Faraday's Law and Induction

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> *Whether Ampere's beautiful theory were adopted, or any other, or whatever reservation were mentally made, still it appeared very extraordinary, that as every electric current was accompanied by a corresponding intensity of magnetic action at right angles to the current, good conductors of electricity, when placed within the sphere of this action, should not have any current induced through them, or some sensible effect produced equivalent in force to such a current.*
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> —Michael Faraday (1791–1867), article 3 in the first series of "Experimental Researches in Electricity," Nov. 24, 1831

10.1 Introduction

In the previous chapters, we found it useful to treat the electric and magnetic phenomena separately. In **Chapters 4** through **7**, we treated electrostatic fields by relying on the two postulates:

$$(\nabla \times \mathbf{E} = 0 \;\; \text{and} \;\; \nabla \cdot \mathbf{D} = \rho) \;\; \text{or} \;\; \left(\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \;\; \text{and} \;\; \oint_s \mathbf{D} \cdot d\mathbf{s} = Q \right)$$

In **Chapters 8** and **9**, some of the basic magnetic phenomena were introduced, now relying on the following two postulates:

$$(\nabla \times \mathbf{H} = \mathbf{J} \;\; \text{and} \;\; \nabla \cdot \mathbf{B} = 0) \;\; \text{or} \;\; \left(\oint_C \mathbf{E} \cdot d\mathbf{l} = I \;\; \text{and} \;\; \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \right)$$

These postulates allowed us to treat a large number of applications and gain insight into the behavior of the electric and magnetic fields governed by the postulates.

Two aspects of this approach should be apparent by now:

(1) The discussion of electrostatic fields was independent of that for magnetostatic fields. Even though we saw in **Chapter 7** that a static electric field can cause a flow of charge in a conductor and, therefore, a current, the electric field does not depend on this current and any of its consequences; that is, the static electric field is uniquely defined from the charge distribution in the system. The current thus generated does, in turn, generate a static magnetic field as defined by Ampere's law. However, this magnetic field, while it may coexist with the electric field, does not affect it.

(2) The discussion was limited to static applications. We have not specifically stated that the postulates of the magnetostatic field are not valid under time-varying conditions, but because the magnetostatic field relied on steady electric currents, time-dependent phenomena could not be included.

The question is: What happens if the fields are time varying? Or, perhaps, a better statement would be: How do we need to modify the fundamental postulates to treat time-varying electric and magnetic fields?

This chapter discusses the question of time dependency in some detail. We will see that the electric and magnetic fields under time-varying conditions are interdependent. A time-varying electric field generates a time-varying magnetic field and vice versa. Thus, the time-dependent magnetic flux density **B** (and therefore **H**) is dependent on the electric field intensity **E** (and therefore on **D**). The above postulates will be modified to account for this dependency. Since now we must treat both the electric

and magnetic fields as coupled vectors, the postulates for both fields must be included. A total of four relations are required to specify the electromagnetic field under time-dependent conditions: the curl of the electric field intensity **E**, the divergence of the electric flux density **D**, the curl of the magnetic field intensity **H**, and the divergence of the magnetic flux density **B**.

Because of the dependency between the electric and magnetic fields, we should use the term electromagnetic field when dealing with time-dependent fields. This term indicates that the two fields cannot be treated separately. The remaining part of this books deals with the electromagnetic field. In some cases, we will find it easier or more useful to treat the electric field or the magnetic field alone, but whichever field we choose to emphasize, it should be remembered that the other field can always be derived if necessary and, more importantly, it always exists.

After introducing Faraday's law as the first basic law governing time-dependent fields, we discuss some applications of Faraday's law. These include traditional power devices like transformers, motors, and generators as well as more recently developed devices like linear motors and levitating mechanisms. Other applications are acceleration of particles, electromagnetic testing of materials, electromagnetic ore prospecting, heating and melting of materials, magnetic braking, and many others.

