

I looked, and lo, a stormy wind came sweeping out of the north—a huge cloud and flashing fire, surrounded by a radiance; and in the center of it, in the center of the fire, a gleam as of amber.

—Ezekiel 1:4

3.1 Introduction

In the previous two chapters, we discussed in some detail the mathematics of electromagnetics: vector algebra and vector calculus. We are now ready to start looking into the physical phenomena of electromagnetics. It will be useful to keep this in mind: The study of electromagnetics is the study of natural phenomena. There are two reasons why it is important to emphasize electromagnetics as an applied science. First, it shows that it is a useful science, and its study leads to understanding of nature and, perhaps most importantly from the engineering point of view, to understanding of the application of electromagnetics to practical and useful designs. Second, all aspects of electromagnetics are based on experimental observations. All laws of electromagnetics were obtained by careful measurements which were then cast in the forms of simple laws. These laws are assumed to be correct simply because there is no evidence to the contrary. This aspect of the laws of electromagnetics should not bother us too much. Although we cannot claim absolute proof of correctness of the laws, experimentation has shown that they are correct and we will view them as such. In the learning process, we will make considerable use of the mathematical tools outlined in **Chapters 1** and **2**. It is easy to forget that the end purpose is physical design; however, every relation and every equation implies some physical quantity or property of the fields involved.

As with the study of any branch of science, we must start with the basics and proceed in a logical fashion. We will start with the study of electrostatic fields. To do so, we need a few assumptions that can be verified easily by experiment. In fact, the basic assumption is the existence of positive and negative electric charges (electrons and protons). Having allowed for their existence, we can then measure forces between charges, and these forces will lead to the definition of the electric field. The electric field is, therefore, merely a manifestation of forces on charges. We may even call it an electric force field. These forces are real forces and are measurable.

The static electric field is an exceedingly useful phenomenon that permeates our lives. The number of applications and effects that rely on electrostatics is vast. From the simplest of capacitors to thunderstorms, and from sand paper deposition to laser printers and memory chips, the use of static fields is the basis either of design of the device or explanation of the effects involved.

Thus, we will try to do two things; one is to state and explain the laws. This will require a mathematical exposition of relations between forces and charges, based on experimental results. At the same time, we will discuss at least a sampling of applications of electrostatic fields.

3.2 Charge¹ and Charge Density

The fundamental electric charge in nature is the charge of the electron. The fact that an electric charge exists was known to the ancient Greeks² who knew that rubbing a piece of amber with fur or silk caused an attraction of particles such as feathers, straw, or lint. Electron is the Greek name for amber.³ It took many years before it was understood what actually happened in this type of experiment or the amount of electric charge associated with the electron was established, but the effects of the charge were all around to be observed. For a body to contain free charge, there must be a way of removing electrons from one body and imparting them to another. The body from which electrons are removed becomes positively charged (because of excess protons) whereas the other body becomes negatively charged (excess electrons).⁴ This is what happened when amber was rubbed with fur: Electrons were removed from the fur and deposited on the amber. The rubbing action supplied the energy required for electrons to be removed. Amber therefore became negatively charged and could attract small bits of material. At the same time, the silk became positively charged. We know that lightning occurs when charge accumulates beyond a certain level. This means that charges must accumulate either in a volume or over some area. Since we know that electric charge causes forces in its vicinity, this force should also be proportional to the amount of charge. Before we can quantify the effects of electric charge, we must establish a few preliminaries. These include the charge of the electron, the unit of charge, and the definitions of point charges and distributed charges.

The unit of charge is called a *coulomb*.⁵ The charge of an electron is denoted by e and equals $e = -1.6019 \times 10^{-19}$ C. That is, one coulomb is equal to the charge of approximately 6.25×10^{18} electrons. The charge of the electron is considered to be the smallest unit of charge and all charges must be multiples of this quantity, although charge can be positive or negative. Charge may be distributed in space or may be concentrated in a small volume or a “point.”

Point Charge A charge that occupies a volume in space may be considered to be a point charge for analysis purposes if this volume is small compared to the surrounding dimensions. The charges of electrons or protons are often assumed to be point charges. For practical purposes, other charges such as the charge of a sphere are often considered to be point charges, provided that we are far from the sphere.

A charge density defines a distributed charge over a body. There are three types of charge densities:

Line Charge Density A charge distributed in a linear fashion such as along a very thin wire is given in charge per unit length. A charge density of 1 C/m means that one coulomb of charge is distributed per each meter length of the device

¹ The charge is a fundamental quantity of nature. It was even proposed as the basic quantity in the SI system of units, although the system as it stands now uses the ampere as the basic electric unit.

² Thales of Miletus (624?–546 B.C.E.), one of the “seven wise men” of ancient Greece. His work was mostly in geometry (for example, the theorem concerning the right angle in a semicircle and at least four others are attributed to him). Thales is thought to be the first to record this phenomenon, although it is almost certain that it was known before him. He himself traveled and studied in many parts of the ancient world, including Egypt. Miletus, in spite of his influence on later natural philosophers, did not write any of his views and findings (or none survived). The references to him come from later writing (primarily from Aristotle who wrote Thales’ record from oral records). The first written record on electricity comes from Theophrastus (371–288 B.C.E.) and dates around 300 B.C.E.

³ Amber has a curious relation to electricity. Although merely a yellowish fossilized tree resin, it has gained some prominence in our view of electricity. The Greek name for amber is electron (ηλεκτρον) (electrum in Latin) and means “bright” or “bright one,” perhaps a reference to the color of amber, a material which was held in very high esteem by the Greeks. Since amber was known to attract bodies when rubbed with fur or cloth, it eventually became a synonym to all electric phenomena, and in particular with electric charge. The actual name “electricity” was coined much later, around 1600, by William Gilbert. We should take some pride in having electricity associated with this material. It is beautiful and rare and precious. Also, it is a very good insulator.

⁴ This explanation was first put forward by Benjamin Franklin (1706–1790). Franklin was, in addition to being a statesman, diplomat, and publisher, a most prolific experimenter in electricity. Whereas the legendary experiment with the kite (June 1752) led to the discovery of atmospheric electricity and the lightning rod, he is also credited with being the first to propose the so-called “one-fluid” theory of electricity. Before him, it was assumed that there are two types of electricity: vitreous (from glass or what we call positive charge) and resinous (from resin like amber, which we call negative electricity or charge). He found that these are the same except that one is excess, whereas the other a deficiency in an otherwise balanced state (this, of course, happened about 130 years before the discovery of the electron). From this, he suggested the law of conservation of charge and coined the terms “positive” and “negative,” as well as the terms “charge” and “conductor.” Franklin held very modern views on electricity and magnetism and his work was the turning point in electromagnetics, a turn that led directly to the modern theory we use today.

⁵ After Charles Augustin de Coulomb (1736–1806). Coulomb was a colonel in the Engineering Corps of the French military who specialized in artillery, a career he abandoned shortly before the French Revolution due to health problems. The naming of the unit of charge after him indicates the importance of this work. Coulomb derived the law of force between charges, which we will investigate shortly.

(such as the wire above). A more exact description of the line charge density is to say that the charge density is the charge Δq distributed over a length Δl as Δl approaches zero:

$$\rho_l = \lim_{\Delta l \rightarrow 0} \frac{\Delta Q}{\Delta l} = \frac{dQ(l)}{dl} \quad \left[\frac{\text{C}}{\text{m}} \right] \quad (3.1)$$

Surface Charge Density A charge distributed over a given surface such as the surface of a sphere or a sheet of paper. A surface charge density of 1 C/m^2 means that one coulomb of charge is distributed over each square meter of the surface. The surface charge density is defined mathematically as

$$\rho_s = \lim_{\Delta s \rightarrow 0} \frac{\Delta Q}{\Delta s} = \frac{dQ(s)}{ds} \quad \left[\frac{\text{C}}{\text{m}^2} \right] \quad (3.2)$$

Volume Charge Density A charge distributed over a volume such as the volume of a cloud. A volume charge density of 1 C/m^3 means that one coulomb of charge is distributed over a one meter cube of volume:

$$\rho_v = \lim_{\Delta v \rightarrow 0} \frac{\Delta Q}{\Delta v} = \frac{dQ(v)}{dv} \quad \left[\frac{\text{C}}{\text{m}^3} \right] \quad (3.3)$$

The charge densities in **Eqs. (3.1)** through **(3.3)** may be uniform or nonuniform over the dimensions given. A uniform charge density means that the charge distributed over any equal section of surface, line, or volume is the same; that is, it is independent of the variables. A nonuniform charge density occurs when the charge in different sections of the charge distribution depends on location. Note, also, that the charge densities given above as examples are rather large densities. Normally, much smaller charge densities are encountered in practical designs.

Example 3.1 Line, Surface, and Volume Charge Density A charge is uniformly distributed over the following three structures such that a charge of $Q = 10^{-9} \text{ C}$ is distributed per unit length (1 m) of the device:

- (a) A very thin wire.
- (b) A conducting wire of radius $d = 10 \text{ mm}$. Assume charge can only exist on the surface of the wire.
- (c) A solid cylindrical, nonconducting material of radius $d = 10 \text{ mm}$ assuming the charge is uniformly distributed throughout the volume of the material. Calculate the charge density in each of the three structures.

Solution:

- (a) Because the wire is very thin, the charge is distributed along the length of the wire (conducting or nonconducting). This is an example of line charge distribution. The line charge density is

$$\rho_l = \frac{Q}{L} = \frac{10^{-9}}{1} = 10^{-9} \quad \left[\frac{\text{C}}{\text{m}} \right]$$

The notation ρ_l for line charge densities will be used throughout this book. **Figure 3.1a** shows this distribution schematically. Line charge densities are typically used for thin conductors or, sometimes, as approximations of thick, charged conductors.

- (b) Now, the same amount of charge is distributed per unit length of a cylindrical conductor of radius d . Since the charge on the conductor can only reside on the surface, the charge distributes itself (in this case uniformly) over the surface of the cylinder. The total external surface of a 1 m length of the cylindrical conductor is $(2\pi d)(1)$. Thus, the surface charge density on the conductor is

$$\rho_s = \frac{Q}{S} = \frac{10^{-9}}{2\pi d} = \frac{10^{-9}}{2\pi \times (0.01)} = 1.59155 \times 10^{-8} \quad \left[\frac{\text{C}}{\text{m}^2} \right]$$

Again, this notation (ρ_s) will be used throughout the book. **Figure 3.1b** shows a surface charge distribution. Typical situations in which surface charge density plays a role are conducting surfaces and interfaces between conducting and nonconducting materials, such as a carpet or the outer shell of a car.

- (c) In this case, charges are distributed throughout the volume. The volume of a cylinder of length 1 m and radius d is (πd^2) (1). The volume charge density is

$$\rho_v = \frac{Q}{v} = \frac{10^{-9}}{\pi d^2} = \frac{10^{-9}}{\pi \times (0.01)^2} = 3.183 \times 10^{-6} \left[\frac{\text{C}}{\text{m}^3} \right]$$

The notation is normally ρ_v . However, because volume charge densities are the most general, we will also use the simpler notation ρ . Unless a charge density is specifically indicated as surface or line, it is implicitly understood to be a volume charge density. **Figure 3.1c** shows a volume charge distribution. A typical example of volume charge density is charge in a cloud, space charge in a vacuum tube, or charge within the volume of a semiconductor device.

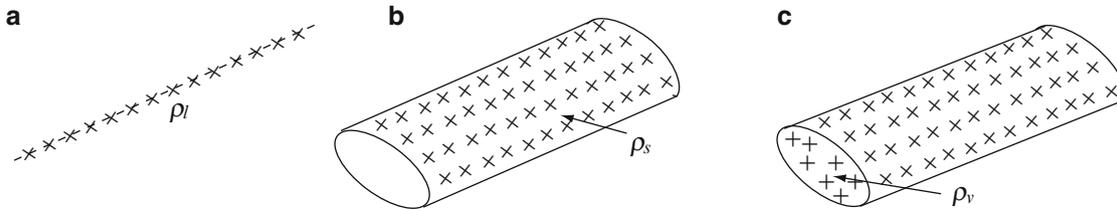


Figure 3.1 (a) Line charge distribution. (b) Surface charge distribution. (c) Volume charge distribution

Example 3.2 Nonuniform Charge Distribution A charge is distributed throughout a spherical volume (such as a cloud). The charge is nonuniformly distributed with a distribution function:

$$\rho_v = \rho_0 R \left[\frac{\text{C}}{\text{m}^3} \right]$$

where $\rho_0 = 10^{-7} \text{ C}$ and R is the distance from the center of the spherical volume. Calculate the total charge in a sphere of radius $R_0 = 10 \text{ m}$.

Solution: To calculate the total charge, we integrate over the sphere of radius R_0 using the charge density ρ_v . The element of volume in spherical coordinates is $dv = R^2 \sin\theta \, dR \, d\theta \, d\phi$. The total charge is therefore

$$Q = \int_v \rho_v dv = \rho_0 \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{R=0}^{R=R_0} R (R^2 \sin\theta dR d\theta d\phi) = \pi \rho_0 R_0^4 \quad [\text{C}]$$

Numerically, this gives 0.0031416 C. Note that if the charge density were uniform, we would simply multiply the charge density by the volume of the sphere.

The remainder of this chapter will discuss point charges and the interaction between them. Distributed charges will be discussed here as assemblages of point charges. In **Chapter 4**, we will view distributed charges in a somewhat different light and will develop methods suited for their treatment.

3.3 Coulomb's Law⁶

Coulomb's law is an experimental law obtained by Charles Augustin de Coulomb that defines quantitatively the force between two charges. It states that:

⁶ At the onset of the French Revolution, Charles Augustin de Coulomb started his work on electricity and magnetism, following a successful military career. In or around 1785, he formulated his now famous law, which he came about in an attempt to verify previous work by Joseph Priestley (1733–1804). However Coulomb's work was much more general than Priestley's and included both attraction and repulsion forces. Coulomb was a meticulous researcher who worked on other problems in science and engineering, including friction, soil mechanics, and elasticity. To perform the necessary experiments, he used a torsion balance, which he invented a year earlier (see **Figure 3.24**). The main advantage of this balance was in its ability to measure very small forces accurately. In addition, it was perhaps the first accurate measuring device used for measurements of electric quantities.

“the force between two point charges Q_1 and Q_2 is proportional to the product of the two charges, inversely proportional to the square of the distance between the two charges, and directed along the line connecting the two charges.”

The mathematical expression of Coulomb's law is

$$\mathbf{F} = \hat{\mathbf{R}} k \frac{Q_1 Q_2}{R^2} \quad [\text{N}] \quad (3.4)$$

where k is a proportionality factor, R is the distance between the two charges, and $\hat{\mathbf{R}}$ is a unit vector pointing from Q_1 to Q_2 if the force on Q_2 due to Q_1 is required, or from Q_2 to Q_1 if the force on Q_1 due to Q_2 is needed. We will expand on this shortly, but, for now, it is sufficient to look at the magnitude of the force. The factor k depends on the material in which the charges are located and is given as

$$k = \frac{1}{4\pi\epsilon} \left[\frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right] \quad (3.5)$$

where ϵ is a material constant. This material constant is called the **dielectric constant** or **permittivity** of the material and we will define it more accurately in **Chapter 4**. For now, it is only important to understand that it has a numerical value that depends on the material in which the charges reside. The magnitude of the force between the two charges is

$$F = \frac{Q_1 Q_2}{4\pi\epsilon R^2} \quad [\text{N}] \quad (3.6)$$

Before continuing, we should take a look at the units involved. Force is measured in **newtons** (N). Charges Q_1 and Q_2 are given in **coulombs** [C]. Thus, the unit of permittivity is given by the relation

$$\epsilon = \frac{Q_1 Q_2}{4\pi F R^2} \left[\frac{\text{C}^2}{\text{N} \cdot \text{m}^2} \right] \quad (3.7)$$

This unit is not normally employed simply because the quantity $[\text{C}^2/\text{N} \cdot \text{m}]$ is known as the **farad** [F]. In this chapter, the permittivity is taken as a given constant, so there is little reason to dwell on its meaning and the meaning of the units involved. These will become obvious in **Chapter 4**, when we talk about material properties. For now, we will simply accept the units of [F/m] or $[\text{C}^2/\text{N} \cdot \text{m}^2]$ for permittivity.

To avoid the need to discuss properties of materials that we have not encountered yet, we will limit our discussion here to charges in free space. For this purpose, the permittivity of free space is used. This is given as

$$\epsilon_0 = 8.8541853 \times 10^{-12} \approx 8.854 \times 10^{-12} \left[\frac{\text{F}}{\text{m}} \right] \quad (3.8)$$

The approximate value in **Eq. (3.8)** will be used throughout this book. Also, because the permittivities of free space and that of air are very close, the permittivity of free space will be used in air. The magnitude of the force between the two charges in free space becomes

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \quad [\text{N}] \quad (3.9)$$

Equation (3.6) or **(3.9)** gives the magnitude of the force between the two charges. Because charges Q_1 and Q_2 can be positive or negative, the force can change direction. To take into account this and the fact that force is a vector quantity, consider **Figure 3.2**:

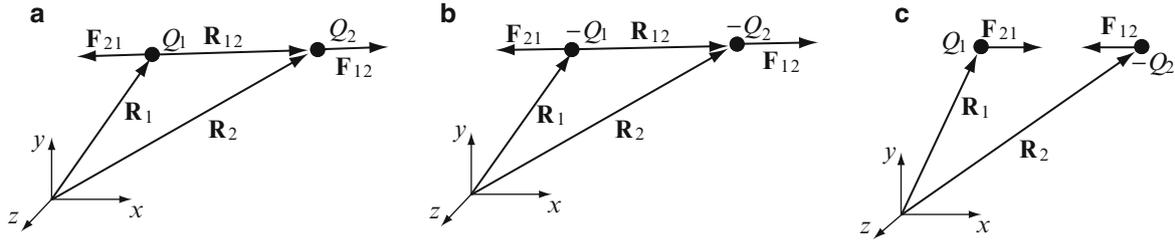


Figure 3.2 Relations between direction of forces and polarity of charge. (a) Two positive charges. (b) Two negative charges. (c) Two charges of opposite polarity

- (1) In **Figure 3.2a**, both charges are positive. According to the results of experimentation, the charges repel each other and the force is along the line connecting the two charges. This means that the direction of forces is away from the charges.
- (2) The force on charge Q_1 due to charge Q_2 is called \mathbf{F}_{21} and is equal in magnitude to the force on charge Q_2 due to charge Q_1 (\mathbf{F}_{12}) as expected from Newton's third law (action and reaction). The directions of these two forces are opposite each other.
- (3) In **Figure 3.2b**, both charges are negative, and, again, the charges repel each other as in **Figure 3.2a**.
- (4) In **Figure 3.2c**, one charge is positive, the other negative and the force is a force of attraction as shown.

