Boundary Value Problems: Analytic Methods of Solution

5

If there is no other use discover'd of Electricity, this, however, is something considerable, that it may help to make a vain man humble.

—Benjamin Franklin (1706–1790)
statesman, scientist, inventor

5.1 Introduction

The relations and methods introduced in **Chapters 3** and **4** dealt primarily with point and distributed charges and the electric fields they produce. If the charges were known, the electric field intensity and potential could be determined. However, many practical situations exist in which the charges are either unknown or are distributed in a complex fashion. The use of the simple formulas for the calculation of fields in these geometries is not always possible. In still other geometries, we have no knowledge of charges but only of fields and potentials. For example, in an overhead transmission line, we may know the potential but not the charge on the line. How can we then calculate the electric field intensity everywhere in space? Similarly, when designing an electric instrument, such as an electrostatic filter, the engineer is not going to calculate "how much charge must be present on the electrode." This information, while important in itself, is not normally a design parameter simply because we do not usually use "charge supply sources" and we are ill equipped to measure charge or charge density. The more common problem in design would be to calculate the required potential on the electrodes of the device to produce the needed effect. This information is important because with it, the power supply required can be designed. Although the principles in **Chapters 3** and **4** and the formulas developed for calculation of fields, potentials, and energy are applicable to these types of problems as well, the difficulty is in applying them.

In this chapter, we look at yet another technique for finding the electric fields and potential differences in a given geometry. The method is that of solution to boundary value problems; that is, we ask ourselves: Given the boundary conditions of an electrostatic problem, what is the potential (as well as electric field intensity and charge distributions) throughout the geometry? In the case of an overhead transmission line, the potentials on the line and perhaps at ground level are known. These are boundary conditions of the geometry. Based on these conditions and the postulates we derived earlier, can we calculate the potential everywhere? The answer is yes, but, as we will see, the actual calculation can be rather involved, depending on the geometry and configuration of potentials. The approach here is straightforward and includes four steps:

(1) Rewrite the electrostatic field postulates in terms of the potential as a second-order partial differential equation.
(2) Find a general solution to this partial differential equation using any suitable method of solution.
(3) Satisfy the general solution in Step (2) for the particular boundary conditions that define the problem to obtain a particular solution.
(4) Calculate any other quantity, such as the electric field intensity and charge density, from the potential.

It is important to mention here that some seemingly simple problems cannot be solved using the methods in the previous chapters and in this, not because the solution does not exist but because the general solutions cannot be satisfied everywhere in the solution domain and on the boundaries of the problem. These types of problems are deferred to the following chapter, in which we discuss numerical methods of solution.

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5.2 Poisson's[1] Equation for the Electrostatic Field

The required postulates for the electrostatic field in linear, homogeneous, isotropic dielectric materials are

$$\nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \tag{5.1}$$

We recall the fact that the curl of **E** is zero (the electrostatic electric field intensity **E** is a conservative field) allowed us to define the electric field intensity in terms of the electric scalar potential as

$$\mathbf{E} = -\nabla V \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{5.2}$$

This is now substituted into the second postulate in **Eq. (5.1)**:

$$\nabla \cdot \mathbf{E} = -\nabla \cdot (\nabla V) = \frac{\rho_v}{\varepsilon} \tag{5.3}$$

We recall from **Chapter 2** [**Eq. (2.132)**] that $\nabla \cdot (\nabla V) = \nabla^2 V$. Substitution of this into **Eq. (5.3)** gives

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \tag{5.4}$$

It is worth looking at this equation in expanded form. For example, in Cartesian coordinates,

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}} \tag{5.5}$$

This is called Poisson's equation. It is a linear, scalar partial differential equation and, most importantly, it is equivalent to solving for the electric field intensity **E** using the basic postulates. Any technique that solves this equation is an appropriate method for calculation of electrostatic fields. Since the left-hand side of **Eq. (5.4)** is the divergence of the gradient of V [i.e., $\nabla \cdot (\nabla V)$], we can immediately obtain Poisson's equation in cylindrical and spherical coordinates. In fact, we have already written the left-hand side of **Eq. (5.4)** in the three systems of coordinates in **Eqs. (2.132), (2.134),** and **(2.135)**. Poisson's equation in cylindrical coordinates is

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}} \tag{5.6}$$

and in spherical coordinates

$$\boxed{\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\varepsilon}} \tag{5.7}$$

[1] Simeon Denis Poisson (1781–1840) was an applied mathematician. Most of his work was in application of mathematics to physics and, in particular, to electrostatics and magnetism. A contemporary of Lagrange, Laplace, and Fourier, he was also an astronomer and is known for his work on probability theory and definite integrals, in addition to electromagnetics. He also contributed to the understanding of Fourier series, the theory of heat, and mechanics.

5.3 Laplace's[2] Equation for the Electrostatic Field

Under charge-free conditions, the right-hand side in **Eq. (5.5)** is zero. In Cartesian coordinates,

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \tag{5.8}$$

This is Laplace's equation, and it is obvious that whenever there are no charge densities in the domain in which a solution is sought, we will use Laplace's equation. Laplace's equation in cylindrical and spherical coordinates is as in **Eqs. (5.6)** and **(5.7)** but with zero on the right-hand side. Laplace's equation applies to a number of other physical problems in addition to electrostatic fields. These include the gravitational field in the absence of mass, the pressure field in the absence of sources, and the temperature field in the absence of heat sources in the solution domain. We will meet both Laplace's and Poisson's equations in the following chapters.

To summarize, the main reasons to try to solve Laplace's or Poisson's equations rather than using the field equations in **Eq. (5.1)** are three:

(1) Both Laplace's and Poisson's equations as used here are scalar equations.
(2) Design parameters are often in terms of potentials on structures rather than charges and fields.
(3) We can take advantage of mathematical techniques that have been developed for solution of second-order partial differential equations.

5.4 Solution Methods

As a general rule, methods of solution for second-order partial differential equations are appropriate here. The first method that comes to mind is the method of separation of variables. However, for Poisson's equation, it is much more difficult to obtain a solution using separation of variables. The method we use is essentially a comparative method: that is, we do not actually solve the partial differential equation but rather obtain a solution by superposition of known solutions, each of which satisfies Poisson's equation. The method is known as the method of images. Both the separation of variables method and the method of images are quite powerful although, as we shall see shortly, limited in scope. In this chapter, we discuss the following three methods:

(1) Direct integration for Laplace's and Poisson's equations.
(2) Method of images for problems described by Poisson's equation.
(3) Separation of variables for problems described by Laplace's equation.

The first of these can only be used in one-dimensional problems: applications in which the potential varies in one dimension in space. The method of images is a rather interesting and quite general method of solution that also has applications beyond the calculation of electrostatic fields.

It is also worth mentioning that any of the solutions we obtained in **Chapter 4**, such as by using Gauss's law, are, necessarily, solutions to Poisson's or Laplace's equations. In particular, the expressions in **Eqs. (4.31)** through **(4.33)** are general solutions of Poisson's equation.

5.4.1 Uniqueness of Solution

Any of the methods mentioned in the previous section and many others can be used to solve electrostatic problems provided that the boundary conditions of the problem can be satisfied. This aspect of solution will become evident shortly, but, before

[2] Pierre Simon Laplace (1749–1827) was a distinguished mathematician and astronomer. Whereas we are aware of his work on the Laplace transform and calculation of the determinant by expansion by minors, Laplace also contributed to the understanding of gravitation, motion of planets, theory of probability, and others. His career was long and varied. In 1799, he was appointed minister of the interior by Napoleon (apparently he was not very successful because this episode only lasted about 6 weeks). After that, he was made a count and, later, a marquis. Laplace's and Poisson's equations and their solutions are very important in understanding electromagnetics.

we do so, it is worthwhile stating that whatever method we use for solution of either Laplace's or Poisson's equations, we must obtain the same solution. In other words, the solution is unique and independent of method.

Uniqueness of solution of Laplace's and Poisson's equations is guaranteed by the ***uniqueness theorem***. The theorem states that if two solutions to either of the equations are obtained, then the solutions must be the same. In formal terms, we must prove the theorem for each equation or at least for Poisson's equation (since Laplace's equation is a special case of Poisson's equation). Instead, we will take this as given and will only resort to substituting a given solution into the equations to show that it satisfies the equation (see **Problems 5.1** and **5.2**).

5.4.2 Solution by Direct Integration

In principle, to solve a differential equation, we need to "integrate" the equation. This, of course, is not possible in general for a second-order partial differential equation. If, however, the potential only varies in one dimension in space, the partial derivatives can be replaced by ordinary derivatives and the equation can be integrated directly. Consider the following one-dimensional Poisson equation in Cartesian coordinates:

$$\frac{d^2V(x)}{dx^2} = -\frac{\rho_v(x)}{\varepsilon} \tag{5.9}$$

Because the potential varies only with x, ordinary derivatives were used to replace the partial derivatives in **Eq. (5.5)**. Integrating both sides of the equation once, we get

$$\frac{dV(x)}{dx} = \int \left[-\frac{\rho_v(x)}{\varepsilon}\right] dx + a \tag{5.10}$$

Integrating again, we get

$$V(x) = \int \left[\int \left[-\frac{\rho_v(x)}{\varepsilon}\right] dx\right] dx + ax + b \quad [\text{V}] \tag{5.11}$$

The notation used here, that of using the indefinite integral and also adding the constants, is unusual, but it indicates the process and allows for space-dependent charge densities. In practice, we start with **Eq. (5.9)** with known charge density on the right-hand side and integrate once. Only then do we proceed to evaluate **Eq. (5.11)**. The two constants of integration a and b must be determined to obtain a particular solution, and this, of course, is obtained from known values of V; the known values are the boundary conditions of the problem.

If instead of solving Poisson's equation, we solve Laplace's equation (i.e., a solution domain in which there are no charge densities), we simply remove the contribution of charge densities and obtain

$$V(x) = ax + b \tag{5.12}$$

and the constants of integration are again evaluated from the boundary conditions of the problem.

Example 5.1 Potential and Fields in a Capacitor The plates of a capacitor are separated a distance d [m] and grounded as in **Figure 5.1**. The dielectric between the plates has permittivity ε [F/m], and a volume charge density $\rho_v(x) = \rho_0 x(x-d)$ [C/m^3] is distributed throughout the volume of the dielectric. Find the potential and electric field intensity everywhere between the plates of the capacitor.

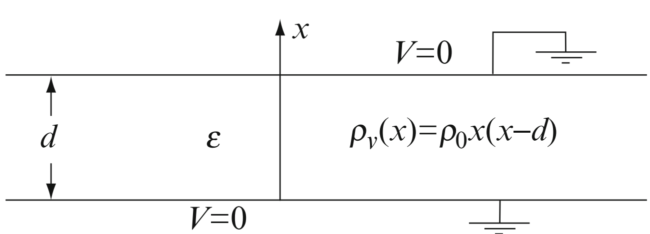


Figure 5.1 A parallel plate capacitor with grounded plates and a charge density between the plates

Solution: Since the charge density only depends on x, this is a one-dimensional problem. Direct integration of Poisson's equation provides the potential. Calculation of the gradient of the potential gives the electric field intensity.

Substitution of the charge density into Poisson's equation gives

$$\frac{d^2V}{dx^2} = -\frac{\rho_0 x(x-d)}{\varepsilon}$$

with boundary conditions $V(x = 0) = 0$ and $V(x = d) = 0$. Integrating twice the general form of the solution is obtained as

$$V(x) = -\frac{\rho_0 x^4}{12\varepsilon} + \frac{\rho_0 x^3 d}{6\varepsilon} + C_1 x + C_2 \quad [\mathrm{V}]$$

With the given boundary conditions, we get

$$V(x=0) = 0 = C_2 \quad \rightarrow \quad C_2 = 0$$

$$V(d) = 0 = -\frac{\rho_0 d^4}{12\varepsilon} + \frac{\rho_0 d^3 d}{6\varepsilon} + C_1 d + 0 \quad \rightarrow \quad C_1 = -\frac{\rho_0 d^3}{12\varepsilon}$$

Thus, the solution to this boundary value problem is

$$V(x) = -\frac{\rho_0 x^4}{12\varepsilon} + \frac{\rho_0 x^3 d}{6\varepsilon} - \frac{\rho_0 x d^3}{12\varepsilon} \quad [\mathrm{V}]$$

This solution satisfies the boundary conditions and the potential is everywhere negative (between the plates).

The electric field intensity is calculated from the gradient of the potential:

$$\mathbf{E}(x) = -\nabla V(x) = -\hat{\mathbf{x}}\frac{dV(x)}{dx} = \hat{\mathbf{x}}\left(\frac{\rho_0 x^3}{3\varepsilon} - \frac{\rho_0 x^2 d}{2\varepsilon} + \frac{\rho_0 d^3}{12\varepsilon}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Note that the electric field intensity is in the negative x direction for any value $x < d/2$, is zero for $x = d/2$, and is in the positive x direction for any value $x > d/2$.

Check Verify this solution by substituting it back into Poisson's equation.

Exercise 5.1 What is the electric field intensity and the electric potential outside the plates in **Example 5.1**? Hint: You must use Gauss's law.

Answer

$$\mathbf{E} = -\hat{\mathbf{x}}\frac{\rho_0 d^3}{12\varepsilon_0} \left[\frac{\mathrm{V}}{\mathrm{m}}\right], \quad V = -\frac{\rho_0 d^3 x}{12\varepsilon_0} \quad [\mathrm{V}], \quad x < 0,$$

$$\mathbf{E} = \hat{\mathbf{x}}\frac{\rho_0 d^3}{12\varepsilon_0} \left[\frac{\mathrm{V}}{\mathrm{m}}\right], \quad V = -\frac{\rho_0 d^3 (x-d)}{12\varepsilon_0} \quad [\mathrm{V}], \quad x > d.$$

Example 5.2 Application: The Stud Sensor A stud sensor is made as a simple capacitor with its two plates located on a plane. The principle is shown in **Figure 5.2a**. The two plates are connected to a simple balanced bridge. The bridge is unbalanced if any material is placed below the plates. When using the device to detect wood studs in a wall,

(continued)

Example 5.2 Application: The Stud Sensor (continued)
the bridge is balanced to zero in the absence of studs. When a stud is detected, the device turns on a light or sound. A simpler configuration is solved in the following example, assuming the plates are semi-infinite in extent.

Two semi-infinite conducting plates are placed on the r–z plane as shown in **Figure 5.2b**. The plates are very close to each other at $r = 0$ but do not touch. The right plate is connected to a potential V_0 and the left to a zero potential. The plates are very thin:

(a) Calculate the electric potential everywhere.
(b) Calculate the electric field intensity everywhere.

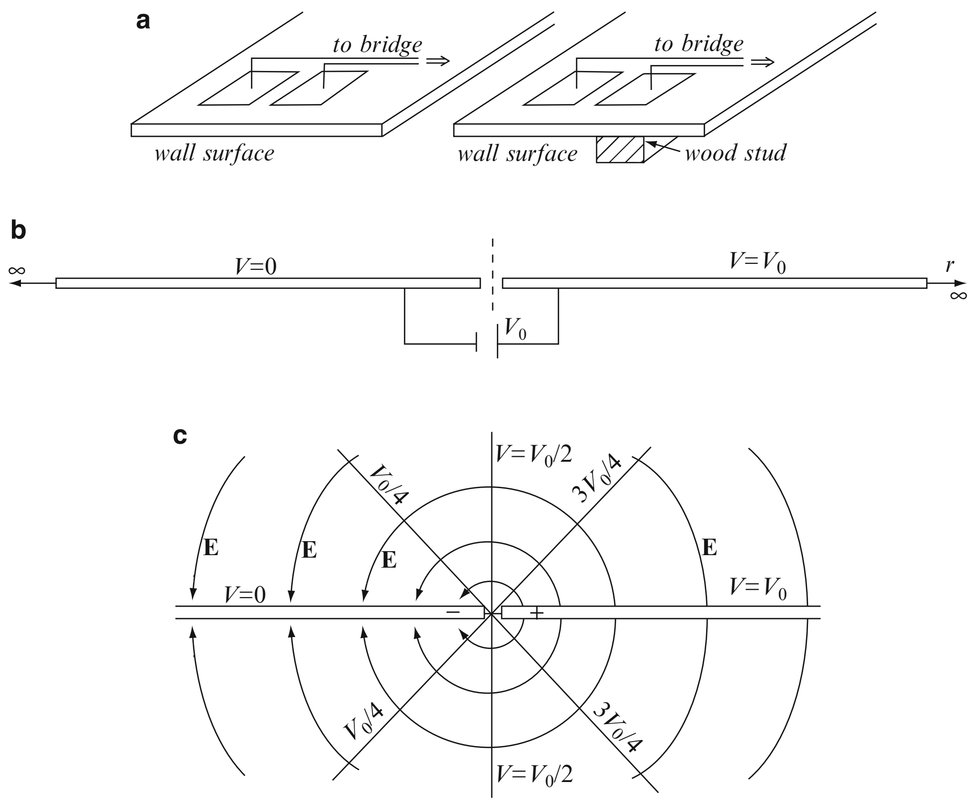


Figure 5.2 The stud sensor. (**a**) Balanced and unbalanced conditions. (**b**) Configuration for solution. (**c**) Plot of electric field intensity and potential

Solution: If we use cylindrical coordinates, both plates are along the r axis but are separated a distance π in the ϕ direction. In this case, we must solve Laplace's equation in cylindrical coordinates (no charges between the plates) and only the ϕ component is nonzero. **Figure 5.2b** shows the arrangement:

(a) From **Eq. (5.6)**, because the potential is constant in the r and z directions, we get

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \rightarrow \quad \frac{d^2 V}{d\phi^2} = 0$$

The solution to this equation is obtained by directly integrating twice:

$$V(\phi) = a\phi + b \quad [\mathrm{V}]$$

From the boundary conditions of the problem, we get:
At $\phi = 0$, $V = V_0$: this gives $b = V_0$.

At $\phi = \pi$, $V = 0$ and we get

$$V(\pi) = a\pi + V_0 = 0 \quad \rightarrow \quad a = -\frac{V_0}{\pi}$$

The solution is therefore

$$V(\phi) = -\frac{V_0}{\pi}\phi + V_0 \quad [\text{V}]$$

The solution is linear with the angle ϕ. For example, at $\pi/2$, the potential equals $V_0/2$.

(b) The electric field intensity is calculated as

$$\mathbf{E} = -\nabla V = -\hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} = -\hat{\boldsymbol{\phi}}\frac{1}{r}\frac{d}{d\phi}\left[-\frac{V_0}{\pi}\phi + V_0\right] = \hat{\boldsymbol{\phi}}\frac{V_0}{\pi r} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

The electric field intensity is circular (in the positive ϕ direction above the plates) and depends on r. As we proceed from $r = 0$ (where the plates are closest to each other), the electric field intensity decreases. A plot of the electric field intensity and potential is shown in **Figure** 5.2c. The solution below the plates is similar, but the electric field intensity is in the opposite direction.

Example 5.3 Potential Distribution in a Coaxial Line Two coaxial conductors are shown in **Figure** 5.3. The conductors are very long. A voltage V_0 [V] is connected across the conductors:

(a) Assuming only the potential difference V_0 is known, find the potential distribution everywhere between the conductors.
(b) What is the electric field intensity between the conductors?

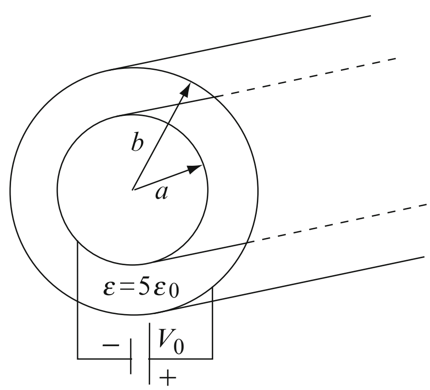


Figure 5.3 Coaxial line connected to a source. The electric field intensity between the conductors is sought

Solution: The geometry is cylindrical and there are no sources in the solution domain enclosed by the two cylinders. Thus, we will use Laplace's equation in cylindrical coordinates. Since the only possible variation in voltage is in the r direction, a one-dimensional Laplace's equation describes the solution:

(a) The equation to solve is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

The latter form is allowable, since in this geometry, $r \neq 0$. Integrating once, we get

$$r\frac{dV}{dr} = C_1 \quad \rightarrow \quad \frac{dV}{dr} = \frac{C_1}{r}$$

Integrating this again gives

$$V(r) = C_1 \ln r + C_2 \qquad [\mathrm{V}]$$

The constants of integration are evaluated from the boundary conditions.
These are $V\ (r = a) = 0$ and $V(r = b) = \mathrm{V}_0$:

$$V(a) = C_1 \ln a + C_2 = 0 \quad \text{and} \quad V(b) = C_1 \ln b + C_2 = V_0 \qquad [\mathrm{V}]$$

Solving for C_1 and C_2 gives

$$C_1 = \frac{V_0}{\ln(b/a)}, \quad C_2 = -\frac{V_0 \ln a}{\ln(b/a)}$$

and the solution for potential everywhere between the conductors is

$$V(r) = \frac{V_0}{\ln(b/a)}(\ln r - \ln a) \quad [\mathrm{V}].$$

(b) The electric field intensity is found from the gradient of the potential. Since the potential only depends on r, the gradient in cylindrical coordinates has an r component only:

$$\mathbf{E}(r) = -\nabla V(r) = -\hat{\mathbf{r}}\frac{\partial}{\partial r}\left[\frac{V_0}{\ln(b/a)}(\ln r - \ln a)\right] = -\hat{\mathbf{r}}\frac{V_0}{r\ln(b/a)} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Note that the solution to this problem can also be obtained using Gauss's law (see **Section 4.3).**

5.4.3 The Method of Images

The method of images is based on the principle that a field produced by a source or system of sources in part of the space can be produced by other sets of sources. Although the uniqueness theorem specifies that any solution obtained using Laplace's or Poisson's equations is unique, which, in turn, means that a given source produces a unique solution, it is possible to produce two identical fields by different source configurations, provided this field is not the total field. By this we mean that different combinations of sources can produce the same solution in a given part of space although they will have completely different solutions in other parts of space. Thus, the method of images solves for the fields in a given problem by replacing it with an equivalent problem, which is easier to solve and which has the same solution in the required domain. In particular, problems that involve point and distributed charges and conducting bodies can often be replaced by equivalent point charges or charge distributions for which the solution is known or can be obtained readily. As an illustration, recall that in using Gauss's law, we could replace a spherical charge distribution with a point charge if all we required was the electric field intensity outside the charge distribution. This provided a correct solution only outside the charge distribution.

The question is, how can we find the equivalent charges or charge distributions? The basic idea is that a charge (point charge or charge distribution) reflects in a conductor much the same as light reflects in a mirror. Under these conditions, the conductor is replaced by the image charge and the field due to the two charges is calculated. The method is usually associated with perfect conductors, but this is not a necessary condition. Image methods can be used with conductors or dielectrics.

In more exact terms, the method of images applies whenever a system of charges can be identified and an equal potential surface produced by these charges exists. As an example, charge Q in **Figure 5.4a** represents any point or distributed charges above a conducting surface. The solution for the electric field intensity or potential is required throughout space. The method of images requires that we remove the conductor (which is at a constant potential) and replace it by the image of the charge Q, as shown in **Figure 5.4b**. By doing so, the surface of the conductor is replaced by an identical constant potential surface and the solution above this surface is the same as in the original problem in **Figure 5.4a**. Note, however, that below the plane (where the conductor was located), the solution is incorrect: The electric field intensity in a conductor is always zero, whereas the electric field intensity for a system of charges is not. This minor difficulty is resolved in a very simple manner: we obtain the solution of **Figure 5.4b** and use it only above the plane whereas the solution below the plane is known ($\mathbf{E} = 0$).