The main difference at this stage, between the time-dependent and time-independent fields, is in the process of induction: A time-dependent field produces an electric field and that, in turn, induces currents in conducting materials. This induction is fundamental to the operation of many very important devices, including transformers and generators. There are two mechanisms of induction: one is due to change in the magnetic flux; the other due to motion in the magnetic field. Both are discussed since they can exist simultaneously in the same device and because both are important in design.

10.2 Faraday's Law

The coupling between the electric and magnetic fields is based on an experimental relation known as Faraday's law. This law was formulated in 1831 by Michael Faraday following a series of experiments. Faraday[1] observed that if he moved a closed loop in the magnetic field of a magnet or if he moved the magnet while the loop remained stationary, a current flowed in the loop. This current was not due to external sources but rather was induced in the loop by the change in the magnetic flux. He also found that the current was proportional to the rate of change of flux.

The current observed by Faraday is due to an induced voltage in the loop. This voltage is called an ***electromotive force*** or ***emf***. The electromotive force produced in this experiment can be written as

$$\text{emf} = -\frac{d\Phi}{dt} \quad [\text{V}] \tag{10.1}$$

where Φ is the flux through the loop. The physical situation is shown in **Figure 10.1**. In general, there may be more than one loop in the same location; therefore, a more general relation for the electromotive force is

$$\text{emf} = -N\frac{d\Phi}{dt} \quad [\text{V}] \tag{10.2}$$

where N is the number of loops. This type of emf is also called a ***transformer action emf*** for reasons we will see shortly.

To find the relation between the electric field intensity and the magnetic flux density, we use the definitions of flux and of electromotive force and substitute these in Faraday's law.

In **Chapter 7** (see, for example, **Section 7.7.1**), we discussed the idea of an electromotive force as the source of steady currents in conductors and in conjunction with Kirchhoff's laws. There, we found that the electromotive force is the closed contour integral of the nonconservative electric field intensity (the closed contour integral of a conservative electric field intensity is always zero):

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[1] Michael Faraday (1791–1867) had little schooling, mostly as self-education. At age 22, he became assistant to Sir Humphry Davy (a well-known chemist and member of the Royal Institution) who helped his education. In 1821, Faraday demonstrated the operation of an electric motor. In 1825 he became director of the Royal Institution Laboratory and, there, in 1831 he discovered what is now known as Faraday's law. His experiment showed that the motion of a magnet near a wire loop produces an electromotive force, and therefore a current, in the closed loop. Faraday experimented in other areas as well. These included dielectrics, electrolysis, and polarization of light. The farad unit of capacitance is named after him in honor of his many and varied achievements. Faraday worked long and hard and is considered the ultimate experimentalist. His work was not all fun. In 1862, he became very ill from inhaling mercury vapors (he was using mercury for contacts) and it took him almost 2 years to recover. Faraday, more than anyone else, laid the foundation for the electromagnetic theory later developed by James Clerk Maxwell into what is today known as Maxwell's theory, and this is summarized by Maxwell's equations. Faraday's law is one of the four equations. Later, when Maxwell unified the electromagnetic theory, he gave credit to many, but mostly to Faraday.
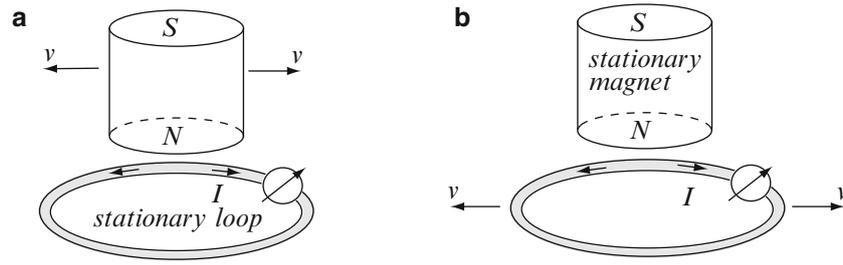
Figure 10.1 Faraday's law: (**a**) Movement of a permanent magnet in the presence of a loop generates an electromotive force and a current in the loop. (**b**) Movement of the loop in the presence of the stationary magnet generates an identical current

$$\text{emf} = \oint_C \mathbf{E} \cdot d\mathbf{l} \quad [\text{V}] \tag{10.3}$$