To formalize these three results, we define the position vectors \mathbf{R}_1 and \mathbf{R}_2 as shown and define a vector that connects the two charges as

$$\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1 \quad \text{and} \quad \mathbf{R}_{21} = \mathbf{R}_1 - \mathbf{R}_2 \quad (3.10)$$

Now we note that the direction of the force on charge Q_1 in **Figures 3.2a** and **3.2b** is in the direction of \mathbf{R}_{21} , whereas the direction of the force on Q_2 is in the direction of \mathbf{R}_{12} . On the other hand, in **Figure 3.2c**, the forces point in opposite directions. Thus, using the vector notation, we can write the forces using the unit vectors as

$$\mathbf{F}_{12} = \hat{\mathbf{R}}_{12} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}, \quad \mathbf{F}_{21} = \hat{\mathbf{R}}_{21} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2}, \quad [\text{N}] \quad (3.11)$$

Using the definition of the unit vector in the direction of \mathbf{R}_{12} or \mathbf{R}_{21} as

$$\hat{\mathbf{R}}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_2 - \mathbf{R}_1}{|\mathbf{R}_2 - \mathbf{R}_1|} \quad \text{and} \quad \hat{\mathbf{R}}_{21} = \frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|} = \frac{\mathbf{R}_1 - \mathbf{R}_2}{|\mathbf{R}_1 - \mathbf{R}_2|} \quad (3.12)$$

we can also write

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 \mathbf{R}_{12}}{4\pi\epsilon_0 |\mathbf{R}_{12}|^3}, \quad \mathbf{F}_{21} = \frac{Q_1 Q_2 \mathbf{R}_{21}}{4\pi\epsilon_0 |\mathbf{R}_{21}|^3}, \quad [\text{N}] \quad (3.13)$$

or, in terms of position vectors,

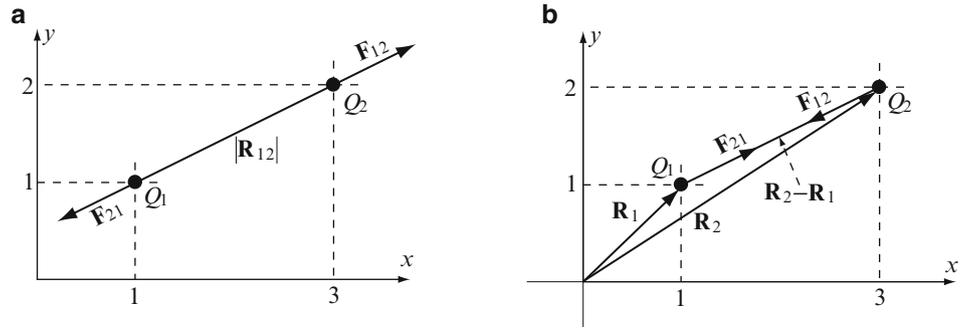
$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\mathbf{R}_2 - \mathbf{R}_1)}{4\pi\epsilon_0 |\mathbf{R}_2 - \mathbf{R}_1|^3}, \quad \mathbf{F}_{21} = \frac{Q_1 Q_2 (\mathbf{R}_1 - \mathbf{R}_2)}{4\pi\epsilon_0 |\mathbf{R}_1 - \mathbf{R}_2|^3}, \quad [\text{N}] \quad (3.14)$$

We have, therefore, three alternative methods of evaluating forces between two charges. **Equation (3.9)** gives only the magnitude of the force. **Equation (3.11)** gives both magnitude and direction using unit vectors, while **Eq. (3.13)** or **(3.14)** gives the force in terms of position vectors. In most cases **Eq. (3.13)** or **(3.14)** is preferable since it gives both direction and magnitude and does not require explicit calculation of the unit vector. Note that these relations also give the correct forces in the case in **Figure 3.2c**, where a negative sign is obtained from the product $Q_1 Q_2$. This negative sign indicates that the forces are in the directions opposite to those in **Eq. (3.13)** or **(3.14)**. From **Eqs. (3.11)**, **(3.13)**, and **(3.14)**, we also note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Example 3.3 Direction of Forces Between Charges Two charges, Q_1 [C] and Q_2 [C], are located at points $P_1(1,1,0)$ and $P_2(3,2,0)$ as shown in **Figure 3.3**:

- (a) Calculate the force on Q_1 and Q_2 if $Q_1 = 2 \times 10^{-9}$ C and $Q_2 = 4 \times 10^{-9}$ C.
 (b) Calculate the force on Q_1 and Q_2 if $Q_1 = 2 \times 10^{-9}$ C and $Q_2 = -4 \times 10^{-9}$ C.

Figure 3.3 (a) Forces between two positive or two negative charges. (b) Forces between a positive and a negative charge



Solution: The force must be calculated as a vector using either **Eq. (3.11)** or **Eq. (3.14)**. To demonstrate both forms, we will use **Eq. (3.11)** for (a) and **Eq. (3.14)** for (b). In the first case, the two charges are positive and the forces are repulsion forces. In the second case, one charge is positive, the other negative, and the charges attract each other:

(a) Consider **Figure 3.3a**. The solution based on **Eq. (3.11)** is

$$\mathbf{F}_{12} = \hat{\mathbf{R}}_{12} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}, \quad \mathbf{F}_{21} = \hat{\mathbf{R}}_{21} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2} \quad [\text{N}]$$

Calculation of the unit vector $\hat{\mathbf{R}}_{12}$ and $\hat{\mathbf{R}}_{21}$: First, we calculate the vector \mathbf{R}_{12} :

$$\mathbf{R}_{12} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) = \hat{\mathbf{x}}(3 - 1) + \hat{\mathbf{y}}(2 - 1) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}1$$

Therefore,

$$\mathbf{R}_{21} = -\mathbf{R}_{12} = -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}1$$

The unit vectors are

$$\hat{\mathbf{R}}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}1}{\sqrt{5}}, \quad \hat{\mathbf{R}}_{21} = \frac{-\hat{\mathbf{x}}2 - \hat{\mathbf{y}}1}{\sqrt{5}}$$

Thus, the force on Q_1 (i.e., the force \mathbf{F}_{21} , exerted by charge Q_2 on Q_1) is

$$\mathbf{F}_{21} = \hat{\mathbf{R}}_{21} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2} = \frac{-\hat{\mathbf{x}}2 - \hat{\mathbf{y}}1}{\sqrt{5}} \left(\frac{2 \times 10^{-9} \times 4 \times 10^{-9}}{4 \times \pi \times 8.854 \times 10^{-12} \times 5} \right) = -\hat{\mathbf{x}}12.86 \times 10^{-9} - \hat{\mathbf{y}}6.43 \times 10^{-9} \quad [\text{N}]$$

From the fact that $\mathbf{F}_{12} = -\mathbf{F}_{21}$, we get the force on Q_2 as

$$\mathbf{F}_{12} = \hat{\mathbf{x}}12.86 \times 10^{-9} + \hat{\mathbf{y}}6.43 \times 10^{-9} \quad [\text{N}]$$

These forces are indicated in their correct directions in **Figure 3.3a**.

(b) In this case, we use position vectors \mathbf{R}_1 and \mathbf{R}_2 and **Eq. (3.14)**. Position vectors \mathbf{R}_1 and \mathbf{R}_2 are (from **Figure 3.3b**)

$$\mathbf{R}_1 = \hat{\mathbf{x}}(x_1 - 0) + \hat{\mathbf{y}}(y_1 - 0) = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}1,$$

$$\mathbf{R}_2 = \hat{\mathbf{x}}(x_2 - 0) + \hat{\mathbf{y}}(y_2 - 0) = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}2$$

The force is now

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\hat{\mathbf{x}}_3 + \hat{\mathbf{y}}_2 - \hat{\mathbf{x}}_1 - \hat{\mathbf{y}}_1)}{4\pi\epsilon_0 |\hat{\mathbf{x}}_3 + \hat{\mathbf{y}}_2 - \hat{\mathbf{x}}_1 - \hat{\mathbf{y}}_1|^3} = \frac{\hat{\mathbf{x}}_2 + \hat{\mathbf{y}}_1}{5\sqrt{5}} \left(\frac{2 \times 10^{-9} \times (-4 \times 10^{-9})}{4 \times \pi \times 8.854 \times 10^{-12}} \right) = -\hat{\mathbf{x}} 12.86 \times 10^{-9} - \hat{\mathbf{y}} 6.43 \times 10^{-9} \quad [\text{N}]$$

and

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \hat{\mathbf{x}} 12.86 \times 10^{-9} + \hat{\mathbf{y}} 6.43 \times 10^{-9} \quad [\text{N}]$$

The forces now are opposite to those in part (a), because Q_2 is negative. The two forces are indicated in **Figure 3.3b**.

Example 3.4 Application: Electrostatic Forces Within the Atom Consider the following model of the helium atom: The atom has two electrons and two protons; assume the electrons are stationary (which they are not) as shown in **Figure 3.4** and the two protons are located at a point. Given: electron mass, $m_e = 9.107 \times 10^{-31}$ kg; distance between nucleus and electron, $R_e = 0.5 \times 10^{-10}$ m; charge of electron, $e = -1.6 \times 10^{-19}$ C:

- Calculate the force between the two electrons and the force between each electron and proton.
- Neglecting all other forces within the atom, what must the angular velocity of the electrons be to stay at the given distance from the nucleus, assuming the two electrons are always in the same relative position?

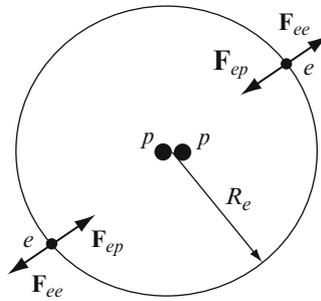


Figure 3.4 A simple model of the helium atom and the electrostatic forces involved

Solution: First, the repulsion forces between the electrons and attraction forces between electrons and protons are calculated. The net force is balanced by the centrifugal force due to orbiting of the electron. This gives the angular velocity required:

- The direction of the electrostatic force is radial. The two electrons repel each other, whereas the electrons are attracted to the protons. The repulsion forces between electrons are

$$\mathbf{F}_{ee} = \hat{\mathbf{R}} \frac{e^2}{4\pi\epsilon_0 (2R_e)^2} = \hat{\mathbf{R}} \frac{(-1.6 \times 10^{-19})^2}{4 \times \pi \times 8.854 \times 10^{-12} \times (2 \times 0.5 \times 10^{-10})^2} = \hat{\mathbf{R}} 2.3 \times 10^{-8} \quad [\text{N}]$$

Attraction force between each electron and the nucleus is that between a negative electron and two positive protons. In **Figure 3.4**, this force is in the negative R direction and is indicated as F_{ep} :

$$\mathbf{F}_{ep} = -\hat{\mathbf{R}} \frac{2e^2}{4\pi\epsilon_0 (R_e)^2} = -\hat{\mathbf{R}} \frac{2 \times (-1.6 \times 10^{-19})^2}{4 \times \pi \times 8.854 \times 10^{-12} \times (0.5 \times 10^{-10})^2} = -\hat{\mathbf{R}} 1.84 \times 10^{-7} \quad [\text{N}]$$

- For the electrons to remain in their orbit (based on the model used here, which does not consider atomic or gravitational forces), the net force on the electron must be balanced by the centrifugal force due to orbiting of the electron. The net electric force is an attraction force on each electron equal to $F_t = F_{ep} + F_{ee} = 1.61 \times 10^{-7}$ N.

The centrifugal force is mv^2/R , where v is the tangential speed of the electron, and is directed radially outward. Equating the two forces and substituting the values given yields

$$\frac{m_e v^2}{R_e} = F_t = F_{ep} + F_{ee} \quad \rightarrow \quad v = \sqrt{\frac{F_t R_e}{m_e}} = \sqrt{\frac{1.61 \times 10^{-7} \times 0.5 \times 10^{-10}}{9.107 \times 10^{-31}}} = 2.973 \times 10^6 \quad \left[\frac{\text{m}}{\text{s}} \right]$$

The angular velocity is found from the relation $v = \omega R$:

$$\omega = \frac{v}{R_e} = \frac{2.973 \times 10^6}{0.5 \times 10^{-10}} = 5.946 \times 10^{16} \quad \left[\frac{\text{rad}}{\text{s}} \right]$$

This translates to about 9.5×10^{15} orbits per second.

Note: There is much more going on in the atom than electrostatic forces, and the dimensions given here are merely assumed, but the calculation does give a flavor of the role electrostatic forces play.

3.4 The Electric Field Intensity

That there is a force acting on charges due to the presence of other charges is by now well understood. We also know how to calculate these forces for point charges. Now, we change our point of view slightly. Consider, for example, a point charge which we tie down so that it cannot be moved. Although nothing has changed in terms of the forces between the charges (remember: charges are assumed to be stationary), it is now more convenient to view the fixed charge as the source of the force acting on the second charge. Using **Figure 3.5**, the force on Q_2 due to Q_1 is

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 \mathbf{R}_{12}}{4\pi\epsilon_0 |\mathbf{R}_{12}|^3} = \left(\frac{Q_1}{4\pi\epsilon_0 |\mathbf{R}_{12}|^3} \right) Q_2 \mathbf{R}_{12} \quad [\text{N}] \quad (3.15)$$

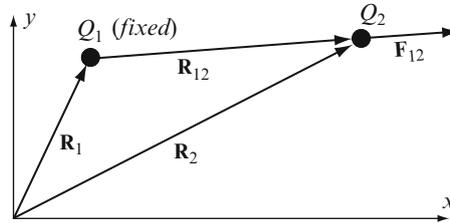


Figure 3.5 Force on charge Q_2 due to the electric field intensity produced by Q_1

If the two sides of the equation are divided by Q_2 , we get

$$\frac{\mathbf{F}_{12}}{Q_2} = \frac{Q_1 \mathbf{R}_{12}}{4\pi\epsilon_0 |\mathbf{R}_{12}|^3} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.16)$$

Inspection of this relation reveals the following:

- (1) The quantity obtained is force per unit charge [N/C].
- (2) The force per unit charge varies as $1/R_{12}^2$.
- (3) The right-hand side depends only on the fixed charge Q_1 and the vector \mathbf{R}_{12} ; that is, the vector field \mathbf{F}_{12}/Q_2 is generated by Q_1 .

Since the vector \mathbf{R}_{12} is arbitrary (i.e., it simply indicates where a charge Q_2 might be located), it is defined everywhere in space. \mathbf{F}_{12}/Q_2 is a vector field that gives the force per unit charge anywhere in space. We call this quantity the *electric field intensity*. In more formal fashion, the electric field intensity is defined as

$$\mathbf{E} = \lim_{Q \rightarrow 0} \frac{\mathbf{F}}{Q} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.17)$$

This indicates that the electric field intensity is a vector in the direction of the force and is proportional to the force. It gives the force per unit charge and has units of [N/C]. Another, more common unit for electric field intensity is *volt/meter* or [V/m]. In this chapter, we will use the [N/C] unit simply because the volt has not been defined yet. However, starting with **Chapter 4**, the [V/m] will be used exclusively.

Now, consider a point charge Q_1 as in **Figure 3.6a**. From **Eq. (3.13)**, the electric field intensity everywhere in space is equal to

$$\mathbf{E} = \frac{Q_1 \mathbf{R}}{4\pi\epsilon_0 |\mathbf{R}|^3} \quad \text{or} \quad \mathbf{E} = \hat{\mathbf{R}} \frac{Q_1}{4\pi\epsilon_0 |\mathbf{R}|^2} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.18)$$

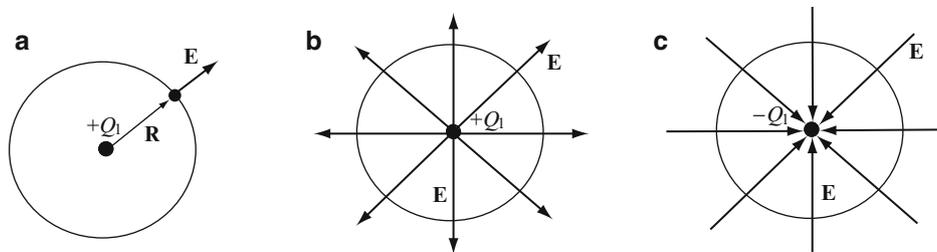


Figure 3.6 Electric field intensity due to a point charge. (a) Direction of the field of a positive point charge. (b) The electric field of a positive point charge. (c) The electric field of a negative point charge

A number of observations are in order here:

- (1) The electric field intensity depends on the charge and the distance R from the charge.
- (2) If the charge Q_1 is positive, the direction of the electric field intensity is from the point charge radially outward, since any point in space is connected with the charge along a radial line (**Figure 3.6b**).
- (3) If the charge Q_1 is negative (**Figure 3.6c**), the direction of the electric field intensity is directed radially toward the charge.
- (4) At equal distances from the charge, the magnitude of the electric field intensity is constant. This can be seen from the formula or simply from symmetry considerations.
- (5) The electric field intensity varies as $1/R^2$ where R is the distance from the charge.

Conclusion The electric field always points away from a positive charge and toward a negative charge. The electric field can be viewed as starting at a positive charge and ending at a negative charge. This is an important point: It will provide us with a reference for electric fields and forces. Based on this simple fact, we can always determine (with or without calculations) the directions of forces and of electric fields.

If we now introduce another point charge in the electric field of the first charge, there will be a force acting on this charge. The force is given as

$$\mathbf{F} = Q_2 \mathbf{E} \quad [\text{N}] \quad (3.19)$$

This is merely an alternative form of Coulomb's law as given in **Eq. (3.13)**. However, this form is conceptually different: Q_1 is viewed as the source of the electric field intensity and this electric field intensity exerts a force on Q_2 . We must hasten to say that this is only a convenience. It is equally acceptable to view Q_2 as the source of the field and the force is then exerted on Q_1 . This symmetry is again due to the fact that the magnitude of the force on each charge (but not the direction) is the same.

Example 3.5 Force Exerted by an Electron An electron is located at a point in space:

- (a) Calculate the electric field intensity everywhere in space. The charge of the electron is e .
 (b) Find the force the electron exerts on a dust particle, charged with a total charge of 3.2×10^{-19} C (two protons) and located at a distance R [m] from the electron.

Solution: The electron is a negative point charge. Assuming it is located at the origin of a spherical system of coordinates, the electric field intensity is radial as in **Figure 3.6c**:

- (a) The electric field intensity at a distance R from an electron is

$$\mathbf{E} = \hat{\mathbf{R}} \frac{e}{4\pi\epsilon_0 R^2} = -\hat{\mathbf{R}} \frac{1.6 \times 10^{-19}}{4 \times \pi \times 8.854 \times 10^{-12} \times R^2} = -\hat{\mathbf{R}} \frac{1.44 \times 10^{-9}}{R^2} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

- (b) The force on the dust particle is calculated from **Eq. (3.19)**:

$$\mathbf{F} = Q\mathbf{E} = -\hat{\mathbf{R}} \frac{3.2 \times 10^{-19} \times 1.44 \times 10^{-9}}{R^2} = -\hat{\mathbf{R}} \frac{4.61 \times 10^{-28}}{R^2} \quad [\text{N}]$$

Although this may seem to be a small force, it can be considerable at very short distances. In the limit, as R tends to zero, the force and the electric field intensity tend to infinity.

