid nineteenth century, a constitutive relation was established for LEHI behavior under small strains. It is,

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] + \beta\Delta T, \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy},\\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] + \beta\Delta T, \qquad \varepsilon_{xz} = \frac{1}{2G}\sigma_{xz},\\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \beta\Delta T, \qquad \varepsilon_{yz} = \frac{1}{2G}\sigma_{yz},
\end{aligned} \tag{2.69}$$

where T is the temperature and E, v, G, and β are material parameters, the specific values of which vary from material to material. In particular, E is called Young's modulus (after T. Young, a physician interested in biomechanics, who, for example, gave lectures in 1808 to the Royal Society of London on the biomechanics of arteries); E is a measure of the extensional stiffness (i.e., change of stress with respect to strain) of a material, which can be inferred by plotting normal stress versus extensional strain in a uniaxial stress test. The parameter v is called Poisson's ratio; it describes a coupling between orthogonal directions and is often defined as $v = -\varepsilon_{\text{lateral}}/\varepsilon_{\text{axial}}$, which is to say that it describes the thinning of a material that is extended. Thermodynamics shows that $-1 < v < \frac{1}{2}$, the value of $\frac{1}{2}$ being associated with an incompressible behavior (see below). G is called the shear modulus; it provides a measure of the resistance to shear. It can be shown that $G = E/2(1+v)$ for LEHI behavior. Finally, β is a coefficient of thermal expansion; it tells us how much the material expands/contracts due to changes in temperature from some reference temperature T_o; that is, $\Delta T = T - T_o$ and thus there is no thermal effect when the material is isothermal at T_o. Although the body regulates temperature very closely at ~37 °C, clinical interventions often involve local warming (e.g., hyperthermia treatment of cancerous cells) or cooling (e.g., cryosurgery). We will focus on isothermal behavior, however. Table A2.1 (see Appendix 2) lists values of some parameters for various materials.

Example 2.7 Given $E = 14$ GPa and $v = 0.32$, which are reasonable values for bone, find the values of strain for a LEHI behavior and the 2-D state of stress in Example 2.1: $\sigma_{xx} = 120$ kPa, $\sigma_{yy} = 150$ kPa, and $\sigma_{xy} = 0$. Note: 1 GPa $= 10^3$ MPa $= 10^6$ kPa $= 10^9$ Pa, where 1 Pa $= 1$ N/m^2.

Solution: From Eq. 2.69,

$$\varepsilon_{xx} = \frac{1}{14 \times 10^6}[120 - 0.32(150 + 0)]\frac{\text{kPa}}{\text{kPa}} = 5.1 \times 10^{-6},$$

$$\varepsilon_{yy} = \frac{1}{14 \times 10^6}[150 - 0.32(120 + 0)] = 8.0 \times 10^{-6}$$

$$\varepsilon_{xy} = \frac{1}{2G}(0) = 0,$$

where $G = E/2(1+v)$. Unless the problem is treated as purely two dimensional,

$$\varepsilon_{zz} = \frac{1}{14 \times 10^6}[0 - 0.32(150 + 120)] = -6.2 \times 10^{-6},$$

which is to say that the material will thin in the z direction due to the (inplane) stresses in the x and y directions. Note, too, that each value of strain is much less than unity, consistent with the small strain requirement for Hooke's law, and positive values denote lengthening, whereas negative values denote shortening. Moreover, strain is unitless (it represents normalized changes in length and changes in angles) and a value of strain times 10^{-6} is often called a microstrain ($\mu\varepsilon$), as, for example, $5.1 \times 10^{-6} \equiv 5.1\ \mu\varepsilon$.

Finally, it is important to note that Eq. (2.69) is called Hooke's law (although it is merely a constitutive relation, not a law) to commemorate R. Hooke's profound observation relating force and extension even though Hooke had no concept of stress or strain. Hooke's law can be derived mathematically because of the assumptions of linearity, elasticity, homogeneity, and isotropy (LEHI), but we did not do so here; we merely listed the final form. Hooke's law can also be established through a comprehensive battery of laboratory observations and experiments; again, we did not provide the associated, detailed information. Herein, therefore, we will focus on its use, not its formulation. Qualitatively, it is also useful to note that for a 1-D state of stress, say $\sigma_{xx} = f/A$, under isothermal conditions, $\varepsilon_{xx} = \sigma_{xx}/E$ and $\varepsilon_{yy} = \varepsilon_{zz} = -\nu(\sigma_{xx})/E = -\nu\varepsilon_{xx}$. Hence, it is easy to see how one could/would design a uniaxial experiment to determine the values of E and ν for a LEHI behavior; the value of G could then be calculated as $E/2(1+\nu)$ and verified via a shear test. For example, E is simply the constant slope (stiffness) in the σ_{xx} versus ε_{xx} plot (cf. Fig. 2.27); that ν is a measure of the lateral thinning relative to the axial extension is seen by taking (for this 1-D state of stress)

$$-\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{(-\nu\sigma_{xx}/E)}{\sigma_{xx}/E} = \nu = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}, \tag{2.70}$$

which reveals how its value can be inferred from experiment.