The magnetic flux used in **Eq. (10.1)** is calculated from the magnetic flux density as

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{Wb}] \tag{10.4}$$

Substituting the relations for flux and emf in **Eq. (10.1)** gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{V}] \tag{10.5}$$

Now, applying Stokes' theorem to the left-hand side of **Eq. (10.5)**,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{V}] \tag{10.6}$$

where s is the surface bounded by the closed contour C. Because the integral on the right-hand side is independent of time, we can perform the differentiation with respect to time inside the integral

$$\int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad [\text{V}] \tag{10.7}$$

and, therefore, for a loop with a constant surface (the surface is not time dependent), the integrands must be equal:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{10.8}$$

This is the differential statement of Faraday's law and it clearly indicates the relation between the electric and magnetic fields. The electric field is nonconservative (the curl of the electric field intensity is not zero). For this reason, the electric field intensity cannot be defined as the gradient of a scalar potential and the definition of electrostatic potential cannot be used here. **Equation (10.8)** also indicates that Faraday's law is general and the loop is not necessary for the induced emf to exist. This aspect of the relation will be used later to define and evaluate induced currents in conducting volumes.

The relation in **Eq. (10.8)** is one of the required postulates for the time-dependent electric field intensity **E**. The remaining postulates, namely, the divergence of **D**, the curl of **H**, and the divergence of **B**, remain unchanged. Therefore, the required postulates for the time-dependent electromagnetic field are as follows.

Differential form	Integral form
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -N\dfrac{d\Phi}{dt}$ [V]
$\nabla \times \mathbf{H} = \mathbf{J}$ [A/m^2]	$\oint_C \mathbf{H} \cdot d\mathbf{l} = I$ [A]
$\nabla \cdot \mathbf{B} = \mathbf{0}$	$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \cdot \mathbf{D} = \rho$ [C/m^3]	$\oint_s \mathbf{D} \cdot d\mathbf{s} = Q$ [C]

The only difference between these relations and those for the static electric and static magnetic fields is in the first equation, but this seemingly small difference has far-reaching implications, as we shall see shortly. For now, we merely observe that Faraday's law couples between the electric and magnetic fields. From now on, we will discuss the electric field and magnetic field as coupled fields.

10.3   Lenz's Law[2]

In **Eq. (10.1)**, the negative sign indicates that if the magnetic flux linking the loop increases, the induced emf produces a current whose flux opposes the increase in the flux linking the loop and if the flux decreases, the emf produces a current whose flux augments the flux linking the loop. To see this, consider first the loop in **Figure 10.2a**. Using the right-hand rule for the flux generated by current I_1 in loop (1), an increase in the current increases the flux through loops (1) and (2). Loop (1) is then the source of the flux. An emf is generated in loop (2), but, because loop (2) is open, there is no current in the loop and therefore no induced flux due to loop (2). Now, consider the situation in **Figure 10.2b** where, again using the right-hand rule, an increase in the flux due to loop (1) produces a negative emf in loop (2). This corresponds to Faraday's law. This induced emf, shown in **Figure 10.2c**, can be viewed as the source of a flux that is in the opposite direction to the flux in **Figure 10.2a**. Thus, the induced emf produces a flux that opposes the flux that generated the emf. This relation between induced electromotive force and the magnetic flux linkage is defined by Lenz's law, which states:

> "*the direction of the electromotive force is such that the flux generated by the induced current opposes the change in flux.*"

As a direct consequence of Lenz's law, the flux linking a circuit tends to maintain its value (magnitude and direction), resisting, as it were, any change. Although Faraday's law deals with electromotive force, and an electromotive force may be generated in an open or closed circuit, the statement of Lenz's law requires a current and, therefore, a closed circuit.
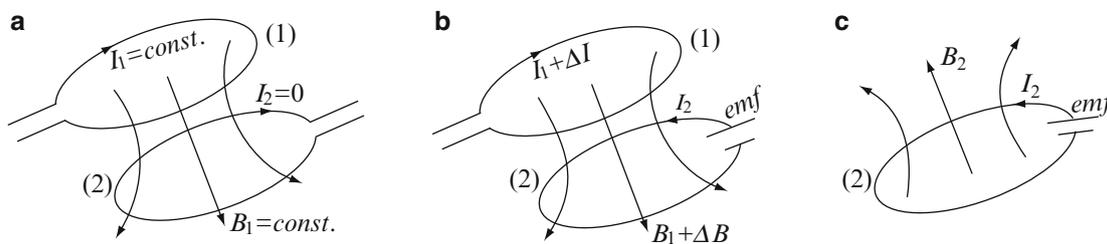