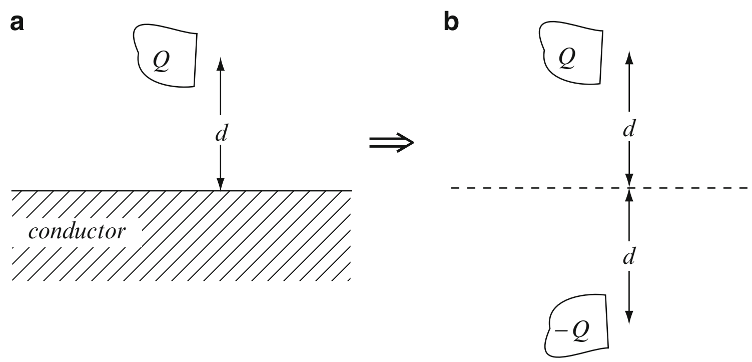


Figure 5.4 The general principle of the method of images. (**a**) The original problem to be solved. (**b**) The equivalent or image problem. The two configurations have the same solution above the conductor

A useful analogy can be obtained by viewing the conducting surface as a mirror. When looking into a mirror, we see an image looking in the opposite direction. If we were to look at an angle into the mirror, so would the image be at an angle, looking at us. We also note that the image itself does not exist: there is nothing behind the mirror. Therefore, only the "solution" in front of the mirror is valid. You may think of it in this sense: A light source in front of the mirror reflects off the mirror. Any object in front of the mirror is lit by two sources: one is the original source, the second is the reflected light from the mirror. If we were to remove the mirror, the reflected light would disappear. If we now place an identical source at the location of the image in the mirror, the light distribution in the area in front of the mirror would be the same as if the mirror were still there. Thus, the image is a convenient artifice. We also note that the name "method of images"[3] comes from the analogy with optical images. From this simple analogy, we can write the following general properties for the method of images in planar geometries (see **Figure 5.4**):

(1) The image is the negative of the source. The magnitude of the image is the same as the source.
(2) The geometry is reflected in the constant potential surface as in a mirror.
(3) The image and source are at the same distance from the mirror.
(4) Multiple sources produce multiple images, again like reflection in a mirror.
(5) Single or multiple point or distributed charges in front of multiple mirrors also produce multiple images.

Important Rules (1) and (3) only apply to planar surfaces. The rest apply in curved surfaces as well.

5.4.3.1 Point and Line Charges

Point_Charges.m

The simplest application of the method of images is that of a single point charge or a single line charge over a conducting plane. These are discussed now, as they serve to demonstrate the method. Using superposition of solutions, these are then extended to multiple charges or lines and multiple conducting surfaces. Consider first a point charge Q over a conducting surface, as shown in **Figure 5.5a**. The conductor is assumed to be at zero potential, but this condition is not necessary since it is only a reference value. In other words, if the conducting surface is not at zero potential, we may assume it is and, after

[3] The method of images is attributed to Lord Kelvin [Sir William Thomson (1824–1907)], who introduced it in 1848 for solution of electric problems. It is, however, a general method that applies equally well to magnetic and electromagnetic problems, as well as to other problems described by Poisson's equation.

solving the problem, add the potential of the conductor. The conductor is now removed and an image charge, equal in magnitude but negative, is located at a distance d below the equipotential surface (which is at the location of the surface of the conductor), as shown in **Figure 5.5b**. The two charges produce a field distribution as shown in **Figure 5.5c**. We note that the potential midway between two opposite charges is zero, as was our assumption. Thus, removing the conductor and replacing it with the image has not changed this condition. We also conclude immediately that the solution above the surface of the conductor is the solution in the upper part of **Figure 5.5c**.

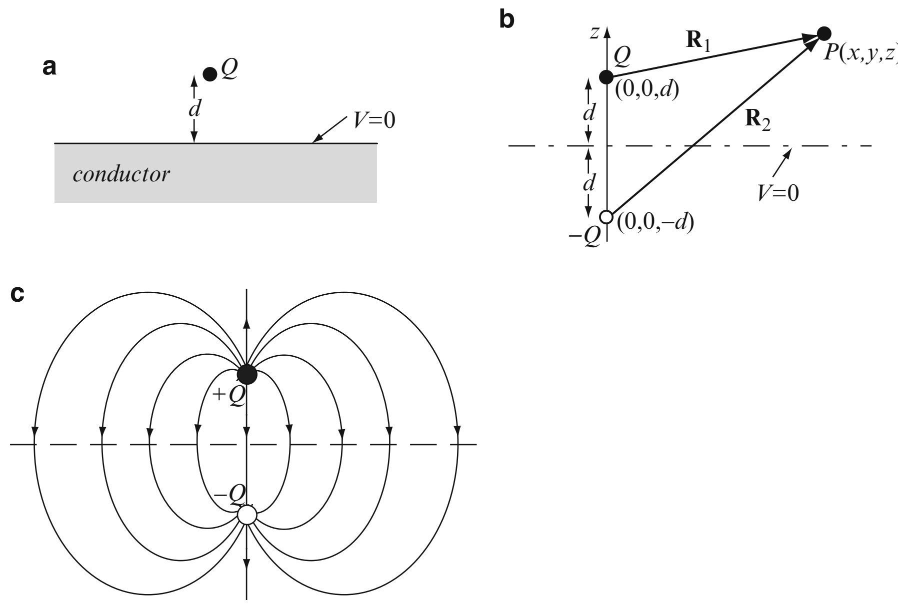


Figure 5.5 (**a**) Point charge over a conducting surface. (**b**) The conducting surface is removed and an image charge equal but opposite in sign is introduced. (**c**) The electric field intensity for the configuration in (**b**)

To find the solution, all we need to do is find the electric field intensity, or the potential, everywhere in space due to two point charges. As a rule, it is easier to calculate the potential whenever there are many point charges and calculate the electric field intensity if we only have a small number of charges. If the potential is calculated, then the electric field intensity must be calculated from the potential through the use of the gradient. If, on the other hand, the electric field intensity is evaluated, the potential is calculated by integration. Both methods will be demonstrated here:

(1) Calculation of Potential. The potential at any point in space $P(x,y,z)$ due to two point charges at points $P_1(0,0,d)$ and $P_2(0,0,-d)$ is (see **Figure 5.5b**)

$$V(x,y,z) = +\frac{Q}{4\pi\varepsilon_0 R_1} - \frac{Q}{4\pi\varepsilon_0 R_2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left[x^2+y^2+(z-d)^2\right]^{1/2}} - \frac{1}{\left[x^2+y^2+(z+d)^2\right]^{1/2}}\right] \quad (\mathrm{V}) \tag{5.13}$$

Note that at $z = 0$, the potential is zero for any point in the x–y plane, as we have assumed. Now, we can calculate the electric field intensity as the gradient of the electric potential:

$$\begin{aligned}\mathbf{E}(x,y,z) &= -\nabla V(x,y,z) = -\hat{\mathbf{x}}\frac{\partial V(x,y,z)}{\partial x} - \hat{\mathbf{y}}\frac{\partial V(x,y,z)}{\partial y} - \hat{\mathbf{z}}\frac{\partial V(x,y,z)}{\partial z} \\ &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]\end{aligned} \tag{5.14}$$

Setting $z = 0$, we obtain the electric field intensity at the surface of the conductor:

$$\mathbf{E}(x, y, 0) = -\hat{\mathbf{z}} \frac{2Qd}{4\pi\varepsilon_0 \left[x^2 + y^2 + d^2\right]^{3/2}} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.15}$$

At the surface of the conductor, the electric field intensity is in the negative z direction and its magnitude depends on the location in the plane as well as on the height d. The electric field intensity above the conducting surface is that given in **Eq. (5.14)**, whereas in the conductor, the solution is zero, resulting in the solution in **Figure 5.6**. Here, the charge density on the surface of the conductor is shown. This charge density is a direct consequence of the fact that the electric field must "end" in a negative charge, indicating that the actual solution is due to the point charge Q and the equivalent induced charges on the surface of the conductor. The induced charge on the surface of the conductor can be easily calculated from the interface conditions at the surface since the electric field intensity only has a normal component at the conducting surface. From the interface conditions between air and the conductor, we get [see **Eq. (4.82)**]

$$D_t = 0, \quad D_n = \rho_s \quad \rightarrow \quad \varepsilon_0 E_n = \rho_s \qquad \left[\frac{\text{C}}{\text{m}^2}\right] \tag{5.16}$$

or using the electric field intensity in **Eq. (5.15)**,

$$\rho_s(x, y) = -\frac{2Qd}{4\pi \left[x^2 + y^2 + d^2\right]^{3/2}} \quad \left[\frac{\text{C}}{\text{m}^2}\right] \tag{5.17}$$

The charge density is negative and distributed on the surface according to location. The maximum charge density is located at ($x = 0$, $y = 0$), directly below the charge Q, as is evident from **Eq. (5.17)**. Also, the total charge on the infinite plane $z = 0$ is equal to $-Q$, as can be shown by direct integration (see **Exercise 5.2**). The charge density distribution is shown in **Figure 5.7a** in the x direction ($y = 0$). The charge distribution on the plane is in the shape of a bell, with its highest point exactly below the point charge, as shown in **Figure 5.7b**.

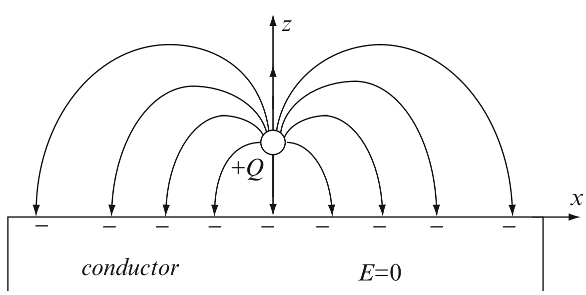


Figure 5.6 Complete solution to the point charge over a conducting plane shown in **Figure 5.5a**

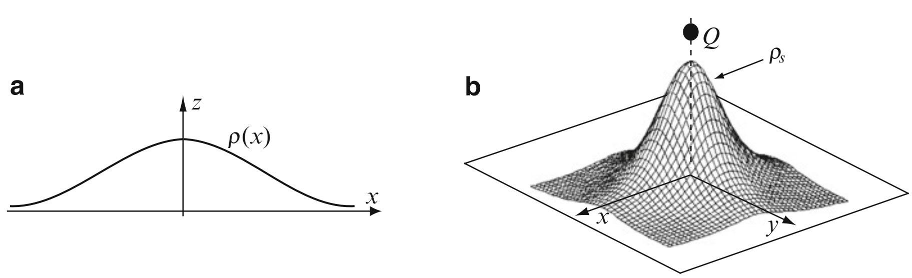


Figure 5.7 (**a**) Induced charge density (magnitude) on the surface of the conductor in **Figure 5.5a**. (**b**) Induced charge density shown as magnitude over the x–y plane

(2) Calculation of the Electric Field Intensity. As mentioned earlier, we can start by calculating the electric field intensity due to the two charges. Starting with **Figure 5.8**, and noting the two position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$, we write

$$\mathbf{E}(x,y,z) = \frac{Q\mathbf{R}_1}{4\pi\varepsilon_0 R_1^3} + \frac{(-Q)\mathbf{R}_2}{4\pi\varepsilon_0 R_2^3} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.18}$$

which, not surprisingly, is the same result as in **Eq. (5.14)**. To calculate the potential everywhere in space, we use the definition of potential. Using **Eq. (5.18)**, we write

$$V = -\int \mathbf{E}(x,y,z)\cdot d\mathbf{l} = -\int \frac{Q\mathbf{R}_1}{4\pi\varepsilon_0 R_1^3}\cdot d\mathbf{R}_1 + \int \frac{Q\mathbf{R}_2}{4\pi\varepsilon_0 R_2^3}\cdot d\mathbf{R}_2 = \frac{Q}{4\pi\varepsilon_0 R_1} - \frac{Q}{4\pi\varepsilon_0 R_2} \quad [\text{V}] \tag{5.19}$$

where the integration is carried out from ∞ to point $P(x,y,z)$. This result is exactly the potential in **Eq. (5.13)**, again as expected.

Now, we can summarize the basic steps involved in the solution by image method in planar geometries:

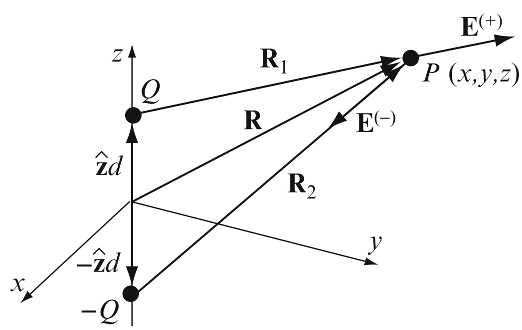


Figure 5.8 The configuration in **Figure 5.5b** as used to calculate the electric field intensity

(1) Find a surface on which the potential is constant. This potential will normally be on the surface of a conductor but does not have to be zero.

(2) Locate the image charge (or charges) equal to the source charge, opposite in sign and at equal distance on the other side of the surface. The conductor itself is removed since its effects have been replaced by the image charge(s).

(3) Find either the electric field intensity (if there are few charges) or the electric potential (if the number of charges is large). In either case the electric field intensity or the potential must be calculated at a general point in space.

(4) Calculate other quantities if required, based on the basic formulas of the electric field intensity or the electric potential as given here and in **Chapters 3** and **4**.

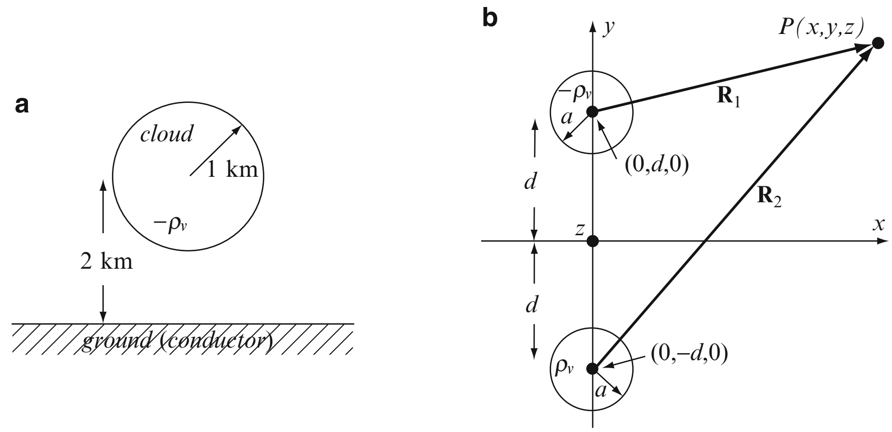
Example 5.4 Application: Electric Field at Ground Level Due to Charged Clouds If the charge distribution in a cloud can be estimated, the electric field intensity can be calculated everywhere, assuming the ground to be a conductor. As a simple example of this type of problem, we consider here a spherical charge distribution as an approximation to a cloud.

A spherical cloud has a uniform negative volume charge density $-\rho_v$ [C/m^3] and is located above ground, as shown in **Figure 5.9a**. The dimensions and charge density are shown in the figure:

(a) Calculate the electric field intensity everywhere at ground level.

(b) What is the largest electric field intensity at ground level? Where does it occur?

Figure 5.9 (**a**) A charged cloud over a conducting ground. (**b**) The image representation of the configuration in (**a**)



Solution: To calculate the electric field intensity, we use the method of images. The ground is removed and an identical sphere, with positive charge, is placed a distance $d = 2$ km below the location of the ground, as shown in **Figure 5.9b**. The potential is calculated outside the spheres as if the spheres were point charges, placed at their center (a result we obtained from Gauss's law). After calculating the potential at a general point $P(x,y,z)$; we find the negative gradient of the potential to calculate the electric field intensity. Setting $y = 0$ gives the electric field intensity on the ground:

(a) The electric potential at a distance R_1 from the upper sphere and R_2 from the lower sphere is

$$V(P) = -\frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} \quad [\text{V}]$$

The total charge Q in each sphere is the volume of the sphere multiplied by the charge density

$$Q = \frac{4\pi a^3 \rho_v}{3} \quad [\text{C}]$$

The distances R_1 and R_2 are $R_1 = (x^2 + (y - d)^2 + z^2)^{1/2}$ and $R_2 = (x^2 + (y + d)^2 + z^2)^{1/2}$. The electric potential at $P(x,y,z)$ is

$$V(x,y,z) = -\frac{a^3\rho_v}{3\varepsilon_0\left(x^2 + (y-d)^2 + z^2\right)^{1/2}} + \frac{a^3\rho_v}{3\varepsilon_0\left(x^2 + (y+d)^2 + z^2\right)^{1/2}} \quad [\text{V}]$$

Now, we calculate the electric field intensity at point P as $\mathbf{E} = -\nabla V$:

$$\mathbf{E}(x,y,z) = -\hat{\mathbf{x}}\frac{\partial V(x,y,z))}{\partial x} - \hat{\mathbf{y}}\frac{\partial V(x,y,z))}{\partial y} - \hat{\mathbf{z}}\frac{\partial V(x,y,z))}{\partial z}$$

$$= \frac{a^3\rho_v}{3\varepsilon_0}\left[-\frac{\hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,(y-d) + \hat{\mathbf{z}}\,z}{\left(x^2 + (y-d)^2 + z^2\right)^{3/2}} + \frac{\hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,(y+d) + \hat{\mathbf{z}}\,z}{\left(x^2 + (y+d)^2 + z^2\right)^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

To find the electric field intensity on the ground plane, we set y = 0:

$$\mathbf{E}(x,0,z) = \hat{\mathbf{y}}\,\frac{2a^3\rho_v}{3\varepsilon_0}\left[\frac{d}{\left(x^2 + d^2 + z^2\right)^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right].$$

(b) From the general expression of the electric field intensity on the ground plane, the intensity is largest at $z = x = 0$ because the denominator is then smallest. The maximum electric field intensity occurs at (0,0,0), exactly under the center of the cloud:

$$\mathbf{E}_{\max} = \mathbf{E}(0,0,0) = \hat{\mathbf{y}}\,\frac{2a^3\rho_v}{3\varepsilon_0 d^2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

This is the same as the electric field intensity midway between two point charges, each of magnitude $4\pi a^3\rho_v/3\varepsilon_0$, separated a distance $2d$ apart and of opposite signs.

For the values given here, the maximum electric field intensity at ground level is

$$\mathbf{E}(0,0,0) = \hat{\mathbf{y}}\,1.885\times 10^{13}\rho_v \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right].$$

Exercise 5.2 In **Example 5.4**:

(a) Calculate the charge density on the ground.
(b) Find the total charge on the ground and show it is equal to the total charge in the cloud but of opposite sign.

Answer **(a)** $\rho_s = \dfrac{2a^3\rho_v}{3}\left[\dfrac{d}{(x^2+d^2+z^2)^{3/2}}\right] \quad \left[\dfrac{\mathrm{C}}{\mathrm{m}^2}\right]$, **(b)** $Q = \dfrac{4\pi a^3\rho_v}{3} \quad [\mathrm{C}]$.

Example 5.5 Charge in Front of a Right-Angle Conductor A point charge q [C] is located in front of an infinitely long, right-angle conductor shown in **Figure 5.10a** in cross section. Calculate:

(a) The electric field intensity everywhere in the plane of the charge ($z = 0$).
(b) The charge density induced on the surfaces of the conductor in the same plane as in part **(a)**.

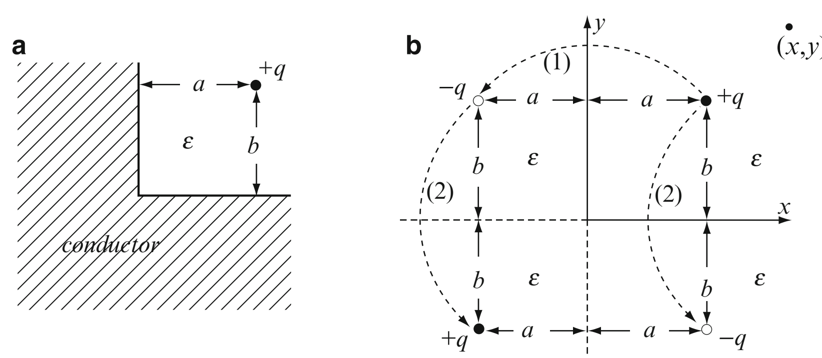


Figure 5.10 (**a**) Point charge in front of a conducting corner. (**b**) Replacement of conducting surfaces by equivalent image charges. The arrowed lines show the sequence of generating the image charges

Solution: First, it is necessary to find the system of images that guarantees zero potential on the planes corresponding to the conductor surface after the conductor is removed. This is shown in **Figure 5.10b**. From this, the potential at a general point in space is calculated, and then the electric field intensity is calculated as $\mathbf{E} = -\nabla V$. The charge density is calculated using the expression $\rho_s = \varepsilon E_n$ at the surface (i.e., at $x = 0$ for the vertical surface or $y = 0$ for the horizontal surface):

(a) The system of charges is found by assuming two perpendicular planes, coinciding with the $x = 0$ and $y = 0$ planes. The image charges shown in **Figure 5.10b** are found by reflecting the charge first about the $x = 0$ plane. This gives an image charge at $(-a,b)$. Then, these two charges are reflected about the $y = 0$ plane to obtain the charges shown in **Figure 5.10b**.

The potential at (x,y) due to the four charges is

$$V(x,y) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\left((x-a)^2+(y-b)^2\right)^{1/2}} - \frac{1}{\left((x-a)^2+(y+b)^2\right)^{1/2}} + \frac{1}{\left((x+a)^2+(y+b)^2\right)^{1/2}} - \frac{1}{\left((x+a)^2+(y-b)^2\right)^{1/2}}\right] \quad [\mathrm{V}]$$

From this, we calculate **E** as

$$\mathbf{E}(x,y) = -\hat{\mathbf{x}}\frac{\partial V(x,y)}{\partial x} - \hat{\mathbf{y}}\frac{\partial V(x,y)}{\partial y}$$

$$= \hat{\mathbf{x}}\frac{q}{4\pi\varepsilon}\left[\frac{x-a}{\left((x-a)^2+(y-b)^2\right)^{3/2}} - \frac{x-a}{\left((x-a)^2+(y+b)^2\right)^{3/2}} + \frac{x+a}{\left((x+a)^2+(y+b)^2\right)^{3/2}} - \frac{x+a}{\left((x+a)^2+(y-b)^2\right)^{3/2}}\right]$$

$$+\hat{\mathbf{y}}\frac{q}{4\pi\varepsilon}\left[\frac{y-b}{\left((x-a)^2+(y-b)^2\right)^{3/2}} - \frac{y+b}{\left((x-a)^2+(y+b)^2\right)^{3/2}} + \frac{y+b}{\left((x+a)^2+(y+b)^2\right)^{3/2}} - \frac{y-b}{\left((x+a)^2+(y-b)^2\right)^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right].$$

(b) To calculate the charge density on the vertical surface, we set $x = 0$ in the expression of the electric field. This gives $\mathbf{E}(0,y)$. Collecting terms, we get

$$\mathbf{E}(0,y) = -\hat{\mathbf{x}}\frac{qa}{2\pi\varepsilon}\left[\frac{1}{\left(a^2+(y-b)^2\right)^{3/2}} - \frac{1}{\left(a^2+(y+b)^2\right)^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

Note that the y component of the field cancels. $E(0,y)$ is the normal component to the vertical surface. Therefore, the charge density on this surface is

$$\rho_s(0,\text{y}) = \varepsilon E(0,y) = -\frac{qa}{2\pi}\left[\frac{1}{\left(a^2+(y-b)^2\right)^{3/2}} - \frac{1}{\left(a^2+(y+b)^2\right)^{3/2}}\right] \quad \left[\frac{\text{C}}{\text{m}^2}\right]$$

The charge density is zero at $y = 0$ and at $y = \infty$. It is maximum at $y = b$ and is negative everywhere since the first term is always larger than the second term in square brackets.