Finally, let us consider Poisson's ratio in more detail. Noting that $\varepsilon_{xx} = \partial u_x/\partial x \cong \partial u_x/\partial X$ where $u_x = x(X) - X$, we can think of (roughly, but not rigorously) an extensional strain ε_{xx} over a small region as a change in length divided by the original length: that is, $\varepsilon_{xx} \sim (\Delta x - \Delta X)/\Delta X = \Delta u_x/\Delta X$, where Δx is the current length and ΔX is the original length. Now, if we consider a cube having initial dimensions ΔX, ΔY, and ΔZ and deformed dimensions Δx, Δy, and Δz, then the current (deformed) volume $\Delta \mathcal{V} = \Delta x \Delta y \Delta z$ can also be computed as

$$\begin{aligned}\Delta\forall &= \Delta x\Delta y\Delta z = \Delta X(1+\varepsilon_{xx})\Delta Y\big(1+\varepsilon_{yy}\big)\Delta Z(1+\varepsilon_{zz})\\ &= (\Delta X\Delta Y\Delta Z)\big[(1+\varepsilon_{xx})\big(1+\varepsilon_{yy}\big)(1+\varepsilon_{zz})\big]\\ &= (\Delta \mathrm{V})\big[1+\varepsilon_{zz}+\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{xx}+\varepsilon_{xx}\varepsilon_{zz}+\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}\big]\\ &= (\Delta \mathrm{V})\big[1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\big]+\text{H.O.T.},\end{aligned} \tag{2.71}$$

or

$$\Delta\forall = \Delta \mathrm{V}+\Delta \mathrm{V}\big(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\big) \rightarrow \frac{\Delta\forall-\Delta \mathrm{V}}{\Delta \mathrm{V}} = \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}, \tag{2.72}$$

where H.O.T. stands for higher-order terms, terms that can be neglected in comparison to other terms (i.e., given that ε_{xx}, ε_{yy}, $\varepsilon_{zz} \ll 1$, quadratic and cubic terms are negligible with respect to linear terms). Now, if we let ε_{xx} be the axial direction strain and ε_{yy} and ε_{zz} be the lateral direction strains, then by the above definition,

$$\varepsilon_{yy} = -\nu\varepsilon_{xx}, \qquad \varepsilon_{zz} = -\nu\varepsilon_{xx}, \tag{2.73}$$

and, thus,

$$\frac{\Delta\forall-\Delta \mathrm{V}}{\Delta \mathrm{V}} = -\nu\varepsilon_{xx}+\varepsilon_{xx}-\nu\varepsilon_{xx} \rightarrow \frac{\Delta\forall-\Delta \mathrm{V}}{\Delta \mathrm{V}} = \varepsilon_{xx}(1-2\nu). \tag{2.74}$$

Hence, if there is no volume change, then $\nu = \frac{1}{2}$ as alluded to earlier, which is to say that the material deforms incompressibly. Determination of a value of a material parameter from a thought experiment is thus possible, albeit uncommon. In most cases, the value of a material parameter must be calculated directly from experimental data. Note, therefore, that with respect to the steps outlined in Sect. 1.7 for formulating a constitutive relation (DEICE), we must *delineate* characteristic behaviors, here manifested as a LEHI behavior; *establish* a theory, here a linearized theory of elasticity in which stress σ and strain ε are related; *identify* the functional form, here a linear relation (Hooke's law); and *calculate* values of the material parameters, here E, ν, and G. *Evaluating* the predictive capability is thus the final step, which is typically performed by comparing computed and measured values of stress or strain for situations not used to formulate the constitutive relation. For example, if we find values of E and ν for a particular material from a uniaxial test, we will want to ensure that these values also provide a good description of the behavior of the material in torsion and bending, particularly if these situations are experienced in service conditions.

2.6.3 Hooke's Law for Transverse Isotropy

We emphasize that "Hooke's law" as stated in Eq. (2.69) holds if the material behavior is isotropic (i.e., the behavior is independent of the direction in which the force is applied at a point within the material). This can be seen, for example, by interchanging the subscripts x, y, and z in the equations, which leaves them unchanged.

Whereas many metals exhibit an isotropic behavior under small strains, many other materials do not. Wood, fiberglass, and other man-made composites as well as tendons, ligaments, skin, bone and most other biological tissues exhibit an anisotropy. Consider, for example, a piece of wood. It is clear that the mechanical response in the direction of the grain is different from that across the grain. The same is true of heart muscle (Fig. 2.24) due to the locally parallel arrangement of the muscle fibers. When a material has a different behavior in one direction compared to all directions in an orthogonal plane, the behavior is said to be *transversely isotropic* (i.e., isotropic in a plane transverse to a preferred or different direction). If the transversely isotropic behavior is otherwise linear, elastic, and homogeneous under small strains, it is describable via a transversely isotropic Hooke's law of the form

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) - \frac{\nu'}{E'}\sigma_{zz}, \qquad & \varepsilon_{xy} &= \frac{1}{2G}\sigma_{xy}, \\
\varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) - \frac{\nu'}{E'}\sigma_{zz}, \qquad & \varepsilon_{xz} &= \frac{1}{2G'}\sigma_{xz}, \\
\varepsilon_{zz} &= \frac{1}{E'}\sigma_{zz} - \frac{\nu'}{E}\left(\sigma_{xx} + \sigma_{yy}\right), \qquad & \varepsilon_{yz} &= \frac{1}{2G'}\sigma_{yz},
\end{aligned} \tag{2.75}$$

where, again,

$$G = \frac{E}{2(1+\nu)}, \tag{2.76}$$

with the z direction (arbitrarily) taken to be the preferred direction. Note that in contrast to the relation for isotropic behavior [Eq. (2.69)], which is described by two independent parameters (E and ν, with G related to these two), this relation for transversely isotropic behavior is described by five independent parameters (two Young's moduli E and E', two Poisson's ratios ν and ν', and a shear modulus G', where G is, again, related to E and ν and thus is not independent). Again, because of the linearity, this relation can be derived theoretically or determined via a complex battery of experiments. We do not focus on either here; we will simply consider in subsequent chapters how one can utilize this relation.