Figure 10.2 (**a**) Flux generated by a constant current in a loop. (**b**) Electromotive force induced in loop (2) due to loop (1). (**c**) Flux in loop (2) due to induced emf opposes the inducing flux. The emf is shown as a DC source to indicate the direction of the induced current but it can be an AC emf as well, depending on how it is generated

[2] Emil Khristianovich Lenz (1804–1865) was a Russian scientist with wide ranging interests. He deduced this law in 1833, but he was involved in many activities in addition to his research in electromagnetics and electrothermal and electrochemical applications. He also worked on measurement methods and developed preliminary laws similar to Kirchhoff's laws before Kirchhoff did so and also contributed to measurements of salinity in the seas as well as contributions to understanding of weather, galvanic cells, and electric machines.

10.4   Motional Electromotive Force: The DC Generator

The force on a charge moving in a constant magnetic field was calculated in **Chapter 9 [Eq. (9.106)]** as

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad [\text{N}] \tag{10.9}$$

Because the electric field intensity is defined as force per unit charge, we can write

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{10.10}$$

The term $\mathbf{v} \times \mathbf{B}$ is therefore an electric field intensity, generated by motion of charges. If we substitute this in **Eq. (10.3)**, we get

$$\text{emf} = \oint_C \mathbf{E} \cdot d\mathbf{l} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \tag{10.11}$$

where C is the contour on which the electromotive force is desired.

To see how these relations apply, we consider here a classical example of induced emf: that of a sliding bar on two parallel rails in a magnetic field as shown in **Figure 10.3a**. The rails are separated a distance d and are shorted together on one side. The magnetic flux density points out of the page. The bar and the shorted rails form a loop $abcd$. If we move the bar to the right at a velocity $\mathbf{v}$, there will be an induced emf in the bar because the loop increases in area and therefore encloses a larger, changing flux. We wish now to calculate this emf for a bar of length d, flux density $\mathbf{B}$, and velocity $\mathbf{v}$, because this calculation leads directly to the idea of the DC generator. There are two ways to calculate the emf in this case. One is from the motional effect; the second is from the change in flux through the loop. The two methods are equivalent, but the reasoning is quite different. We look at both solution methods and show that they are equivalent:
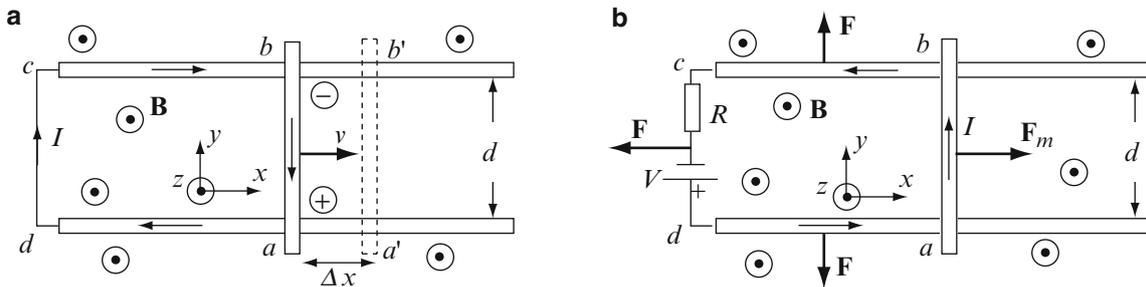

Figure 10.3 (**a**) Force on a conducting bar which carries current perpendicular to the magnetic field. (**b**) A bar sliding on rails at velocity $\mathbf{v}$, perpendicular to the magnetic field