The charge density on the horizontal surface is calculated in exactly the same way, but setting $y = 0$ in the general expression for **E**. After simplification, we get

$$\mathbf{E}(x,0) = -\hat{\mathbf{y}}\frac{qb}{2\pi\varepsilon}\left[\frac{1}{\left((x-a)^2+b^2\right)^{3/2}} - \frac{1}{\left((x+a)^2+b^2\right)^{3/2}}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The electric field intensity on the horizontal surface is normal to the surface. The charge density on this surface is

$$\rho_s(x,0) = -\frac{qb}{2\pi}\left[\frac{1}{\left((x-a)^2+b^2\right)^{3/2}} - \frac{1}{\left((x+a)^2+b^2\right)^{3/2}}\right] \quad \left[\frac{\text{C}}{\text{m}^2}\right]$$

The charge density is zero at $x = 0$ and at $x = \infty$ and is maximum at $x = a$. A plot of the charge distribution is shown in **Figure 5.11a** and for the electric field intensity in **Figure 5.11b**.

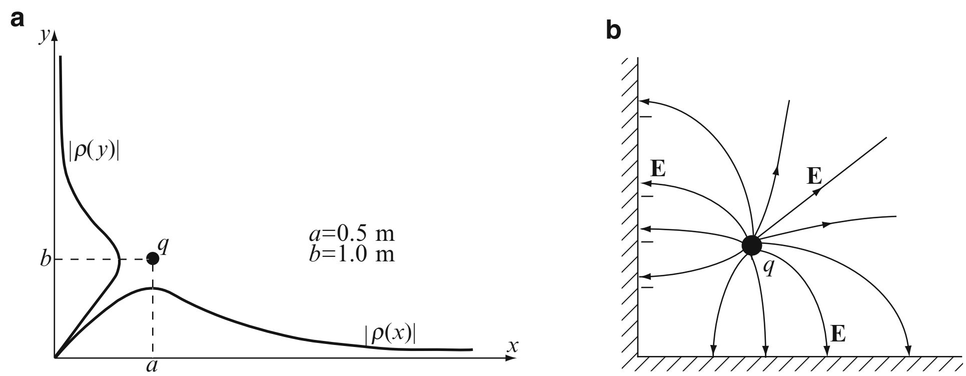


Figure 5.11 (**a**) Plot of the charge distribution on the surfaces of the conductor in **Figure 5.10a**. (**b**) Plot of the electric field intensity in **Figure 5.10a**

Exercise 5.3 Repeat **Example 5.5** and calculate the electric field intensity everywhere in space (not only on the x–y plane) and the charge density on the conducting surfaces as it depends on x, y, and z. Assume the charge q is placed at $(a,b,0)$.

Answer

$$\mathbf{E}(x,y,z) = \hat{\mathbf{x}}\frac{q}{4\pi\varepsilon}\left[\frac{x-a}{\left((x-a)^2+(y-b)^2+z^2\right)^{3/2}} - \frac{x-a}{\left((x-a)^2+(y+b)^2+z^2\right)^{3/2}} + \frac{x+a}{\left((x+a)^2+(y+b)^2+z^2\right)^{3/2}} - \frac{x+a}{\left((x+a)^2+(y-b)^2+z^2\right)^{3/2}}\right]$$
$$+\hat{\mathbf{y}}\frac{q}{4\pi\varepsilon}\left[\frac{y-b}{\left((x-a)^2+(y-b)^2+z^2\right)^{3/2}} - \frac{y+b}{\left((x-a)^2+(y+b)^2+z^2\right)^{3/2}} + \frac{y+b}{\left((x+a)^2+(y+b)^2+z^2\right)^{3/2}} - \frac{y-b}{\left((x+a)^2+(y-b)^2+z^2\right)^{3/2}}\right]$$
$$+\hat{\mathbf{z}}\frac{q}{4\pi\varepsilon}\left[\frac{z}{\left((x-a)^2+(y-b)^2+z^2\right)^{3/2}} - \frac{z}{\left((x-a)^2+(y+b)^2+z^2\right)^{3/2}} + \frac{z}{\left((x+a)^2+(y+b)^2+z^2\right)^{3/2}} - \frac{z}{\left((x+a)^2+(y-b)^2+z^2\right)^{3/2}}\right] \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

$$\rho_s(0,y,z) = -\frac{qa}{2\pi}\left[\frac{1}{\left(a^2+(y-b)^2+z^2\right)^{3/2}} - \frac{1}{\left(a^2+(y+b)^2+z^2\right)^{3/2}}\right] \quad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right],$$

$$\rho_s(x,0,z) = -\frac{qb}{2\pi}\left[\frac{1}{\left((x-a)^2+b^2+z^2\right)^{3/2}} - \frac{1}{\left((x+a)^2+b^2+z^2\right)^{3/2}}\right] \quad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right].$$

Example 5.6 Two Surfaces at Any Angle Point_Charges.m

Two conducting planes intersect at an angle smaller than 180°. Both surfaces are at zero potential. A point charge q [C] is placed midway between the two planes at a distance a [m] from the intersection, as shown in **Figure 5.12a**:

(a) Show the image charges.
(b) How many charges are needed for an angle that is an integer divisor of 180°?
(c) How many charges are needed for an angle that is a non-integer divisor of 180°?
(d) What happens if the charge is not midway between the two planes?

Solution: An image charge is created for any charge reflecting in a conducting surface. Multiple surfaces generate multiple charges. In the case of two surfaces, an infinite number of charges are always generated. The number of charges is finite only when multiple images happen to fall on each other. If multiple reflections at the same location are of the same sign, the problem has an image solution. If not, the problem cannot be solved using the method of images:

(a) To find the images, reflect the charge in each surface in turn. In the first step, charge q is reflected about the upper and lower surfaces, generating image charges $q_1' = -q$ and $q_1'' = -q$ as shown in **Figure 5.12b**. Now, charge q_1' is reflected by the upper surface, generating charge $q_2' = +q$, and charge q_1'' is reflected about the lower surface, generating image charge $q_2'' = +q$. Note that the two planes are extended to show that the lines connecting the charge with its image always intersect the plane, or its extension, at right angles. The sequence now is repeated indefinitely: charge $+q_2'$ is reflected about the upper surface to generate image charge $q_3' = -q$ and charge $+q_2''$ reflects about the lower surface to generate image charge $q_3'' = -q$, and so on. Note that all image charges as well as the original charge are located on a circle of radius a. Careful drawing of the charges should convince you of that. The example in **Figure 5.12b** was done for a 60° angle between the plates. Therefore, the last image charge is q_3''. For other angles, the number of charges would be different as discussed in parts **(b)**, **(c)**, and **(d)** of this example.

(b) In part **(a)**, we used a 60° angle between the plates. Reflection of the charge generated a sequence of image charges until we reached $q_3' = -q$ and $q_3'' = -q$. These two images fell on the same point, and since they are both of the same sign, this completes the generation of images since continuing the reflection process does not generate new images. The number of image charges is 5. The divisor for 60° is $n = 180/60 = 3$. Thus, the number of image charges is $2n - 1 = 5$. For any integer divisor of 180°, n, there are $2n - 1$ images, for a total of $2n$ charges necessary to produce zero potential on planes A and B.

(c) If the angle is not an integer divisor of 180°, the image charges cannot satisfy the zero potential condition on the two planes. One of two situations can happen: either the number of charges is infinite (including charges of opposite sign falling on top of each other) and therefore the sequence of generating the charges cannot be completed or the image charges eventually fall on the plates or their extensions. In either case, the zero potential condition cannot be satisfied. As an example, consider $n = 1.5$. In this case, the angle between the plates is 120°. The first two images are the reflection about plates A and B in **Figure 5.12c** and are shown as q_1' and q_1''. These images are negative and lie on the extension of the planes themselves, and therefore, the potential on each of the two planes varies with position on the planes, contradicting the requirement of constant potential. Therefore, the problem cannot be solved using the method of images.

(d) If the angle is an integer divisor of 180°, the answer is as in **(b)**, but the charges are nonuniformly spaced on the circle (see **Exercise 5.4**). If it is a non-integer divisor, the answer is as in **(c)**.

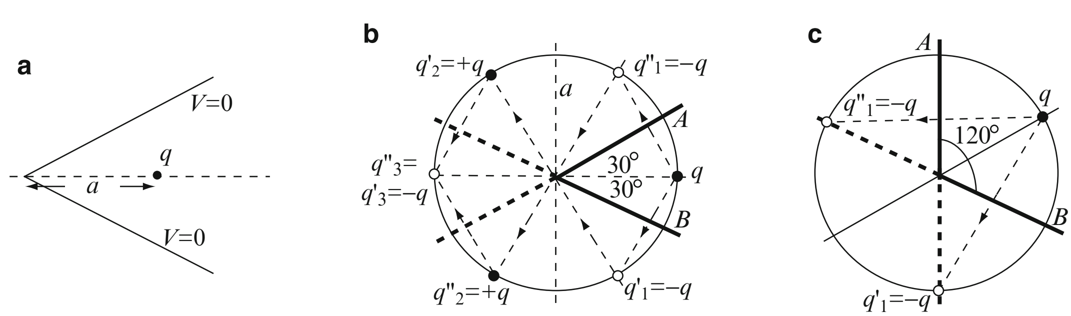


Figure 5.12 (**a**) Point charge in front of two intersecting conducting planes. (**b**) System of images equivalent to (**a**). (**c**) System of images for a 120° angle between the planes

Exercise 5.4 Find the location and number of image charges if the charge in **Figure 5.12a** is placed at a distance d [m] from the upper plate and a distance b [m] from the lower plate while the angle between the plates is 30°.

Answer 11 image charges. See **Figure 5.13** for location of the charges.

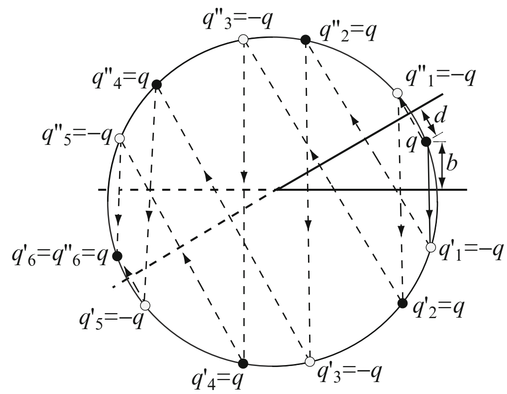


Figure 5.13 Location of image charges for **Exercise 5.4**

Exercise 5.5

(a) Find the electric potential at a general point in the x–y plane for a charge placed as in **Figure 5.14** with the angle between the plates equal to 60°. Assume the intersection between the plates occurs at $x = 0$, $y = 0$, the upper plane coincides with the y axis, and the charge q [C] is on the bisector (at 30° from the y axis), at a distance a [m] from the intersection of the plates.

(b) Find the electric field intensity on the vertical plate in the x–y plane.

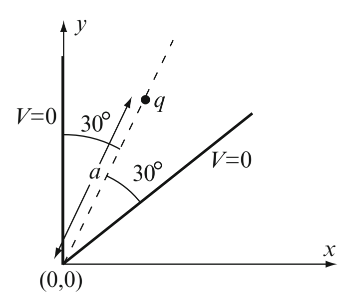


Figure 5.14 Configuration for **Exercise 5.5**

Answer

$$V(x,y)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a/2)^2+\left(y-a\sqrt{3}/2\right)^2}}-\frac{1}{\sqrt{(x+a/2)^2+\left(y-a\sqrt{3}/2\right)^2}}-\frac{1}{\sqrt{(x-a)^2+y^2}}+\frac{1}{\sqrt{(x+a)^2+y^2}}\right.$$
$$\left.+\frac{1}{\sqrt{(x-a/2)^2+\left(y+a\sqrt{3}/2\right)^2}}-\frac{1}{\sqrt{(x+a/2)^2+\left(y+a\sqrt{3}/2\right)^2}}\right]\quad[\mathrm{V}]$$

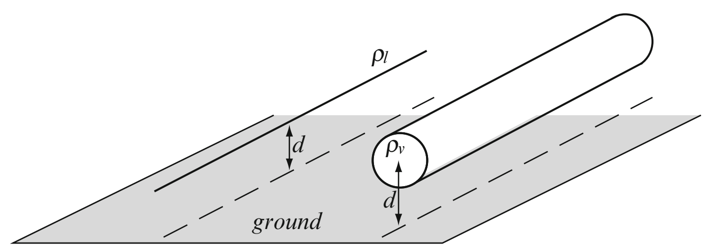
$$\mathbf{E}(0,y)=\hat{\mathbf{x}}\frac{qa}{4\pi\varepsilon_0}\left[\frac{2}{\left(a^2+y^2\right)^{3/2}}-\frac{1}{\left(a^2/4+\left(y-a\sqrt{3}/2\right)^2\right)^{3/2}}-\frac{1}{\left(a^2/4+\left(y+a\sqrt{3}/2\right)^2\right)^{3/2}}\right]\quad\left[\frac{\mathrm{V}}{\mathrm{m}}\right].$$

5.4.3.2 Charged Line over a Conducting Plane

A problem similar to that of a point charge over a plane is that of a charged line over a plane. In fact, if the charged conductor is long, we may be able to assume it is infinite and the problem becomes a two-dimensional problem (the electric field intensity or potential does not vary along the length of the conductor). A physical application of this type is a cable or a pair of wires over a conducting ground or a conducting strip over a conducting plane in a printed circuit board.

The basic configurations we consider are shown in **Figure 5.15**. The thin line is analyzed here, whereas the thick and finite-length lines are shown in the examples. For ease of solution, we will again assume zero potential on the ground (a fixed potential can always be added as a reference potential). The problem described here was once proposed as a solution to telephone communication. In the 1830s, when all telephone communication required copper wires, it was proposed to use a single wire for each telephone line and use the ground as a common, return conductor.[4] The idea was eventually abandoned, primarily because of "noise" in the system (the sources of which will be understood in later chapters) and the difficulty of making good ground connections, but was used for a while as a practical system.

Figure 5.15 Thin and thick charged lines over a conducting plane



As with the point charge in **Figure 5.5**, the conducting plane is removed and replaced with a constant potential plane and an image, equal in magnitude, shape, and dimensions and opposite in sign, placed at a distance d below the constant potential surface, as shown in **Figure 5.16a**. Now, the solution is obtained as the solution due to two line charges, one positive and one negative, at a distance $2d$ apart. Again, we can choose to use the potential or the electric field intensity as solution variables.

[4] The idea of using the ground as the return conductor is sometimes attributed to Joseph Henry who is known to have used a system of this type in the early 1830s to communicate from his home to his laboratory on the Campus of Albany Academy in Albany, NY. During the same period, and apparently preceding Henry, C.A. Steinhil produced the same type of telegraph and installed working devices in Munich, Germany.

In this case, because the lines are long, it is more convenient to use the electric field intensity since we can apply Gauss's law to each line separately and add the solutions together using superposition. With the dimensions in **Figure 5.16a**, we calculate the electric field intensity at point $P(x, y)$ as if the lower (negative) line does not exist. The electric field intensity is constant on a circle of radius r_1 and in the direction of r_1. Using Gauss's law we can write for the electric field intensity at a distance r_1 from the positive line,

$$\int_s \mathbf{E} \cdot d\mathbf{s} = E 2\pi r_1 L = \frac{\rho_1 L}{\varepsilon_0} \tag{5.20}$$

where L is an arbitrary length of the line (see **Chapter 4**), $\rho_l L$ is the total charge on a length L of the line, and $2\pi r_1 L$ is the cylindrical surface of the cylinder of length L. Thus, the electric field intensity (considering the direction r_1) is

$$\mathbf{E}^+ = \hat{\mathbf{r}}_1 \frac{\rho_l}{2\pi r_1 \varepsilon_0} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.21}$$

Similarly, for the lower line of charge, the direction is in the $-\hat{\mathbf{r}}_2$ direction:

$$\mathbf{E}^- = -\hat{\mathbf{r}}_2 \frac{\rho_l}{2\pi r_2 \varepsilon_0} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.22}$$

The total electric field intensity is the sum of the two electric fields:

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{\hat{\mathbf{r}}_1}{r_1} - \frac{\hat{\mathbf{r}}_2}{r_2}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.23}$$

A more explicit expression is obtained by substituting the position vectors $\mathbf{r}_1 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}(y - d)$ and $\mathbf{r}_2 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}(y + d)$. Writing the magnitudes of the vectors as $r_1 = \sqrt{x^2 + (y-d)^2}$ and $r_2 = \sqrt{x^2 + (y+d)^2}$ and the unit vectors as $\hat{\mathbf{r}}_1 = \mathbf{r}_1/r_1$ and $\hat{\mathbf{r}}_2 = \mathbf{r}_2/r_2$, the electric field intensity in **Eq. (5.23)** becomes

$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y-d)}{x^2 + (y-d)^2} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(y+d)}{x^2 + (y+d)^2}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.24}$$

To check that this solution is correct, we note that the electric field intensity tends to zero at infinity (either in x or y or both). Also, at $y = 0$, we get

$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}(-d)}{x^2 + d^2} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}\,d}{x^2 + d^2}\right) = -\hat{\mathbf{y}}\frac{\rho_l d}{\pi\varepsilon_0}\frac{1}{(x^2 + d^2)} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{5.25}$$

Thus, on the conducting surface, the electric field is perpendicular, as required.

We can now easily calculate the electric potential everywhere in space and the charge density on the surface of the conductor as for the point charge over a plane. The charge density on the surface of the conductor is again negative (because the source charge, i.e., the line over the plane in **Figure 5.16a**, is positive). The charge density is

$$\rho_s = \varepsilon_0 E_n = -\frac{\rho_l d}{\pi}\frac{1}{(x^2 + d^2)} \quad \left[\frac{\text{C}}{\text{m}^2}\right] \tag{5.26}$$

where the fact that at the surface there is only a normal component of the electric field intensity in the negative y direction was used. Note that the charge density only varies with the x dimension. This is a consequence of the fact that the line is assumed to be infinitely long. We must also remember that ρ_l is a line charge density [C/m] and ρ_s is a charge per unit area [C/m^2]. If we integrate the surface charge density on the conducting surface from $x = -\infty$ to $x = +\infty$, the result is $-\rho_l$; that is,

$$-\int_{x=-\infty}^{x=+\infty} \frac{\rho_l d}{\pi}\frac{1}{(x^2 + d^2)}dx = -\frac{\rho_l}{\pi}\tan^{-1}\frac{x}{d}\bigg|_{-\infty}^{+\infty} = \frac{\rho_l}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = -\rho_l \quad \left[\frac{\text{C}}{\text{m}}\right] \tag{5.27}$$

The negative sign indicates that the charge per unit length of the conducting surface is the opposite of the charge on the line. The solution now is as shown in **Figure 5.16b**, where the electric field distribution above the plane is that obtained in **Eq. (5.24)** and the surface charge density is that in **Eq. (5.26)**. The solution below the surface (in the conductor) is zero, as required, since all electric field lines that start on the line of charge end on the surface of the conductor. Note the similarity of this solution to that in **Figure 5.5c**. However, the two solutions are not the same; the surface charge distributions on the conductor are completely different.

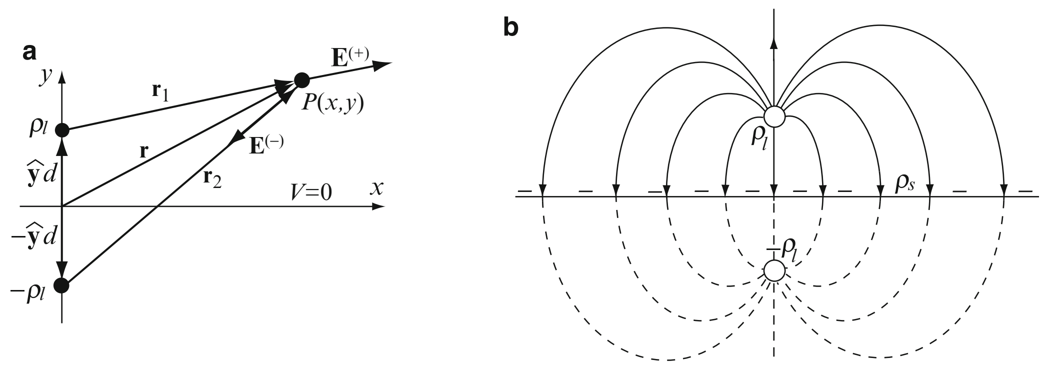


Figure 5.16 (**a**) The charged line and its image with the conducting plane removed. (**b**) The electric field intensity due to a charged line over a conducting plane

Example 5.7 Overhead Transmission Lines Two overhead wires carry line charge densities as shown. Find the electric field intensity at point P (magnitude and direction) in **Figure 5.17a**.

Solution: The method of images may be used to remove the conducting surface and replace its effect with image lines. Now, there are four charged lines as shown in **Figure 5.17b**. Calculate the x and y components of the electric field intensity at P due to the four lines.

The electric field intensities of the four lines are

$$\mathbf{E}_1 = \frac{\hat{\mathbf{r}}_1 \rho_l}{2\pi r_1 \varepsilon_0}, \quad \mathbf{E}_2 = -\frac{\hat{\mathbf{r}}_2 \rho_l}{2\pi r_2 \varepsilon_0}, \quad \mathbf{E}_3 = \frac{\hat{\mathbf{r}}_3 \rho_l}{2\pi r_3 \varepsilon_0}, \quad \mathbf{E}_4 = -\frac{\hat{\mathbf{r}}_4 \rho_l}{2\pi r_4 \varepsilon_0} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

where

$$r_1 = \sqrt{c^2 + (a-b)^2}, \quad r_2 = c - a, \quad r_3 = \sqrt{(c-a)^2 + 4a^2}, \quad r_4 = \sqrt{(a+b)^2 + c^2}$$

Also

$$E_{1x} = E_1 \cos\alpha_1 = E_1 \frac{c}{r_1}, \qquad E_{1y} = E_1 \sin\alpha_1 = E_1 \frac{b-a}{r_1} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$E_{2x} = E_2, \quad E_{2y} = 0$$

$$E_{3x} = E_3 \cos\alpha_3 = E_3 \frac{c-a}{r_3}, \qquad E_{3y} = E_3 \sin\alpha_3 = E_3 \frac{2a}{r_3} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$E_{4x} = E_4 \cos\alpha_4 = E_4 \frac{c}{r_4}, \qquad E_{4y} = E_4 \sin\alpha_4 = E_4 \frac{2b+a}{r_4} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The x and y components are

$$E_x = E_{1x} + E_{3x} - E_{2x} - E_{4x}, \quad E_y = -E_{1y} + E_{3y} - E_{4y}$$

Substituting the values of r_i in the expressions for the electric field intensities and separating them into their x and y components, we get

$$E_x = \frac{\rho_l}{2\pi\varepsilon_0}\left[\frac{c}{c^2+(a-b)^2} + \frac{c-a}{(c-a)^2+4a^2} - \frac{1}{c-a} - \frac{c}{(a+b)^2+c^2}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$E_y = \frac{\rho_l}{2\pi\varepsilon_0}\left[-\frac{b-a}{c^2+(a-b)^2} + \frac{2a}{(c-a)^2+4a^2} - \frac{b+a}{(a+b)^2+c^2}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The electric field intensity may now be written as $\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y$ [V/m].