3.4.1 Electric Fields of Point Charges

Point_Charges.m

3.4.1.1 Superposition of Electric Fields

Before proceeding, it is well to establish the fact that superposition applies in the case of the electric field intensity. We might, of course, have suspected that it does since all relations discussed so far were linear. In our case, superposition means that the field of a charge is unaffected by the existence of other charges or by the electric fields these charges generate. For this reason, the electric field intensity at a point in space due to a number of charges is the **vector sum** of the electric field intensities of individual charges, each calculated in the absence of all other charges. Considering two charges Q_1 and Q_2 , located at points P_1 and P_2 as shown in **Figure 3.7**, the electric field intensity at a point P_3 is calculated as

$$\mathbf{E}_3 = \hat{\mathbf{R}}_{13} \frac{Q_1}{4\pi\epsilon_0 |\mathbf{R}_{13}|^2} + \hat{\mathbf{R}}_{23} \frac{Q_2}{4\pi\epsilon_0 |\mathbf{R}_{23}|^2} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.20)$$

and the individual electric field intensities are those of each charge, calculated as if the other charge does not exist. These two electric field intensities have different directions in space and must be added vectorially.

The same principle applies to forces. Placing a third charge Q_3 at point P_3 , the force on this charge is

$$\mathbf{F}_3 = Q_3 \mathbf{E}_3 = \hat{\mathbf{R}}_{13} \frac{Q_1 Q_3}{4\pi\epsilon_0 |\mathbf{R}_{13}|^2} + \hat{\mathbf{R}}_{23} \frac{Q_2 Q_3}{4\pi\epsilon_0 |\mathbf{R}_{23}|^2} \quad [\text{N}] \quad (3.21)$$

as expected. Note that in addition to this force, which the electric field of charges Q_1 and Q_2 exert on charge Q_3 , there is also a force between charges Q_1 and Q_2 according to Coulomb's law. This force between Q_1 and Q_2 does not affect the force on Q_3 . The expressions above can now be generalized for any number of point charges: Assuming n charges, the electric field intensity at a general point P_k is

$$\mathbf{E}_k = \sum_{\substack{i=1 \\ i \neq k}}^n \frac{Q_i (\mathbf{R}_k - \mathbf{R}_i)}{4\pi\epsilon_0 |\mathbf{R}_k - \mathbf{R}_i|^3} = \sum_{\substack{i=1 \\ i \neq k}}^n \hat{\mathbf{R}}_{ik} \frac{Q_i}{4\pi\epsilon_0 |\mathbf{R}_{ik}|^2} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.22)$$

and for force is

$$\mathbf{F}_k = Q_k \mathbf{E}_k = Q_k \sum_{\substack{i=1 \\ i \neq k}}^n \frac{Q_i (\mathbf{R}_k - \mathbf{R}_i)}{4\pi\epsilon_0 |\mathbf{R}_k - \mathbf{R}_i|^3} = Q_k \sum_{\substack{i=1 \\ i \neq k}}^n \hat{\mathbf{R}}_{ik} \frac{Q_i}{4\pi\epsilon_0 |\mathbf{R}_{ik}|^2} \quad [\text{N}] \quad (3.23)$$

When using this form of the field of multiple charges, it is important to remember that this is a summation of vectors. Each electric field intensity \mathbf{E}_i or electric force \mathbf{F}_i is directed in a different direction in space as indicated by its unit vector. If a total vector field is required, the components of the fields must be calculated and added to produce a single vector field.

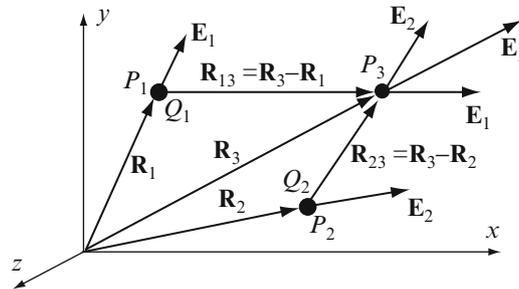


Figure 3.7 Electric field intensity at a general point in space, due to two point charges

3.4.1.2 Electric Field Lines

In an attempt to visualize the electric field, it is customary to draw the electric field intensity in terms of field lines. These are imaginary lines that show the direction of force on an infinitesimal positive point charge if it were placed in the field. The electric field intensity is everywhere tangential to field lines. Field lines can also be called force lines. Plots of field lines are quite useful in describing, qualitatively, the behavior of the electric field and of charges in the electric field. The electric field intensity of the point charge in **Figures 3.6b** and **3.6c** shows the electric field lines for a positive and a negative point charge. Similar sketches of more complicated field distributions help in understanding the field distribution in space. For example, **Figure 3.8** shows the field lines of two equal but opposite point charges. The following should be noted from this description of the field:

- (1) Field lines begin at positive charges and end in negative charges. If only one type of charge exists, the lines start or end at infinity (**Figures 3.6b** and **3.6c**).
- (2) Field lines show the direction of force on a positive point charge if it were placed in the field and, therefore, also show the direction of the electric field intensity. The arrows help in showing the direction of force and field.
- (3) Field lines are imaginary lines; their only purpose is to visualize the electric field.

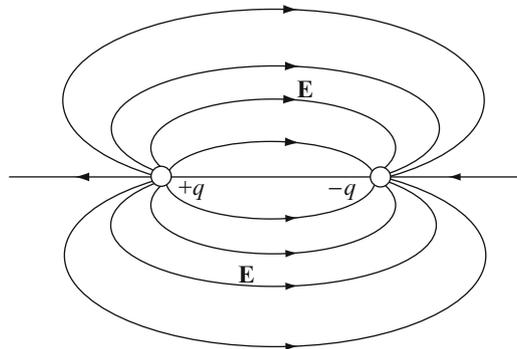
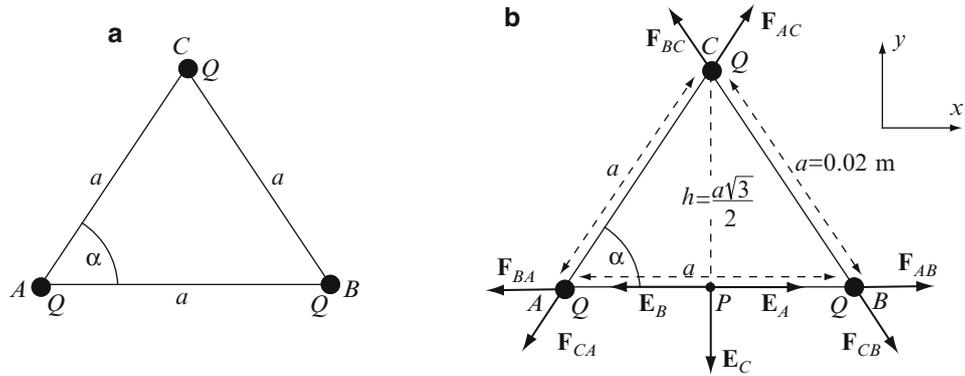


Figure 3.8 Field line representation of the electric field of two point charges

Example 3.6 Forces in a System of Charges Three equal point charges Q [C] are located as shown in **Figure 3.9a**. Each two charges are connected with a very thin string to hold them in place. The string is designed to break when a force of 0.1 N is applied.

- (a) Calculate the charge Q required to break the strings, if $a = 20$ mm.
- (b) What is the electric field intensity at the center of the string $A-B$?

Figure 3.9 (a) Three point charges connected by strings. (b) Forces on the charges and the electric field intensity at P



Solution: Each charge applies a force on each other charge. There are, therefore, six forces, two at each vertex of the triangle. The tension on a string is due to all forces acting along the string. These are shown in **Figure 3.9b**. The electric field intensity at the center of string $A-B$ is due to the charge at C only since charges at A and B produce electric field intensities in opposite directions at point P and they cancel each other:

- (a) The force on any string is composed of two parts: One is the force which acts along the string and the other is the projection of the second force onto the direction of the first. For example, the total force (magnitude) on the string connecting charge A and B is $F_{BA} + F_{CA}\cos\alpha$. This force acts along the string as shown. The magnitude of any of the forces \mathbf{F}_{AB} , \mathbf{F}_{BA} , \mathbf{F}_{AC} , \mathbf{F}_{CA} , \mathbf{F}_{BC} , and \mathbf{F}_{CB} is

$$|\mathbf{F}_{ij}| = \frac{Q_i Q_j}{4\pi\epsilon_0 |\mathbf{R}_{ij}|^2} = \frac{Q^2}{4\pi\epsilon_0 a^2} \quad [\text{N}]$$

The force acting on any string is therefore

$$F_t = F_{ij} + F_{ij}\cos\alpha = \frac{Q^2}{4\pi\epsilon_0 a^2} (1 + \cos 60^\circ) = \frac{3Q^2}{8\pi\epsilon_0 a^2} \quad [\text{N}]$$

For the string to break, this force must be larger than 0.1 N:

$$\frac{3Q^2}{8\pi\epsilon_0 a^2} \geq 0.1\text{N} \quad \rightarrow \quad Q \geq a\sqrt{\frac{0.8\pi\epsilon_0}{3}} = 5.45 \times 10^{-8} \quad [\text{C}]$$

- (b) The electric field intensity at point P is in the negative y direction (away from charge C), whereas the electric field intensities E_A and E_B due to charges A and B cancel as shown in **Figure 3.9b**:

$$\mathbf{E}_C = -\hat{\mathbf{y}} \frac{Q^2}{4\pi\epsilon_0 h^2} = -\hat{\mathbf{y}} \frac{Q^2}{\pi\epsilon_0 3a^2} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

Example 3.7 Electric Field Due to a System of Charges Three charges are arranged as shown in **Figure 3.10a**:

- (a) Calculate the electric field intensity everywhere on the x axis.
 (b) What are the points on the axis at which the electric field intensity is zero (other than at infinity)?

Solution: The electric field intensity is the superposition of the electric field intensity of the three charges. To find the location of zero electric field intensity, we assume a location on the axis, say $+x$, calculate the electric field intensity, and set it to zero. However, there are four distinct domains that must be considered: Two are $x > a$ and $x < -a$. The other two domains are $0 < x < a$ and $-a < x < 0$. Because the configuration of charges is symmetric, only two of these four domains need to be considered. Therefore, we solve for the domains $x > a$ and $0 < x < a$:

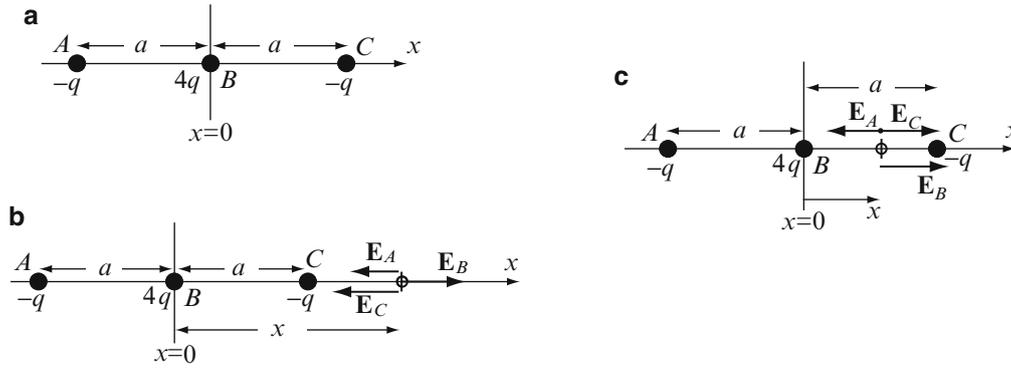


Figure 3.10 (a) Three point charges on a line. (b) The electric field intensity at $x > a$. (c) The electric field intensity at $x < a$

(a) Consider a point at a distance $+x$ from the charge $+4q$ such that $x > a$. Since the three charges are on the axis, the electric field will also be directed along the axis. The electric field intensity for any value of $x > a$ is (see **Figure 3.10b**)

$$\mathbf{E} = \hat{\mathbf{x}} \left(\frac{4q}{4\pi\epsilon x^2} - \frac{q}{4\pi\epsilon(x-a)^2} - \frac{q}{4\pi\epsilon(x+a)^2} \right), \quad x > a \quad \left[\frac{\text{N}}{\text{C}} \right]$$

From symmetry considerations, the electric field intensity for $x < -a$ is

$$\mathbf{E} = -\hat{\mathbf{x}} \left(\frac{4q}{4\pi\epsilon x^2} - \frac{q}{4\pi\epsilon(x-a)^2} - \frac{q}{4\pi\epsilon(x+a)^2} \right), \quad x < -a \quad \left[\frac{\text{N}}{\text{C}} \right]$$

For a point between $x = 0$ and $x = a$, the configuration is as in **Figure 3.10c**. The electric field intensity is

$$\mathbf{E} = \hat{\mathbf{x}} \left(\frac{4q}{4\pi\epsilon x^2} + \frac{q}{4\pi\epsilon(a-x)^2} - \frac{q}{4\pi\epsilon(x+a)^2} \right), \quad 0 < x < a \quad \left[\frac{\text{N}}{\text{C}} \right]$$

and for $-a < x < 0$,

$$\mathbf{E} = -\hat{\mathbf{x}} \left(\frac{4q}{4\pi\epsilon x^2} + \frac{q}{4\pi\epsilon(a-x)^2} - \frac{q}{4\pi\epsilon(x+a)^2} \right), \quad -a < x < 0 \quad \left[\frac{\text{N}}{\text{C}} \right]$$

(b) To find the points at which the electric field intensity is zero, we set the fields in (a) to zero and find a solution for x . For $x > a$: Taking only the magnitude of \mathbf{E} , we get

$$E = \frac{4q}{4\pi\epsilon x^2} - \frac{q}{4\pi\epsilon(x-a)^2} - \frac{q}{4\pi\epsilon(x+a)^2} = 0 \quad \rightarrow \quad \frac{4}{x^2} - \frac{1}{(x-a)^2} - \frac{1}{(x+a)^2} = 0$$

or

$$\frac{4}{x^2} - \frac{2(x^2 + a^2)}{(x^2 - a^2)^2} = 0 \quad \rightarrow \quad (x^2)^2 - 5a^2(x^2) + 2a^4 = 0$$

This is a quadratic equation in x^2 : Solving for x^2 and then taking the square root to find x gives

$$x = \pm 2.13578a \quad \text{or} \quad x = \pm 0.66215a \quad [\text{m}]$$

The last two solutions are not valid (because we assumed $x > a$). Thus, the first two solutions are correct and the electric field intensity is zero at $x = \pm 2.13578a$.

For $0 < x < a$: Following steps similar to the previous case:

$$\frac{4q}{4\pi\epsilon x^2} + \frac{q}{4\pi\epsilon(a-x)^2} - \frac{q}{4\pi\epsilon(x+a)^2} = 0 \quad \rightarrow \quad x^4 + ax^3 - 2a^2x^2 + a^4 = 0$$

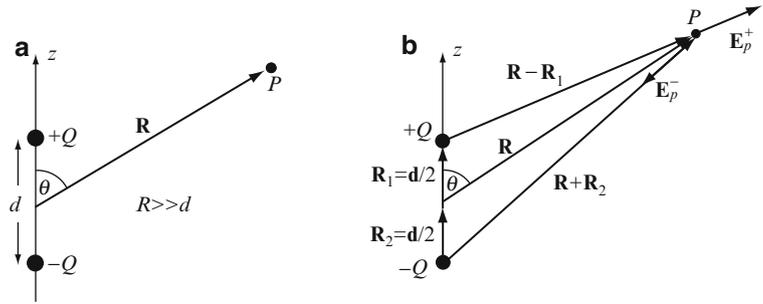
The two real solutions (the other two are imaginary) to this equation are $x = -0.671a$ [m] and $x = -1.905a$ [m]. Since neither is in the required domain (and the other two solutions are imaginary), the electric field intensity cannot be zero in the given domain. The solution in the domain $-a < x < 0$ leads to similar conclusions. Thus, the only locations at which the electric field intensity is zero are $x = \pm 2.13578a$ [m].

Note: That the electric field intensity cannot be zero between $x = 0$ and $x = a$ (or $x = 0$ and $x = -a$) can be seen from the fact that, for example, E_A must counter $E_B + E_C$ (**Figure 3.10c**), both of which are larger than E_A .

3.4.1.3 The Electric Dipole

A configuration of practical importance is that of two charges, separated a very short distance apart as shown in **Figure 3.11a**. If, in addition, the electric field intensity at a distance $R \gg d$ is needed, the configuration is called an *electric dipole*. The electric dipole is often encountered when dealing with electric fields of atoms and since these fields are fundamental in our attempt to understand the behavior of dielectrics, we shall now discuss the electric field intensity of the electric dipole.