2.6.4 Hooke's Law for Orthotropy

Given the complexity of the microstructure of many materials in their solid phase, it should not be surprising that there are many different types of anisotropy. In addition to isotropy and transverse isotropy, however, the other most common type of material symmetry is orthotropy. As the name implies, an *orthotropic* response is one that differs in three orthogonal directions. It is thought, for example, that an artery exhibits an orthotropic response: Its behavior differs in the axial (due to axially oriented adventitial collagen), circumferential (due to the nearly circumferentially oriented smooth muscle in the media), and radial directions. Bone, too, tends to exhibit an orthotropic response, albeit nearly transversely isotropic in some cases. When the response is otherwise linear, elastic, and homogeneous under small strains, Hooke's law can be generalized to account for the orthotropy via

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E_1}\sigma_{xx} - \frac{\nu_{21}}{E_2}\sigma_{yy} - \frac{\nu_{31}}{E_3}\sigma_{zz}, & \varepsilon_{xy} &= \frac{1}{2G_{12}}\sigma_{xy},\\
\varepsilon_{yy} &= \frac{1}{E_2}\sigma_{yy} - \frac{\nu_{12}}{E_1}\sigma_{xx} - \frac{\nu_{32}}{E_3}\sigma_{zz}, & \varepsilon_{xz} &= \frac{1}{2G_{13}}\sigma_{xz},\\
\varepsilon_{zz} &= \frac{1}{E_3}\sigma_{zz} - \frac{\nu_{13}}{E_1}\sigma_{xx} - \frac{\nu_{23}}{E_2}\sigma_{yy}, & \varepsilon_{yz} &= \frac{1}{2G_{23}}\sigma_{yz},
\end{aligned} \tag{2.77}$$

wherein there are now nine independent material parameters: three Young's moduli E_1, E_2, and E_3, three shear moduli G_{12}, G_{13}, and G_{23}, and six Poisson's ratios ν_{12}, ν_{21}, ν_{13}, ν_{31}, ν_{23}, and ν_{32}, only three of which are independent; that is, it can be shown that

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \qquad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \qquad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}. \tag{2.78}$$

2.6.5 Other Coordinate Systems

It is essential to recognize that Hooke's law relates stress to strain at each *point* with respect to a given coordinate system. Whereas Eqs. (2.69), (2.75), and (2.77) are written in terms of Cartesians, they could also be written for cylindrical coordinates. For example, Eq. (2.69) for LEHI behavior can be written as

$$\begin{aligned}
\varepsilon_{rr} &= \frac{1}{E}[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})] + \beta\Delta T, \qquad \varepsilon_{r\theta} = \frac{1}{2G}\sigma_{r\theta}, \\
\varepsilon_{\theta\theta} &= \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})] + \beta\Delta T, \qquad \varepsilon_{rz} = \frac{1}{2G}\sigma_{rz}, \\
\varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})] + \beta\Delta T, \qquad \varepsilon_{\theta z} = \frac{1}{2G}\sigma_{\theta z}.
\end{aligned} \tag{2.79}$$

and similarly for sphericals and so forth.

Likewise, Hooke's law can be written with respect to coordinate systems that are transformed relative to one another. For example [cf. Eq. (2.69)], for isotropy we may have, relative to (x', y', z'),

$$\begin{aligned}
\varepsilon'_{xx} &= \frac{1}{E}\left[\sigma'_{xx} - \nu\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \qquad \varepsilon'_{xy} = \frac{1}{2G}\sigma'_{xy}, \\
\varepsilon'_{yy} &= \frac{1}{E}\left[\sigma'_{yy} - \nu\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \qquad \varepsilon'_{xz} = \frac{1}{2G}\sigma'_{xz}, \\
\varepsilon'_{zz} &= \frac{1}{E}\left[\sigma'_{zz} - \nu\left(\sigma'_{xx} + \sigma'_{yy}\right)\right], \qquad \varepsilon'_{yz} = \frac{1}{2G}\sigma'_{yz}.
\end{aligned} \tag{2.80}$$

Indeed, see Exercise 2.23, which asks that you prove this.

Finally, note from Eq. (2.69) that a 2-D state of stress necessarily requires a 3-D state of strain and vice versa for a 2-D state of strain. Indeed, even a 1-D state of stress (e.g., an axial force which induces a stress $\sigma_{xx} = E\varepsilon_{xx}$) will generally induce a 3-D state of strain (an extensional strain ε_{xx} plus thinning in two orthogonal directions, given by ε_{yy} and ε_{zz}). Hence, we must be careful when describing the dimension of a problem. A truly 1-D or 2-D problem is thus one wherein we simply ignore the effects in certain directions, which can be useful in some cases. For example, in a purely 1-D problem, σ_{xx} and ε_{xx} alone may exist.

Here, however, let us define a state of *plane stress* as one where

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}, \tag{2.81}$$

whereas a state of *plane strain* is defined by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.82}$$

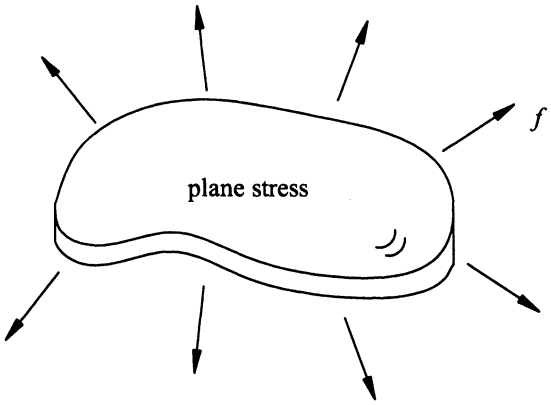


FIGURE 2.28 Schema of a state of plane stress, which is characterized by in-plane stresses only. Such states of stress also exist locally in curved membranes such as the pericardium, urinary bladder, and saccular aneurysms. Indeed, to a first approximation, many tissues (e.g., even skin in some situations) can be considered to be in a state of plane stress.