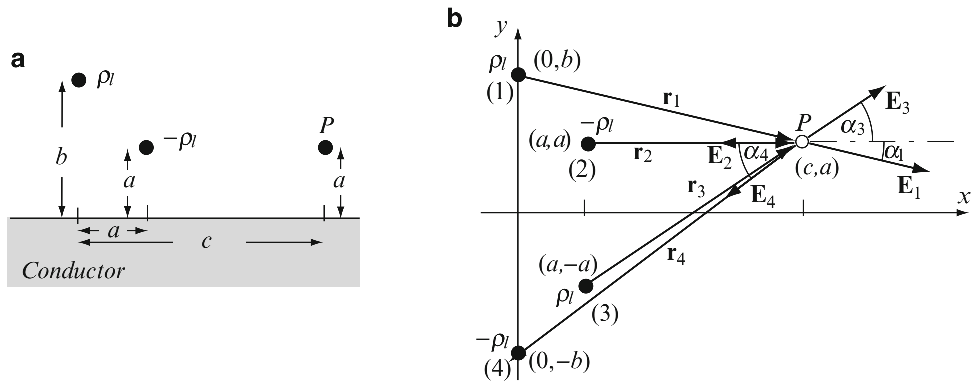


Figure 5.17 (**a**) Two charged lines above a conducting plane. (**b**) The system of images necessary to calculate the electric field in (**a**)

Example 5.8 Application: Thick Cylinder Above a Conducting Surface A very long cylinder (you may assume it is infinitely long) of radius a [m] made of a perfect dielectric has a charge density uniformly distributed on its surface equal to ρ_s [C/m^2]. The cylinder is located at a height d [m] above a very thick conductor, as shown in **Figure 5.18a**. Calculate the electric field intensity everywhere in space assuming the presence of the conductor does not disturb the charges on the surface of the cylinder. Assume permittivity of the dielectric is ε_0 [F/m].

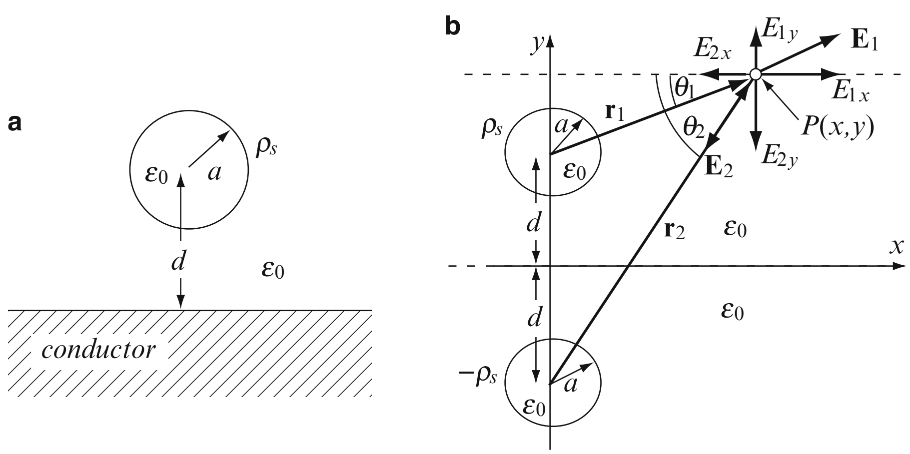


Figure 5.18 (**a**) A charged dielectric cylinder over a conducting surface. (**b**) The cylinder in (**a**) and its image with the conducting surface removed

Solution: The conductor is replaced by an image cylinder with equal and negative surface charge density. The cylinders are nonconducting; therefore, the charge density remains on the surface and remains uniformly distributed. The electric field intensity of each cylinder is calculated separately and the results superimposed to obtain the solution.

Refer to **Figure 5.18b** for the images and coordinates. The electric fields are separated into the x and y components as shown. The electric field intensities are

$$\mathbf{E}_1 = \hat{\mathbf{r}}_1 \frac{a\rho_s}{\varepsilon_0 r_1}, \quad \mathbf{E}_2 = -\hat{\mathbf{r}}_2 \frac{a\rho_s}{\varepsilon_0 r_2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

where a Gaussian surface for each cylinder was used. r_1 is the distance between the center of the positively charged cylinder to point $P(x,y)$ and r_2 is the distance from the center of the negatively charged cylinder to point $P(x,y)$. The x and y components of the electric field intensity are

$$E_{1x} = \frac{a\rho_s}{\varepsilon_0 r_1}\cos\theta_1 = \frac{a\rho_s x}{\varepsilon_0 r_1^2}, \quad E_{2x} = -\frac{a\rho_s}{\varepsilon_0 r_2}\cos\theta_2 = -\frac{a\rho_s x}{\varepsilon_0 r_2^2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$E_{1y} = \frac{a\rho_s}{\varepsilon_0 r_1}\sin\theta_1 = \frac{a\rho_s(y-d)}{\varepsilon_0 r_1^2}, \quad E_{2y} = -\frac{a\rho_s}{\varepsilon_0 r_2}\sin\theta_2 = -\frac{a\rho_s(y+d)}{\varepsilon_0 r_2^2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

where r_1 and r_2 are

$$r_1 = \sqrt{x^2 + (y-d)^2}, \quad r_2 = \sqrt{x^2 + (y+d)^2}$$

The solution anywhere outside the cylinders is

$$E_x = E_{1x} + E_{2x} = \frac{a\rho_s x}{\varepsilon_0}\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right], \quad E_y = E_{1y} + E_{2y} = \frac{a\rho_s}{\varepsilon_0}\left[\frac{y-d}{r_1^2} - \frac{y+d}{r_2^2}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y = \hat{\mathbf{x}}\frac{a\rho_s x}{\varepsilon_0}\left[\frac{1}{x^2+(y-d)^2} - \frac{1}{x^2+(y+d)^2}\right] + \hat{\mathbf{y}}\frac{a\rho_s}{\varepsilon_0}\left[\frac{y-d}{x^2+(y-d)^2} - \frac{y+d}{x^2+(y+d)^2}\right] \quad \left[\frac{\text{V}}{\text{m}}\right]$$

This solution is valid everywhere above ground ($y > 0$) except inside the upper cylinder. Inside the upper cylinder, the field due to itself is zero (because the charge is only distributed on the surface). The solution inside the upper cylinder is

$$E_x = E_{2x} = -\frac{1}{r_2^2}\frac{a\rho_s x}{\varepsilon_0}, \quad E_y = E_{2y} = -\frac{a\rho_s}{\varepsilon_0}\frac{(y+d)}{r_2^2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$\mathbf{E} = -\hat{\mathbf{x}}\frac{a\rho_s x}{\varepsilon_0\left(x^2+(y+d)^2\right)} - \hat{\mathbf{y}}\frac{a\rho_s}{\varepsilon_0}\frac{y+d}{\left(x^2+(y+d)^2\right)} \quad \left[\frac{\text{V}}{\text{m}}\right].$$

Note: The electric field intensity below ground ($y < 0$) is zero.

Example 5.9 Short Line Segment over a Conducting Plane A line of charge with uniform charge density ρ_l [C/m] and length d [m] is placed at a height h [m] above a thick conducting plane, as shown in **Figure 5.19a**:

(a) Calculate the electric potential everywhere in space.
(b) Calculate the electric field intensity at the surface of the conducting plane, below the center of the wire.

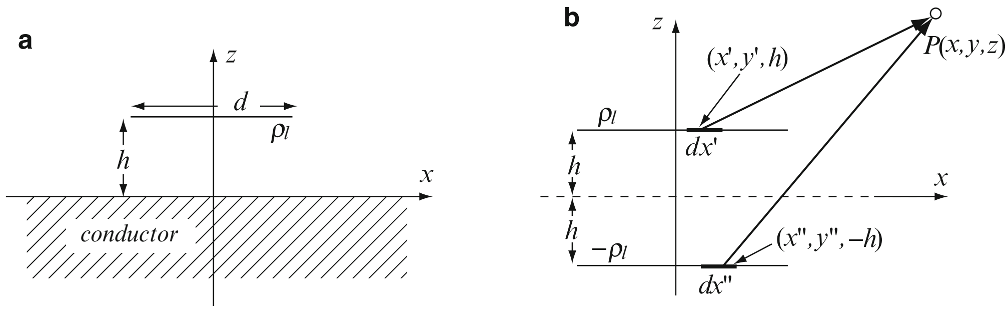


Figure 5.19 (**a**) A short, charged line segment over a conducting plane. (**b**) The line segment and its image with the conducting plane removed

Solution: The conductor is replaced by an image of the line segment as in **Figure 5.19b**. The line segments are placed in the x–z plane. The solution now is that of two short wires with opposite charge densities as shown in **Figure 5.19b**. Using $dl' = dx'$ for the upper wire and $dl'' = dx''$ for the lower wire, a point $(x',0,h)$ on the upper wire and a point $(x'',0,-h)$ on the lower wire, the potential at a general point (x,y,z) is

(a) Due to the upper wire:

$$V_P^+ = \frac{1}{4\pi\varepsilon_0}\int_{x'=-d/2}^{x'=d/2} \frac{\rho_l dx'}{\left[(x-x')^2+(y-0)^2+(z-h)^2\right]^{1/2}} \quad [\text{V}]$$

Due to the lower wire:

$$V_P^- = -\frac{1}{4\pi\varepsilon}\int_{x''=-d/2}^{x''=d/2} \frac{\rho_l dx''}{\left[(x-x'')^2+(y-0)^2+(z+h)^2\right]^{1/2}} \quad [\text{V}]$$

The total potential is

$$V_P = V_P^+ + V_P^- = \frac{\rho_l}{4\pi\varepsilon_0}\left[\int_{x'=-d/2}^{x'=d/2} \frac{dx'}{\left[(x-x')^2+(y-0)^2+(z-h)^2\right]^{1/2}} - \int_{x''=-d/2}^{x''=d/2} \frac{dx''}{\left[(x-x'')^2+(y-0)^2+(z+h)^2\right]^{1/2}}\right.$$

$$= \frac{\rho_l}{4\pi\varepsilon_0}\ln\left[2\sqrt{(x-x')^2+y^2+(z-h)^2}+2x'-2x\right]_{x'=-d/2}^{x'=d/2} - \frac{\rho_l}{4\pi\varepsilon_0}\ln\left[2\sqrt{(x-x'')^2+y^2+(z+h)^2}+2x''-2x\right]_{x''=-d/2}^{x''=d/2}$$

$$= \frac{\rho_l}{4\pi\varepsilon_0}\ln\frac{\left[2\sqrt{(x-d/2)^2+y^2+(z-h)^2}+d-2x\right]}{\left[2\sqrt{(x+d/2)^2+y^2+(z-h)^2}-d-2x\right]} + \frac{\rho_l}{4\pi\varepsilon_0}\ln\frac{\left[2\sqrt{(x+d/2)^2+y^2+(z+h)^2}-d-2x\right]}{\left[2\sqrt{(x-d/2)^2+y^2+(z+h)^2}+d-2x\right]} \quad [\text{V}]$$

Note that $y' = 0$ and $y'' = 0$ since the two wires are in the xz plane. However, $y \neq 0$ because we seek a solution at any point in space.

(b) The electric field intensity is required at $x = 0$, $y = 0$, $z = 0$. At this point, the electric field intensity points in the negative z direction; therefore, there is no need to calculate the x and y components of $\mathbf{E}$. The z component of the electric field intensity at a general point in space is

$$E_z(x,y,z) = -\frac{\partial V_P(x,y,z)}{\partial z} = -\frac{\rho_l\left[2(z-h)\left((x-d/2)^2+y^2+(z-h)^2\right)^{-1/2}\right]}{4\pi\varepsilon_0\left[2\sqrt{(x-d/2)^2+y^2+(z-h)^2}+d-2x\right]}$$
$$+\frac{\rho_l\left[2(z-h)\left((x+d/2)^2+y^2+(z-h)^2\right)^{-1/2}\right]}{4\pi\varepsilon_0\left[2\sqrt{(x+d/2)^2+y^2+(z-h)^2}-d-2x\right]}-\frac{\rho_l\left[2(z+h)\left((x+d/2)^2+y^2+(z+h)^2\right)^{-1/2}\right]}{4\pi\varepsilon_0\left[2\sqrt{(x+d/2)^2+y^2+(z+h)^2}-d-2x\right]}$$
$$+\frac{\rho_l\left[2(z+h)\left((x-d/2)^2+y^2+(z+h)^2\right)^{-1/2}\right]}{4\pi\varepsilon_0\left[2\sqrt{(x-d/2)^2+y^2+(z+h)^2}+d-2x\right]} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

At $x = 0$, $y = 0$, $z = 0$, the electric field intensity becomes

$$\mathbf{E}(0,0,0) = -\hat{\mathbf{z}}\,\frac{\rho_l h}{\pi\varepsilon_0}\left[\frac{\sqrt{d^2/4+h^2}}{2\sqrt{d^2/4+h^2}-d}-\frac{\sqrt{d^2/4+h^2}}{2\sqrt{d^2/4+h^2}+d}\right] \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right].$$

5.4.3.3 Charges Between Parallel Planes

In **Section 5.4.3.1** we saw that if there are multiple conducting planes in which charges reflect, the situation is similar to two or more mirrors, which, of course, produce multiple reflections. Similarly, if we have multiple charges, and perhaps multiple surfaces, the number of image charges may be rather high. However, the method of treating the charges and their reflections is systematic.

The method we used so far was essentially one of finding the reflections of the charges due to the conducting surfaces. Once all image charges were established, a general solution in the space between the conducting surfaces was found.

An alternative way to look at the same process is as follows: Since all conducting surfaces must be at constant potentials, assume all are at zero potential (any other potential can be added later as a reference potential). Removing the conductors, we must now add charges that will ensure that the potential on all surfaces which previously coincided with conducting surfaces is zero. Any number of charges, positive or negative and of any magnitude, can be added, provided the zero potential condition is satisfied. The location of the charges can also be adjusted if necessary. Note also that this particular way of looking at the problem does not require that conductors be flat and, in fact, they can be curved, as we will see shortly.

Example 5.10 Point Charge Between Parallel Plates A point charge is placed midway between two very large, grounded, conducting surfaces, as shown in **Figure 5.20a**. Find the potential at P.

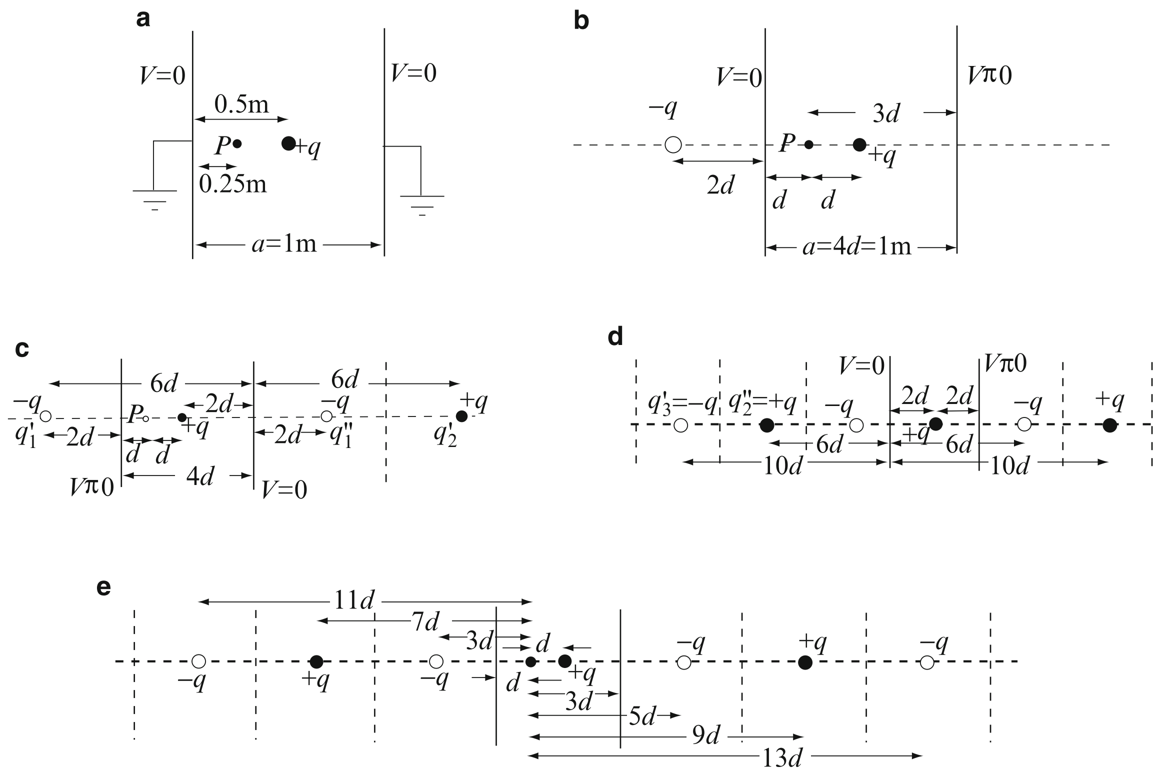


Figure 5.20 (**a**) Point charge between two infinite planes. (**b**) An image charge $q_1{}'$ about the left plane ensures zero potential on the left plane. (**c**) Reflection of q and its image $q_1{}'$ results in zero potential on the right plane. (**d**) Reflection of $q_1{}''$ and $q_2{}'$ about the left plane restores zero potential on the left plane. (**e**) "Final" system of charges required to replace the conducting planes

Solution: First, we must find the image charges that will produce a zero potential on the two plates when the plates are removed. This is done by first reflecting the charge q about the left plate to produce charge $q_1' = -q$. This produces zero potential on the left plate but not on the right plate (see **Figure 5.20b**). The charge q is then reflected by the right plate to produce image charge $q_1'' = -q$, and charge $-q_1'$ is reflected by the right plate to produce image charge $q_2' = +q$. Now, the potential is zero on the right plate, but the potential on the left plate is disturbed (see **Figure 5.20c**). The next step is to reestablish zero potential on the left plate. This is done by reflecting both charges on the right-hand side of the plates by the left plate to produce image charges $q_2'' = +q$ and $q_3' = -q$ (**Figure 5.20d**). Since there are three charges on each side of the left plane, the potential is now zero on the left plate but not on the right plate. This process repeats indefinitely, but each step takes us further away from the plates, reducing the disturbance we introduce. The net effect is shown in **Figure 5.20e**: There is an infinite number of charges, each placed at the center between two imaginary plates indicated by the dotted lines. The charges alternate in sign in both directions starting from the original charge and are all equal in magnitude.

To calculate the electric potential at point P, we use the dimensions in **Figure 5.20e** with d 0.25m and a 4d:

$$V_P = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{d} - \frac{1}{3d} - \frac{1}{5d} + \frac{1}{7d} + \frac{1}{9d} - \frac{1}{11d} - \frac{1}{13d} + \frac{1}{15d} + \frac{1}{17d} - \frac{1}{19d} - \frac{1}{21d} + \frac{1}{23d} + \frac{1}{25d} - \frac{1}{27d} - \frac{1}{29d} + \cdots \right. \quad [\mathrm{V}]$$

To evaluate this expression, we rewrite it as follows:

$$V_P = \frac{q}{4\pi\varepsilon_0 d}\left[1 - \sum_{i=1}^{N}\frac{1}{8i-5} - \sum_{i=1}^{N}\frac{1}{8i-3} + \sum_{i=1}^{N}\frac{1}{8i-1} + \sum_{i=1}^{N}\frac{1}{8i+1}\right] \quad [\mathrm{V}]$$

where $N = \infty$ for an exact solution. The latter can be computed for any finite number of terms. The accuracy of the result depends on how many charges we use. The following shows the results for $N = 1, 2$:

$$V_P = \frac{q}{4\pi\varepsilon_0 d}0.7206 = \frac{q}{4\times\pi\times 8.854\times 10^{-12}\times 0.25}0.7206 = 2.591\times 10^{10}q \quad [\text{V}], \quad N = 1$$

$$V_P = \frac{q}{4\pi\varepsilon_0 d}0.6783 = \frac{q}{4\times\pi\times 8.854\times 10^{-12}\times 0.25}0.6783 = 2.438\times 10^{10}q \quad [\text{V}], \quad N = 2$$

The following table shows the solution for a few additional values of N, up to $N = 10{,}000$.

N	*V* [V]	*Relative error* [%]
1	$2.591 \times 10^{10}q$	
2	$2.438 \times 10^{10}q$	–5.9 %
10	$2.287 \times 10^{10}q$	–6.2 %
100	$2.248 \times 10^{10}q$	–1.7 %
1,000	$2.244 \times 10^{10}q$	–0.18 %
10,000	$2.244 \times 10^{10}q$	0

After about $N = 1{,}000$, the solution changes very little and, therefore, may be regarded as an "exact" solution. Note that using $N = 10$ means a total of 43 charges. The result with $N = 10$ is less than 1.92% below the exact solution. This value is calculated as

$$\frac{V_P(N = 10{,}000) - V_P(N = 10)}{V_P(N = 10{,}000)}\times 100\% = \frac{2.244\times 10^{10}q - 2.287\times 10^{10}q}{2.244\times 10^{10}q}\times 100\% = -1.92\%$$

The relative error in the third column is calculated as follows:

$$e = \frac{V_P(\text{current}) - V_P(\text{previous})}{V_P(\text{previous})}\times 100\%$$

where "current" refers to the current row (say for $N = 100$) and "previous" to the previous row (say for $N = 10$). For the values given ($d = 0.25$ m, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m), the potential at P is $2.244 \times 10^{10}q$ [V].

Note: The results above were obtained for a specific value of d. It should be remembered that the method outlined here can be used to calculate the potential at any point P between the plates (and only between the plates).

Example 5.11 Point Charge in a Conducting Box Point_Charges.m

A point charge is placed at the center of a grounded, conducting box as shown in **Figure 5.21a**. Find the system of charges required to calculate the potential inside the box.

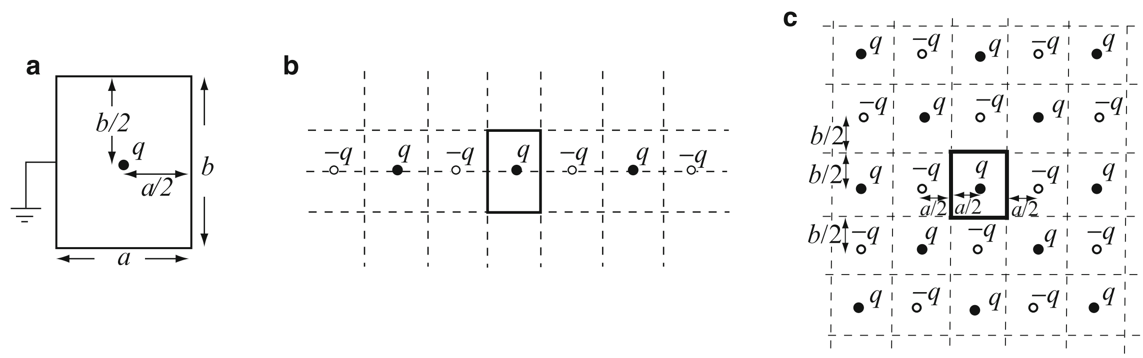


Figure 5.21 (**a**) A point charge in a conducting box. (**b**) Reflection of charges about the vertical planes. (**c**) Reflection of the charges obtained in (**b**) about the horizontal planes

Solution: The box in **Figure 5.21a** may be viewed as the intersection of two sets of parallel planes, one horizontal and one vertical. We obtained the system of charges for the vertical plates in **Figure 5.20e**. Now, we can view this system of charges as being placed at the center of the horizontal plates as shown in **Figure 5.21b**. These charges are all reflected by the two horizontal surfaces with the net result given in **Figure 5.21c**. Again, the number of charges is infinite, their signs alternate, and each is inside an imaginary box identical to the original box.