Figure 3.11 The dipole.
(a) Configuration. (b) The electric field intensity at P



The electric field intensities of the negative and positive point charges are calculated using the position vectors indicated in **Figure 3.11b**:

$$\mathbf{E}_p^- = -\frac{Q(\mathbf{R} + \mathbf{R}_2)}{4\pi\epsilon_0|\mathbf{R} + \mathbf{R}_2|^3} = -\frac{Q(\mathbf{R} + \mathbf{d}/2)}{4\pi\epsilon_0|\mathbf{R} + \mathbf{d}/2|^3} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.24)$$

$$\mathbf{E}_p^+ = \frac{Q(\mathbf{R} - \mathbf{R}_1)}{4\pi\epsilon_0|\mathbf{R} - \mathbf{R}_1|^3} = \frac{Q(\mathbf{R} - \mathbf{d}/2)}{4\pi\epsilon_0|\mathbf{R} - \mathbf{d}/2|^3} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.25)$$

These two field intensities are in different directions in space (see **Figure 3.11b**). Their vector sum is

$$\mathbf{E}_d = \mathbf{E}_p^+ + \mathbf{E}_p^- = \frac{Q}{4\pi\epsilon_0} \left(\frac{(\mathbf{R} - \mathbf{d}/2)}{|\mathbf{R} - \mathbf{d}/2|^3} - \frac{(\mathbf{R} + \mathbf{d}/2)}{|\mathbf{R} + \mathbf{d}/2|^3} \right) \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.26)$$

Although **Eq. (3.26)** is exact, we can obtain an approximate, simplified form which is often more useful, by using the fact that $R \gg d$, together with the use of the binomial expansion. First we write

$$\frac{1}{|\mathbf{R} + \mathbf{d}/2|^3} = \left[\left(\mathbf{R} + \frac{\mathbf{d}}{2} \right) \cdot \left(\mathbf{R} + \frac{\mathbf{d}}{2} \right) \right]^{-3/2} = \left[R^2 + \mathbf{R} \cdot \mathbf{d} + \frac{d^2}{4} \right]^{-3/2} \quad (3.27)$$

$$\frac{1}{|\mathbf{R} - \mathbf{d}/2|^3} = \left[\left(\mathbf{R} - \frac{\mathbf{d}}{2} \right) \cdot \left(\mathbf{R} - \frac{\mathbf{d}}{2} \right) \right]^{-3/2} = \left[R^2 - \mathbf{R} \cdot \mathbf{d} + \frac{d^2}{4} \right]^{-3/2} \quad (3.28)$$

Taking the term $(R^2)^{-3/2}$ outside the brackets and neglecting the term $d^2/4$ (because $d \ll R$),

$$\left| \mathbf{R} + \frac{\mathbf{d}}{2} \right|^{-3} \approx R^{-3} \left[1 + \frac{\mathbf{R} \cdot \mathbf{d}}{R^2} \right]^{-3/2} \quad \text{and} \quad \left| \mathbf{R} - \frac{\mathbf{d}}{2} \right|^{-3} \approx R^{-3} \left[1 - \frac{\mathbf{R} \cdot \mathbf{d}}{R^2} \right]^{-3/2} \quad (3.29)$$

The binomial expansion states

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + \frac{n(n-1)(n-2)\dots(n-k)x^k}{k!} + \dots, \quad |x| < 1, n \text{ real} \quad (3.30)$$

Using $x = \mathbf{R} \cdot \mathbf{d}/R^2$, $n = -3/2$, and neglecting all terms with orders of x larger than 1 in the expression for $|\mathbf{R} + \mathbf{d}/2|^{-3}$ (x is small because $d \ll R$), we get

$$\left| \mathbf{R} + \frac{\mathbf{d}}{2} \right|^{-3} \approx R^{-3} \left(1 - \frac{3 \mathbf{R} \cdot \mathbf{d}}{2 R^2} \right) \quad (3.31)$$

Similarly, using $x = -\mathbf{R} \cdot \mathbf{d}/R^2$, $n = -3/2$, and neglecting all terms with orders of x larger than 1 in the expression for $|\mathbf{R} - \mathbf{d}/2|^{-3}$ gives

$$\left| \mathbf{R} - \frac{\mathbf{d}}{2} \right|^{-3} \approx R^{-3} \left(1 + \frac{3 \mathbf{R} \cdot \mathbf{d}}{2 R^2} \right) \quad (3.32)$$

Substituting the approximations in **Eqs. (3.31)** and **(3.32)** into **Eq. (3.26)**, we get

$$\mathbf{E}_d \approx \frac{1}{4\pi\epsilon_0 R^3} \left[\left(\mathbf{R} - \frac{\mathbf{d}}{2} \right) \left(1 + \frac{3 \mathbf{R} \cdot \mathbf{d}}{2 R^2} \right) - \left(\mathbf{R} + \frac{\mathbf{d}}{2} \right) \left(1 - \frac{3 \mathbf{R} \cdot \mathbf{d}}{2 R^2} \right) \right] = \frac{Q}{4\pi\epsilon_0 R^3} \left(3 \frac{\mathbf{R} \cdot \mathbf{d}}{R^2} \mathbf{R} - \mathbf{d} \right) \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.33)$$

The common terms in this expression are Q and \mathbf{d} . For convenience we define a new vector:

$$\mathbf{p} = Q\mathbf{d} \quad [\text{C} \cdot \text{m}] \quad (3.34)$$

which we call the *electric dipole moment*. With this, the electric field intensity may be written as

$$\mathbf{E}_d \approx \frac{1}{4\pi\epsilon_0 R^3} \left(3 \frac{(\mathbf{R} \cdot \mathbf{p})}{R^2} \mathbf{R} - \mathbf{p} \right) \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.35)$$

In the configuration in **Figure 3.11b**, the dipole is along the z axis, and, therefore, the electric dipole moment is in the z direction ($\mathbf{p} = \hat{\mathbf{z}}p = \hat{\mathbf{z}}Qd$). If we transform this into spherical coordinates [see **Section 1.5.3**, and **Eq. (2.45)**], we get

$$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \quad \rightarrow \quad \mathbf{p} = p(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta) \quad (3.36)$$

and

$$\mathbf{R} \cdot \mathbf{p} = [\hat{\mathbf{R}}R] \cdot [p(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)] = R p \cos\theta \quad (3.37)$$

Substituting these into the expression for the electric field intensity of the dipole gives

$$\mathbf{E}_d \approx \frac{1}{4\pi\epsilon_0 R^3} \left[3 \frac{R p \cos\theta}{R^2} \hat{\mathbf{R}} R - p(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta) \right] = \frac{p}{4\pi\epsilon_0 R^3} (\hat{\mathbf{R}} 2\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta) \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.38)$$

Note that the electric field intensity of a dipole varies as $1/R^3$, in contrast to that of a point charge, which varies as $1/R^2$.

A plot of the electric field intensity of the dipole is shown in **Figure 3.12**. Note, in particular, that all field lines are closed through the charges and that the field distribution is symmetric about the dipole axis.

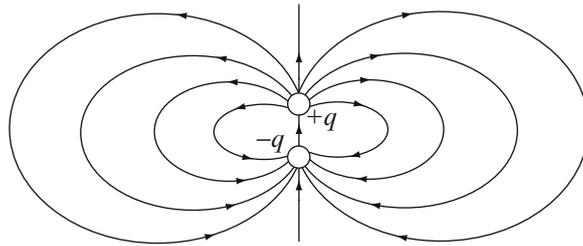


Figure 3.12 The electric field intensity of an electric dipole

Exercise 3.1 Calculate the electric field intensity of a configuration as in **Figure 3.11a** but with the positive charge twice as large as the negative charge.

Answer

$$\mathbf{E}_d \approx \frac{1}{4\pi\epsilon_0 R^3} \left[\hat{\mathbf{R}} \left(RQ + 3Qd\cos\theta - \frac{3Qd^2\cos^2\theta}{4R} \right) + \hat{\boldsymbol{\theta}} \left(\frac{3}{2}Qd\sin\theta + \frac{3Qd^2\sin\theta\cos\theta}{4R} \right) \right] \quad \left[\frac{\text{N}}{\text{C}} \right]$$

3.4.2 Electric Fields of Charge Distributions

As mentioned above, point charges are only one type of charge possible. Line, surface, and volume charge distributions are also quite commonly encountered, and the question that arises naturally is how do we calculate electric fields and forces due to distributed charges. Charge distributions were defined in **Section 3.2** as charges spread over a given domain such as a volume, a surface, or a line. We may argue that all charge distributions are composed of point charges, since charges exist in multiples of the charge of electrons (or protons). However, the charge of the electron is so small that for practical purposes, we can view a charge distribution as a continuous distribution.

To treat charge distributions and the electric fields they produce, we will use the ideas of point charges and superposition; a differential point charge is defined as an elemental point charge, and the contributions of all elemental charges are summed up to produce the net effect such as the electric field intensity or force due to a charge distribution.

3.4.2.1 Line Charge Distributions

Consider **Figure 3.13** in which a charge is distributed over a line. Designating a differential length dl' at a point (x', y', z') , the total charge on this element is $\rho_l dl'$.

This charge can be viewed as an equivalent point charge $dQ = \rho_l dl'$. The electric field intensity at a point in space due to this elemental point charge is

$$d\mathbf{E} = \frac{(\mathbf{r} - \mathbf{r}')\rho_l dl'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3} \begin{bmatrix} \text{N} \\ \text{C} \end{bmatrix} \quad (3.39)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ is the vector connecting the point charge with the point at which the electric field intensity is required. This is integrated along the line of charge to obtain the field due to a segment of the line or due to the whole line

$$\mathbf{E} = \int_{l'} \frac{(\mathbf{r} - \mathbf{r}')\rho_l dl'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3} \begin{bmatrix} \text{N} \\ \text{C} \end{bmatrix} \quad (3.40)$$

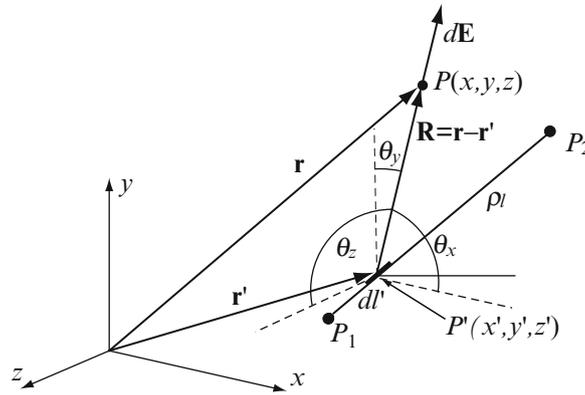


Figure 3.13 Electric field intensity due to a charged line element

The direction of the electric field intensity is in the direction of $\mathbf{r} - \mathbf{r}'$, as shown in **Figure 3.13**. Since as we integrate along the line, this direction changes, it is easier to separate the electric field intensity into its three components using the three angles indicated in the figure. To do so, we note that the angles that $d\mathbf{E}$ makes with the three coordinates are the same as those made by the vector $\mathbf{r} - \mathbf{r}'$. The latter are written from **Figure 3.13**:

$$\frac{(dE_x)}{(dE)} = \frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|} = \cos\theta_x, \quad \frac{(dE_y)}{(dE)} = \frac{(\mathbf{r} - \mathbf{r}')_y}{|\mathbf{r} - \mathbf{r}'|} = \cos\theta_y, \quad \frac{(dE_z)}{(dE)} = \frac{(\mathbf{r} - \mathbf{r}')_z}{|\mathbf{r} - \mathbf{r}'|} = \cos\theta_z \quad (3.41)$$

where $(\mathbf{r} - \mathbf{r}')_x$, $(\mathbf{r} - \mathbf{r}')_y$, and $(\mathbf{r} - \mathbf{r}')_z$ are the scalar components of the vector $\mathbf{r} - \mathbf{r}'$ in the x , y , and z directions, respectively. With $dE_x = dE\cos\theta_x$, $dE_y = dE\cos\theta_y$, $dE_z = dE\cos\theta_z$, and $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$, the electric field intensity at point $P(x, y, z)$ is

$$\begin{aligned} d\mathbf{E} &= \hat{\mathbf{x}} dE\cos\theta_x + \hat{\mathbf{y}} dE\cos\theta_y + \hat{\mathbf{z}} dE\cos\theta_z \\ &= \hat{\mathbf{x}} dE \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} + \hat{\mathbf{y}} dE \frac{y - y'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &\quad + \hat{\mathbf{z}} dE \frac{z - z'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \end{aligned} \quad (3.42)$$

Now, from the relation for $d\mathbf{E}$ from Eq. (3.39), we get

$$dE = |d\mathbf{E}| = \frac{\rho_l dl'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} = \frac{\rho_l dl'}{4\pi\epsilon_0 \left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)} \quad (3.43)$$

Substitution of this in Eq. (3.42) and integration over the length of the line gives

$$\begin{aligned} \mathbf{E} = & \hat{\mathbf{x}} \frac{\rho_l}{4\pi\epsilon_0} \int_{P_1}^{P_2} \frac{(x - x') dl'}{\left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)^{3/2}} + \hat{\mathbf{y}} \frac{\rho_l}{4\pi\epsilon_0} \int_{P_1}^{P_2} \frac{(y - y') dl'}{\left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)^{3/2}} \\ & + \hat{\mathbf{z}} \frac{\rho_l}{4\pi\epsilon_0} \int_{P_1}^{P_2} \frac{(z - z') dl'}{\left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)^{3/2}} \quad \left[\frac{\text{N}}{\text{C}} \right] \end{aligned} \quad (3.44)$$

This is a rather lengthy but simple expression. It indicates that to obtain each component of the vector, we must integrate along the line. Note also that the points P_1 and P_2 are general and that the same type of result can be obtained in any system of coordinates.

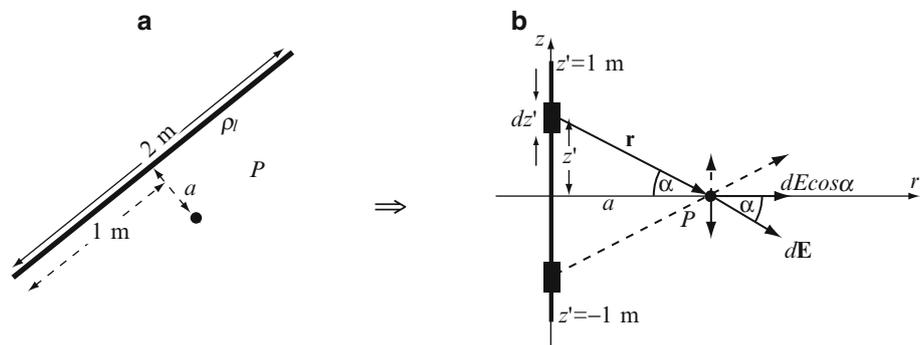
Important Note: Throughout the derivation in this section, we used primed coordinates for the location of the element of charge dq . This is called the *source point*. Unprimed coordinates are used for the location at which the field is calculated. This is called the *field point*. This distinction between source and field points will be followed throughout the book. In practical terms, the integration required to find the field (in this case the electric field intensity) is on the primed coordinates, whereas the field point coordinates remain constant.

Example 3.8 Electric Field Intensity Due to a Charged Line Segment A thin line segment is 2 m long and charged with a uniform line charge density ρ_l [C/m]:

- Find the electric field intensity at a distance $a = 1$ m from the center of the segment.
- Find the electric field intensity at a distance a from the line if the line is infinite in length.

Solution: The segment is placed in a system of coordinates as shown in Figure 3.14. The cylindrical system shown is chosen because of the cylindrical nature of the segment. We set up an element of length dz' and, therefore, an equivalent point charge $dq = \rho_l dz'$ and calculate the distance between the element of length dz' and point P in terms of the coordinate z' and the known distance to point P . Integrating along the line from $z' = -1$ to $z' = +1$ gives the result:

Figure 3.14 (a) A charged segment in space. (b) The same segment in a cylindrical system of coordinates



- (a) The distance from a general point z' on the line and point P is $|\mathbf{r}| = \sqrt{a^2 + z'^2}$. Because of symmetry, the only field component is in the r direction (the z components from the lower and upper halves of the line cancel each other as shown in **Figure 3.14b**). Using $\cos\alpha = a/|\mathbf{r}|$,

$$\mathbf{E} = \hat{\mathbf{r}} \int_{z'=-1}^{z'=+1} \frac{\rho_l \cos\alpha dz'}{4\pi\epsilon_0 (a^2 + z'^2)} = \hat{\mathbf{r}} \int_{z'=-1}^{z'=+1} \frac{\rho_l a dz'}{4\pi\epsilon_0 (a^2 + z'^2)^{3/2}} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

Integrating gives

$$\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l a z'}{4\pi\epsilon_0 a^2 (a^2 + z'^2)^{1/2}} \Bigg|_{z'=-1}^{z'=+1} = \hat{\mathbf{r}} \frac{\rho_l}{4\pi\epsilon_0} \left[\frac{1}{a(a^2 + 1)^{1/2}} + \frac{1}{a(a^2 + 1)^{1/2}} \right] = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 a (a^2 + 1)^{1/2}} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

For the values given

$$\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 \sqrt{2}} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

- (b) Solution for the infinite line follows the same method except that the integration is between $z' = -\infty$ and $z' = +\infty$:

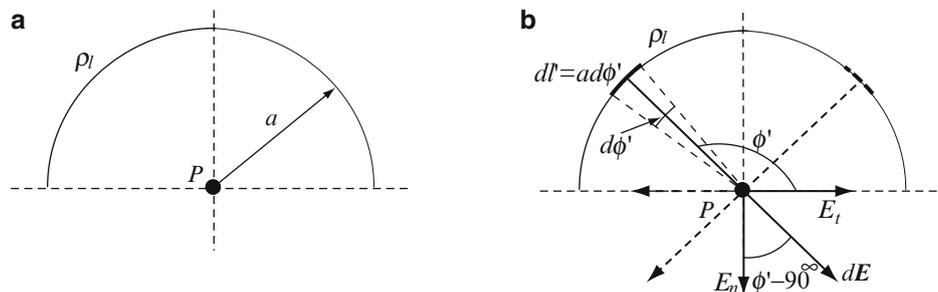
$$\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l a}{4\pi\epsilon_0} \int_{z'=-\infty}^{z'=+\infty} \frac{dz'}{(a^2 + z'^2)^{3/2}} = \hat{\mathbf{r}} \frac{\rho_l a z'}{4\pi\epsilon_0 a^2 (a^2 + z'^2)^{1/2}} \Bigg|_{z'=-\infty}^{z'=+\infty} = \hat{\mathbf{r}} \frac{\rho_l}{4\pi\epsilon_0 a (a^2/z'^2 + 1)^{1/2}} \Bigg|_{z'=-\infty}^{z'=+\infty} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon_0 a} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

Example 3.9 Electric Field Intensity Due to a Charged Half-Loop A wire is bent in the form of a half-loop of radius $a = 10$ mm and charged with a line charge density $\rho_l = 10^{-9}$ C/m. Calculate the electric field intensity at the center of the loop (**Figure 3.15a**).

Solution: First, we establish an elemental point charge due to a differential arc length of the loop and calculate the electric field intensity at the center of the loop due to this elemental point charge. Because of symmetry, only a component pointing straight down may exist at the center of the loop. These aspects of calculation are shown in **Figure 3.15b**. Integration is on the angle ϕ' .

The point charge is $\rho_l ad\phi'$, where $ad\phi'$ is the differential arc length in cylindrical coordinates. The electric field intensity at point P is in the r direction:

Figure 3.15 (a) A charged semicircular loop. (b) The electric field intensity due to a differential segment on the loop



$$d\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l a d\phi'}{4\pi\epsilon_0 a^2} \rightarrow d\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l d\phi'}{4\pi\epsilon_0 a}$$

The horizontal components cancel because for each element of the arc on the left half of the loop, there is an identical element on the right half with the horizontal component in the opposite direction (see **Figure 3.15b**). The normal components sum up and we get

$$dE_n = 2dE \cos(\phi' - 90^\circ) = 2dE \cos(-(90^\circ - \phi')) = 2dE \sin\phi' = \frac{\rho_l \sin\phi' d\phi'}{2\pi\epsilon_0 a}$$

The total electric field intensity is found by integrating over one-quarter of the loop (between $\phi' = 0$ and $\phi' = \pi/2$)

$$E_n = \frac{\rho_l}{2\pi\epsilon_0 a} \int_{\phi'=0}^{\pi/2} \sin\phi' d\phi' = -\frac{\rho_l}{2\pi\epsilon_0 a} \cos\phi' \Big|_0^{\pi/2} = \frac{\rho_l}{2\pi\epsilon_0 a} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

For the given values, we get

$$E_n = \frac{10^{-9}}{2 \times \pi \times 8.854 \times 10^{-12} \times 0.01} = 1797.55 \quad \left[\frac{\text{N}}{\text{C}} \right]$$

Exercise 3.2 Calculate the electric field intensity at the center of a very thin ring of radius a [m] if a charge density ρ_l [C/m] is uniformly distributed on the ring.