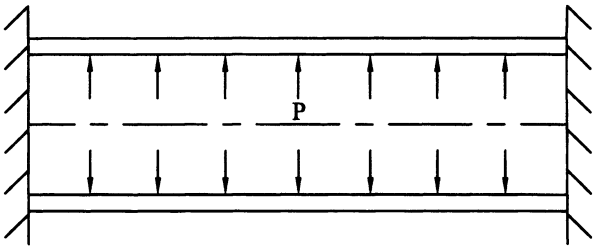


FIGURE 2.29 Schema of a state of plane strain, characterized by nonzero values in a single plane (often normal to the long axis of a prismatic structure). Although many have assumed that arteries and airways are in a state of plane strain, given that they deform primarily in the radial and circumferential directions due to internal pressurization, these tissues are actually prestretched and thus the axial strain is not zero; they are in a fully 3-D state of strain.

A state of plane stress is realized easily in thin planar structures that are loaded only in-plane (Fig. 2.28) whereas a state of plane strain is realized easily in long straight members that are constrained from deforming in the axial direction (Fig. 2.29). Although we will not go into these cases in detail, note that they each afford certain simplifications in formulation and solution (Timoshenko and Goodier 1970).

2.7 Mechanical Properties of Bone

Whereas most soft tissues (e.g., skin, tendons, arteries, lung tissue, myocardium) exhibit nonlinear material behaviors over finite (large) strains, teeth and bones tend to exhibit a linearly elastic behavior over small strains. Hooke's law is thus applicable and the associated stress analysis is easier than that for soft tissues. Therefore, let us consider bone in some detail.

According to *Dorland's Medical Dictionary*, bone is

> the hard form of connective tissue that constitutes the majority of the skeleton of most vertebrates; it consists of an organic component (the cells and matrix) and an inorganic, or mineral component; the matrix contains a framework of collagen fibers and is impregnated with the mineral component, chiefly calcium phosphate (85 percent) and calcium carbonate (10 percent).

Specifically, the type I collagen fibers tend to be organized in layers, locally parallel within a layer with the orientation varying approximately 90° from layer to layer. This layering may suggest a local transverse isotropy with the preferred direction changing from layer to layer or, more grossly, an overall orthotropic behavior at each point. Whereas the collagen endows bone with its tensile stiffness, the embedded calcium endows it with a high compressive stiffness.

Two primary cell types within mature bone are responsible for growth and remodeling: the osteoblasts, which secrete bone matrix, and the osteoclasts, which degrade it. These cells thus allow for a continuous turnover of the matrix material (Alberts et al. 2008) (i.e., a continuous maintenance or, in times of altered loading, a mechanism for adaptation). It is for this reason that bedridden patients and astronauts each suffer bone atrophy, particularly in the legs and arms, whereas athletes may have a buildup of bone. Indeed, as noted in Chap. 1, it was the work of Meyer, Wolff, and Roux in the late nineteenth century on bone that revealed a strong relationship between mechanical factors and biological growth and remodeling. For more recent work in this important and active research area, see Mow and Hayes (1991), Cowin (2001) or Carter and Beaupré (2001).

Bone typically consists of two to three layers, depending on its location within the body: an outer, dense cortical layer, a middle trabecular layer, and, in certain regions, an innermost layer of bone marrow. It is the marrow that forms blood cells. Bone thus serves several important, diverse mechanical and physiological functions: It supports and protects soft tissues and organs and it serves as a primary store of calcium and producer of blood cells. We discuss the associated microstructure further in Chap. 4.

The 206 distinct bones that constitute the adult human skeleton are often classified into one of five groups according to their shape (see Table 1.2 from Nigg and Herzog (1994)): long (e.g., femur), short (e.g., carpal), flat (e.g.,

TABLE 2.1 Physical properties of bone.

Variable	Bone	Value
Density	Cortical	1,700–2,000 kg/m^3
	Lumbar vertebra	600–1,000 kg/m^3
Mineral content	All	60–70 %
Elastic modulus	Femur	5–28 GPa
Tensile strength	Femur	80–150 MPa
	Tibia	95–140 MPa
Compressive strength	Femur	131–224 MPa
	Tibia	106–200 MPa

sternum), irregular (pubis), and sesamoid (e.g., patella). Table 2.1 lists some of the physical properties of bone. In particular, the order of magnitude of the stiffness (Young's modulus) is ~16 GPa for cortical bone and ~1 GPa for cancellous bone. As noted earlier, however, bone does not exhibit an isotropic behavior. Rather, its linear, elastic, nonhomogeneous, and orthotropic response is better described by Eq. (2.77) with values of the parameters on the order of

$$\begin{aligned} E_1 &= 6.9\,\text{GPa}, \quad \nu_{12} = 0.49, \quad \nu_{21} = 0.62, \quad G_{12} = 2.41\,\text{GPa}, \\ E_2 &= 8.5\,\text{GPa}, \quad \nu_{13} = 0.12, \quad \nu_{31} = 0.32, \quad G_{13} = 3.56\,\text{GPa}, \\ E_3 &= 18.4\,\text{GPa}, \quad \nu_{23} = 0.14, \quad \nu_{32} = 0.31, \quad G_{23} = 4.91\,\text{GPa} \end{aligned} \tag{2.83}$$

for the tibia and

$$\begin{aligned} E_1 &= 12.0\,\text{GPa}, \quad \nu_{12} = 0.376, \quad \nu_{21} = 0.422, \quad G_{12} = 4.53\,\text{GPa}, \\ E_2 &= 13.4\,\text{GPa}, \quad \nu_{13} = 0.222, \quad \nu_{31} = 0.371, \quad G_{13} = 5.61\,\text{GPa}, \\ E_3 &= 20.0\,\text{GPa}, \quad \nu_{23} = 0.235, \quad \nu_{32} = 0.350, \quad G_{23} = 6.23\,\text{GPa} \end{aligned} \tag{2.84}$$

for the femur, where 1 denotes the radial direction, 2 the circumferential, and 3 the axial (Cowin 2001). Separate values for the cortical and cancellous portions can also be found in this reference. For more on the mechanobiology and in vivo loading of bone, see Chap. 4.