Once a satisfactory number of image charges have been identified and properly placed (a number which depends on the accuracy required), the electric field intensity or potential everywhere within the box can be calculated, although hand calculation is not usually feasible. It should also be remembered that only the solution inside the original box is valid.

5.4.3.4 Images in Curved Geometries

The solution of Poisson's equation in curved geometries for a number of important physical applications can also be carried out using the method of images. An example in which this might be important is in the calculation of the electric field and voltage distribution in an underground cable, as shown in **Figure 5.22**. The outer conducting shield is a surface at constant (zero) potential, and the voltage distribution inside the cable depends on the charges or potentials on the two conductors. This application is rather difficult to analyze by other methods, but it is relatively simple with the method of images:

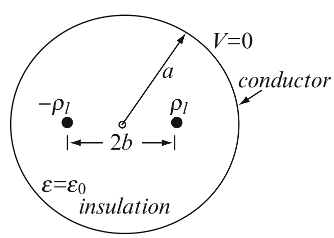


Figure 5.22 Shielded underground cable: example for use of the method of images

(1) Line Charge Outside a Conducting Cylinder. Before solving the problem in **Figure 5.22**, we will look at a simpler configuration: that of a thin line of charge in the vicinity of a cylindrical conductor.

First, we invoke the optical equivalent: A line source (such as a thin filament) reflected by a cylindrical mirror produces an image similar to that obtained by a flat mirror, but the distance between the surface of the mirror and the image is different than between the source and the surface (see **Figure 5.23a**). In fact, we know that the image must be at a location on the r axis, at a distance $0 < b < a$, simply from the optical experience. As the source approaches the conducting surface, so does the image and vice versa, but the image is always in the range given.

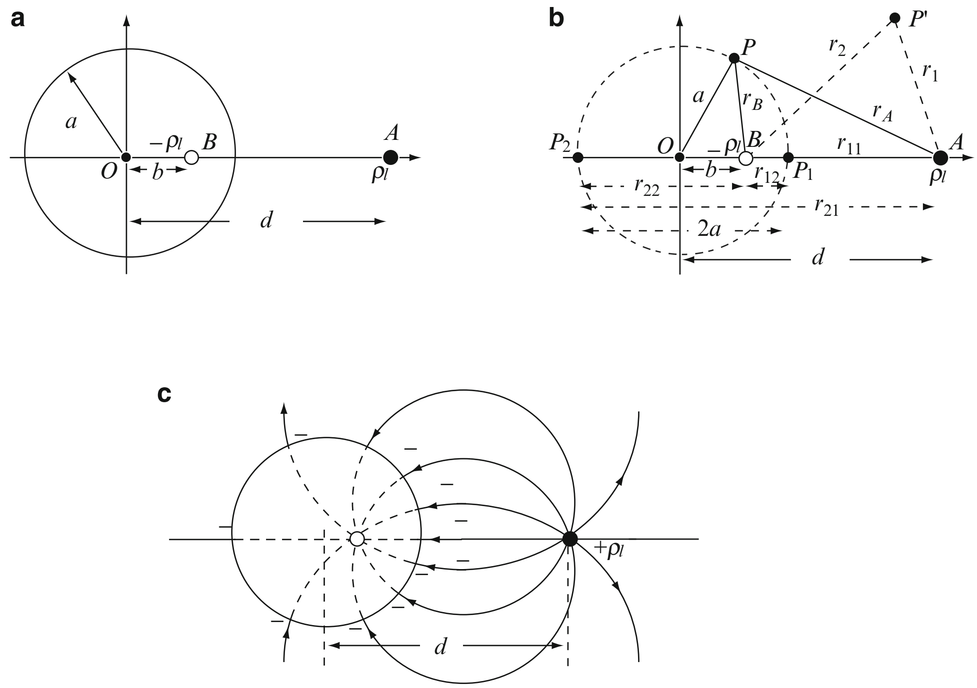


Figure 5.23 (**a**) A charged line and its image required to produce constant potential on a parallel cylindrical conductor. (**b**) Calculation of location of the image line and potential on the cylinder. (**c**) The electric field intensity outside the cylinder

To solve the problem, we must place the image at a position b such that the potential at the location of the surface (dashed circle in **Figure 5.23b**) is constant. Here, we specifically require a constant potential rather than a zero potential; the latter cannot be satisfied on a cylindrical surface with only two equal line charges but the former can. The magnitude of the image is assumed to be equal to the source but negative ($-\rho_l$ [C/m])

We wish now to calculate the potential at a general point P' in space. Once we have the general expression, we use this to calculate the potential at the location of the cylinder using the electric field intensity of the charged line at a distance r from the line. This was calculated in **Section 3.4.2 (**see **Example 3.8)** and is equal to

$$\mathbf{E} = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon_0 r} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{5.28}$$

However, we are interested in the potential. To calculate the potential difference between two points r_1 and r_0, we integrate the electric fields using the definition

$$V^+ = -\int_{r_0}^{r_1} \mathbf{E}\cdot d\mathbf{l} = -\int_{r_0}^{r_1} \frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r}\cdot d\mathbf{r} = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_1}{r_0} \qquad [\mathrm{V}] \tag{5.29}$$

This calculation gives the voltage at a distance r_1 from the positive line, with reference (zero potential) at a distance r_0 from the positive line. r_0 is, of course, arbitrary, but unlike most problems we solved in **Chapters 3** and **4**, r_0 cannot be at infinity because the line itself is infinite. We will choose this distance in a way that simplifies the solution, but it should be emphasized again that any value will do, as long as the logarithmic function can be evaluated and the potential we obtain is physically correct (thus, $r_0 = 0$, $r_0 = \infty$, and $r_0 < 0$ are not appropriate choices).

Similarly, the potential at a distance r_2 from the negative line of charge is

$$V^- = -\int_{r_0}^{r2} \frac{(-\rho_l)}{2\pi\varepsilon_0 r} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_2}{r_0} \qquad [\mathrm{V}] \tag{5.30}$$

where we have chosen the same reference point for both potentials. The total potential at point P' is

$$V_{P'} = V^+ + V^- = \frac{\rho_l}{2\pi\varepsilon_0}\left(\ln\frac{r_2}{r_0} - \ln\frac{r_1}{r_0}\right) = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1} \quad [\text{V}] \tag{5.31}$$

Note that the potential at P' is nowhere zero except for $r_2 = r_1$. This can be anywhere on the plane midway between the two lines of charge.

The potential on the cylinder cannot, in fact, be calculated unless we know the exact location of the line charges, which we do not. However, we know that whatever the potential on this surface, it must be constant. To calculate the location of the image charge, we use this property and equate the two potentials at points P_1 and P_2. Using the expression above, with r_1 and r_2 as shown in **Figure 5.23b**, we get

$$V_{P1} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_{12}}{r_{11}} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a-b}{d-a} \quad [\text{V}] \tag{5.32}$$

$$V_{P2} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_{22}}{r_{21}} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a+b}{a+d} \quad [\text{V}] \tag{5.33}$$

Equating the two gives

$$V_{P1} = V_{P2} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a-b}{d-a} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a+b}{a+d} \quad \rightarrow \quad \frac{a-b}{d-a} = \frac{a+b}{a+d} \tag{5.34}$$

or

$$\boxed{b = \frac{a^2}{d} \quad [\text{m}]} \tag{5.35}$$

Now that we know the location of the image charge, we can calculate the actual potential on the surface of the cylinder by substituting this result into the expression for V_{P1} or V_{P2}:

$$V_{P1} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a-b}{d-a} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{da-a^2}{d(d-a)} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d} \quad [\text{V}] \tag{5.36}$$

We must also show that this holds true for any point on the cylinder, not only at points P_1 and P_2. To do so, we calculate the potential at point P, which is a general point on the cylinder. This potential is

$$V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_B}{r_A} \quad [\text{V}] \tag{5.37}$$

To evaluate this potential, we note that r_B/r_A must be a constant. Also, from the geometry in **Figure 5.23a**, triangles *OBP* and *OPA* must be similar triangles to ensure that the circular surface is an equal potential surface. To satisfy this condition we select b so that the angles *OPB* and *OAP* are equal. Similarity of the triangles gives

$$\frac{b}{a} = \frac{a}{d} = \frac{r_B}{r_A} \tag{5.38}$$

Thus, the potential on the surface of the cylinder is

$$\boxed{V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d} \quad [\text{V}]} \tag{5.39}$$

Our assumption was correct and the potential is the same everywhere on the cylinder.

Now we can also justify the assumption that the location of the image line must be between $0 < b < a$ since as d approaches a (the source line approaches the surface of the cylinder), b approaches a. Similarly, for d approaching infinity, b approaches zero [**Eq. (5.35)**].

The configuration in **Figure 5.23b** is that of a line of charge ρ_l [C/m] and its image $-\rho_l$ [C/m]. The potential and electric field intensity due to this configuration were calculated in **Section 5.4.3.2** and that solution applies here as well, at any location outside the conducting cylinder. Inside the cylinder the electric field intensity is zero and the potential is constant and given by **Eq. (5.39)**. A plot of the electric field is shown in **Figure 5.23c**. Note that the charge density on that part of the surface closer to the positive line of charge is higher than on the opposite side, indicating a higher normal electric field intensity on that section of the conducting cylinder. The dotted lines inside the cylinder indicate the solution obtained from the two charged lines and the fact that only the solid parts of the lines exist in the physical configuration which, in this case, was that of a line of charge outside the conducting cylinder. The dotted lines are the solution to the complementary problem: that of a line of (negative) charge inside a conducting shell.

It is also worth noting again the fact that we could not reasonably assume zero potential on the surface of the cylindrical conductor. From hindsight, this is clear: two equal, opposite charged lines only produce zero potential on the plane midway between the two.

(2) Point Charge Outside a Conducting Sphere. Another problem that can be solved using the method of images is that of a point charge outside a conducting sphere. We start first by looking at a point charge outside a grounded sphere (sphere at zero potential) and then extend the method to spheres at given potentials or with a given charge density on their surface. The sphere is assumed to be conducting.

The geometry and the charge are shown in **Figure 5.24a**. The charge q' is an image charge and is shown in **Figure 5.24b**, but we do not know its magnitude and location; we will calculate both. From experience with the cylindrical geometry of the previous section, we know it must be negative if q is positive and must lie somewhere between $b = 0$ and $b = a$. The means of calculation is the potential on the surface of the sphere. Calculating the potential at two points like P_1 and P_2 in **Figure 5.24b** gives the necessary relations. The potential at P_1 is

$$V_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d-a} + \frac{q'}{a-b}\right) = 0 \tag{5.40}$$

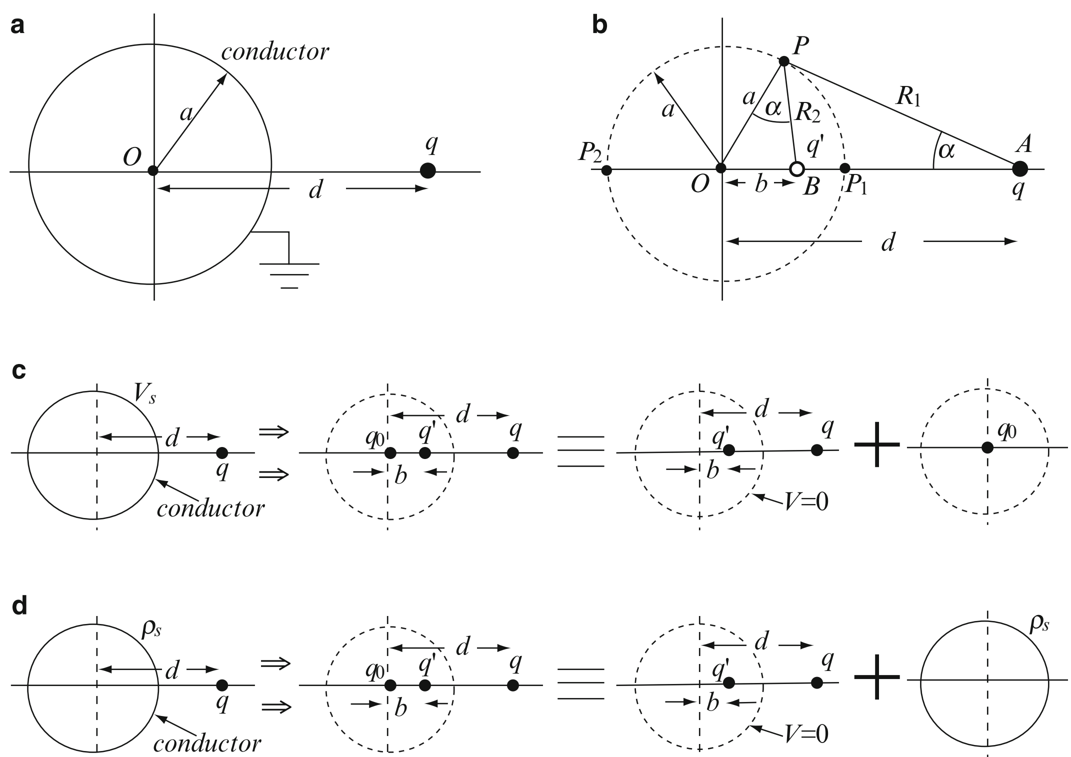


Figure 5.24 (**a**) Point charge outside a conducting sphere held at zero potential. (**b**) Assumed magnitude and location of the image point charge. (**c**) Application of superposition to take into account a constant potential on the surface. (**d**) Application of superposition to take into account surface charge density

Note that the potential everywhere on the sphere can be zero since the magnitude of the image charge is assumed to be different than that of q. The potential at P_2 similarly is

$$V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d+a} + \frac{q'}{a+b}\right) = 0 \tag{5.41}$$

Solving these two equations for q and b gives

$$\boxed{b = \frac{a^2}{d} \quad [\text{m}], \quad q' = -\frac{qa}{d} \quad [\text{C}]} \tag{5.42}$$

The potentials V_1 and V_2 are zero, but is the potential zero everywhere else on the sphere? To show that it is, and therefore our solution is correct, we calculate the potential at a general point P.
Using the law of cosines, we calculate first the distances R_1 and R_2 from the charge and its image to point P:

$$R_1 = \left(d^2 + a^2 + 2da\cos\alpha\right)^{1/2}, \quad R_2 = \left(b^2 + a^2 + 2ba\cos\alpha\right)^{1/2} \tag{5.43}$$

where the equality of angles BPO and OAP was used. The potential at P is therefore

$$V_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left(d^2 + a^2 + 2ad\cos\alpha\right)^{1/2}} - \frac{qa}{d\left(\left(a^2/d\right)^2 + a^2 + 2\left(a^2/d\right)a\cos\alpha\right)^{1/2}}\right) = 0 \tag{5.44}$$

where b and q' from **Eq. (5.42)** were substituted.
Now that we have established the location, magnitude, and sign of the image charge, the electric field intensity and potential may be calculated anywhere in space. This solution is, again, only valid outside the conducting sphere. The electric field intensity inside the conducting sphere is zero and its potential is constant and equals the potential on the surface.

A charged sphere can be analyzed in a similar manner. We start by first assuming the potential on the sphere is zero. Then, after finding the image charge as above, we add a point charge at the center to account for the charge on the surface of the sphere. The method is merely an application of superposition as shown in **Figure 5.24c** or **Figure 5.24d**. From Gauss's law, we know that the point charge at the center of the sphere must be equal to the total charge on the sphere. For a charge density ρ_s [C/m^2], the equivalent point charge is $q_0 = 4\pi a^2\rho_s$ [C/m^2]. If, on the other hand, the surface charge density is not specified, but, instead, the potential V_s on the surface of the sphere is known, then again from Gauss's law, the charge q_0 must be that point charge at the center of the sphere that will produce the given potential at its surface. From the definition of potential of a point charge, it must equal $q_0 = 4\pi\varepsilon_0 aV_s$ [C].

Example 5.12 Point Charge Inside a Conducting, Hollow Sphere A point charge is placed in a spherical cavity in a conductor as shown in **Figure 5.25**. Calculate the electric field intensity inside the cavity and the charge density at point $P(0,0,0)$. Assume the conductor is grounded (zero potential).

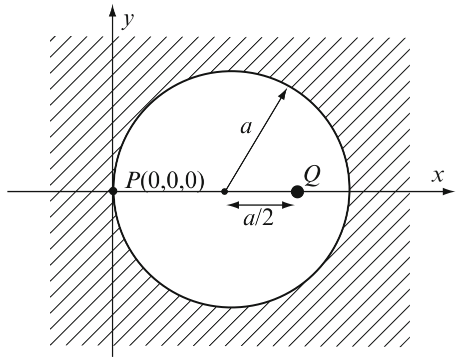


Figure 5.25 Point charge inside a hollow spherical cavity with conducting walls

Solution: Consider first the complementary problem, that of a charge outside a conducting sphere, as shown in **Figure 5.26a**.

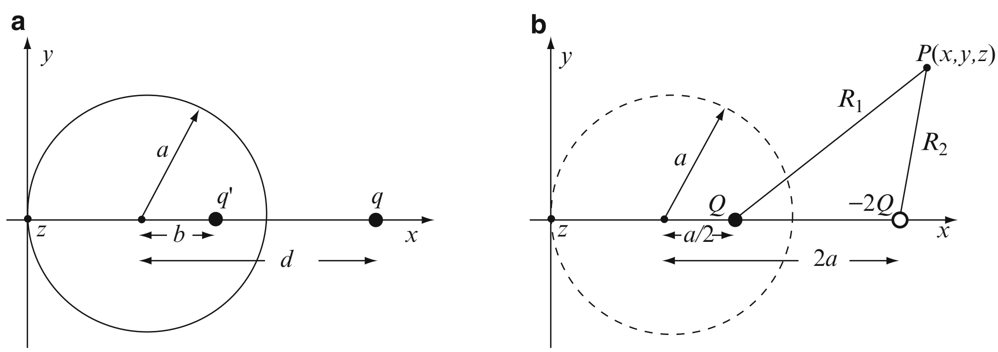


Figure 5.26 (**a**) The complementary problem to that in **Figure 5.25**: a point charge q outside a conducting sphere and its image q'. (**b**) The configuration in **Figure 5.25** with its image charge and conductor removed

This is the same configuration considered in **Figure 5.24**, in which q' was the image charge due to the external charge q. For this configuration, the charge and its location were found in **Eq. (5.42)**:

$$q' = -\frac{qa}{d} \quad [\text{V}], \quad b = \frac{a^2}{d} \quad [\text{m}]$$

where $b = a^2/d = a/2$. For the given values,

$$b = \frac{a^2}{d} = \frac{a}{2} \quad \rightarrow \quad d = 2a \quad [\text{m}]$$

However, in the configuration in **Figure 5.26b**, the charge q' is given ($q' = Q$). Therefore, we view q' as the actual charge and calculate q as its image outside the conducting sphere. Comparing **Figure 5.26a** with **Figure 5.24b**, we write

$$Q = q' = -\frac{qa}{d} \quad \rightarrow \quad q = -\frac{Qd}{a} = -2Q \quad [\text{C}]$$

The configuration is now shown in **Figure 5.26b**. The solution due to this system of charges inside the sphere is identical to the solution inside the spherical cavity in **Figure 5.25**. The potential at a general point $P(x,y,z)$ anywhere in space is

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1} - \frac{2}{R_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left[(x-3a/2)^2 + y^2 + z^2\right]^{1/2}} - \frac{2}{\left[(x-3a)^2 + y^2 + z^2\right]^{1/2}}\right] \quad [\text{V}]$$

The electric field intensity anywhere in space is calculated through the gradient. After collecting terms,

$$\mathbf{E}(x,y,z) = -\nabla V(x,y,z) = -\hat{\mathbf{x}}\frac{\partial V(x,y,z)}{\partial x} - \hat{\mathbf{y}}\frac{\partial V(x,y,z)}{\partial y} - \hat{\mathbf{z}}\frac{\partial V(x,y,z)}{\partial z}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{\hat{\mathbf{x}}(x-3a/2)+\hat{\mathbf{y}}\,y+\hat{\mathbf{z}}\,z}{\left[(x-3z/2)^2+y^2+z^2\right]^{3/2}} + \frac{\hat{\mathbf{x}}\,2(x-3a)+\hat{\mathbf{y}}\,2y+\hat{\mathbf{z}}\,2z}{\left[(x-3a)^2+y^2+z^2\right]^{3/2}}\right) \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The electric charge density at $x = 0$, $y = 0$, $z = 0$ is found from the fact that at the surface of a conductor, the relation between the normal component of the electric field intensity and charge density is

$$|E_n(x,y,z)| = \frac{\rho}{\varepsilon_0} \quad \rightarrow \quad \rho = \varepsilon_0|E_n(x,y,z)| \quad \left[\frac{\text{V}}{\text{m}}\right]$$

at any point (x,y,z) on the internal surface of the cavity. Although the calculation gives the magnitude of the charge density, we know the charge density must be negative since the charge that induces it is the point charge Q inside the sphere, which is positive. At (0,0,0), the normal component of the electric field intensity is in the negative x direction. Taking the x component of **E**, substituting $x = 0$, $y = 0$, $z = 0$ and multiplying by ε_0 gives the charge density at point $P(0,0,0)$:

$$\rho_s(0,0,0) = -\frac{Q}{18\pi a^2} \quad \left[\frac{\text{C}}{\text{m}^2}\right]$$

The solution to this problem inside the spherical cavity is shown in **Figure** 5.27. The dashed lines are the solution outside the sphere, which, although a correct solution to the complementary problem (that of a charge outside a conducting sphere), is not valid here.

Exercise 5.6 A point charge of 10 nC is placed outside a conducting sphere of radius 1 m and at a distance 0.1 m from the surface of the sphere. Calculate the potential midway between the surface of the sphere and the charge on the line connecting the charge and the center of the sphere.

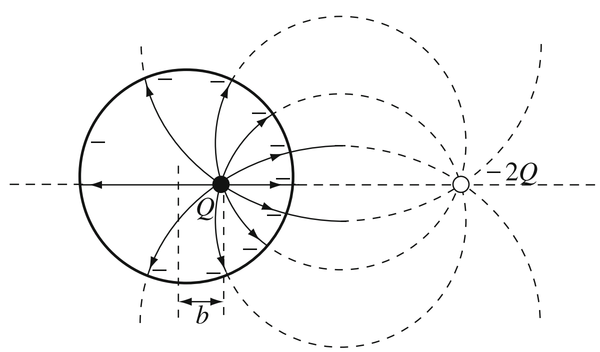


Figure 5.27 The electric field intensity inside the spherical cavity of **Figure** 5.25

Answer 1218.13 [V].

Example 5.13 Application: Underground Cable Find the electric field intensity inside the cable shown in **Figure** 5.22. The two internal lines are charged with charge densities ρ_l [C/m] and – ρ_l [C/m] and are symmetric about the center of the shield. The shield is conducting and has radius a [C/m]. Assume the lines are very long, thin, and separated a distance $2b$ [m] apart ($b < a$).

Solution: This is the application with which we started the section of image charges in curved surfaces, and we solve it here by superposition of solutions using **Eqs. (5.35)** and **(5.28)**. Taking first one of the conductors as a line of charge, we find the image line of charge outside the surface of the shield. Then, repeating the process for the second line of charge, we obtain a system of four charged lines, two positive and two negative, which produce a solution identical to that of the two lines and shield. The solution is only valid inside the shielded cable.