(**a**) **Calculation of emf from motional considerations**. As the bar moves at a velocity $\mathbf{v}$, the electrons in the bar move with it at the same velocity. This motion creates a force on the electrons and, therefore, an electric field intensity $\mathbf{v} \times \mathbf{B}$ in the bar. The induced emf is, therefore,

$$\text{emf} = \int_a^b (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = Bvd \quad [\text{V}]. \tag{10.12}$$

(b) Calculation of emf from change in flux. The total flux enclosed by the loop is

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} = BS \quad [\text{Wb}] \tag{10.13}$$

where S is the area of the loop *abcd* and $\mathbf{B}$ is constant throughout the loop.

Now, if the bar moves at a velocity $\mathbf{v}$, the location of the bar after a time Δt will be at a distance Δx from the previous point (dashed bar in **Figure 10.3a**). The flux through the loop is now

$$\Phi' = \int_s \mathbf{B} \cdot d\mathbf{s} = BS' = B(S + \Delta S) = B(S + \Delta xd) \quad [\text{Wb}] \tag{10.14}$$

Therefore, the change in flux due to this motion is

$$\Delta\Phi = \Phi - \Phi' = -Bd\Delta x \quad [\text{Wb}] \tag{10.15}$$

Using the magnitude of velocity as $v = \Delta x/\Delta t$ and taking the limit as $\Delta t \to 0$, we get

$$d\Phi = -Bd(v\,dt) \quad [\text{Wb}] \tag{10.16}$$

and the emf is found from the definition in **Eq. (10.1)**:

$$\text{emf} = -\frac{d\Phi}{dt} = B\,vd \quad [\text{V}] \tag{10.17}$$

This is the same result as previously obtained, but there is a distinction between the two results. In the first case, the emf is associated with the moving bar itself. In the second, it is associated with the whole loop. **Figure 10.3a** also shows the polarity of the bar and the direction of current in the loop caused by the generated emf.

We can view this device as a linear generator since application of a mechanical force produces an emf. How about the opposite: Does the connection of a source produce motion of the bar and, therefore, produce a motor effect? The answer is yes. To see this, consider **Figure 10.3b**, which shows the bar and rail discussed above, but now a source V and a resistance (representing the internal resistance of the source and that of the rails and bar) are connected as shown. We wish to calculate the force, if any, on the bar. The current in the rail and bar is $I = V/R$ and flows as shown. The force on a current-carrying conductor in a magnetic field is given by the following relation [see **Eq. (9.114)**]:

$$\mathbf{F} = I\oint_C d\mathbf{l}' \times \mathbf{B} \quad [\text{N}] \tag{10.18}$$

where $d\mathbf{l}'$ is taken in the direction of the current. Each segment of the loop will experience a force. The forces are shown in **Figure 10.3b** and each is equal to BIl where l is the length of the corresponding segment. These forces tend to expand the loop, but, since only the bar is free to move, it will move to the right. The force moving the bar is

$$\mathbf{F} = \hat{\mathbf{x}}BId = \hat{\mathbf{x}}B\frac{V}{R}d \quad [\text{N}] \tag{10.19}$$

However, as the loop expands, the length of the rail increases and the resistance increases. This, in turn, decreases the force with the expansion of the loop. This is in addition to the effect of the emf induced due to the motion of the bar (also called ***back emf***) which generates a force that tends to decrease the applied force on the bar (see **Example 10.2**).

The relations above deal primarily with constant motion in a DC magnetic field and hence the emf generated is a DC emf. However, if the magnetic field is time dependent, or if motion is not constant (for example, if a conductor moves back and forth in a DC magnetic field), the emf will also be time dependent. That is, motion action can produce either DC or AC emfs.