Observation 2.3. When two or more materials are bonded together, *delamination* becomes a possible mechanism of failure. Simply put, delamination is a load-induced separation between two mechanically distinct materials or layers. One way to prevent, or at least to minimize, delamination is to create a 3-D interaction (e.g., weave) at the interface. In the case of bone–metal interfaces, for example, the surface of the metal implant is often made porous to allow ingrowth of the bone. Delamination often occurs due to interfacial shear stresses

and thus there is a need to design experiments that impose shear stresses. One simple experiment is a so-called "pull-out" test. Briefly, one material is bonded to the inside of a hollow sample of the second material. The outer material is then fixed in place and the inner material subjected to an axial load through its centroid. A free-body diagram of the inner material reveals that the axial load must be supported by the integrated manifestation of all the shear stresses acting on its outer surface. Although the magnitude of these shears may vary from point to point, one can determine the mean shear stress at which delamination initiates. Subsequent design would then seek to protect the bonded surface from experiencing damaging values of shear stresses.

Chapter Summary

Mechanics is the study of responses by materials and structures to applied loads, the most familiar example of which is perhaps the simple linear spring, first studied in detail by R. Hooke in the late seventeenth century. As we recall from physics, Hooke said, as the force, so the extension (i.e., $f = k\delta$, where f is force, k the spring stiffness, and δ the displacement at the end of the spring). Yet, during the period from the mid-eighteenth to the early nineteenth century, savants such as L. Euler and A. Cauchy showed that it was much more appropriate in continuum mechanics to work with the concepts of stress and strain, not force and extension. Strictly speaking, stress is a mathematical quantity that transforms an outward unit normal vector on a differential area of interest into an associated traction vector in the direction of the applied force but having units of force per area. Strain can similarly be defined in terms of a transformation of a differential position vector from a reference to a current configuration. See Humphrey (2001, 2002) for detailed derivations and descriptions of the utility of the concepts of stress and strain in biology, including a discussion of the fundamental utility of tensor calculus in continuum biomechanics.

Herein, however, we introduced the concept of stress intuitively, not mathematically, and we noted that a component of stress represents the magnitude of a component of force acting over an oriented area (i.e., stress is a force intensity). We also noted that, relative to a single coordinate system (defined by an origin and basis), one can define up to nine different components of stress that act on a generic infinitesimal cube of material, which when reduced in a limiting process becomes a point. Moreover, relative to each coordinate system, one can associate nine different components of strain (defined by particular combinations of displacement gradients, that is, changes in displacement with position) with these nine different components of stress at each point. Albeit

often not emphasized, it is vital to remember that there is nothing special per se about a particular component of stress or strain; rather, each component results from our choice of coordinate system, which we should select simply to render the overall mathematical solution easier. Fortunately, transformation relations allow components of stress or strain relative to one coordinate system to be related to those of other coordinate systems. We found, for example, that a judicious selection of coordinate system can yield the maximum/minimum (e.g., principal) values of stress or strain at any point. Hence, our overall strategy should be (1) determine components of stress and strain by selecting that coordinate system which renders easiest the solution of the initial or boundary value problem at hand and (2) transform these components of stress or strain to those components that are most meaningful for experimental, theoretical, or biological purposes.

Finally, recall that the aforementioned relation $f = k\delta$ for a linear spring is a simple example of a structural constitutive relation, that is, it describes the response (extension) of a spring (structure) to an applied load (force) under conditions of interest (e.g., ranges of loading that allow the spring to recoil elastically when unloaded). The spring constant k is thus the constitutive parameter for this simple 1-D linear spring. Whereas the five basic postulates of continuum mechanics (e.g., balance of linear momentum) hold for all continua, constitutive relations hold for individual materials or structures under particular conditions of interest (e.g., temperatures or time scales). We introduced the 3-D Hooke's law for linearly elastic material behaviors under small strain, which can be a useful descriptor for materials ranging from stainless steel to bone. It is essential, however, that one understand the limitations of each relation. Because the formulation of appropriate constitutive relations is perhaps the most important and challenging aspect of modern continuum biomechanics, the reader should be especially attentive to related discussions throughout.

Appendix 2: Material Properties

The properties of many materials can be found in textbooks on material science (e.g., Askeland 1994) or biomaterials (e.g., Ratner 2003) as well as many handbooks. Here, we simply tabulate a few of the properties that may be useful in the examination of example and exercise problems in this book.

TABLE A2.1 Physical properties of common engineering materials.

Material	Young's modulus (GPa)	Shear modulus (GPa)	Density (kg/m^3)	Yield strength (MPa)		Ultimate strength (MPa)	
				Tension	Shear	Tension	Shear
Aluminum							
2024-T4	73	27.6	2,770	300	170	414	220
6061-T6	70	25.9	2,770	241	138	262	165
Steel							
0.2 % Carbon	200	83	7,830	250	165	450	330
0.6 % Carbon	200	83	7,830	415	250	690	550

Exercises

2.1 Find a general relation for σ'_{xx} [Eq. (2.13)] when $\alpha = \alpha_s$.

2.2 Show that $\alpha_p = \alpha_s \pm 45°$. In addition, show that Eq. (2.25) for α_p can also be determined via $d\sigma'_{yy}/d\alpha = 0$.