Answer $\mathbf{E} = 0$.

3.4.2.2 Surface Charge Distributions

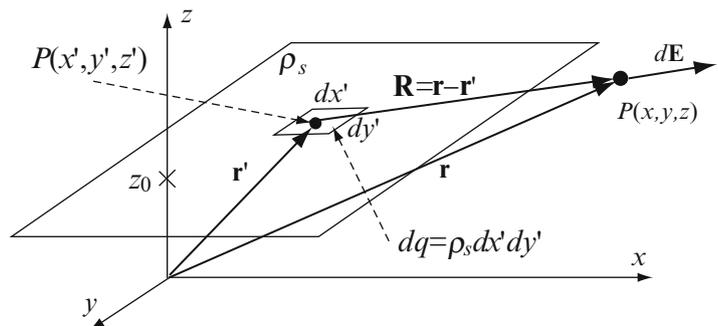
As with line distributions, surface charge densities can be uniform or nonuniform, and since charges are involved, the charge distribution generates an electric field in space. Again using the idea of a point charge, we view an element of surface as containing an elemental charge which is considered to be a point charge, as in **Figure 3.16**:

$$dq = \rho_s ds' \quad (3.45)$$

The electric field intensity due to this point charge at a distance $R = |\mathbf{r} - \mathbf{r}'|$ from the point is

$$d\mathbf{E} = \hat{\mathbf{R}} \frac{\rho_s ds'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} = \frac{\rho_s ds' (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3} \quad (3.46)$$

Figure 3.16 A surface charge distribution and the electric field intensity due to an element of charge on the surface. The surface is parallel to the x - y plane



Now, the electric field intensity due to a general surface distribution can be written as

$$\mathbf{E} = \int_{s'} \frac{(\mathbf{r} - \mathbf{r}')\rho_s}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3} ds' \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.47)$$

This expression is quite general, but it is of little use at this point. Before we can actually calculate the electric field intensity, it is first necessary to define the vectors \mathbf{r} and \mathbf{r}' in terms of the coordinates of the source point (x', y', z') and field point (x, y, z) . Although it is possible to do so in general coordinates, it is much easier to understand the steps involved through examples. However, it is useful to view this relation as the counterpart of **Eq. (3.44)**. In this sense, the calculation of electric fields due to surface distributions is essentially the same as that for line charge distributions. The differences are in the charge density itself and in the integration.

To see the general relations involved, consider again the flat surface in **Figure 3.16**. The orientation of the surface in space is not important. Assuming that the surface charge density is uniform and is given, an equivalent point charge is defined as shown. The electric field intensity at point $P(x, y, z)$ due to the differential surface charge is given in **Eq. (3.46)**. Now, before performing the integration, we separate the electric field intensity into its components. Since the separation into components in **Eq. (3.40)** was independent of the type of charge distribution we used (it only had to do with the infinitesimal point charge), we can use the same relations here. Using these together with surface integration, we get

$$\mathbf{E} = \hat{\mathbf{x}} \frac{\rho_s}{4\pi\epsilon_0} \int_{s'} \frac{(x - x') ds'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{y}} \frac{\rho_s}{4\pi\epsilon_0} \int_{s'} \frac{(y - y') ds'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{z}} \frac{\rho_s}{4\pi\epsilon_0} \int_{s'} \frac{(z - z') ds'}{|\mathbf{r} - \mathbf{r}'|^3} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.48)$$

where $|\mathbf{r} - \mathbf{r}'| = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}$. The expression in **Eq. (3.48)** looks simple, but we still need to evaluate the surface integral. This may or may not be easy to do, depending on the surface on which the integration must be performed. In general, for surfaces that lie in a plane, it is relatively easy to evaluate the integral. Although actual examples will be given shortly, it is worth rewriting the expression in **Eq. (3.48)** for a surface that lies parallel to the x - y plane, as in **Figure 3.16**. Under these conditions, we replace ds' by $dx'dy'$ and the surface integral becomes a double integral over the x' and y' coordinates:

$$\begin{aligned} \mathbf{E} = & \hat{\mathbf{x}} \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=y_1}^{y'=y_2} \left[\int_{x'=x_1}^{x'=x_2} \frac{(x - x') dx'}{|\mathbf{r} - \mathbf{r}'|^3} \right] dy' + \hat{\mathbf{y}} \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=y_1}^{y'=y_2} \left[\int_{x'=x_1}^{x'=x_2} \frac{(y - y') dx'}{|\mathbf{r} - \mathbf{r}'|^3} \right] dy' \\ & + \hat{\mathbf{z}} \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=y_1}^{y'=y_2} \left[\int_{x'=x_1}^{x'=x_2} \frac{(z - z') dx'}{|\mathbf{r} - \mathbf{r}'|^3} \right] dy' \quad \left[\frac{\text{N}}{\text{C}} \right] \end{aligned} \quad (3.49)$$

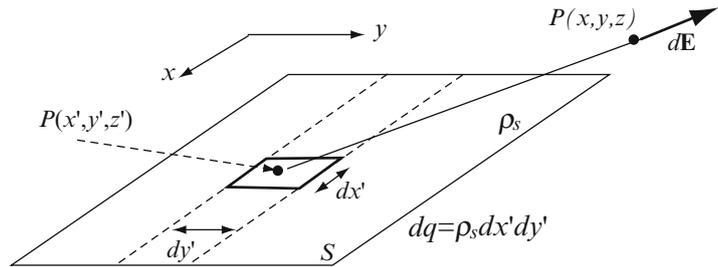
It is also possible, and sometimes useful, to view the surface distribution as an assembly of lines of charges as shown in **Figure 3.17**. This approach is a direct consequence of **Eq. (3.49)**. Because the inner integral for each component is identical to that for a line integral, the expression can be written as follows:

$$\mathbf{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=y_1}^{y'=y_2} \left[\hat{\mathbf{x}} \int_{x'=x_1}^{x'=x_2} \frac{(x - x') dx'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{y}} \int_{x'=x_1}^{x'=x_2} \frac{(y - y') dx'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{z}} \int_{x'=x_1}^{x'=x_2} \frac{(z - z') dx'}{|\mathbf{r} - \mathbf{r}'|^3} \right] dy' \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.50)$$

This may seem to be a rather minor point but it shows the use of superposition and helps understand the process of double integration because each step can be performed separately. Returning now to **Figure 3.17**, the inner integral is simply that over a line of charge with charge density $\rho_l' = \rho_s dy'$ (integrated over dx'), whereas the outer integral sums up all lines of charge making up the surface.

Example 3.10 Electric Field Intensity Due to Surface Charge Densities A very thin plate of size $2a \times 2b$ [m²] is charged with a uniform charge density ρ_s [C/m²]. Calculate the electric field intensity parallel to the surface at the center of the plate.

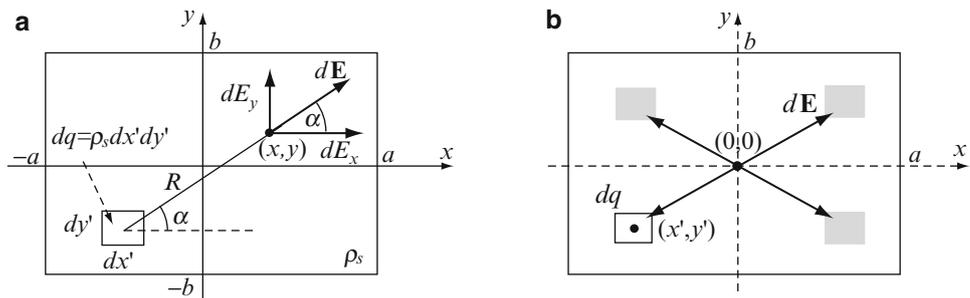
Figure 3.17 Calculation of the electric field of a surface charge density



Solution: To calculate the electric field intensity, we define the elemental charge on an area $dx'dy'$ at a point (x',y') . The differential components of the electric field intensity in the x and y directions at a generic point on the surface of the plate (x,y) are then calculated (see **Figure 3.18a** for the relations involved). Now, with the general relation for the electric field intensity, we substitute the required values for x and y at the center of the plate to find the electric field intensity. The distance between the two points is

$$R = \sqrt{(x - x')^2 + (y - y')^2}$$

Figure 3.18 (a) Calculation of the electric field intensity at a general point on the plate. **(b)** The horizontal electric field intensity at the center of the plate from symmetry consideration



The electric field intensity due to an element of charge $dq = \rho_s dx'dy'$ at point (x',y') at a general point (x,y) (see **Figure 3.18a**) is

$$d\mathbf{E} = \hat{\mathbf{R}} \frac{\rho_s dx'dy'}{4\pi\epsilon_0 [(x - x')^2 + (y - y')^2]}$$

The electric field intensity at (x,y) is now found by first separating $d\mathbf{E}$ into its x and y components:

$$dE_x = dE \cos\alpha = \frac{\rho_s (x - x') dx' dy'}{4\pi\epsilon_0 [(x - x')^2 + (y - y')^2]^{3/2}}$$

$$dE_y = dE \sin\alpha = \frac{\rho_s (y - y') dx' dy'}{4\pi\epsilon_0 [(x - x')^2 + (y - y')^2]^{3/2}}$$

where $\cos\alpha = (x - x')/R$ and $\sin\alpha = (y - y')/R$. Now, we can use these relations to find the electric field intensity anywhere on the plate by integrating over x' and y' . In the case discussed here, we need to find the electric field intensity at the center of the plate. By placing the plate as shown in **Figure 3.18a**, $x = 0$, $y = 0$, and the limits of integration are between $-a$ and $+a$ in the x direction and between $-b$ and $+b$ in the y direction. Substituting these in dE_x and dE_y above and integrating over the surface gives the x and y components of the electric field intensity as

$$E_x = -\frac{\rho_s}{4\pi\epsilon_0} \int_{y'=-b}^{y'=+b} \int_{x'=-a}^{x'=+a} \frac{x' dx' dy'}{[x'^2 + y'^2]^{3/2}} = \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=-b}^{y'=+b} \frac{1}{\sqrt{x'^2 + y'^2}} \Bigg|_{x'=-a}^{x'=+a} = \frac{\rho_s}{4\pi\epsilon_0} \int_{y'=-b}^{y'=+b} \left(\frac{1}{\sqrt{a^2 + y'^2}} - \frac{1}{\sqrt{a^2 + y'^2}} \right) dy' = 0$$

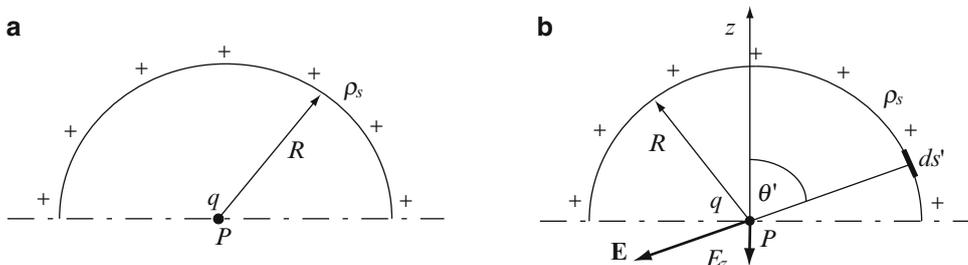
$$E_y = -\frac{\rho_s}{4\pi\epsilon_0} \int_{x'=-a}^{x'=+a} \int_{y'=-b}^{y'=+b} \frac{y' dx' dy'}{[x'^2 + y'^2]^{3/2}} = \frac{\rho_s}{4\pi\epsilon_0} \int_{x'=-a}^{x'=+a} \frac{1}{\sqrt{x'^2 + y'^2}} \Bigg|_{y'=-b}^{y'=+b} dx' = 0$$

Thus, the parallel electric field intensity at the center of the plate is zero. We could have anticipated this result from symmetry considerations. For any element of charge dq , there are three additional, symmetric elements of charge, as shown in **Figure 3.18b**. As can be seen, the electric field intensities due to the four elements cancel at the center.

Example 3.11 Force Due to Surface Charge Densities A hemisphere of radius R [m] has a uniformly distributed surface charge density ρ_s [C/m²]. Calculate the force on a very small, positive charge q [C], placed at the center of the hemisphere (point P in **Figure 3.19**).

Solution: Because of symmetry about point P , only the vertical component of the electric field intensity (which we call the z component) is nonzero, as shown in **Figure 3.19b**. Integration is done over the surface of the hemisphere in spherical coordinates. After the electric field intensity is known, the force is found by multiplying by the charge at point P .

Figure 3.19 (a) A charged hemisphere. (b) The electric field intensity at point P due to a differential surface ds'



The elemental point charge on the surface of the sphere of radius R is $dQ = \rho_s R^2 \sin\theta' d\theta' d\phi'$ [C]. The electric field intensity due to this charge is

$$d\mathbf{E} = \hat{\mathbf{R}} \frac{dQ}{4\pi\epsilon_0 R^2} = \frac{\rho_s}{4\pi\epsilon_0} \sin\theta' d\theta' d\phi' \left[\frac{\text{N}}{\text{C}} \right]$$

The electric field intensity has a vertical component in the negative z direction and a horizontal component. The horizontal components due to two elements of surface which are symmetric about the vertical axis cancel each other. Thus, the only nonzero field points in the negative z direction. Its magnitude equals

$$dE_z = dE \cos\theta' = \frac{\rho_s}{4\pi\epsilon_0} \sin\theta' \cos\theta' d\theta' d\phi' \left[\frac{\text{N}}{\text{C}} \right]$$

The electric field intensity is found by integrating over θ' from 0 to $\pi/2$ and over ϕ' from 0 to 2π :

$$E_z = \frac{\rho_s}{4\pi\epsilon_0} \int_{\theta'=0}^{\theta'=\pi/2} \int_{\phi'=0}^{\phi'=2\pi} \sin\theta' \cos\theta' d\theta' d\phi' = \frac{\rho_s}{4\pi\epsilon_0} \int_{\theta'=0}^{\theta'=\pi/2} 2\pi \frac{\sin 2\theta'}{2} d\theta' = \frac{\rho_s}{4\epsilon_0} \left(-\frac{\cos 2\theta'}{2} \right) \Big|_0^{\pi/2} = \frac{\rho_s}{4\epsilon_0} \left[\frac{\text{N}}{\text{C}} \right]$$

where the identity $\sin\theta' \cos\theta' = (\sin 2\theta')/2$ was used. The electric field intensity points downward (negative z direction). The force is therefore

$$\mathbf{F} = q\mathbf{E}_z = -\hat{\mathbf{z}} \frac{q\rho_s}{4\epsilon_0} \quad [\text{N}]$$

and since the point charge is positive, the force is in the same direction as \mathbf{E} (a repulsion force).

3.4.2.3 Volume Charge Distributions

Treatment of volume charge distributions follows steps identical to those for surface and line charge distributions. The electric field due to an element of charged volume is shown in **Figure 3.20**. For an element of volume, the equivalent element of charge is $dq = \rho_v dv'$ and the electric field intensity due to this point charge at a distance R from the point is

$$d\mathbf{E} = \hat{\mathbf{R}} \frac{\rho_v dv'}{4\pi\epsilon_0 R^2} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.51)$$

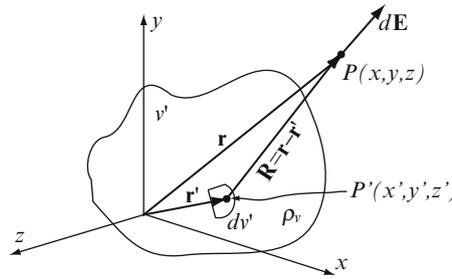


Figure 3.20 Electric field intensity due to an element of charged volume

Integrating and using the position vectors in **Figure 3.20**, we get

$$\mathbf{E} = \int_{v'} \frac{(\mathbf{r} - \mathbf{r}') \rho_v}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3} dv' \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.52)$$

As in the previous cases, the integration over the volume may or may not be easy to perform, but the principle of setting up the solution is straightforward. Using the same steps as for the line charge, we get for a uniform charge density ρ_v

$$\mathbf{E} = \hat{\mathbf{x}} \frac{\rho_v}{4\pi\epsilon_0} \int_{v'} \frac{(x - x') dv'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{y}} \frac{\rho_v}{4\pi\epsilon_0} \int_{v'} \frac{(y - y') dv'}{|\mathbf{r} - \mathbf{r}'|^3} + \hat{\mathbf{z}} \frac{\rho_v}{4\pi\epsilon_0} \int_{v'} \frac{(z - z') dv'}{|\mathbf{r} - \mathbf{r}'|^3} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.53)$$

For a volume with limits as in **Figure 3.21**, we can write $dv' = dx' dy' dz'$, and the element of volume dv' is located at the source point (x', y', z') , whereas the electric field intensity is calculated at the field point (x, y, z) . Also, $|\mathbf{r} - \mathbf{r}'| = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}$, as was indicated earlier. Each of the three terms of **Eq. (3.53)** is integrated over the volume. This gives

$$\mathbf{E} = \hat{\mathbf{x}} \frac{\rho_v}{4\pi\epsilon_0} \int_{y'=-z_0}^{y'+z_0} \left\{ \int_{x'=-x_0}^{x'+x_0} \frac{(x-x')dx'}{|\mathbf{r}-\mathbf{r}'|^3} \right\} dy' dz' + \hat{\mathbf{y}} \frac{\rho_v}{4\pi\epsilon_0} \int_{z'=-z_0}^{z'+z_0} \left\{ \int_{y'=-y_0}^{y'+y_0} \left[\int_{x'=-x_0}^{x'+x_0} \frac{(y-y')dx'}{|\mathbf{r}-\mathbf{r}'|^3} \right] dy' \right\} dz' + \hat{\mathbf{z}} \frac{\rho_v}{4\pi\epsilon_0} \int_{z'=-z_0}^{z'+z_0} \left\{ \int_{y'=-y_0}^{y'+y_0} \left[\int_{x'=-x_0}^{x'+x_0} \frac{(z-z')dx'}{|\mathbf{r}-\mathbf{r}'|^3} \right] dy' \right\} dz' \quad \left[\frac{\text{N}}{\text{C}} \right] \tag{3.54}$$

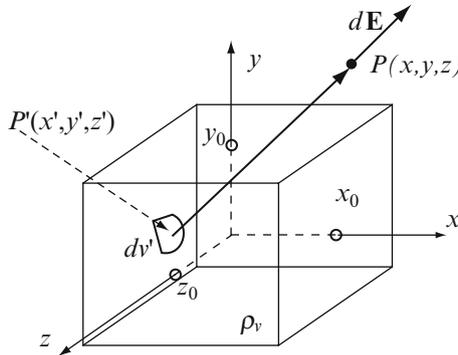


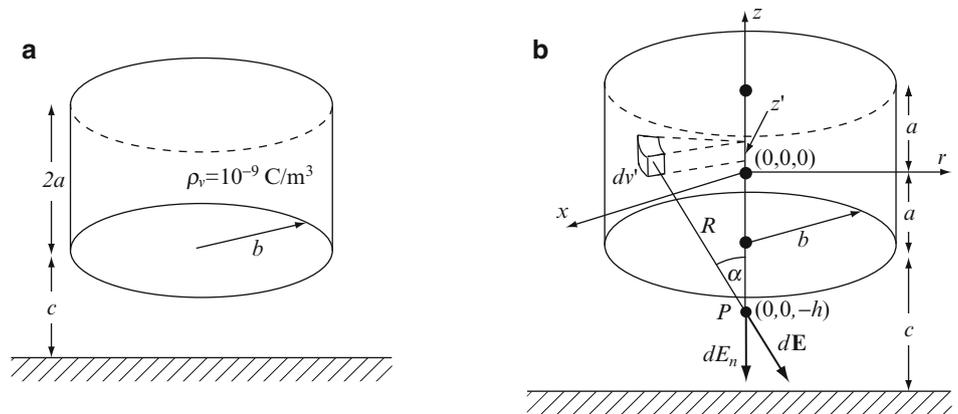
Figure 3.21 Electric field intensity due to an element of charge inside the cube

Example 3.12 Application: Electric Field Intensity In and Around a Cloud As an approximation of the electric field in and around a thundercloud, consider a thundercloud in the form of a cylinder of radius $b = 1,000$ m, height $2a = 4,000$ m, and with its bottom $c = 1,000$ m above ground, as shown in **Figure 3.22a**. The cloud has a charge density $\rho_v = 10^{-9}$ C/m³ uniformly distributed throughout its volume:

- (a) Calculate the electric field intensity at ground level, below the center of the cloud.
- (b) Calculate the electric field intensity at the bottom of the cloud, on its axis.