Taking first the positive line of charge, we obtain a situation similar to that in **Figure 5.23a**, except that the signs of the two lines of charge are reversed. From **Eq.** (**5.35**), the location of the image line of charge outside the cylindrical shield is

$$b = \frac{a^2}{d} \quad \rightarrow \quad d = \frac{a^2}{b} \quad [\mathrm{m}]$$

The image line of charge is negative and situated at a distance a^2/b from the center of the cable. Taking now the negative line of charge inside the shield, the image due to this line is positive and situated at the same distance as the image to the positive line but to the left of the cable. The four charged lines are shown in **Figure 5.28**.

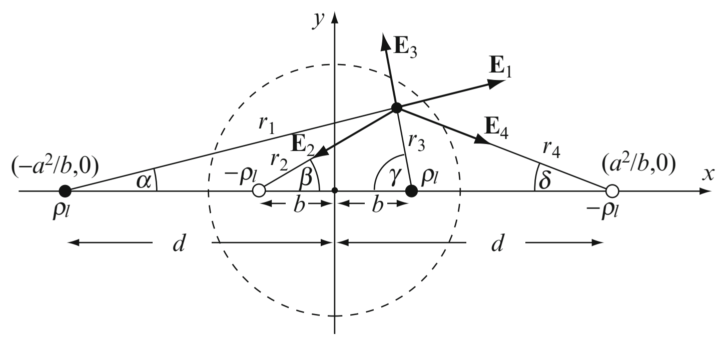


Figure 5.28 Complete system of charges needed for analysis of the shielded cable in **Figure 5.22**. Dimensions and distances to a general point inside the shield are also shown

To find the electric field intensity inside the cable, we may first find the potential, but since the number of lines is small, the electric field intensity may be found directly using Gauss's law. Writing the field intensity in terms of components and using the formula for the electric field intensity of a charged line, we get (see **Figure 5.28**)

$$\mathbf{E} = \hat{\mathbf{x}}\left(\frac{\rho_l \cos\alpha}{2\pi\varepsilon_0 r_1} - \frac{\rho_l \cos\beta}{2\pi\varepsilon_0 r_2} + \frac{\rho_l \cos\gamma}{2\pi\varepsilon_0 r_3} - \frac{\rho_l \cos\delta}{2\pi\varepsilon_0 r_4}\right) + \hat{\mathbf{y}}\left(\frac{\rho_l \sin\alpha}{2\pi\varepsilon_0 r_1} - \frac{\rho_l \sin\beta}{2\pi\varepsilon_0 r_2} + \frac{\rho_l \sin\gamma}{2\pi\varepsilon_0 r_3} - \frac{\rho_l \sin\delta}{2\pi\varepsilon_0 r_4}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Writing r_1, r_2, r_3, r_4, $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$, $\sin\gamma$, $\cos\gamma$, $\sin\delta$, and $\cos\delta$ in terms of the coordinates of a general point (x,y) and the coordinates of the four charged lines, we get from **Figure 5.28**

$$r_1 = \sqrt{(x + a^2/b)^2 + y^2}, \quad r_2 = \sqrt{(x + b)^2 + y^2}, \quad r_3 = \sqrt{(x - b)^2 + y^2}, \quad r_4 = \sqrt{(x - a^2/b)^2 + y^2}$$

$$\cos\alpha = \frac{x + a^2/b}{\sqrt{(x + a^2/b)^2 + y^2}}, \quad \cos\beta = \frac{x + b}{\sqrt{(x + b)^2 + y^2}}, \quad \cos\gamma = \frac{x - b}{\sqrt{(x - b)^2 + y^2}}, \quad \cos\delta = \frac{x - a^2/b}{\sqrt{(x - a^2/b)^2 + y^2}}$$

$$\sin\alpha = \frac{y}{\sqrt{(x + a^2/b)^2 + y^2}}, \quad \sin\beta = \frac{y}{\sqrt{(x + b)^2 + y^2}}, \quad \sin\gamma = \frac{y}{\sqrt{(x - b)^2 + y^2}}, \quad \sin\delta = \frac{y}{\sqrt{(x - a^2/b)^2 + y^2}}$$

Substituting these into the expression for the electric field intensity above gives

$$\mathbf{E}(x,y) = \hat{\mathbf{x}}\,\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{x+a^2/b}{(x+a^2/b)^2+y^2} - \frac{x+b}{(x+b)^2+y^2} + \frac{x-b}{(x-b)^2+y^2} - \frac{x-a^2/b}{(x-a^2/b)^2+y^2}\right)$$
$$+\,\hat{\mathbf{y}}\,\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{y}{(x+a^2/b)^2+y^2} - \frac{y}{(x+b)^2+y^2} + \frac{y}{(x-b)^2+y^2} - \frac{y}{(x-a^2/b)^2+y^2}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

The electric field intensity obtained here is that of four line charges in the absence of the shield. This solution is only valid inside the shield. Outside the shield, the electric field intensity is zero. The reason for this is that the charged lines produce induced charges on the shield and the combination of the fields of the line charges and those of the induced charges on the shield produce zero electric field intensity outside the shield.

(3) Cylindrical Distributions: Calculation of Capacitance. Most of the applications discussed in the previous subsection dealt with point sources or thin line charge distributions. The method of images can be easily applied to other geometries. One particularly important class of applications is the calculation of capacitance per unit length of long or infinite conductors in the presence of conducting surfaces. The following example shows how this can be done.

Example 5.14 Application: Capacitance of Overhead Line A cylindrical wire of radius $a = 10$ mm runs parallel to the ground at a height $h = 10$ m. Calculate the capacitance per unit length between the wire and ground. This problem is important in determining capacitance per unit length of transmission lines and, therefore, coupling between wires and conducting surfaces such as cables inside instruments.

Solution: To calculate the capacitance per unit length, we assume that a line charge density ρ_l [C/m] exists on the conductor and calculate the potential difference between the charged line and its image, thus removing the ground. However, the potential between the two lines is twice the potential between line and ground. Thus, the capacitance between the conductor and its image will be half the capacitance between line and ground.

The geometry is shown in **Figure 5.29a**. The potential at ground level is zero, whereas the potential of the line is some constant value, yet to be determined. The first step in the solution is to replace the ground by an image of the line, as shown in **Figure 5.29b**. Since the image line has opposite sign charge, the potential at ground level remains zero. Now, we remove the conducting surfaces of both cylindrical wires and replace them by line charges, so that the potential on the upper line remains constant and positive, while the potential on the lower line remains constant and negative. The sequence is as follows: Starting with the upper conductor, we view the lower line charge as an external line charge to the upper line. Accordingly, the image inside the upper line is ρ_l and is placed a distance $b = a^2/d$ from the center of the line as shown in **Figure 5.29c**. Viewing the charge on the upper line as external to the lower line, we obtain the situation shown in **Figure 5.29d**. Combining **Figure 5.29c** and **Figure 5.29d**, the two conductors are replaced by the two line charges shown in **Figure 5.29e**. These two line charges produce constant positive potential on the conductor, constant negative potential on its image (lower conductor), and zero potential midway between them (at ground level).

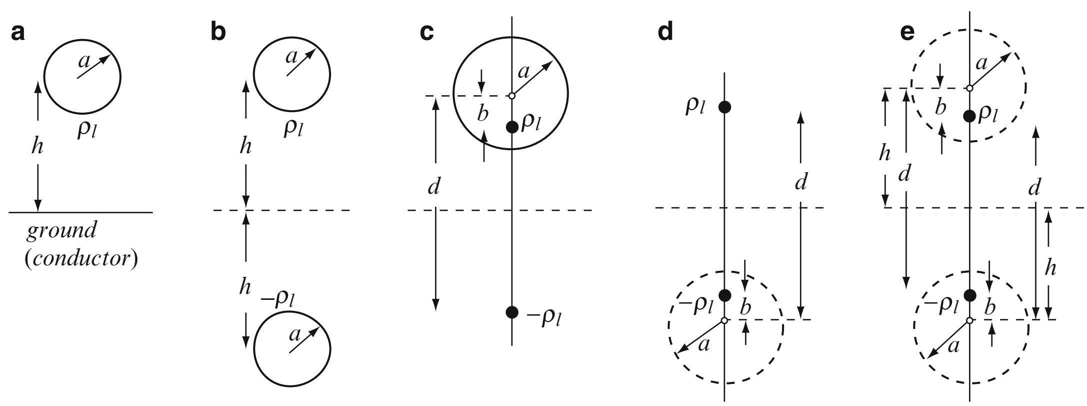


Figure 5.29 (**a**) Thick overhead conductor above ground. (**b**) The conductor and its image. (**c**) Replacement of the image by a thin line of charge that maintains the potential at the locations of the image conductor. (**d**) Replacement of the upper conductor by an equivalent thin charged line. (**e**) The equivalent configuration used for calculation

The two equivalent lines of charge are at a distance $b = a^2/d$ from the center of each of the conductors. Thus, the two line charges are a distance $d - a^2/d$ or $2h - 2b$ apart. Rewriting d in terms of b gives $d = a^2/b$. In terms of the known distance between ground and conductor h, we get

$$2h = b + d = b + \frac{a^2}{b} \quad \rightarrow \quad b^2 - 2hb + a^2 = 0$$

The two possible solutions for b are

$$b_1 = h + \sqrt{h^2 - a^2} \text{ and } b_2 = h - \sqrt{h^2 - a^2} \quad [\text{m}]$$

In this example, $h = 10$ m and $a = 0.01$ m. Thus, b_1 is ignored because it gives the location of the charge outside the conductor. With $b = b_2$, the distance d is (from **Figure 5.29e**)

$$d = 2h - b = h + \sqrt{h^2 - a^2} \quad [\text{m}]$$

From **Eq. (5.39)**, the potential on the upper conductor is

$$V_{uc} = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d} = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{h + \sqrt{h^2 - a^2}} \quad [\text{V}]$$

This potential is positive because $a < d$. On the lower conductor, the charge is negative. Therefore, again from **Eq. (5.39)**, the potential on the lower conductor is

$$V_{lc} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{h + \sqrt{h^2 - a^2}} \quad [\text{V}]$$

The potential difference between the two conductors is the potential on the upper conductor minus that on the lower conductor:

$$V_t = V_{uc} - V_{lc} = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{h + \sqrt{h^2 - a^2}} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{h + \sqrt{h^2 - a^2}} = \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{h + \sqrt{h^2 - a^2}}{a} \quad [\text{V}]$$

The total charge per unit length of the cable is ρ_l [C/m] (by assumption). Thus, the capacitance per unit length of the cable is

$$C = \frac{\rho_l}{V_t} = \frac{\rho_l}{\frac{\rho_l}{\pi\varepsilon_0}\ln\frac{h + \sqrt{h^2 - a^2}}{a}} = \frac{\pi\varepsilon_0}{\ln\left(\frac{h + \sqrt{h^2 - a^2}}{a}\right)} \quad \left[\frac{\text{F}}{\text{m}}\right]$$

This is the capacitance per unit length of two cylindrical conductors, separated a distance $2h$ apart. We are interested here in the capacitance between the upper conductor and ground. Because the potential difference between the upper conductor and ground is half the potential calculated above whereas the charge per unit length remains the same, the capacitance of the conductor-ground arrangement is twice that calculated above:

$$C = \frac{2\rho_l}{V_t} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{h + \sqrt{h^2 - a^2}}{a}\right)} \quad \left[\frac{\text{F}}{\text{m}}\right]$$

For the values given above ($a = 0.01$ m, $h = 10$ m), the capacitance per unit length is

$$C = \frac{2\pi\varepsilon_0}{\ln\left(\frac{h + \sqrt{h^2 - a^2}}{a}\right)} = \frac{2 \times \pi \times 8.854 \times 10^{-12}}{\ln\left(\frac{10 + \sqrt{10^2 - 0.01^2}}{0.01}\right)} = 7.32 \quad \left[\frac{\text{pF}}{\text{m}}\right]$$

Note: In this solution, we implicitly assumed that the charge distribution on one wire is not affected by the charge on the ground (or the second wire). This is a good assumption if $h \gg a$.

5.4.4 Separation of Variables: Solution to Laplace's Equation

Another method of solving Laplace's equation is the general method of separation of variables. It is a common method of solution for second-order partial differential equations and should be familiar from calculus. We repeat here the basic steps because this will give us a better insight into the physical solution. The method provides general solutions to Laplace's equation. In fact, this generality will turn out to be somewhat of a disappointment because although a general solution is rather easy to obtain, a particular solution for a particular problem is not. Nevertheless, the method is one of the most useful methods for the solution of electrostatic problems as well as others. Separation of variables will only be shown in Cartesian and cylindrical coordinates. Separation of variables in spherical coordinates requires additional mathematical tools and since its utility is limited will not be discussed here.

5.4.4.1 Separation of Variables in Cartesian Coordinates

Electric_Potential.m

To outline the method, consider first Laplace's equation in Cartesian coordinates:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{5.45}$$

where V indicates any scalar function including the electric scalar potential. Assuming linear behavior, the solution $V(x,y,z)$ can be written as the product of three separate solutions:

$$V(x,y,z)=X(x)Y(y)Z(z) \tag{5.46}$$

where $X(x)$ is only dependent on the x variable, $Y(y)$ on the y variable, and $Z(z)$ on the z variable. This dependence on a single variable will allow the separation of Laplace's equation into three scalar equations, each dependent on a single variable. Substitution of the general solution into **Eq. (5.45)** gives

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2}+X(x)Z(z)\frac{d^2Y(y)}{dy^2}+X(x)Y(y)\frac{d^2Z(z)}{dz^2}=0 \tag{5.47}$$

where the partial derivatives were replaced by ordinary derivatives because of the dependency on a single variable. Dividing both sides by $V(x,y,z)$ gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0 \tag{5.48}$$

In this form, each term depends on a single variable and, therefore, can be separated. For the separation to be valid, each term must be equal to a constant to be determined. The equation can be written as

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-k_x^2 \tag{5.49}$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-k_y^2 \tag{5.50}$$

$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=-k_z^2 \tag{5.51}$$

where the three constants must satisfy

$$\boxed{k_x^2+k_y^2+k_z^2=0} \tag{5.52}$$

and the negative sign in **Eqs. (5.49)**, through **(5.51)** as well as the use of the square in **Eq. (5.52)** are arbitrary; that is, any combination of constants can be chosen, provided **Eq. (5.52)** is satisfied. This choice will be convenient later when we evaluate the constants for specific applications. We also note that only two of the constants can be chosen independently and the third is defined by **Eq. (5.52)**. The following three differential equations are obtained from **Eqs. (5.49)** through (**5.51**):

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0 \tag{5.53}$$

$$\frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0 \tag{5.54}$$

$$\frac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0 \tag{5.55}$$

Now, the three equations are completely independent and can be solved separately. Further, these three equations have standard solutions which we can now use. The first two equations are identical in form and, therefore, have the same form of solution. Assuming that k_x^2 and k_y^2 are positive, the general solutions for $X(x)$ and $Y(y)$ are

$$\boxed{X(x) = A_1 \sin(k_x \sin)(k_x x) + A_2 \cos(k_x x) = B_1 e^{jk_x x} + B_2 e^{-jk_x x}} \tag{5.56}$$

$$\boxed{Y(y) = A_3 \sin(k_y y) + A_4 \cos(k_y y) = B_3 e^{jk_y y} + B_4 e^{-jk_y y}} \tag{5.57}$$

To find the solution for the third term, we first note from **Eq. (5.52)** that $k_z^2 = -(k_x^2 + k_y^2)$. Therefore, because k_z^2 is negative, **Eq. (5.55)** has the following standard solution:

$$\boxed{Z(z) = A_5 \sinh(|k_z|z) + A_6 \cosh(|k_z|z) = B_5 e^{|k_z|z} + B_6 e^{-|k_z|z}} \tag{5.58}$$

where the constants A_i or B_i must be evaluated to obtain a particular solution. These are evaluated from the boundary conditions of the problem. Either the harmonic form of solution or the exponential form may be used, depending on which form is most appropriate or most convenient. Thus, the general solution for the three-dimensional Laplace equation is written as

$$\boxed{\begin{aligned} V(x,y,z) = X(x)Y(y)Z(z) &= [A_1 \sin(k_x x) + A_2 \cos(k_x x)] \\ &\times [A_3 \sin(k_y y) + A_4 \cos(k_y y)][A_5 \sinh(|k_z|z) + A_6 \cosh(|k_z|z)] \end{aligned}} \tag{5.59}$$

or

$$\boxed{V(x,y,z) = \left[B_1 e^{jk_x x} + B_2 e^{-jk_x x}\right]\left[B_3 e^{jk_y y} + B_4 e^{-jk_y y}\right]\left[B_5 e^{|k_z|z} + B_6 e^{-|k_z|z}\right]} \tag{5.60}$$

First, we note that any problem described in terms of Laplace's equation will have a solution in terms of spatial sine, cosine, hyperbolic sine, and hyperbolic cosine functions. Second, any of the constants A_i or B_i can be zero, depending on the boundary condition of the problem. Finally, although the solution appears in terms of nine unknowns ($A_1, A_2, A_3, A_4, A_5, A_6, k_x, k_y, k_z$), only six unknowns are independent. This can be verified in general, but instead of doing so, we will show this in **Example 5.16**.

The solution in **Eq. (5.59)** or (**5.60**) is a solution in three dimensions. If the potential does not vary in one of the dimensions, the problem becomes two-dimensional and the general solution is the product of two variables. In two dimensions, for example, in the x–z plane, $\partial V(x,y,z)/\partial y = 0$ and $k_y = 0$. Thus, the solutions are essentially identical with the middle term (the term $Y(y)$) removed and with $k_z^2 = -k_x^2$. The general solution is

$$\boxed{V(x,y) = X(x)Z(z) = [A_1 \sin(k_x x) + A_2 \cos(k_x x)]\left[A_5 \sinh\left(\left|k_z\right| z\right) + A_6 \cosh\left(\left|k_z\right| z\right)\right]} \tag{5.61}$$

$$\boxed{V(x,z) = \left[B_1 e^{jk_x x} + B_2 e^{-jk_x x}\right]\left[B_5 e^{\left|k_x\right| z} + B_6 e^{-\left|k_z\right| z}\right]} \tag{5.62}$$

with

$$\boxed{k_z^2 = -k_x^2}$$

Of course, if the solution is in the x–y plane rather than in the x–z plane, z is replaced by y in the equations above without any other change being necessary. We also note that if any of the constants k_x or k_y are zero (k_z can only be zero if k_x and k_y are zero in three dimensions or if one of the constants is zero in two dimensions), the corresponding equation has a simple linear solution. If, for example, $k_x = 0$, the solution for $X(x)$ is $X(x) = A_1 x + A_2$, as can be shown by direct integration of **Eq.** (5.53) after setting $k_x = 0$.

Example 5.15 Potential Inside a Two-Dimensional Box Two parallel conducting surfaces, separated a distance a and held at zero potential, are placed above but insulated from a third conducting surface that is held at a constant potential V_0 [V] as shown in **Figure** 5.30a. Assuming the potential at infinity is zero, calculate the potential everywhere in the channel defined by the three surfaces.

Solution: The geometry described here is a two-dimensional problem since the potential does not vary in the z direction. Thus, we can view the geometry as an infinitely large conducting sheet folded at $x = 0$ and $x = a$ [m]. (See **Figure** 5.30a). The solution therefore occurs in the cross section. The solution is obtained by use of **Eq.** (5.61) and **Eq.** (5.63), but with the y variable replacing the z variable.

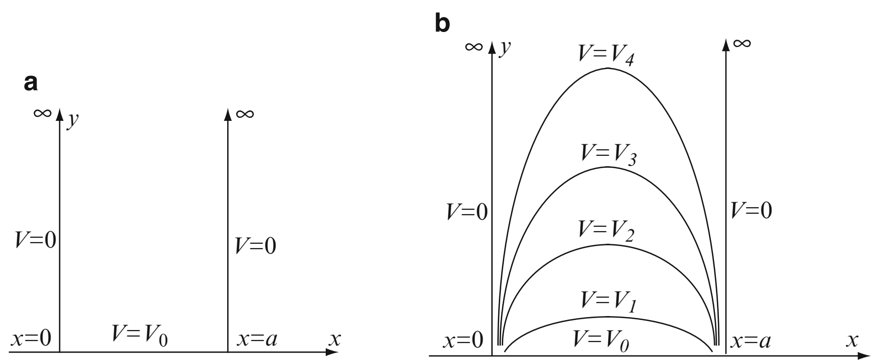


Figure 5.30 (**a**) Infinite box defined by $x = 0$, $x = a$, $y = 0$, $y \to \infty$. (**b**) Potential distribution inside the box

The general solution is

$$V(x,y) = (A_1 \sin kx + A_2 \cos kx)\left(B_1 e^{ky} + B_2 e^{-ky}\right) \qquad [\text{V}]$$

where $k_y{}^2 = -k_x{}^2 = -k^2$ was used. For the solution in the y direction, we used the exponential form. This is because we anticipate using values of y that tend to infinity. For such values, exponential forms are more convenient than the hyperbolic forms. To satisfy the boundary conditions, we write

(1) At $x = 0, \to V(0,y) = A_2(B_1 e^{ky} + B_2 e^{-ky}) = 0 \to A_2 = 0$.

(2) At $x = a, \to V(a,y) = (A_1 \sin ka)(B_1 e^{ky} + B_2 e^{-ky}) = 0 \to A_1 \sin ka = 0$

This gives

$$ka = m\pi \quad \rightarrow \quad k = \frac{m\pi}{a} \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

where m is any integer, including zero. We will, however, exclude $m = 0$ from the solution because it leads to $k = 0$ and a linear solution of the form $Ax + B$.

Similarly, the negative values of m need not be considered because negative m will only change the sign of the solution. The general solution at this stage looks like

$$V(x,y) = A_1 \sin \frac{m\pi}{a} x\left(B_1 e^{m\pi y/a} + B_2 e^{-m\pi y/a}\right) \quad [\text{V}]$$

(3) At $y = \infty$, $\rightarrow$ $V(x,\infty) = A_1 \sin \frac{m\pi x}{a}\left(B_1 e^{m\pi\infty/a}\right) = 0 \rightarrow B_1 = 0$

The solution at this stage is

$$V(x,y) = C \sin \frac{m\pi x}{a} e^{-m\pi y/a} \quad [\text{V}]$$

where $C = A_1 B_2$.