2.3 Rederive the transformation equation for σ'_{yy} using the result for σ'_{xx} and the observation that σ'_{yy} exists on a face at an angle $\pi/2 + \alpha$ from the x direction.

2.4 Show that for a 2-D state of stress,

$$\sigma_{xx} + \sigma_{yy} = \sigma'_{xx} + \sigma'_{yy} \quad \forall \alpha.$$

This combination of the normal stresses is called an *invariant*; that is, its numerical value at any point is independent of the coordinate system even though its value will differ, in general, from point to point in a body and, of course, with changes in load. Invariants have been found to be useful in modeling material behavior, which, by definition, must be independent of man and his coordinate systems.

2.5 Show that Eq. (2.80) can be determined directly from Eqs. (2.69) and the transformation relations (2.13) and (2.57).

2.6 The results for the max/min normal stresses can also be found using matrix equations. Using ideas from linear algebra, show that the 2-D eigenvalue problem for the matrix equation

$$\det \begin{bmatrix} \sigma_{xx} - \Lambda_p & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} - \Lambda_p \end{bmatrix} = 0$$

yields eigenvalues $\Lambda_1 \equiv \sigma_1$ (with $p = 1$) and $\Lambda_2 \equiv \sigma_2$ (with $p = 2$). Hint: Solve the quadratic equation for Λ_p, the two roots of which correspond to $p = 1$ and $p = 2$. Also, if familiar with linear algebra, find the eigendirections $\boldsymbol{n}^p$, where $|\boldsymbol{n}^p| = 1$, and discuss their relationship to α_p.

2.7 Given the state of stress in Example 2.1, $\sigma_{xx} = 120$ kPa, $\sigma_{yy} = 150$ kPa, and $\sigma_{xy} = 0$ kPa, compute the values of σ'_{yy} for all values of α from 0° to 90° and plot as a function of α. Compare the values of α at which σ'_{yy} is max/min versus those found using the formula for α_p. Repeat for σ'_{xy} and compare the value of α at which the shear is max/min versus that using the formula for α_s.

2.8 A state of pure shear is one in which the normal stresses are zero. Consider $\sigma_{xx} = 0$, $\sigma_{yy} = 0$, and $\sigma_{xy} = \sigma_{yx} = 5$ MPa. Find the values of the principal stresses and denote them on an infinitesimal element with orientation given by α_p.

2.9 Given a hydrostatic state of stress, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, where p is a pressure, we computed σ'_{xx}, σ'_{yy} and σ'_{xy} for all α in Example 2.4. Likewise, compute the principal stresses σ_1 and σ_2 [i.e., $\sigma'_{xx})_{\text{max/min}}$ and $\sigma'_{yy})_{\text{max/min}}$] as well as the maximum shear $\sigma'_{xy})_{\text{max/min}}$ using the explicit formulas in the text. Discuss your findings.

2.10 Given $\sigma_{xx} = 3$ MPa, $\sigma_{yy} = 1$ MPa, and $\sigma_{xy} = 2$ MPa, find the values of the principal stresses and the maximum shear stress. What are the associated values of α_p and α_s? Draw a 2-D representation of the stress at a point p relative to each set of coordinates (x, y) and those for α_p and α_s.

2.11 Given $\sigma_{xx} = 3$ MPa and $\sigma_{yy} = -3$ MPa, find the maximum shear stress and the plane on which it acts. Draw the 2-D representation of stress about a point p.

2.12 Given $u_x = (\Lambda - 1)X$ and $u_y = 0$, compute and compare the exact (E_{XX}) and the approximate/linearized (ε_{xx}) strains for $\Lambda = 1.001$, 1.01, 1.1, 1.5, and 2.0. Calculate the error introduced by the linearization in each case and determine those values of Λ for which the approximation is reasonable.

2.13 Let

$$u_x = (X + 0.001Y) - X,$$
$$u_y = Y - Y.$$

Compute the values of the components of the 2-D Green strain E_{XX}, E_{YY}, and E_{XY} and compare to those for the linearized strain ε_{xx}, ε_{yy}, and ε_{xy}. Repeat with the value premultiplying Y in the expression for u_x being 0.8.

2.14 Calculate the values of E_{YY} and E_{XY} for the rigid-body motion given by Eq. (2.54) and compare to the results for ε_{yy} and ε_{xy}.

2.15 The transformation relations for strain [Eq. (2.57)] can be found directly via coordinate transformations; recall Eqs. (2.62–2.68). Hence, if we recall from calculus that

$$x' = x\cos\alpha + y\sin\alpha, \qquad y' = -x\sin\alpha + y\cos\alpha$$

and note that similar relations hold for the displacement vector,

$$u_x' = u_x \cos\alpha + u_y \sin\alpha, \qquad u_y' = -u_x \sin\alpha + u_y \cos\alpha,$$

then show that

$$\varepsilon_{yy}' = \varepsilon_{xx} \sin^2\alpha - 2\varepsilon_{xy} \sin\alpha \cos\alpha + \varepsilon_{yy} \cos^2\alpha,$$
$$\varepsilon_{xy}' = 2 \sin\alpha \cos\alpha \left(\frac{\varepsilon_{yy} - \varepsilon_{xx}}{2}\right) + (\cos^2\alpha - \sin^2\alpha)\varepsilon_{xy}.$$

Hint: Note that the angle α, which relates the two coordinate systems, is very different from the angle ϕ used in the text to represent a rigid-body rotation. Moreover, for the linearized strain,

$$\varepsilon_{yy}' = \frac{\partial u_y'}{\partial y'} = \frac{\partial u_y'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u_y'}{\partial y}\frac{\partial y}{\partial y'}.$$

2.16 For the delta strain gauge rosette ($\alpha_1 = 0$, $\alpha_2 = 60°$, $\alpha_3 = 120°$), show that

$$\varepsilon_{xy} = \frac{1}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°}).$$

2.17 For the 0–45°–90° strain rosette of Example 2.6, find general expressions for the principal strains and maximum shear strains in terms of the measurable values ε_0, $\varepsilon_{45°}$, and $\varepsilon_{90°}$.