Note: Thunderclouds tend to be cylindrical in shape. The charge density within the cloud is not uniform and not of the same polarity. However, this example does give an idea of the quantities involved and the method of calculation of the electric field in volume charge distributions.

Figure 3.22 (a) Approximation to a thundercloud. **(b)** Calculation of the electric field below the cloud



Solution: We place the cloud in a cylindrical system of coordinates as shown. The center of the cylinder is at $(0,0,0)$. A differential of volume at a general point within the cloud (r',ϕ',z') is defined as $dv'r'dr'd\phi'dz'$. The electric field intensity is then calculated at a point $P(0,0,-h)$ on the axis, as shown in **Figure 3.22b**. Because of symmetry, only a normal component may exist at point P . The normal component is calculated and the result integrated over the primed coordinates to obtain the total field at P :

(a) The magnitude of the electric field intensity due to the elemental charge is

$$dE = \frac{\rho_v r' dr' d\phi' dz'}{4\pi\epsilon_0 R^2} = \frac{\rho_v r' dr' d\phi' dz'}{4\pi\epsilon_0 (r'^2 + (h+z')^2)} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

where h is the distance from the center of the cloud to the field point and z' is the z coordinate of the source point, as shown in **Figure 3.22b**. The vertical distance from the element of charge to the field point is $h+z'$ and the scalar normal component of the electric field intensity is

$$dE_n = dE \cos\alpha = \frac{\rho_v r' dr' d\phi' dz'}{4\pi\epsilon_0 (r'^2 + (h+z')^2)} \frac{h+z'}{R} = \frac{\rho_v (h+z') r' dr' d\phi' dz'}{4\pi\epsilon_0 (r'^2 + (h+z')^2)^{3/2}} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

Since the only component is in the negative z direction, we may also write for the total electric field intensity at P :

$$\begin{aligned} \mathbf{E} &= -\hat{\mathbf{z}} \frac{\rho_v}{4\pi\epsilon_0} \int_{z'=-a}^{z'=a} \left[\int_{r'=0}^{r'=b} \left[\int_{\phi'=0}^{\phi'=2\pi} \frac{r' (h+z')}{(r'^2 + (h+z')^2)^{3/2}} d\phi' \right] dr' \right] dz' \\ &= -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \int_{z'=-a}^{z'=a} \left[\int_{r'=0}^{r'=h} \frac{r' (h+z')}{(r'^2 + (h+z')^2)^{3/2}} dr' \right] dz' \quad \left[\frac{\text{N}}{\text{C}} \right] \end{aligned}$$

where the fact that the integral over ϕ' results in 2π was used. The inner integral is evaluated as follows:

$$\int_{r'=0}^{r'=b} \frac{r' (h+z')}{(r'^2 + (h+z')^2)^{3/2}} dr' = \left[-\frac{(h+z')}{(r'^2 + (h+z')^2)^{1/2}} \right]_{r'=0}^{r'=a} = 1 - \frac{(h+z')}{(b^2 + (h+z')^2)^{1/2}}$$

Substituting this result in the electric field intensity, we are left with

$$\begin{aligned} \mathbf{E} &= -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \int_{z'=-a}^{z'=a} \left[1 - \frac{h+z'}{(b^2 + (h+z')^2)^{1/2}} \right] dz' \\ &= -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \left[z' - h \ln \left(2\sqrt{z'^2 + 2hz' + h^2 + b^2 + 2z' + 2h} \right) \right. \\ &\quad \left. - \sqrt{b^2 + (h+z')^2} + h \ln \left(2\sqrt{z'^2 + 2hz' + h^2 + b^2 + 2z' + 2h} \right) \right]_{z'=-a}^{z'=a} \\ &= -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \left[z' - \sqrt{b^2 + (h+z')^2} \right]_{z'=-a}^{z'=a} = -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \left[a - \sqrt{b^2 + (h+a)^2} + a + \sqrt{b^2 + (h-a)^2} \right] \\ &= -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \left[2a + \sqrt{b^2 + (h-a)^2} - \sqrt{b^2 + (h+a)^2} \right] \quad \left[\frac{\text{N}}{\text{C}} \right] \end{aligned}$$

For the given values, the electric field intensity at ground level ($h = c + a = 3,000$ m) is

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{10^{-9}}{2 \times 8.854 \times 10^{-12}} \left[4000 + \sqrt{1000^2 + 1000^2} - \sqrt{1000^2 + 5000^2} \right] = -\hat{\mathbf{z}} 17,799.53 \left[\frac{\text{N}}{\text{C}} \right]$$

(b) At the bottom of the cloud, $h = a = 2,000$ m and we get

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{\rho_v}{2\epsilon_0} \left[2a + b - \sqrt{b^2 + 4a^2} \right] = -\hat{\mathbf{z}} \frac{10^{-9}}{2 \times 8.854 \times 10^{-12}} \left[4000 + 1000 - \sqrt{1000^2 + 4 \times 2000^2} \right] = -\hat{\mathbf{z}} 49,519 \left[\frac{\text{N}}{\text{C}} \right]$$

Exercise 3.3 A very thin disk of radius $a = 50$ mm is charged with a uniform surface charge density $\rho_s = 10^{-7}$ C/m². (a) Calculate the electric field intensity at a distance $h = 0.1$ m above the center of the disk. Assume the disk is placed in the r - ϕ plane. (b) Calculate the electric field intensity below and above the disk at $h = 0$ (on the r - ϕ plane at the center of the disk).

Answer

$$(a) \mathbf{E} = -\hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{h}{\sqrt{a^2 + h^2}} \right] = \hat{\mathbf{z}} 596.2 \left[\frac{\text{N}}{\text{C}} \right].$$

$$(b) \mathbf{E}_{\text{below}} = -\hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0} = -\hat{\mathbf{z}} 5,647 \left[\frac{\text{N}}{\text{C}} \right] \quad \mathbf{E}_{\text{above}} = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0} = \hat{\mathbf{z}} 5,647 \left[\frac{\text{N}}{\text{C}} \right].$$

3.5 The Electric Flux Density and Electric Flux

In all relations used in the preceding sections, we made use of permittivity of materials without actually discussing what permittivity is or, for that matter, how it influences the electric field, except to say that it is material dependent. This was of relatively little concern since Coulomb's law was given as an experimental relation and any relation derived from it is therefore also experimental. In other words, dependency of the electric field or force on some material property was established through experiment. In **Chapter 4**, we will discuss this again and at that point properly define permittivity and its meaning as well as its utility and applications. Simply from a convenience point of view, we observe that if we multiply the electric field intensity \mathbf{E} by permittivity ϵ , we get a new vector:

$$\mathbf{D} = \epsilon \mathbf{E} \left[\frac{\text{C}}{\text{m}^2} \right] \quad (3.55)$$

This vector has the same direction as \mathbf{E} but unlike the expression for \mathbf{E} , it is independent of ϵ and therefore of material properties. In terms of units, the electric field intensity has units of [N/C] and permittivity has units of [C²/N · m²]. The units of \mathbf{D} are [C/m²] as indicated above. Thus, because this is a density and because it is derived from the electric field intensity, we will call it the *electric flux density*. The use of the term "flux" will become apparent in the following chapter. We merely want to comment here that the use of this new vector is sometimes more convenient since the electric flux density is independent of material properties [ϵ is not involved in calculation as shown in **Eq. (3.55)**] whereas the electric field intensity is material dependent. In terms of usage, the electric flux density is used in a manner similar to the electric field intensity. For example, the electric flux density of a point charge Q_1 at a distance R is

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \frac{\epsilon_0 Q_1 \mathbf{R}}{4\pi \epsilon_0 |\mathbf{R}|^3} = \frac{Q_1 \mathbf{R}}{4\pi |\mathbf{R}|^3} \left[\frac{\text{C}}{\text{m}^2} \right] \quad (3.56)$$

Thus, the calculation of the electric flux density (and from it, the electric field intensity) is simplified because the material is not taken into account explicitly. For example, the permittivity of water is about 80 times larger than that of free space. Therefore, the electric field intensity in water due to a given charge is 80 times smaller than the electric field intensity for the same charge in free space, but the electric flux density is the same regardless of permittivity.

On the other hand, the calculation of forces requires knowledge of permittivity. Since $\mathbf{F} = q\mathbf{E}$, the use of the electric flux density does not eliminate the need to consider permittivity. We may use either of the following expressions to calculate the force on a charge q due to the field of charge Q_1 , and in each, permittivity appears explicitly:

$$\mathbf{F} = q\mathbf{E} = \frac{qQ_1\mathbf{R}}{4\pi\epsilon_0|\mathbf{R}|^3} = \frac{q\mathbf{D}}{\epsilon_0} \quad [\text{N}] \quad (3.57)$$

Once we defined the electric flux density, it is only obvious that we should also define electric flux. We note from Eq. (3.55) that the electric flux density is a surface density. Therefore, integrating the electric flux density over a surface provides the electric flux through that surface. We write the electric flux as:

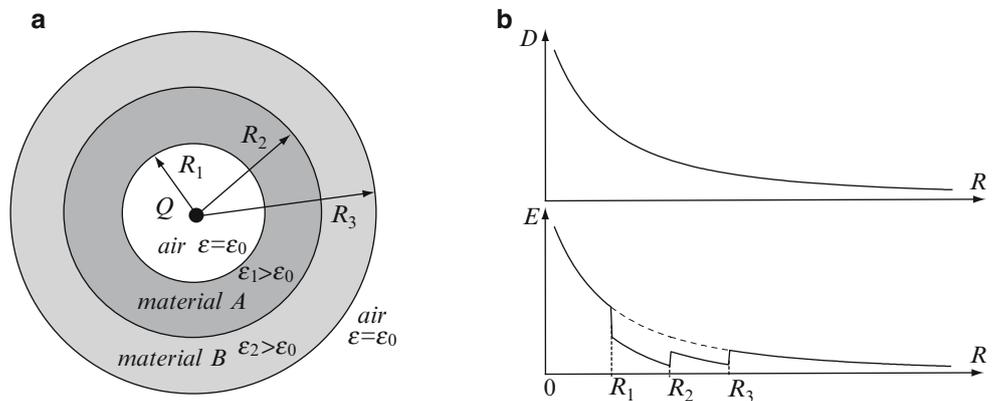
$$\Phi = \int_s \mathbf{D} \cdot d\mathbf{s} \quad [\text{C}] \quad (3.58)$$

Note that the electric flux is given in units of coulombs [C] as expected.

Example 3.13 Calculation of Electric Flux Density, Electric Field Intensity, and Electric Flux in Layered Materials A point charge Q [C] is located at a point in space. The charge is surrounded by two spherical layers of materials as shown in Figure 3.23a. Both materials have permittivities different than free space, as indicated:

- Calculate the electric flux density and field intensity everywhere in space.
- Calculate the electric flux through a spherical surface of radius R .
- Plot the electric field intensity and electric flux density everywhere in space.

Figure 3.23 (a) A spherical layered configuration. (b) Plot of the electric flux density and electric field intensity as a function of distance from the center



Solution: After calculating the electric flux density using Eq. (3.56), the electric field intensity can be calculated from Eq. (3.55):

- The electric flux density in all materials is radially oriented and equals

$$\mathbf{D} = \hat{\mathbf{R}} \frac{Q}{4\pi R^2} \quad \left[\frac{\text{C}}{\text{m}^2} \right]$$

where R is the distance from the point charge Q . The electric field intensity in any material is

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \hat{\mathbf{R}} \frac{Q}{4\pi\varepsilon R^2} \quad \left[\frac{\text{N}}{\text{C}} \right]$$

where ε is the permittivity of the corresponding material. Thus, we have

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{R}} \frac{Q}{4\pi\varepsilon_0 R^2} \quad \left[\frac{\text{N}}{\text{C}} \right], \quad 0 < R \leq R_1$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_1} = \hat{\mathbf{R}} \frac{Q}{4\pi\varepsilon_1 R^2} \quad \left[\frac{\text{N}}{\text{C}} \right], \quad R_1 < R \leq R_2$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_2} = \hat{\mathbf{R}} \frac{Q}{4\pi\varepsilon_2 R^2} \quad \left[\frac{\text{N}}{\text{C}} \right], \quad R_2 < R \leq R_3$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{R}} \frac{Q}{4\pi\varepsilon_0 R^2} \quad \left[\frac{\text{N}}{\text{C}} \right], \quad R_3 < R < \infty$$

To be noted is the fact that the electric field intensity in one material is not affected by the presence of other materials; that is, the electric field intensity in, say, material B behaves as if the whole space were made of material B . However, we should not jump to conclusions regarding these observations since this is a unique example in that the electric field intensity in all materials is normal to the material boundaries everywhere.

- (b) The electric flux density calculated in (a) is integrated over the spherical surface of radius R . The element of area is $ds = \hat{\mathbf{R}} R^2 \sin\theta d\theta d\phi$. The flux is

$$\Phi = \int_s \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}} \frac{Q}{4\pi R^2} \cdot \hat{\mathbf{R}} R^2 \sin\theta d\theta d\phi = \frac{Q}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta d\theta d\phi = \frac{Q}{2} \int_{\theta=0}^{\pi} \sin\theta d\theta = Q \quad [\text{C}]$$

Of course, since the flux density is constant on any spherical surface of radius R , the integration is trivial and we could just as well multiply the electric flux density by the area of the sphere and obtain the same result.

Note that the flux is independent of R .

- (c) **Figure 3.23b** shows the plots of the values obtained above. Note in particular how the electric field intensity varies with permittivity, while the electric flux density is independent of permittivity.

3.6 Applications

Application: Coulomb's Torsional Balance In the experiments leading to development of the law bearing his name, Coulomb used a special torsional balance he devised. The balance is shown in **Figure 3.24**. It consists of a thin, insulated rod, suspended on a wire. A small conducting ball is attached to each end of the rod. One is charged while the other is used as a counterbalance weight. If a charge is now placed in the vicinity of the charged end and kept stationary, the charged end of the rod will move toward or away from the test charge. When moving, the rod twists the wire. The angle it twists is then a measure of the force between the two charges. After calibration (which could be accomplished by applying known mechanical forces), the force can be read directly (the wire acts as a spring that balances the force due to charges). This instrument was particularly useful in Coulomb's experiments because it was simple and accurate and could measure extremely small forces. To ensure accurate measurements, the balance was used in a repulsion mode and was placed in an enclosed container to minimize the effects of air movement. More modern torsional balances of this type make use of an evacuated container.

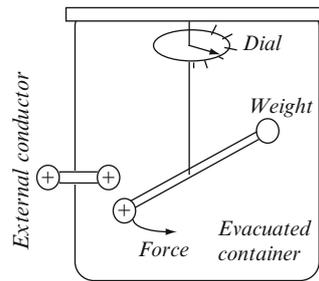


Figure 3.24 Coulomb's torsional balance

Application: Electrostatic Spraying—Spraying of Paints and Pesticides Electrostatic spraying of liquids is based on imparting a charge to the liquid to be sprayed which is forced out through a capillary opening, as shown in **Figure 3.25**. The sharp electrode generates a high electric field intensity, sufficient to charge the liquid. This is then pulled out by the effect of the anode (which is the article being sprayed). As the fluid moves out, it streams as a continuous conical jet. This jet breaks up into individual droplets because of instability in flow (depending on the charge given to the fluid) which now accelerate toward the anode, coating it. A diffuser is usually used to scatter the droplets. Spraying of pesticides with electrostatic guns can also reduce the amount of pesticides needed, since these tend to adhere to plants rather than the ground (foliage is closer to the gun than the ground). Also, because they adhere equally well on all surfaces of the plant, there is better coverage and less wash-off due to rain. Droplets can be as small as individual ions and there is minimal fogging, since all droplets are accelerated. Interestingly, the theory behind electrostatic spraying goes back to the last decade of the nineteenth century when Lord Rayleigh discussed the ideas involved. One interesting application of this principle is one type of electrostatic air cleaner designed for home use. Droplets moving through the cleaner are passed through a negatively charged screen. As they leave the device, they are still charged and are eventually drawn to the walls or furniture in the room where they tend to cling. In more acceptable designs, the droplets are trapped and those that leave the cleaner are neutralized. Electrostatic spraying is useful in continuous and automatic spraying of articles and is used extensively in the automobile and appliance industries. Because of the electrostatic force exerted on the paint droplets, the paint tends to adhere better and be more uniform, with less running than other methods.