(4) At $y = 0$, $V(x,0) = V_0$: To satisfy this condition, we cannot simply substitute $y = 0$ in the general solution. If we did, the solution would be sinusoidal in the x direction and no constant C can satisfy the boundary condition. However, the solution may also be written as a superposition of solutions of the above form. We write

$$V(x,y) = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} e^{-m\pi y/a} \quad [\text{V}]$$

Now, we substitute $y = 0$:

$$V(x,0) = V_0 = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} \quad [\text{V}]$$

The latter form is a Fourier sine series which, in effect, approximates the pulse $V(x,0) = V_0$, $0 \le x \le a$, by an infinite series. In this sense, C_m are the amplitudes of the coefficients of the series. To obtain C_m, we multiply both sides by $\sin(p\pi x/a)$, where p is an integer, and integrate both sides from *zero* to a. This is a general technique we will use again and is due to Fourier himself:

$$\int_{x=0}^{a} V_0 \sin \frac{p\pi x}{a} dx = \int_{x=0}^{a} \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{a} \sin \frac{p\pi x}{a} dx = \sum_{m=1}^{\infty} \int_{x=0}^{a} C_m \sin \frac{m\pi x}{a} \sin \frac{p\pi x}{a} dx$$

where the integration and the sum were interchanged. Each side of the relation is integrated separately. The left-hand side gives

$$\int_{x=0}^{a} V_0 \sin \frac{p\pi x}{a} dx = \begin{cases} \dfrac{2aV_0}{p\pi} & \text{for } p \text{ odd} \\ 0 & \text{for } p \text{ even} \end{cases}$$

For the right-hand side, we integrate each integral in the sum. For any value of m, we get

$$\int_{x=0}^{a} C_m \sin \frac{m\pi x}{a} \sin \frac{p\pi x}{a} dx = \begin{cases} \dfrac{C_m a}{2} & \text{for } p = m \\ 0 & \text{for } p \neq m \end{cases}$$

To satisfy both conditions above, m must be odd and $p = m$. Any other value yields zero. Thus,

$$C_m = \frac{4V_0}{m\pi}, \quad m = 1, 3, 5, \ldots.$$

If we substitute this in the general solution, we obtain the general solution inside the box:

$$V(x, y) = \frac{4V_0}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m} \sin \frac{m\pi x}{a} e^{-m\pi y/a} \quad [\mathrm{V}]$$

The potential distribution in the box is shown in **Figure 5.30b**.

Example 5.16 Potential in a Box A hollow cube with conducting surfaces is given in **Figure 5.31**. Five of the walls are at zero potential, whereas the wall at $z = c$ [m] is insulated from the others and connected to a constant potential $V = V_0$ [V]. Calculate the potential everywhere inside the box.

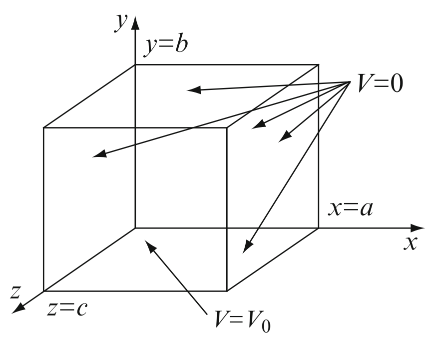


Figure 5.31

Solution: The general solution is given in **Eq. (5.59)**:

$$V(x, y, z) = [A_1 \sin(k_x x) + A_2 \cos(k_x x)]\left[A_3 \sin\left(k_y y\right) + A_4 \cos\left(k_y y\right)\right]\left[A_5 \sinh\left(|k_z| z\right) + A_6 \cosh\left(|k_z| z\right) \quad [\mathrm{V}]$$

$$\text{at } x = 0, \quad V(0, y, z) = 0 \quad \text{at } x = a, \quad V(a, y, z) = 0$$
$$\text{at } y = 0 \quad V(x, 0, z) = 0 \quad \text{at } y = b, \quad V(x, b, z) = 0$$
$$\text{at } z = 0 \quad V(z, y, 0) = 0 \quad \text{at } z = c, \quad V(x, y, c) = V_0$$

We start by satisfying the boundary conditions at $x = 0$, $x = a$, $y = 0$, $y = b$, and $z = 0$:

(1) At $x = 0$:

$$V(0, y, z) = [A_2]\left[A_3 \sin\left(k_y y\right) + A_4 \cos\left(k_y y\right)\right]\left[A_5 \sinh\left(|k_z| z\right) + A_6 \cosh\left(|k_z| z\right)\right] = 0 \rightarrow A_2 = 0$$

(2) At $x = a$:

$$V(a, y, z) = A_1 \sin(k_x a)\left[A_3 \sin\left(k_y y\right) + A_4 \cos\left(k_y y\right)\right]\left[A_5 \sinh\left(|k_z| z\right)\right] + A_6 \cosh\left(|k_z| z\right)\right] = 0$$

This condition can be satisfied if

$$\sin(k_x a) = 0 \quad \rightarrow \quad k_x a = m\pi \quad \rightarrow \quad k_x = \frac{m\pi}{a} \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad m = 1, 2, 3, \ldots$$

(3) At $y = 0$:

$$V(x,0,z) = A_1 \sin\left(\frac{m\pi}{a}x\right)[A_4]\left[A_5 \sinh(|k_z|z) + A_6 \cosh(|k_z|z)\right] = 0 \quad \rightarrow \quad A_4 = 0$$

(4) At $y = b$:

$$V(x,b,z) = A_1 \sin\left(\frac{m\pi}{a}x\right)A_3 \sin(k_y y)\left[A_5 \sinh(|k_z|z) + A_6 \cosh(|k_z|z)\right] = 0$$

This condition can be satisfied if

$$\sin(k_y b) = 0 \quad \rightarrow \quad k_y b = n\pi \quad \rightarrow \quad k_y = \frac{n\pi}{b} \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad n = 1, 2, 3, \ldots$$

(5) At $z = 0$:

$$V(x,y,0) = A_1 \sin\left(\frac{m\pi}{a}x\right)A_3 \sin\left(\frac{n\pi}{b}x\right)[A_6] = 0 \quad \rightarrow \quad A_6 = 0$$

Note: Both m and n can be zero, but this leads to a linear solution (see **Example 5.15**). Also, m and n can be negative, but since negative values merely change the sign of the solution, we exclude these values. Now, since $k_z^2 = -(k_x^2 + k_y^2)$ [see **Eq.** (5.52)] and combining the constants A_1, A_3, and A_5 into a single constant, we can write the general solution at this stage as

$$V(x,y,z) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{k_x^2 + k_y^2}\,z\right) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\pi z\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\right) \quad [\text{V}]$$

First, we note that there is only one condition to be satisfied: the sixth and final boundary condition at $z = c$. Before using this condition, we note that any superposition of solutions is also a solution to the original equation. The following sum may be taken as a general solution (see discussion in **Example 5.15**):

$$V(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\pi z\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\right) \quad [\text{V}]$$

(6) The sixth boundary condition is substituted into this sum in order to evaluate the constant C_{mn}. For $z = c$, $V(x,y,c) = V_0$, and using the short form notation for $|k_z|$

$$k_z = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

we get the general solution as

$$V(x,y,c) = V_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sinh(k_z c) \quad [\text{V}]$$

As in **Example 5.15**, this is a Fourier series, but now it is a series in two variables and applies anywhere on the plane $z = c$ ($0 \leq x \leq a$, $0 \leq y \leq b$). To find the constant C_{mn}, we perform two operations. First, we multiply both sides by the term $\sin(p\pi x/a)\ \sin(q\pi y/b)$, where p and q are integers (see **Example 5.15**). This gives

$$V_0\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)\sinh(k_z c)$$

In the second step, both sides of the expression are integrated on the plane $z = c$ in the limits $0 \leq x \leq a$, $0 \leq y \leq b$. To simplify the evaluation, we perform the integration on the left-hand side and right-hand side separately. For the left-hand side,

$$\int_{y=0}^{b}\int_{x=0}^{a} V_0\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right) = V_0\left[\int_{x=0}^{a}\sin\left(\frac{p\pi x}{a}\right)dx\right]\left[\int_{y=0}^{b}\sin\left(\frac{q\pi y}{b}\right)dy\right]$$

$$= \left\{\begin{array}{ll} V_0\left[\dfrac{2a}{p\pi}\right]\left[\dfrac{2b}{q\pi}\right] = \dfrac{4abV_0}{pq\pi^2} & \text{for}\, p,\ q,\ \text{odd} \\ 0 & \text{for}\, p,\ q,\ \text{even} \end{array}\right\}$$

For the right-hand side, interchanging the summation and integration and evaluating each of the integrals in the sum separately yields

$$\int_{x=0}^{a}\int_{y=0}^{b} C_{mn}\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)\right]\left[\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi y}{b}\right)\right]\sinh(k_z c)dx\,dy$$

$$= C_{mn}\sinh(k_z c)\left[\int_{x=0}^{a}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx\right]\left[\int_{y=0}^{b}\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi y}{b}\right)dy\right]$$

$$= C_{mn}\sinh(k_z c)\left[\int_{x=0}^{a}\frac{1}{2}\left[\cos\left(\frac{(m-p)\pi x}{a}\right) - \cos\left(\frac{(m+p)\pi x}{a}\right)\right]dx\right]$$

$$\times\left[\int_{y=0}^{b}\frac{1}{2}\left[\cos\left(\frac{(n-q)\pi y}{b}\right) - \cos\left(\frac{(n+q)\pi y}{b}\right)\right]dy\right] = \left\{\begin{array}{ll} \dfrac{C_{mn}ab\sinh(k_z c)}{4} & \text{for}\quad m = p,\ n = q \\ 0 & m \neq p,\ n \neq q \end{array}\right\}$$

Now, satisfying both the conditions $m = p$, $n = q$, and m and n, odd, we get

$$C_{mn} = \frac{16V_0}{mn\pi^2\sinh(k_z c)}$$

Substituting this back into the general solution, we get by equating the left- and right-hand sides

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\ \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{mn\sinh\pi\left(\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\,c\right)}$$

$$\times\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\pi\left(\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\,z\right) \qquad [\text{V}]$$

Not a simple result but doable. In particular, note the similarities between this and the previous example: the difference is that here we used a Fourier series in two variables to approximate the solution.

Finally, it is worth reiterating that to obtain the solution, we used six boundary conditions. This means that only six independent constants needed to be evaluated. In this case, these were A_2, k_x, A_4, k_y, A_6, and C_{mn}.

5.4.4.2 Separation of Variables in Cylindrical Coordinates

An essentially identical process to that in the previous section can be followed in cylindrical coordinates. We can start with the general Laplace equation in cylindrical coordinates and proceed in exactly the same manner by assuming a general solution and separating the equation into three independent equations. However, the equations themselves are expected to be different and, in fact, they are more difficult to solve as they require, in addition to harmonic functions, knowledge of Bessel functions. For this reason, we will not pursue here the solution of the general Laplace equation in cylindrical coordinates. Rather, we solve a simplified form of Laplace's equation in two dimensions, which only requires harmonic functions. In spite of this simplification, the process is illustrative of the general method and has important practical application in solution of real problems.

The type of physical configuration we look at is any physical problem in which the potential does not vary with the z dimension. In other words, the partial derivative with respect to z is zero. With this condition, Laplace's equation in cylindrical coordinates becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0 \tag{5.64}$$

This equation represents physical applications in which the third dimension is large (or perhaps infinite) and the potential does not vary in this dimension. For example, a coaxial capacitor or a coaxial transmission line is of this type. Following the steps of the previous section, we write a general solution as the product of two independent solutions:

$$V(r,\phi)=R(r)\Phi(\phi) \tag{5.65}$$

and, as before, the implicit assumption is that the solutions are linear. Substitution of the general solution in **Eq.** (**5.64**) gives

$$\frac{\Phi(\phi)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{R(r)}{r^2}\frac{\partial^2\Phi(\phi)}{\partial \phi^2}=0 \tag{5.66}$$

The equation is now divided by $V(r, \phi)$:

$$\frac{1}{rR(r)}\frac{1}{\partial r}\frac{\partial}{}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{1}{r^2\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial \phi^2}=0 \tag{5.67}$$

This equation is still not separable since both terms depend on r. To eliminate this dependence, we multiply both sides of the equation by r^2 and obtain

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial \phi^2}=0 \tag{5.68}$$

Now the first term depends on r only whereas the second term depends only on ϕ. Choosing a constant k_r^2 for the first term and k_ϕ^2 for the second term, we can write

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)+k_r^2=0 \tag{5.69}$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}+k_\phi^2=0 \tag{5.70}$$

and

$$\boxed{k_\phi^2+k_r^2=0} \tag{5.71}$$

The choice here is to use a positive constant for k_ϕ^2 since, then, the solution is identical in form to that in **Eq.** (**5.56**). The solution for $\Phi(\phi)$ is

$$\boxed{\Phi(\phi) = A_1\sin(k\phi) + A_2\cos(k\phi) = B_1e^{jk\phi} + B_2e^{-jk\phi}} \tag{5.72}$$

where the term k was used as $k^2 = k_\phi^2 = -k_r^2$. The equation for $R(r)$ can now be written as

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - k^2R(r) = 0 \tag{5.73}$$

Also, because the solution for $\Phi(\phi)$ is periodic, k must be an integer. The solution of **Eq.** (5.73) is

$$R(r) = B_3r^k + B_4r^{-k} \tag{5.74}$$

and the solution for $V(r,\phi)$ is

$$\boxed{V(r,\phi) = R(r)\Phi(\phi) = [A_1\sin(k\phi) + A_2\cos(k\phi)]\left[B_3r^k + B_4r^{-k}\right]} \tag{5.75}$$

or

$$\boxed{V(r,\phi) = R(r)\Phi(\phi) = \left[B_1e^{jk\phi} + B_2e^{-jk\phi}\right]\left[B_3r^k + B_4r^{-k}\right]} \tag{5.76}$$

Again, we obtained a general solution for any problem that can be described in $r - \phi$ coordinates. As previously, the constants are found by satisfying the equation for known boundary conditions. We also note that we must calculate a total of four independent constants (i.e., k and three constant coefficients).

If $k = 0$, **Eqs.** (**5.69**) and (**5.70**) become Laplace equations and each can be integrated directly as was done in **Section** 5.4.2. In this case, the solutions are (see **Section** 5.4.2)

$$\boxed{V(r) = A\ln r + B} \quad \boxed{V(\phi) = C\phi + D} \tag{5.77}$$

Example 5.17 Application: The Electrostatic Precipitator An electrostatic precipitator is a device designed to collect both positively and negatively charged ash in the stack of a power plant. The stack is high and is built as shown in **Figure** 5.32a. The central conductor is in the form of a mesh to allow particles to move around. With the dimensions and potentials as given, calculate the potential distribution everywhere in the stack.

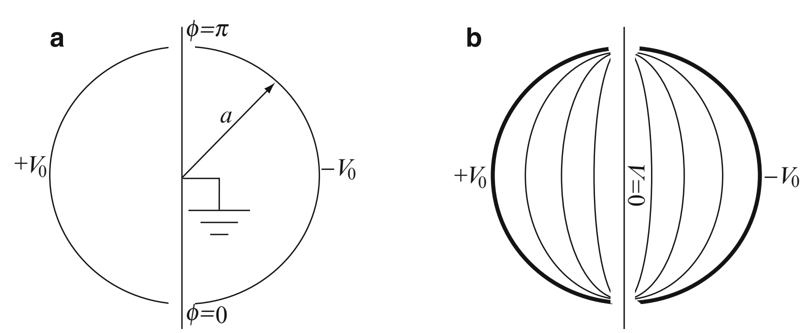


Figure 5.32 (**a**) Cross section of an electrostatic precipitator. (**b**) Potential distribution inside the precipitator

Solution: Because the stack is circular, a cylindrical system of coordinates is used. If the system is placed with the z axis along the axis of the stack, the potential can only vary in the r and ϕ directions. Because the stack is high, we may assume that any edge effects at the bottom and top of the stack can be neglected. The equation describing the potential is **Eq.** (**5.64**) and the general solution is given in **Eq.** (**5.75**):

$$V(r,\phi) = R(r)\Phi(\phi) = [A_1\sin(k\phi) + A_2\cos(k\phi)]\left[B_3r^k + B_4r^{-k}\right]$$

The boundary conditions are

$$V(a,\phi) = -V_0 \quad \text{for} \quad 0 < \phi < \pi \text{ and } V(a,\phi) = V_0 \quad \text{for} \quad \pi < \phi < 2\pi$$

Because the potential must be finite everywhere in the stack, the term containing r^{-k} must be zero (i.e., $B_4 = 0$); otherwise, as we approach the center of the stack, the potential would approach infinity. The solution therefore becomes

$$V(r,\phi) = R(r)\Phi(\phi) = B_3r^k[A_1\sin(k\phi) + A_2\cos(k\phi)] \quad [\text{V}]$$

Now, we note that the solution $V(r,\phi)$ must be an odd function of ϕ. This can be seen from the fact that the boundary condition is negative for $0 < \phi < \pi$, $2\pi < \phi < 3\pi$, etc., and positive for $\pi < \phi < 2\pi$, $3\pi < \phi < 4\pi$, etc. Therefore, the cosine term must also vanish everywhere ($A_2 = 0$). The solution is

$$V(r,\phi) = Cr^k\sin(k\phi) \quad [\text{V}]$$

This solution, while correct, is not sufficient to satisfy the problem and its boundary conditions. To ensure that the problem is satisfied everywhere, we take an infinite sum of all possible solutions of the above form:

$$V(r,\phi) = \sum_{k=1}^{\infty} C_kr^k\sin(k\phi) \quad [\text{V}]$$

To evaluate the constants C_k, we use the method in **Example 5.15**. First, we note that at $r = a$, the conditions are

$$V(a,\phi) = -V_0 = \sum_{k=1}^{\infty} C_ka^k\sin(k\phi) \quad \text{for} \quad 0 < \phi < \pi$$
$$V(a,\phi) = V_0 = \sum_{k=1}^{\infty} C_ka^k\sin(k\phi) \quad \text{for} \quad \pi < \phi < 2\pi$$

Now, we follow the process discussed in **Example 5.15** and multiply both sides by $\sin(p\pi\phi/\pi) = \sin(p\phi)$ and integrate between $\phi = 0$ and $\phi = \pi$. Taking the first of these expressions, we get

$$-\int_{\phi=0}^{\pi} V_0\sin(p\phi)d\phi = \sum_{k=1}^{\infty}\int_{\phi=0}^{\pi} C_ka^k\sin(k\phi)\sin(p\phi)d\phi$$

From the left-hand side,

$$-\int_{\phi=0}^{\pi} V_0\sin(p\phi)d\phi = \begin{cases} -\dfrac{2V_0}{p} & \text{for } p \text{ odd} \\ 0 & \text{for } p \text{ even} \end{cases}$$

From the right-hand side,

$$-\int_{\phi=0}^{\pi} C_ka^k\sin(k\phi)\sin(p\phi)d\phi = \begin{cases} \dfrac{\pi C_ka^k}{2} & \text{for } p = k \\ 0 & \text{for } p \neq k \end{cases}$$

Thus, k must be odd and $p = k$. The constant C_k is therefore

$$C_k = -\frac{4V_0}{k\pi a^k}$$

Substituting this in the general solution, we get

$$V(r,\phi) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{-4V_0}{k\pi a^k} r^k \sin(k\phi) \quad [\mathrm{V}], \quad 0 < \phi < 2\pi$$

A plot of the solution is shown in **Figure 5.32b**.

Exercise 5.7 The configuration in **Example 5.17** is given again. Using the method shown, calculate and plot the potential outside the stack. **Hint**: Outside the stack the term containing r^k must be zero so that the potential decreases to zero at infinity, but the term containing r^{-k} must be retained.

Answer

$$V(r,\phi) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{-4V_0 a^k}{k\pi r^k} \sin(k\phi) \qquad [\mathrm{V}], \quad 0 < \phi < 2\pi$$

5.5 Experiments: The Method of Images

Experiment 1 (Demonstrates: Images Due to Point Charges and Flat Surfaces). Take a flat mirror and place a light source (or any other object) in front of it. The reflection is the optical image. If this object were charged, the image charge would be the negative of the source charge, located at the same distance from the mirror, behind it.

Experiment 2 (Demonstrates: Images Due to Distributed Charges and Flat Surfaces). Place a long stick in front of the mirror in an arbitrary orientation. Mark the two ends of the stick so you can distinguish them. Note the relative position of the source (stick) and image.

Experiment 3 (Demonstrates: Images Due to Point Charges and Multiple, Parallel Flat Surfaces):

(a) Take two flat mirrors. Position them at a 90° angle and position an object in front of the mirrors. Draw the system of charges you obtain. Note in particular the image at the corner between the mirrors: this is a true image in that left and right in the mirror are in the same relation as in the object.

(b) Place the two mirrors parallel to each other with some distance between the two. Position an object (a light source is best) at a point between the two mirrors. Note the system of image charges. How many charges do you get? Where are these charges located? What about polarity? Sketch the system of charges.

(c) Change the angle of the mirrors. What happens to the system of charges?

Experiment 4 (Demonstrates: Images Due to Point Charges and Curved Surfaces):

(a) Repeat Experiments 1 and 2 in front of concave and convex mirrors. Note the size and location of images. Do they remain the same size? Where are they located?

(b) If you can find two spherical mirrors (for example, polished metal spheres, spherical door knobs, etc.), place a source midway between the two spheres. Note the location and size of images. How many images do you get? Where are these images located?

5.6 Summary

Chapter 5 continues discussion of methods of solution for the electrostatic field. The methods are called collectively boundary value problems because they essentially solve Poisson's (or Laplace's) equation under given boundary conditions.

Poisson's equation for the electric potential is the basis of the methods that follow:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}, \qquad \text{in Cartesian coordinates}: \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon} \qquad (5.4) \text{ and } (5.5)$$

If the right-hand side is zero, we get ***Laplace's equation***. The equation can be written in cylindrical and spherical coordinates as well [**Eqs.** (**5.6**) and (**5.7**)]. Solution provides the potential and, if necessary, the electric field intensity.

Direct integration of either Poisson's or Laplace's equations is possible if the potential only depends on one variable (one-dimensional problem). If direct integration can be used, the process is as follows:

(1) Integrate both sides of the equation twice to obtain the potential as a function of the variable.
(2) Substitute known values of the potential (boundary conditions) to obtain the constants of integration.
(3) Substitute the constants into the function in **(1)** to obtain the particular solution for V.

Method of Images The method solves for fields and potentials due to charges (point charges or charge distributions) in the presence of conductor(s) by removing the conductor(s) and replacing them with image charges that maintain the potential on the surface(s) unchanged. Then the solution outside the conductor(s) is due to the original charges and the image charges.

Flat Conducting Surfaces

(1) Image charges are equal in magnitude to the original charges, opposite in signs, and placed at the same distance below the surface.
(2) The number of image charges depends on the number of conducting surfaces.
(3) For two parallel conducting plates with point charges between them, the number of image charges is infinite.
(4) For conducting surfaces at an angle α, if $n = 180°/\alpha$ is an integer, the number of image charges is $2n - 1$. If n is not an integer, the method cannot be used (see **Example 5.6**).

Line, surface, or volume charge distributions in the presence of conductors behave the same way as point charges since they are assemblies of point charges. The images are then lines, surfaces, or volumes of the same geometric shape and size, mirrored about the conducting surface(s).

Cylindrical Surfaces Lines of charge parallel to conducting cylindrical surfaces reflect as follows:

(1) The image line of charge is parallel and equal in magnitude to the original line of charge. The sign of the image line of charge is opposite (**Figure 5.23**).
(2) A line of charge at a distance d outside a cylindrical conductor of radius $a < d$ produces an image charge at a distance b from the center of the conductor such that

$$b = \frac{a^2}{d} \quad [\text{m}]. \tag{5.35}$$

Point Charges and Conducting Spherical Surfaces A point charge q at a distance d from the center of a conducting sphere of radius $a < d$ creates an image with magnitude q' inside the sphere at a distance b from its center (**Figure 5.26**):

$$\boxed{b = \frac{a^2}{d} \quad [\text{m}], \quad q' = -\frac{qa}{d} \quad [\text{C}]} \tag{5.42}$$

Notes:

(1) In all cases, the images do not actually exist—they are artificially postulated for calculation purposes. They replace the effect of conductors.
(2) The fields found are only valid outside conductors.
(3) The potential of the conducting surface is assumed to be constant and, except for cylindrical surfaces, to be zero.
(4) The effect of any other potential, charge, or charge distribution must be added to the solution found by the method of images.

Separation of Variables Laplace's equation is a second-order partial differential equation and can be solved using the method of separation of variables in any system of coordinates. The general solution is the following.