2.18 Whereas Eqs. (2.69), (2.75), and (2.77) are called strain–stress relations, Hooke's law can also be written as stress–strain relations. For example, for isotropy, we have

$$\sigma_{xx} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx}, \qquad \sigma_{xy} = 2\mu\varepsilon_{xy},$$
$$\sigma_{yy} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{yy}, \qquad \sigma_{yz} = 2\mu\varepsilon_{yz},$$
$$\sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz}, \qquad \sigma_{zx} = 2\mu\varepsilon_{zx},$$

where λ and μ are called Lamé constants (material parameters), after the French scientist G. Lamé (1795–1870). Show that

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \equiv G,$$

where E and ν are the Young's modulus and Poisson ratio, respectively.

2.19 Note from the previous exercise and Eq. (2.69) that λ multiplies the first invariant of strain $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$, which is a measure of volume change. Show that

$$e = \frac{1-2\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

for isotropy. Note that there is no change in volume (i.e., $e = 0$) if $\nu = \frac{1}{2}$. Moreover, if a cube of material is subjected to a hydrostatic pressure, then $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$. In this case, note that

$$e = -\frac{3(1-2\nu)}{E}\,p \rightarrow -\frac{p}{e} = \frac{E}{3(1-2\nu)} \equiv K,$$

where K is the so called *bulk modulus*; it represents the ratio of the hydrostatic compressive stress to the decrease in volume.

2.20 For a LEHI behavior, show that a plane state of stress requires that

$$\sigma_{zz} = 0 = \frac{E}{(1+\nu)(1+2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right],$$

or

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}\left(\varepsilon_{xx} + \varepsilon_{yy}\right).$$

2.21 For a LEHI behavior, show that the principal stresses in a plane state of stress can be written as

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2), \qquad \sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1),$$

where ε_1 and ε_2 are the principal strains.

2.22 Given $\sigma_{xx} = 20$ MPa, $\sigma_{yy} = -10$ MPa, and $\sigma_{xy} = -20$ MPa, find the principal stresses and principal strains with LEHI behavior and $E = 16$ GPa and $\nu = 0.325$.

2.23 Starting with Eq. (2.80) and using Eqs. (2.13), (2.17), (2.21), and (2.57), show that you recover Eq. (2.69). Note: We assume a rotation about the z-axis thus $\sigma'_{zz} \equiv \sigma_{zz}$.

2.24 Given reasonable values of the material parameters for bone, estimate the axial stress in your femur due to standing, walking, and running. Toward this end, estimate the increase in the applied load (in terms of body weight and in comparison to the load due to standing) due to walking and running. Once done, note that even though we did not discuss it, bone exhibits viscoelastic, not just elastic, behavior under certain conditions. In particular, the Young's modulus increases with increases in strain rate. It has been estimated, for example, that $E \sim c\dot{\varepsilon}^d$, where $\dot{\varepsilon}$ is the extensional strain rate and c and d are material parameters. If $E = 16$ GPa at $\dot{\varepsilon}$

$= 0.001\ \text{s}^{-1}$ (slow walking) and $d \sim 0.06$, find the value of c. Next, compute the value of E for vigorous activity, with $\dot{\varepsilon} \sim 0.01\ \text{s}^{-1}$, and discuss how this would effect your first estimate for stress in your femur.

2.25 Research the different constitutive relations used to describe the behavior of water in its solid, liquid, and gaseous phases (i.e., different conditions of interest). Write a two-page report on your findings, showing explicitly the different equations and discussing how the different characteristic behaviors dictate the need to establish different theoretical frameworks (DEICE).

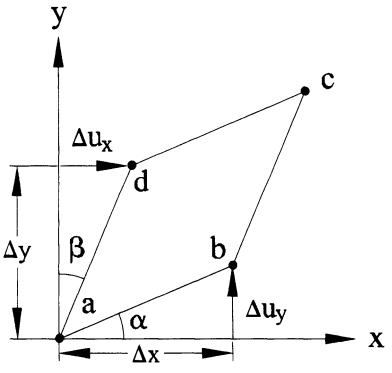
2.26 Referring to Fig. 2.23a, note that the material can return to its original configuration by releasing the energy that is stored in it due to deformation. This "strain energy" W can be computed (per initial volume) as the area under the stress–strain curve. For the 1-D test in Fig. 2.23a, $\sigma_{xx} = E\varepsilon_{xx}$ and the stored energy is $\frac{1}{2}$ (base)(height) $= \frac{1}{2}\,\varepsilon_{xx}\sigma_{xx} = \frac{1}{2}\,\varepsilon_{xx}E\varepsilon_{xx}$. Show, therefore, that the stress can be determined as the change in energy with respect to changes in strain (i.e., $\sigma_{xx} = \partial W/\partial \varepsilon_{xx}$), whereas the stiffness can be computed as the change in stress with respect to the change in strain ($\partial \sigma_{xx}/\partial \varepsilon_{xx}$). Plot this stiffness as a function of stress and comment.