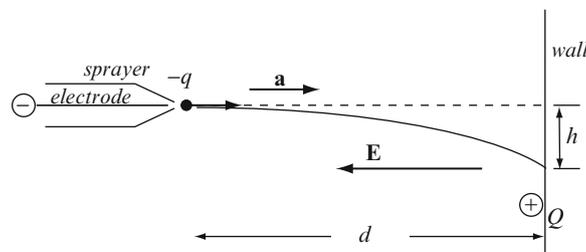
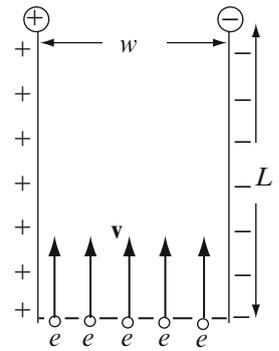


Figure 3.25 Electrostatic spraying

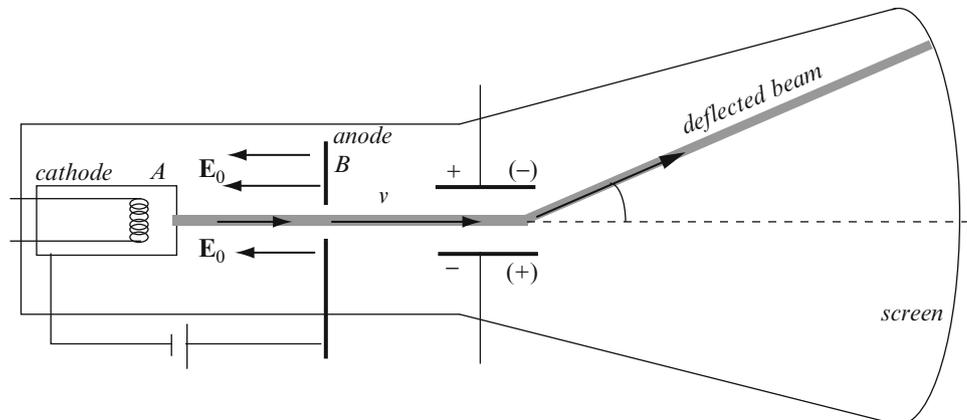
Application: Electrostatic Cleaners (Electrostatic Separators, Scrubbers, and Precipitators) A simple method of cleaning air is the electrostatic precipitator. It consists of a chamber through which the air is forced by means of a fan. An electric field intensity is generated inside the chamber (**Figure 3.26**). As the air moves through a charged screen, dust (or other particles such as ash and smoke) acquire a charge (usually negative). As they move up, the charged particles are attracted to the positively charged electrode and stick to it, accumulating until they are physically removed. In small, household cleaners, this is accomplished by washing the electrode or by removing a disposable liner on the electrode. In large scrubbers in coal-fired power plants, the cleaners are installed in flue-like structures in the smoke stacks, and cleaning is usually accomplished by “shaking” the particles from the electrodes and removing them physically from a collection pit at the bottom of the stack. As might be expected, the amount of particles collected in these types of scrubbers is huge.

Figure 3.26 An electrostatic air cleaner



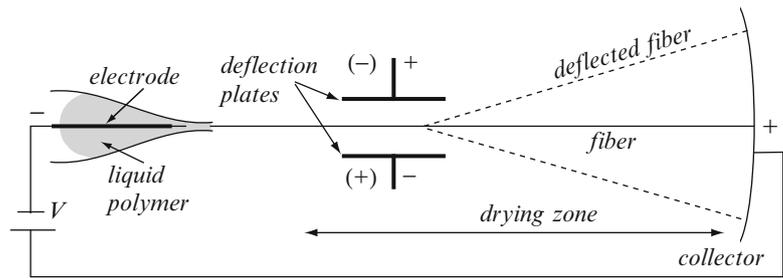
Application: Electrostatic Deflection—The Electron Gun and Deflection in Oscilloscopes, the Cathode Ray Tube The principle of electrostatic deflection is widely used in instruments such as oscilloscopes. As the cathode is heated, a beam of electrons is released and accelerated toward a screen, as shown in **Figure 3.27**. The electric field intensity between cathode and anode is of the order of a few thousand N/C. Beyond the anode, the electrons move straight toward the screen. When hitting the screen, their kinetic energy is released, which, in turn, excites the screen material (such as a phosphorous compound) to release photons. This is the image we see on the screen. To allow movement of the beam on the screen, two sets of plates are used to produce an electric field intensity between each two plates. One moves the beam up and down, the second sideways. The deflection depends on the electric field intensity between the plates. Positive charge on the upper plate will move the beam upward. A negative charge will move it downward. Note: This type of deflection is commonly used in oscilloscopes. In TV tubes, the deflection is by magnetic means.

Figure 3.27 Electrostatic deflection in a cathode ray tube



Application: Electrostatic Deflection—Processing of Polymers by Electrostatic Extrusion An interesting extension of the electrostatic deflection mechanism described above is electrostatic spinning of fibers. The electron gun is now replaced with a tube with a very thin capillary opening. The tube is filled with the polymer solution which, when pulled and dried, makes the fibers (such as polyesters used in synthetic fabrics). The negative electrode is introduced into the liquid material and the anode is at some distance away. Electrostatic force pulls the fluid into a continuous stream, and as it moves toward the anode it dries, forming the fiber. This fiber is deposited on the anode or spooled in its vicinity. Fibers as thin as $1 \mu\text{m}$ are possible. This method has some promise of spinning very fine, uniform fibers for a variety of applications such as specialty filters. Unlike other methods, it can be used to spin the fibers on demand at the location they are needed rather than in plants. **Figure 3.28** shows the system including the deflection mechanism used to move the fiber (since it is negatively charged, it can be deflected at will).

Figure 3.28 Electrostatic spinning of polymers



Application: Ion Beam Etching and Milling Ion beams as small as $0.1 \mu\text{m}$ can be focused onto a material. With the high energy densities in these highly concentrated ion beams, they can be used to etch microcircuits, for ion lithography, and to produce localized ion implantation for production of very high-density microcircuits. The ions are accelerated using large electric fields and are often focused by electrostatic lenses to a narrow point for precision machining. **Figure 3.29** shows an ion beam etching installation. Note in particular the function of the electrostatic lens shown as part of the arrangement in **Figure 3.29**. It is essentially a charged tube which squeezes the charged ions together, focusing them into a narrower beam.

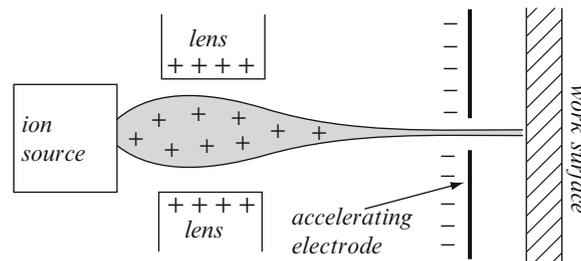


Figure 3.29 Ion beam working of materials

Application: Xerography The xerographic process as used for copying and in laser printers is a six-step electrostatic process. The method relies on, first, uniformly charging a plate or a drum in preparation for copying. This can be done in a number of ways, including discharging charge from a high-voltage source through sharp points to create a corona. The sharp point or points are moved over the surface to charge the whole surface (step 1: charging). The effect is a uniformly charged surface as shown in **Figure 3.30a**. The plate or the drum is made of a photoconducting material such as selenium, usually coated on a base conducting material. A photoconducting material becomes conducting when exposed to light. Although we have not yet discussed conductivity, conductors are understood to be materials that allow motion of charges. The conducting surface below the selenium is negatively charged, but positive charges and negative charges cannot move because selenium is normally nonconducting or poorly conducting. The material to be copied is projected onto the plate as you would project a slide on a screen (step 2: exposure). This is shown in **Figure 3.30b**. The photoconducting material becomes more conducting where light strikes (lighter areas of the image), and, therefore, some or all the charges in this area will combine with the negative charges by passing through the selenium. The effect is an electrostatic image on the plate. **Figure 3.30b** shows a black arrow on a white page. The only positive charges left are those in the area of the arrow. All other charges have diffused through the material and combined with the negative charges. In the next step, very fine, negatively charged carbon particles combined with synthetic binding materials are sprinkled onto the plate (step 3: toner). The particles will only be attracted to positively charged regions (black arrow in this case). Now, we have a layer of negative carbon particles on top of the arrow as shown in **Figure 3.30c**. Next, a positively charged page passes above the plate (or drum). The negatively charged particles are attracted to the positively charged page (step 4: transfer). To fix the image, the paper is heated to fuse the particles onto the page (step 5: fusing). After this, the drum is cleaned in preparation for the next image (step 6: cleaning).

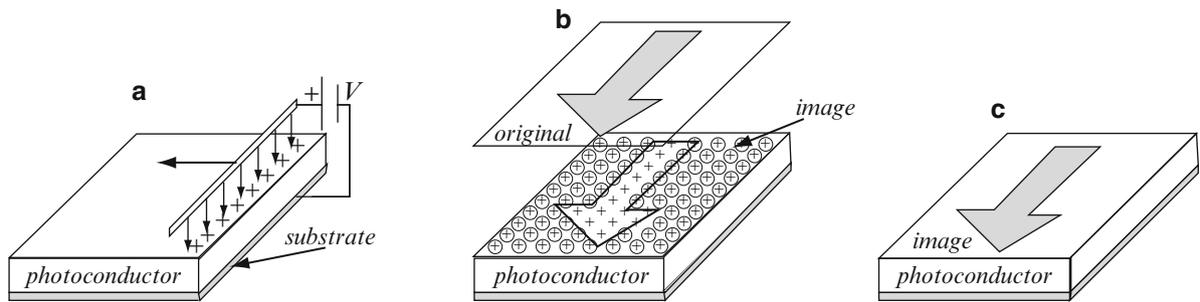
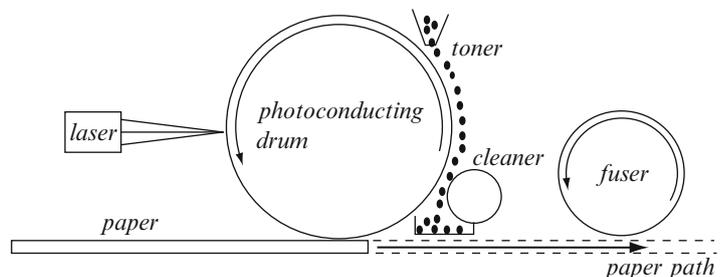


Figure 3.30 Xerography. (a) Charging of the photoconducting surface with positive charge. (b) Exposure to the original removes charges from the light portions of the original (circled charges recombine). (c) Negatively charged carbon particles (toner) adhere to the dark (positively charged) areas

The process described above is rather complex and its operation depends on many parameters, including reaction time of the photoconducting material (i.e., how long it takes charges to diffuse into the photoconductor), uniformity of charge, quality of the cleaning process, quality and size of particles, and many others. Nevertheless, this is one of the most useful and extensive uses of electrostatics.

Application The Laser Printer The laser printer is a xerographic copier with one exception. Instead of exposing the image using a light source and lens, the image is created electronically and transferred onto the photoconducting drum by laser pulses. Essentially, for each white dot on the page, the laser beam (produced by a laser diode) is switched on, exposing this dot (or pixel), and then switched off, and moved to the next dot of the image. Typical laser printers produce an image with 600 to 1,200 dpi (dots per inch) or about 24 to 48 dots/mm. The size of a dot is of the order of 0.04 mm to 0.01 mm in diameter. A typical page requires 14.4 to 57.6 million dots and all of them must be scanned for each page. Scanning can be done optically. A laser printer that produces 10 pages per minute, scans the page in less than about 5 s. After this, the process is the same as for the copier, including the use of the same type of particles and fusing process. **Figure 3.31** shows a schematic view of the laser printer. In this scheme, the drum rotates twice: once for scanning and once for transferring the image. The fuser heats the page to fix the image. The laser printer is particularly well adapted for digital imaging and the resolution depends only on the number of dots. Laser printers with 48 dots/mm (1200 dpi and higher) are common, including color printers. Higher-resolution laser printers exist and are commonly used in publishing.

Figure 3.31 The principle of the laser printer



3.7 Experiments

Experiment 1 (Induction of charge. Demonstrates: Generation of charges by friction, induction of charges, Coulomb's Law). Take a balloon and inflate it well with air (not helium). Rub it against your hair a few times. Throw it to the ceiling. The balloon will stick to the ceiling.

Explanation By rubbing the balloon, which is made of latex (a good insulator), it acquires a charge. When approaching the ceiling, this charge causes charges in the insulating ceiling to be induced on the surface. These charges are of opposite polarity which cause the balloon to stick. Since the balloon is light, the attraction force between balloon and ceiling is

sufficient to hold it. Note: If the balloon cannot be charged by rubbing against your hair, any dry fabric (wool, synthetic fibers, etc.) will do. Try to make it stick to other materials such as a window pane. Will it stick to a metallic surface? Explain.

Experiment 2 (Force between two charges. Demonstrates: Coulomb's Law) Take two balloons and repeat Experiment 1. Suspend the two on very thin strings and bring them in close proximity. The balloons will repel each other.

Explanation The two balloons acquire identical charges, thus the repulsion. You can try rubbing one balloon with different materials until the two balloons attract each other, indicating opposite sign charges.

Experiment 3 (Charging of conductors. Demonstrates: Induction of charges, charging of conducting and nonconducting bodies). Suspend two plastic balls by strings as shown in **Figure 3.32**. Use ping-pong balls (or small Styrofoam balls) and thin strings to keep weight to a minimum. If you can get a glass rod (such as an old thermometer), rub the rod against your hair, your cats or dog's fur, or on a carpet. The glass rod will acquire electric charge (polarity of charge depends on the material with which it is rubbed, but for most materials, it is positive). A plastic pen should do just as well, but now the charge on the pen will be negative. Rub the rod and draw it slowly toward the suspended ball. The ball will draw toward the rod. Negative charge (for the glass rod) is induced on the ball and the two are attracted. If you touch the ball, some of the charge from the rod will be transferred to the ball and the ball will be repelled.

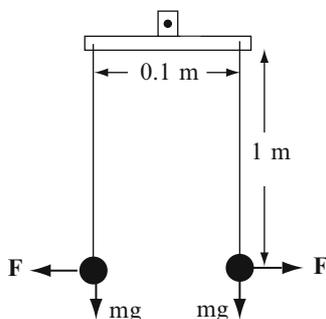


Figure 3.32

Coat the suspended ball with a thin layer of conducting aluminum foil. Foil from a candy or chocolate wrapper will do. Make sure the coating is uniform and thin to reduce weight. Rub the rod and touch the conducting ball. Charge will be transferred to the ball. Rub the rod again and approach the suspended ball with the rod. Since both rod and ball are of the same polarity, the ball will be repelled.

Instead of using a rod, you can also draw the conducting or nonconducting balls near a TV screen or a computer monitor screen. Because the charge on these screens is rather large (particularly on older TV screens), the effect is much larger. Also, try to put the back of your palm parallel and near a TV screen. You should be able to “feel” the electric field as a tingling sensation which is due to low currents caused by discharge through your body. Crackling noises may also accompany this experiment if done on an older TV screen. These are due to small discharges between the screen and your hand. Hair on the back of your hand or arm (if you have any) will bristle in the electric field intensity produced by the screen and the induced charge in the hair.

3.8 Summary

The starting relation in electromagnetics is *Coulomb's law*, describing the force between two point charges. It may be written in different forms but **Eq. (3.11)** is representative:

$$\mathbf{F}_{12} = \hat{\mathbf{R}}_{12} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}, \quad \mathbf{F}_{21} = \hat{\mathbf{R}}_{21} \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2} \quad [\text{N}] \quad (3.11)$$

where ϵ_0 is the permittivity (in this case that of free space). From Coulomb's law we defined the *electric field intensity* in **Eq. (3.17)** as force per unit charge. The forms in **Eq. (3.18)** are more useful:

$$\mathbf{E} = \frac{Q\mathbf{R}}{4\pi\epsilon_0|\mathbf{R}|^3} \quad \text{or} \quad \mathbf{E} = \hat{\mathbf{R}} \frac{Q}{4\pi\epsilon_0|\mathbf{R}|^2} \quad \left[\frac{\text{N}}{\text{C}} \right] \quad (3.18)$$

The relations for point charges were extended to distributed charge densities by defining elemental, differential charges based on line, surface, and volume charge distributions. Viewing the elemental charges as point charges, Coulomb's law provides a general way of calculating the electric field intensity due to distributed charges as

$$d\mathbf{E} = \frac{(\mathbf{r} - \mathbf{r}')\rho_{\Omega'}}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}d\Omega', \quad \mathbf{E} = \int_{\Omega'} \frac{(\mathbf{r} - \mathbf{r}')\rho_{\Omega'}}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}d\Omega' \quad \left[\frac{\text{N}}{\text{C}} \right]$$

\mathbf{r}' is the position vector of the elemental point charge (source point), \mathbf{r} is the position vector of the field point at which the field is calculated (see **Figures 3.13, 3.16, and 3.20**), and Ω' stands for l' , s' , or v' [see **Eqs. (3.40), (3.47), and (3.52)**].

Finally, the **electric flux density** \mathbf{D} is introduced, for now simply as a means of eliminating the dependence of the electric field intensity on permittivity from calculation:

$$\mathbf{D} = \epsilon\mathbf{E} \quad [\text{C}/\text{m}^2] \quad (3.55)$$

In effect, all of this chapter deals with the electric field intensity of point charges and the forces between them—either physical point charges such as isolated point charges defined on line, surface, and volume distributions.

Important reminders:

1. Note the distinction between field points and source points (denoted with a (')).
2. Pay attention to symmetries—they can help in understanding and in solving problems.
3. Superposition of fields and forces can be used to calculate fields of complex distributions.
4. The electric field intensity and the force are vectors. When summing up various contribution to either of these, separation into components simplifies the task.
5. An appropriate choice of coordinates can often simplify calculations.