In Cartesian coordinates (3 dimensions):

$$V(x,y,z) = [A_1\sin(k_x x) + A_2\cos(k_x x)]\left[A_3\sin(k_y y) + A_4\cos(k_y y)\right][A_5\sinh(|k_z|z) + A_6\cosh(|k_z|z)] \quad [\text{V}] \tag{5.59}$$

or

$$V(x,y,z) = \left[B_1 e^{jk_x x} + B_2 e^{-jk_x x}\right]\left[B_3 e^{jk_y y} + B_4 e^{-jk_y y}\right]\left[B_5 e^{|k_z|z} + B_6 e^{-|k_z|z}\right] \quad [\text{V}] \tag{5.60}$$

where

$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{5.52}$$

The constants A_i and B_i are evaluated from known boundary conditions (see **Example 5.15**).
In Cartesian coordinates (2 dimensions in x–z), with $k_z^2 = -k_x^2$:

$$V(x,z) = [A_1 \sin(k_x x) + A_2 \cos(k_x x)][A_5 \sinh(|k_z|z) + A_6 \cosh(|k_z|z)] \quad [\text{V}] \tag{5.61}$$

or

$$V(x,z) = \left[B_1 e^{jk_x x} + B_2 e^{-jk_x x}\right]\left[B_5 e^{j|k_z|z} + B_6 e^{-j|k_z|z}\right] \quad [\text{V}] \tag{5.62}$$

In 2-dimensional cylindrical coordinates (variation in the r–ϕ plane, no variation of the field in the z direction):

$$V(r,\phi) = [A_1 \sin(k\phi) + A_2 \cos(k\phi)]\left[B_3 r^k + B_4 r^{-k}\right] \quad [\text{V}] \tag{5.75}$$

or

$$V(r,\phi) = [A_1 e^{jk\phi} + A_2 e^{-jk\phi}][B_3 r^k + B_4 r^{-k}] \quad [\text{V}] \quad (5.76), \quad k^2 = k_\phi{}^2 = -k_r{}^2$$

The main difficulty in using these general solutions is the evaluation of the constants A_i and B_i from the boundary conditions. The task is highly dependent on geometry and the given boundary conditions.

Problems

Laplace's and Poisson's Equations

5.1 Solution to Laplace's Equation. The solution of Laplace's equation is known to be $V = f(x)g(y) = fg$, where f and g are scalar functions, f is a linear function of x alone, and g is a linear function of y alone. Which of the following is also a solution to Laplace's equation?

(a) $V = 2fg$.
(b) $V = fg + x + yg$.
(c) $V = 2f + cg$, where c is a constant.
(d) $V = f^2 g$.

5.2 Solution to Laplace's Equation. Given a potential $V(x,y,z) = 5xy + y^3 z + 5kz^2$, find an expression for k so that $V(x,y,z)$ satisfies Laplace's equation. Is the solution unique? Explain.

Direct Integration

5.3 Potential Distribution in a Capacitor. A parallel plate capacitor with distance between plates of 2 mm contains between its plates a dielectric with permittivity $\varepsilon = 4\varepsilon_0$ [F/m]. The potential difference between the plates is 5 V. Suppose the plate at $x = 0$ is at zero potential, the plate at $x = 0.002$ m is at 5 V. Find $V(x)$ everywhere assuming there are no edge effects.

5.4 Uniform Space Charge Density in a Capacitor. The parallel plate capacitor in **Figure 5.33** contains a dielectric with permittivity ε_0 and a constant charge density $\rho_v = \rho_0$ [C/m^3] distributed uniformly throughout the dielectric. Assume the plates are infinite and calculate the potential everywhere between the plates.

5.5 Nonuniform Space Charge in a Capacitor. A parallel plate capacitor with plates separated a distance $d = 2$ mm is connected to a 100 V source as in **Figure 5.33**. The space between the plates is air, but there is also a charge distribution between the plates given as $\rho_v = 10^{-6}x(x - d)$ [C/m^3] where x is the distance from the zero voltage plate. Assume the plates are large and calculate the potential everywhere between the plates.

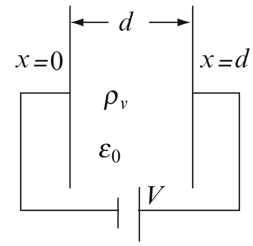


Figure 5.33

5.6 Potential and Field in Coaxial Cables. Coaxial cables used for cable TV also distribute power to amplifiers and other devices on the lines. Suppose a standard coaxial cable with an inner conductor made of a solid wire 0.5 mm in diameter and an outer shell 8 mm in diameter is used. A DC voltage of 64 V is connected with the positive pole connected to the inner conductor. Calculate:

(a) The potential distribution everywhere in the coaxial cable.
(b) The electric field intensity everywhere in the coaxial cable.
(c) Plot the potential and the electric field intensity as a function of distance from the center of the cable.

5.7 Potential Due to Large Planes. The planes $x = 0$ and $y = 0$ are conducting and are connected to potentials as shown in **Figure** 5.34. The two planes do not meet at the origin (a small gap exists between them). If a potential V_0 is given on the positive x axis, calculate the potential everywhere in the first quadrant. **Hint**: Use cylindrical coordinates.

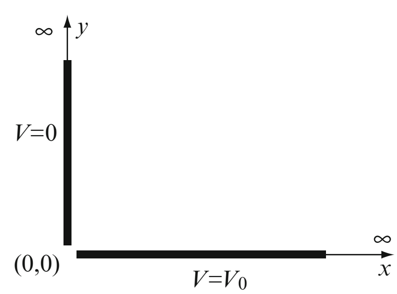


Figure 5.34

5.8 Potential and Field in Spherical Capacitor. A spherical capacitor is made of two concentric shells. The inner shell is 10 mm in diameter. The outer shell is 10.1 mm in diameter. A potential $V_0 = 50$ V is connected across the two shells so that the positive pole of the battery is connected to the inner shell. The space between the shells is filled with a dielectric with a relative permittivity of 2.25. Calculate:

(a) The potential everywhere between the shells. Does the potential depend on permittivity? Explain.
(b) The electric field intensity everywhere between the shells.
(c) Plot the potential and electric field intensity as a function of distance from the inner shell.

Method of Images: Point and Line Charges in Planar Configurations

5.9 Point Charge Above a Conducting Plane. A 5 nC point charge lies 2 m above a conducting plane. Find the surface charge density:

(a) Directly below the point charge, on the conducting plane.
(b) 1 m from the point in **(a)** (sideway).

5.10 Point Charges Above a Conducting Plane. Two point charges are located above a conducting plane as shown in **Figure 5.35**. Calculate the electric field intensity:

(a) In the space above the plane.
(b) At the surface of the conductor.

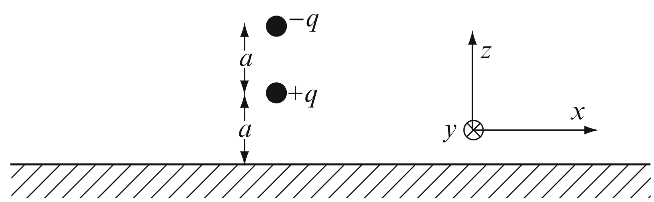


Figure 5.35

5.11 Charged Line Above a Conducting Plane. A very long charged line is located above a perfectly conducting plane as shown in **Figure 5.36**. The charge density on the line is ρ_l [C/m]:

(a) Find the surface charge density on the conducting plane.
(b) Show that the total charge per unit depth of the conducting surface equals $-\rho_l$ [C/m].

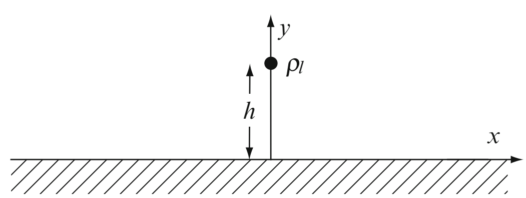


Figure 5.36

5.12 Sheet of Charge Above a Conducting Plane. An infinite sheet of charge with surface charge density ρ_s [C/m^2] is located at a distance d [m] above a very thick conductor, as in **Figure 5.37**. Calculate the electric field intensity everywhere.

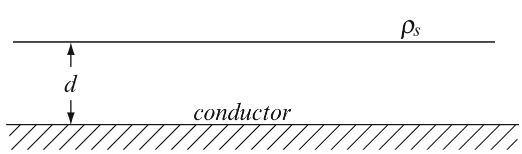


Figure 5.37

5.13 Charged Line and Multiple Conducting Planes. An infinitely long line charged with a line charge density ρ_l [C/m] is located at a distance d [m] from a very large conductor, as in **Figure 5.38**:

(a) Calculate the electric field intensity and the potential on the dotted line shown.
(b) A second conducting surface is now placed to fill the space to the right of the dotted line. What are now the electric field intensity and potential on this line? Compare with the results in **(a)**.

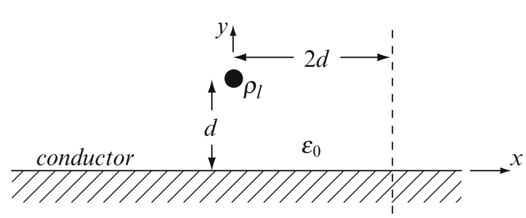


Figure 5.38

Method of Images: Multiple Planes

5.14 Point Charge Between Conducting Planes. A point charge Q [C] is located midway between two conducting plates. The plates are at 45° to each other:

(a) Show that the electric field intensity is perpendicular to the plates everywhere.
(b) Calculate the charge density induced on the plates if the radial distance from the intersection of the plates to the point charge is d [m].

5.15 Point Charge Between Intersecting Conducting Planes. Two conducting planes intersect at 30°, as in **Figure 5.39**. Both surfaces are at zero potential. A point charge q [C] is placed midway between the two planes at point A, as shown.

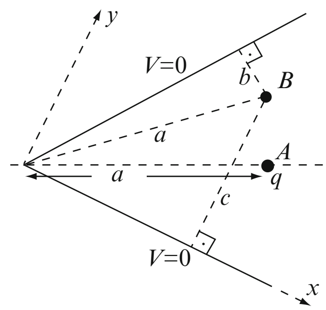


Figure 5.39

(a) Draw the system of image charges and calculate the electric field intensity everywhere between the planes.
(b) Show by means of a drawing what happens if the charge is moved from the center plane toward the upper plane so that it is at a distance b [m] from the upper plane and a distance c [m] from the lower plane while the radial distance is maintained (point B).
(c) Show by means of a drawing what happens if the angle is not an integer divisor of 180°. Use an angle of 75° as an example.

5.16 Point Charge Between Parallel Conducting Planes. A point charge is placed between two infinite planes as in **Figure 5.40**:

(a) Show the location and magnitude of the first few image charges.
(b) Place a system of coordinates so that the upper plate is at $x = 0$ and the point charge is at ($x = 3d/4$, $y = 0$), and calculate the potential at the middle point between the plates ($x = d/2$,$y = 0$) using the first six image charges (plus the original point charge).

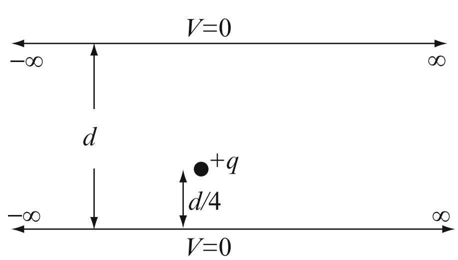


Figure 5.40

5.17 Point Charge Between Parallel Conducting Planes. A point charge is placed between two infinite planes, as in **Figure 5.40**:

(a) Place a system of coordinates so that the upper plate is at $x = 0$ and the point charge is at ($3d/4$,0). Find an expression for the potential at any point between the two plates as a sum on N image charges.

(b) Optional: Write a computer program that will evaluate the sum in **(a)** for $N = 7$, $N = 100$, $N = 1{,}000$, and $N = 100{,}000$, at $x = 0.05$ m, $y = 2.0$ m using $q = 1 \times 10^{-12}$ C, $d = 0.5$ m. Compare the results and provide an indication of how many image charges are necessary for a maximum relative error of 1%.

5.18 Point Charge Between Parallel Conducting Planes. Consider a point charge embedded in a dielectric material as shown in **Figure 5.41**. The dielectric is bound by two parallel plates, each held at a constant potential V_0 [V]:

(a) Can you use the method of images to find the field inside the dielectric. How?
(b) If so, find the first four image charges and their location and calculate the potential at any point between the plates. Place a system of coordinates so that the upper plate is at $x = 0$ and the point charge is at $(d_2,0)$.

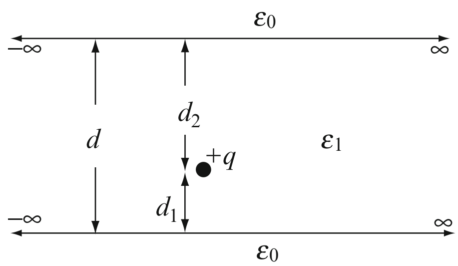


Figure 5.41

5.19 Arbitrary Charge Between Parallel Conducting Planes. A line charge distribution is placed as in **Figure 5.42** between two parallel (infinite) plates. The potential difference between the plates is V_1 [V]:

(a) Find the location, sign, and size of the first four image line charges.
(b) Explain why this is a correct way to solve the problem.
(c) How do you take into account the potential difference between the plates?

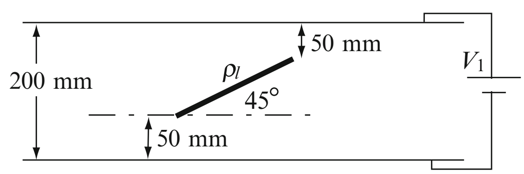


Figure 5.42

5.20 Point Charge in Infinite Closed Channel. Figure 5.43 shows a hollow cavity in a conducting medium. The cavity is infinite in the dimension perpendicular to the plane shown:

(a) Show the location of the image charges necessary to calculate the electric field intensity inside the cavity.
(b) Use the nearest four images and the charge q [C] to calculate an approximate value for the electric field intensity at the center of the box, in the plane shown. Place a system of coordinates so that the lower left corner of the cavity is at (0,0).

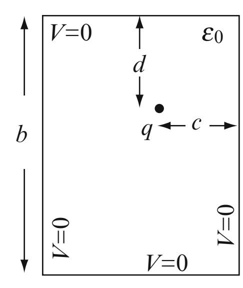


Figure 5.43

5.21 Point Charge in Infinite Closed Channel. The cavity in **Figure 5.43** is given again. The cavity is infinite in the dimension perpendicular to the plane shown:

(a) Show the image charges locations.
(b) Place a system of coordinates so that the lower left corner of the cavity is at (0, 0) and find a general expression for the potential at a general point within the cavity.
(c) Evaluate the expression in **(b)** using the closest eight image charges (plus the original charge) at the center of the cavity.
(d) Optional: Write a computer program that will evaluate the sum in **(b)** at $x = 1.0$ m, $y = 2.0$ m, for the closest $N = 9$, $N = 100$, $N = 1{,}000$, and $N = 100{,}000$ charges. Use $q = 1 \times 10^{-9}$ C, $a = 5$m, $b = 10$ m, $c = 2.5$ m, $d = 2.5$ m. Compare the results and provide an indication on how many image charges are necessary for a maximum incremental error of 1%.

Method of Images in Curved Geometries

5.22 Application: Cable in a Tunnel. A cable runs through a mine shaft suspended from the ceiling as shown in **Figure 5.44**. The cable is thin and carries a charge of 10 nC per meter. The shaft is 10 m in diameter. Calculate the electric field intensity at the center of the shaft (magnitude and direction). Assume ground is conducting, and the air in the shaft has permittivity of free space.

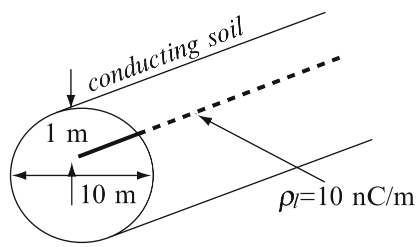


Figure 5.44

5.23 Two Charged Wires Next to a Conducting Cylinder. Two wires, one charged with a positive line charge density and one with a negative line charge density, run parallel to a thick conducting pipe as shown in **Figure 5.45**. Calculate the electric field intensity everywhere.

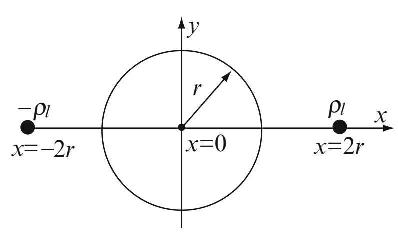


Figure 5.45

5.24 Point Charge in a Conducting Shell. A point charge + q [C] is located inside a very thin conducting spherical shell halfway between the center and the shell shown in **Figure 5.46**. Assume the conducting shell is at zero potential:

(a) Calculate the electric field intensity at the center of the sphere.
(b) What is the field outside the sphere?
(c) Plot the electric field intensity inside the sphere by plotting the field lines (approximate plot is sufficient).

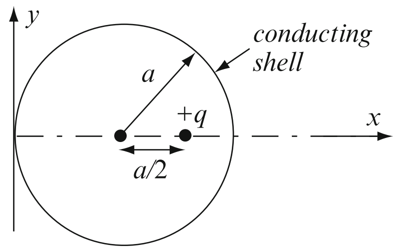


Figure 5.46

5.25 Point Charge Outside a Conducting Sphere. For the conducting sphere and point charge shown in **Figure 5.47**, find:

(a) The image charge (magnitude and location).
(b) The electric field intensity on the surface of the sphere.
(c) The induced charge density on the surface of the sphere.
(d) Sketch the electric field intensity outside the sphere.

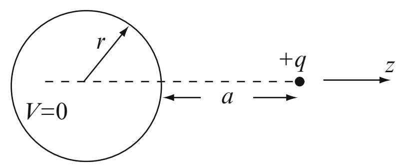


Figure 5.47

5.26 Point Charge Inside Hollow, Charged Conducting Sphere. Solve **Problem 5.24** if the shell is at a constant potential V_0 [V]. Sketch the electric field intensity outside the shell.

5.27 Point Charge Outside Charged Conducting Sphere. Solve **Problem 5.25** if the shell has a positive surface charge density ρ_s [C/m^2]. The potential on the shell need not be zero.

5.28 Two Point Charges Outside Grounded Conducting Sphere. A grounded (zero potential) conducting sphere and two point charges are located as shown in **Figure 5.48**:

(a) Calculate the electric field intensity at a general point outside the sphere.
(b) What is the electric field intensity inside the sphere? Explain.

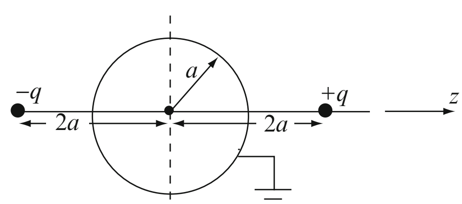


Figure 5.48

Separation of Variables in Planar Geometries

5.29 Potential in Infinite Channel. The geometry in **Figure 5.49** is made of a semi-infinite channel bounded by the planes $x = 0, x = \infty, y = 0, y = b$. Three sides are held at zero potential and the fourth is isolated from the others (between $y = 0$ and $y = b$) at potential V_0 [V]. Find the potential within the channel.

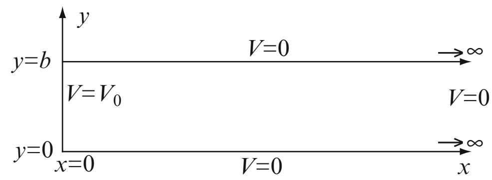


Figure 5.49

5.30 Potential in Infinite Channel. The infinite channel shown in **Figure** 5.50 is made of three conducting walls connected to a zero potential, and a fourth is an isolated wall (top cover) connected to a 10 V potential. The cover and sides are close to each other but are not touching. The material in the channel is free space:

(a) Calculate the potential everywhere in the channel.
(b) Plot the potential as a sequence of constant potential lines inside the box.

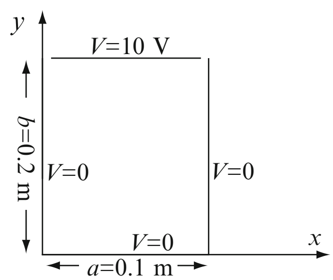


Figure 5.50

5.31 Potential in Infinite Channel. **Figure** 5.51 shows the cross section through an infinitely long channel. Three sides are connected to ground potential, and the top is insulated from the rest of the structure and connected to a potential that varies sinusoidally across the top as shown. The channel is filled with a material with permittivity ε. Calculate the potential everywhere inside the cross section.

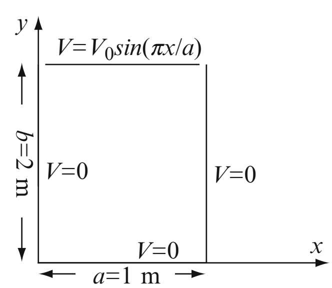


Figure 5.51

5.32 Potential in Infinite Channel. A two-dimensional geometry consists of four infinitely long plates, shown in cross section in **Figure** 5.52. For the given plate potentials, calculate the electric potential between the plates.

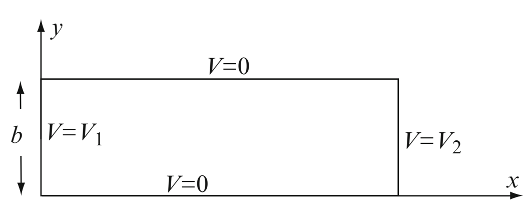


Figure 5.52

5.33 Potential in a Box with Conducting Walls. A three-dimensional cubic box, with dimensions $a \times a \times a$ [m^3], is grounded on four sides. The two sides at $z = a$ [m] and at $x = a$ [m] are connected to a constant potential as shown in **Figure 5.53**. Calculate the potential everywhere inside the box.

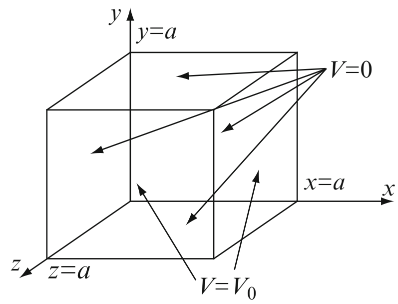


Figure 5.53

Separation of Variables in Cylindrical Geometries

5.34 Application: Electrostatic Precipitator. An electrostatic precipitator is made by lining the interior of a smokestack with two half-shells as shown in cross section in **Figure 5.54**. Assume the stack is very long and a potential difference of 100 kV is connected as shown. Calculate:

(a) The potential inside and outside the stack.
(b) The electric field intensity inside and outside the stack.

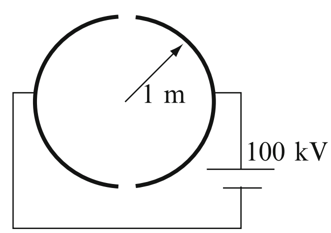


Figure 5.54

5.35 Application: Electrostatic Precipitator. The designers of the precipitator in **Problem 5.34** found that they cannot make two half-shells as needed, but they can make the precipitator of four quarter-shells and connect them as shown in **Figure 5.55**. All other data remain the same as in **Problem 5.34**. Calculate:

(a) The potential inside and outside the stack.
(b) The electric field intensity inside and outside the stack.

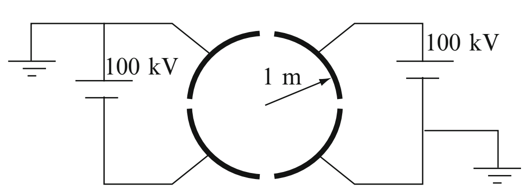


Figure 5.55