2.27 Although we chose not to derive the linearized strains directly, it is common to relate them (for illustrative purposes) to changes in length and changes in angle. The former was used to show that Poisson's ratio $\nu = ½$ if the behavior is incompressible. Here, note the following for shear. Referring to Fig. 2.30, let point b displace upward an amount Δu_y. With point d displacing rightward by Δu_x, we call this a pure shear. Note, therefore, that the angles α and β are given by

$$\alpha = \tan\left(\frac{\Delta u_y}{\Delta x}\right) \approx \frac{\Delta u_y}{\Delta x}, \qquad \beta = \tan\left(\frac{\Delta u_x}{\Delta y}\right) \approx \frac{\Delta u_x}{\Delta y}$$

for which we used the small-angle approximation for the tangent, and thus the mean value is

FIGURE 2.30



$$\frac{1}{2}(\alpha+\beta)=\frac{1}{2}\left(\frac{\Delta u_y}{\Delta x}+\frac{\Delta u_x}{\Delta y}\right)\rightarrow \lim_{\Delta x,\Delta y\rightarrow 0}\frac{1}{2}(\alpha+\beta)=\frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right),$$

which we recognize equals ε_{xy}. Repeat this exercise for the y-z plane.

2.28 Common experimental setups include uniaxial extension or compression of a rod, biaxial extension of a sheet, tension–torsion of a cylinder, inflation–extension of a hollow cylinder, and inflation of an axisymmetric membrane (Fig. 2.31). Identify tissues that would be appropriately tested using these potential setups without excessive dissection following removal from the body.

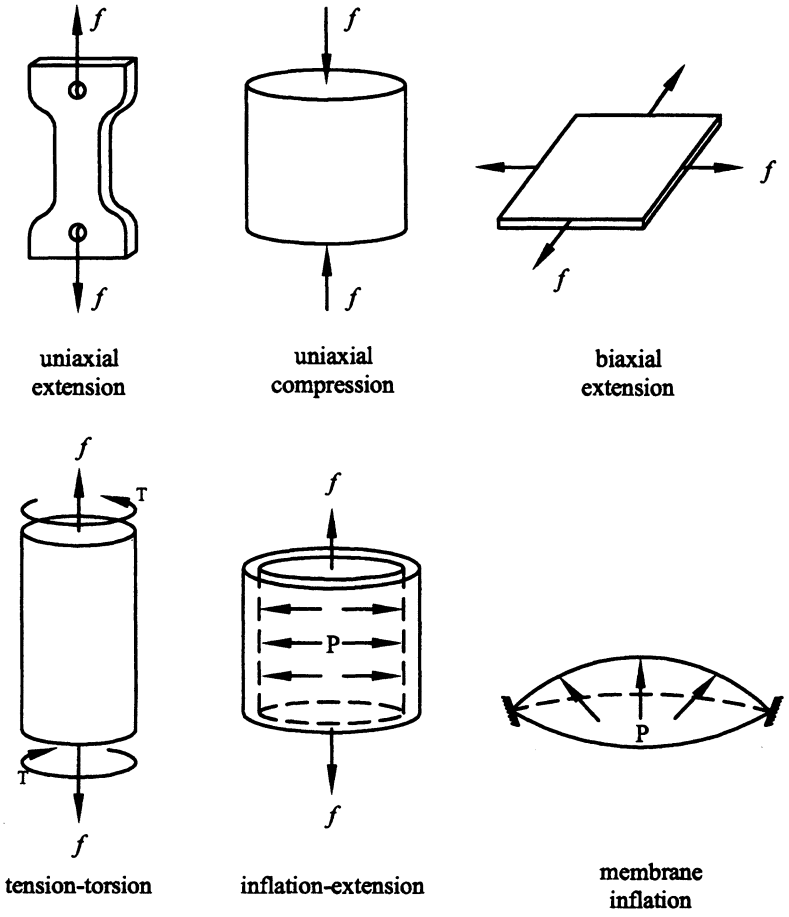


FIGURE 2.31

2.29 Using Eqs. (2.13) and (2.21), it can be shown that, in two dimensions, $\sigma_{xx} + \sigma_{yy} = \sigma'_{xx} + \sigma'_{yy}$. Show, in addition, that the principal stresses $\sigma_1 \equiv \sigma'_{xx}(\max)$ and $\sigma_2 \equiv \sigma'_{yy}(\max)$ simultaneously add to yield

$$\sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2.$$

2.30 A special 2-D state of stress is called an *equibiaxial stress*. It is defined by $\sigma_{xx} = \sigma_{yy} = \sigma_o$ and $\sigma_{xy} = \sigma_{yx} = 0$. Find the principal stresses and max/min shear stresses in this case. Note that equibiaxial stretching tests are particularly useful in determining the anisotropy of a planar tissue (membrane). Why?

2.31 A uniaxial test was performed on a bone specimen having a central (gauge) region initially 6 mm long and 2 mm in diameter. Five data points were recorded:

Axial force (N)	94	190	284	376	440
Change in length (mm)	0.009	0.018	0.027	0.050	0.094

Plot the associated stress–strain relation, calculate a Young's modulus, and show that the yield stress is ~118 MPa. Recall that the yield stress reveals the transition from elastic to plastic (cf. Fig. 2.25). Data from Özkaya and Nordin (1999).

2.32 Data from a uniaxial tension test to failure (data point 4) for a human cortical bone are

Stress (MPa)	0	85	114	128
Strain (mm/mm)	0	0.005	0.010	0.026

Plot the data and interpret. Estimate the Young's modulus, yield stress, and ultimate stress (cf. Fig. 2.26). Clearly, much more data are useful in general. Data from Özkaya and Nordin (1999).

2.33 Data from a uniaxial tension test in the elastic region for a bone sample are

Stress (MPa)	0	60	120	180
Strain	0	0.0034	0.0066	0.0100

Referring to Exercise 2.32, were these tests performed on the same type of bone? Compare the Young's moduli.