Useful quantities:

Permittivity of free space: $\epsilon_0 = 8.854853 \times 10^{-12}$ [F/m]

Charge of the electron: $e = -1.602129 \times 10^{-19}$ [C]

Problems

Simple Point Charges, Forces/Electric Field

- 3.1 Electric and gravitation forces.** Two planets are 1,000,000 km apart (about the same as between the Earth and Mars). Their mass is the same, equal to 10^{24} kg (of the same order of magnitude as the Earth):
- (a) What must be the amount of free charge on each planet for the electrostatic force to equal the gravitational force? The gravitational force is $F_g = GM_1M_2/R^2$, where $G = 6.67 \times 10^{-11}$ [$\text{N} \cdot \text{m}^2/\text{kg}^2$]. M_1 and M_2 are the masses of the planets in [kg] and R is the distance between the planets in [m]. Because of the very large distance, you may assume the planets behave like point charges, and the total charge (magnitude) on the two planets is the same.
 - (b) What must be the signs of the charges so that they cancel the gravitational force?
- 3.2 Atomic forces.** In fusion of two hydrogen nuclei, each with a charge $q = 1.6 \times 10^{-19}$ C, the nuclei must be brought together to within 10^{-20} m. Calculate the external force necessary to do so. View each nucleus as a point charge.
- 3.3 Repulsion force.** A very thin tube contains at its bottom a small plastic ball charged with a charge $Q = 0.1 \mu\text{C}$ as shown in **Figure 3.33**. A second, identical ball with identical charge is inserted from above. Assume there is no friction and the glass does not affect the charges on the balls. If each ball has a mass $m = 1$ g and the permittivity of the tube may be assumed to be the same as that of air (free space) calculate:
- (a) The distance between the balls if the tube is vertical (**Figure 3.33a**).
 - (b) The distance between the balls if the tube is tilted so it makes an angle $\alpha = 30^\circ$ with the horizontal (**Figure 3.33b**).

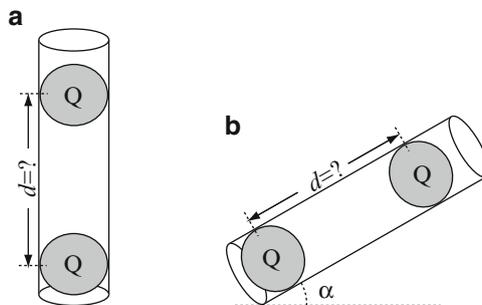


Figure 3.33

3.4 Repulsion forces. Two point charges, each with charge $q = 10^{-9}$ C, are suspended by two strings, 0.2 m long, and connected to the same point as in **Figure 3.34**. If each charge has a mass of 10^{-4} kg, find the horizontal distance between the two charges under the assumption that α is small.

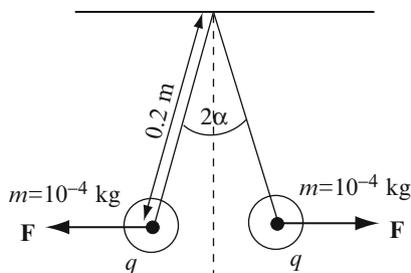


Figure 3.34

3.5 Application The electrometer. An electrometer is a device that can measure electric field intensities or charge. One simple implementation is shown in **Figure 3.34**. Two very small conducting balls are suspended on thin conducting wires. To apply the charge, the test charge is placed at the point of contact of the two wires. The balls repel each other since half the charge is distributed on each ball.

Suppose you need to design an instrument of this sort. Each wire is 0.2 m long and the mass of each ball is 10^{-4} kg. Assume each ball acquires half the measured charge and the balls are small enough to be considered as points:

- (a) If the minimum distance measurable is 0.5 mm along the circumference of the circle the ball describes as it is deflected (arclength), what is the lowest amount of charge this instrument can measure?
- (b) Can you also calculate the largest amount of charge measurable? Explain.

3.6 Force on point charge. Two positive point charges, each equal to Q [C], are located a distance d [m] apart. An electron, of charge e [C] and mass m [kg], is held in position on the centerline between them and at a distance x [m] from the line connecting the two charges as shown in **Figure 3.35**. The electron is now released:

- (a) Calculate the acceleration the charge is experiencing (direction and magnitude). Where is the acceleration maximum?
- (b) If there are no losses and no external forces, what is the path the electron describes?

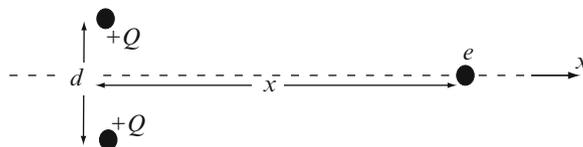


Figure 3.35

3.7 Force on point charge. Two point charges are located at a distance d [m] apart. One charge is $Q_1 = +q$ [C], the other is $Q_2 = +2q$ [C], as in **Figure 3.36**. A third charge, $+q$ [C], is placed somewhere on the line connecting the two charges:

- Assuming the third charge can only move on the line connecting Q_1 and Q_2 , where will the charge move?
- From this stationary point, the charge is given a small vertical push (up or down) and allowed to move freely. Describe qualitatively the motion of the charge as it moves away from the two stationary charges.

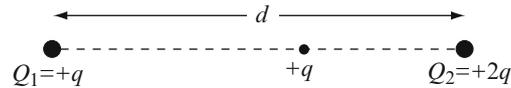


Figure 3.36

3.8 Electric field of point charges. Two point charges are located as in **Figure 3.37**:

- Sketch the electric field intensity of this arrangement using field lines.
- Find a point (on the axis) where the electric field intensity is zero.

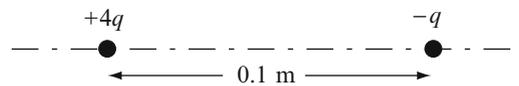


Figure 3.37

3.9 Electric field of point charges. Four charges are arranged as shown in **Figure 3.38**:

- Find the electric field intensity at any point in space.
- Show that the electric field intensity is perpendicular to the two dotted lines shown (at 45°).

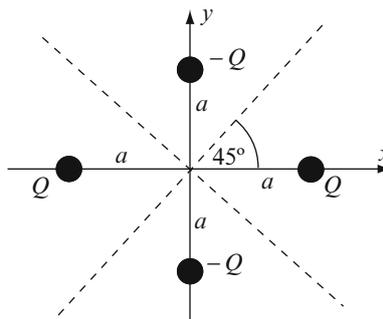


Figure 3.38

3.10 The quadrupole. Three point charges are located as shown in **Figure 3.39**. The outer charges equal $-q$ [C] and the center charge equals $+2q$ [C]. This arrangement is called a quadrupole because it consists of two dipoles side by side:

- Find the electric field intensity at an arbitrary point P a distance R [m] from the positive charge.
- Find the electric field intensity at P if $R \gg d$. Use the method and approximations of **Section 3.4.1.3**.
- Sketch the field lines of the quadrupole and compare with those of the dipole in **Section 3.4.1.3** (**Figure 3.12**).

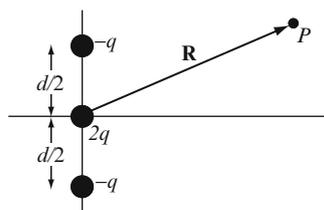


Figure 3.39

3.11 Electric field and forces due to point charges. Two positive point charges each equal to Q [C] and two negative charges each equal to $-Q$ are placed at the corners of the base of a pyramid as shown in **Figure 3.40**. The base is a square $a \times a$ m² in dimensions. A 5th charge $-Q$ is placed at the pinnacle of the pyramid. If all edges of the pyramid equal to a , calculate:

- The electric field intensity at the pinnacle of the pyramid.
- The electric field intensity at the center of the base of the pyramid.
- The force on the negative charge at the pinnacle.

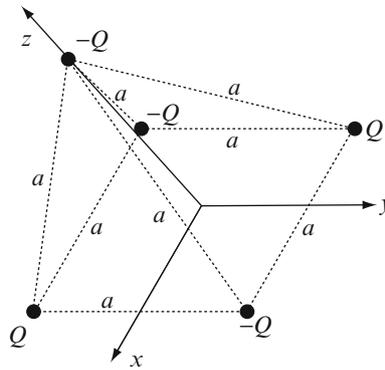


Figure 3.40

3.12 Application: Electrostatic levitation—Repulsion mode. The repulsion force between charges can be used to levitate bodies by ensuring the repulsion force is equal to the weight of the levitated body. Suppose four small plastic balls are placed at the corners of a rigid square, a [m] on the side, and charged each to $+100$ nC. Four identically charged balls are embedded into a fixed plane, with the distance between the balls also a [m]. Assume the levitation h is small compared to the length a ($h \ll a$), and the weight of the frame is negligible:

- If the movable frame is placed with its charges exactly above the four stationary charges and if the mass of each ball is 10^{-4} kg, calculate the elevation at which the movable frame levitates.
- If the movable frame is allowed to rotate but not to translate, what is the rest position of the charges? Do not calculate the position; just describe it.
- Suppose you push the movable frame downward slightly from the position in (b) and release it. What happens to the frame?
- Suppose you move the frame downward until the movable charges are in the same plane as the stationary charges. Disregarding mechanical questions of how this can be done, what happens to the frame now?
- If you continue pushing until the frame is below the plane, what will be the stationary position of the frame?

3.13 Application: Electrostatic levitation—Attraction mode. The situation in **Problem 3.12** is given again, but the stationary charges are positive and the movable charges are negative and of the same magnitude:

- Find the position at which the frame is stationary (below the plane). Assume the distance h [m] between the frames is small compared to the frame's size ($h \ll a$) and the frame is free to rotate in its plane.
- If the frame is pushed either downward or upward slightly, what happens?
- Compare with **Problem 3.12**. Which arrangement would you choose?

3.14 Force on charge in an electric field. Two stationary point charges, each equal in magnitude to q [C], are located a distance d [m] apart. A third point charge is placed somewhere on the line separating the two charges and is allowed to move. The charge is free to move in any direction:

- If the stationary charges are positive and the moving charge is positive, it will move to the midpoint between the two stationary charges. Is this a stable position? If not, where will the charge eventually end up if its position is disturbed?
- What is the answer to (a) if the stationary charges are positive but the moving charge is negative?
- What is the answer to (a) and (b) if the stationary charges are negative?

3.15 Application: Accumulation of charge. One of the common mechanisms for charge to accumulate in clouds is through friction due to raindrops. In the process, each drop acquires a positive charge, while the cloud tends to become more negative. Suppose each raindrop is 2 mm in diameter and each acquires the charge of one proton. If it rains at a rate of 10 mm/h, what is the rate of transfer of charge to earth per unit area?

3.16 Electric field of point charges. Three point charges are placed at the vertices of an equilateral triangle with sides of length L [m]:

- (a) Each charge is equal to q [C]. Find the points in space at which the electric field intensity is zero.
- (b) Two charges are equal to q [C], the third to $2q$ [C]. Show that the electric field intensity is zero only at infinity.

Hint: To find the answers you will need to solve a transcendental equation. Use a graphical method to find the roots of the equation.

3.17 Application: Millikan's experiment—Determination of the charge of electrons. A very small oil drop has a mass of 1 μg . The oil drop is placed between two plates as shown in **Figure 3.41**. A uniform electric field intensity \mathbf{E} is applied between the two plates, pointing down:

- (a) What is the electric field intensity that will keep the drop suspended without moving up or down if the drop contains one free electron?
- (b) What is the next (lower) value of field possible and what is the charge on the drop?

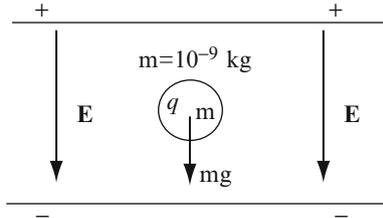


Figure 3.41

3.18 Application: Attraction of dust to TV screens. A TV screen produces a uniform electric field intensity of 600 kN/C in the immediate vicinity of its screen. The screen is negatively charged because of excess electrons produced by the electron gun. A dust particle, with mass $m = 10^{-19}$ kg and charge of one proton, is drawn toward the screen, starting from rest at a distance of 100 mm away from the screen:

- (a) With what speed does it strike the screen?
- (b) Is there any danger the particle may damage the screen? Explain.

Note: TV screens tend to accumulate a considerable amount of dust due to this attraction. Screens need frequent cleaning to remove the dust mechanically. This should be done after the TV has been turned off and allowed to discharge.

3.19 Force in a uniform field. A uniform electric field intensity of 1,000 N/C is directed upwards. A very small plastic ball of mass 10^{-4} kg is charged with a charge such that the ball is suspended motionless in the field:

- (a) Calculate the charge on the ball assuming a point charge. Find the magnitude and sign of the charge.
- (b) If the charge is doubled and the ball allowed to move starting from rest, find the acceleration of the ball.
- (c) If the ball starts from rest, in a frictionless environment, find the time it takes the ball to reach a speed equal to half the speed of light under the condition given in (b).

Line Charge Densities

3.20 Field due to line charge density. A short line of length L [m] is charged with a line charge density ρ_l [C/m]. Calculate the electric field intensities at the three points shown in **Figure 3.42**.

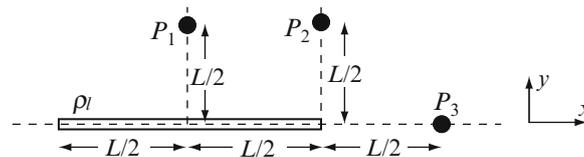


Figure 3.42

3.21 Field due to a charged ring. A thin ring of radius a [m] is charged so that a uniform line charge density of ρ_l [C/m] exists on the ring:

- Calculate the electric field intensity at a height h [m] above the center of the ring, on its axis.
- Show that at very large distances ($h \gg a$), the electric field intensity is that of a point charge equal to the total charge on the ring.

3.22 Force on line charge density. A line charge 1 m long is charged with a line charge density of 1 nC/m. A point charge is placed 10 mm away from its center. Calculate the force acting on the point charge if its charge equals 10 nC.

3.23 Electrostatic forces. Two thin, segments, each 1 m long, are charged with line charge densities of 10 nC/m and -10 nC/m and are placed a distance 10 mm apart, parallel to each other. Calculate the total force one segment exerts on the other.

3.24 Force between two short charged segments. Two charged lines are colinear as shown in **Figure 3.43**. The total charge on line 1 is Q_1 [C]; the total charge on line 2 is Q_2 [C]. On each line the charge is uniformly distributed over the length of the line. Given the lengths of the lines (L_1 [m] and L_2 [m]) and the distance between them as a [m], calculate the force one line exerts on the other if both are in free space.

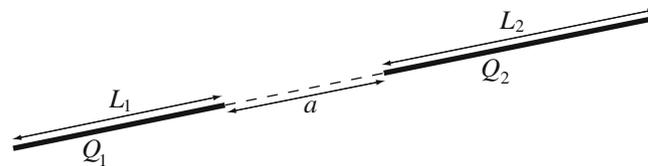


Figure 3.43

Surface Charge Densities

3.25 Electric field due to surface charge density. A disk of radius a [m] is charged with a uniform surface charge density $\rho_s = \rho_0$ [C/m²]. Calculate the electric field intensity at a distance h from the center of the disk on the axis (see **Figure 3.44**).

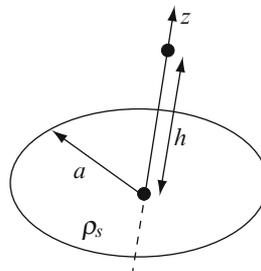


Figure 3.44

- 3.26 Electric field due to surface charge density.** A disk of radius a [m] is charged with a nonuniform charge density $\rho_s = \rho_0 r$ [C/m²], where r [m] is the distance from the center of the disk. Calculate the electric field intensity at a distance h [m] from the center of the disk, on the axis (see **Figure 3.44**).
- 3.27 Electric field of infinite surface.** Calculate the electric field intensity in free space at a distance d [m] above an infinite plane charged with a uniform surface charge density ρ_s [C/m²]. Show that the electric field intensity is independent of d and perpendicular to the plane.
- 3.28 Electric force due to hollow, charged cylindrical surface.** A very thin-walled cylindrical tube of length L [m] and radius a has a surface charge density ρ_s [C/m²] uniformly distributed as shown in **Figure 3.45**. A point charge Q is placed at point P on the axis of the cylinder. Calculate the force on the charge Q (magnitude and direction). The tube is drawn in axial cross section.

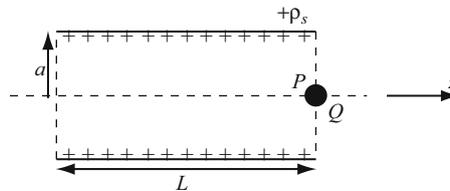


Figure 3.45

Volume Charge Densities

- 3.29 Field due to volume charge density.** A short plastic cylinder of length L [m] and diameter $L/2$ [m] has a uniform volume charge density ρ_v [C/m³] distributed throughout its volume. Calculate the electric field intensity at P_1 , P_2 , and P_3 (on the axis of the cylinder) shown in **Figure 3.46**.

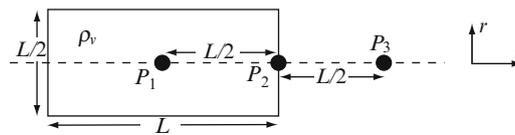


Figure 3.46

- 3.30 Electric field of the electron.** The structure of the electron is not defined (i.e., its dimensions cannot be identified with certainty). Its charge, however, is considered to be a point charge. Consider the following:
- Suppose the electron is a point charge equal to $e = -1.602 \times 10^{-19}$ C. Calculate the electric field intensity it produces everywhere in space.
 - Suppose now that the charge of the electron is distributed uniformly over a spherical volume of radius $R_0 = 2 \times 10^{-13}$ m. Calculate the electric field intensity everywhere in space outside the electron (that is, for $R > R_0$). Show that the result is the same as in (a). **Hint:** To simplify solution, assume that the sphere is made of a stack of disks of varying radii and differential thickness dz' and calculate the electric field intensity on the axis of the disk (see, for example, **Problem 3.25**).
- 3.31 Electric field due to conducting shell.** A spherical shell of radius a [m] is charged with a uniform surface charge density ρ_s [C/m²]. Find the electric field intensity inside and outside the spherical shell. Show that the electric field inside is zero, whereas outside, it is equal to the electric field of a point charge, located at the center, and equal in magnitude to the total charge on the sphere.
- 3.32 Electric field of small volume charge density.** A small sphere of radius a [m] has a nonuniform volume charge density given as $\rho = \rho_0 r(r - a)/a$ [C/m³] where r is the distance from the center of the sphere. Find the electric field intensity at a very large distance $R \gg a$. What are the assumptions you must make?

The Electric Flux Density

3.33 Electric flux density due to point charges. Two point charges are separated a distance d [m] apart. Each charge is 10 nC and both are positive:

- (a) Calculate the force between the two charges.
- (b) Both charges, while still at the same distance, are immersed in distilled water ($\epsilon = 81\epsilon_0$). What is the force between the charges in water? Explain the difference.
- (c) What can you say about the electric field intensity and electric flux density in air and water?

3.34 Electric flux density in dielectrics. A point charge is located in free space. The charge is surrounded by a spherical dielectric shell, with inner diameter d_1 [m], outer diameter d_2 [m], and permittivity ϵ [F/m]. Calculate:

- (a) The electric flux density everywhere in space.
- (b) The electric field intensity everywhere in space.
- (c) The total electric flux passing through the outer surface of the dielectric shell.
- (d) The total flux passing through the inner surface of the dielectric.
- (e) The total flux through a spherical surface of radius $R > d_2/2$ in air.
- (f) What is your conclusion from the results in (c), (d), and (e)?