Example 10.1 Induced emf Due to Motion: Motional Electromotive Force A straight, thin conducting bar of length $L = 1$ m moves in a constant magnetic field at a speed $v = 20$ m/s. The magnitude of the magnetic field intensity is $B_0 = 0.5$ T. Calculate the induced emf in the bar under the following conditions:

(a) The bar moves perpendicular to the magnetic field as shown in **Figure 10.4a**. Calculate the emf and show its polarity.
(b) The bar moves at $30°$ to the x axis as shown in **Figure 10.4b**. Calculate the emf and show its polarity.
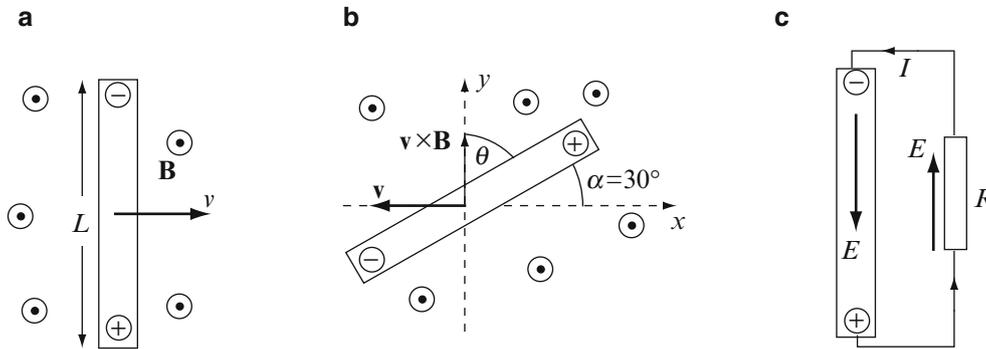
Figure 10.4 The emf in a moving conductor. (**a**) Conductor moves perpendicular to the magnetic field and velocity is perpendicular to the conductor. (**b**) Conductor and velocity vector at an angle α. (**c**) Equivalent circuit of the bar in (**a**) as a DC generator, showing polarity of the emf

Solution: The solution is based on calculation of the electric field intensity at the location of the bar, $\mathbf{E} = \mathbf{v} \times \mathbf{B}$. We can either create a system of coordinates to use the vector notation or, since the induced emf is required in the bar, use the bar as a reference axis and calculate all quantities with respect to this axis:

(a) The cross-product $\mathbf{v} \times \mathbf{B}$ is found to be in the direction of the bar, pointing downward. This is the electric field intensity in the bar: $\mathbf{E} = \mathbf{v} \times \mathbf{B}$. The emf is therefore

$$\text{emf} = \int_l \mathbf{E} \cdot d\mathbf{l} = \int_l (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vB_0L = 20 \times 0.5 \times 1 = 10 \quad [\text{V}].$$

The potential difference (emf) between the ends of the bar is 10 V. The polarity is as shown in **Figure 10.4c**. The polarity can be deduced from the direction of the field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ or from the fact that the magnetic field exerts a force on the electrons which moves them upward. This bar can now be used as a source in a circuit, and in this sense, the motion in the field makes the bar a generator.

(b) The solution in this case is the same except that the cross-product $\mathbf{v} \times \mathbf{B}$ and the bar are at $\theta = 90 - \alpha = 60°$ to each other. The induced emf is found in two steps. First, we find the vector product $\mathbf{E} = \mathbf{v} \times \mathbf{B}$. This gives the electric field intensity in the direction perpendicular to $\mathbf{v}$ and $\mathbf{B}$. However, we are only interested in that component of the electric field intensity that is parallel to the bar. Thus, the second step is to calculate the projection of E on the bar:

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} = -\hat{\mathbf{x}}v \times \hat{\mathbf{z}}B = \hat{\mathbf{y}}vB \quad [\text{V/m}]$$

The direction of this field is straight up. The component parallel to the bar is

$$E_b = vB\cos(90 - \alpha) = vB\sin\alpha \quad [\text{V/m}]$$

and the induced electromotive force is now

$$\text{emf} = \int_l \mathbf{E} \cdot d\mathbf{l} = vB_0 L \sin\alpha = 20 \times 0.5 \times 1 \times 0.5 = 5 \quad [\text{V}]$$

The smaller the angle α, the lower the induced emf. Polarity of the emf is opposite that in (**a**) and is shown in **Figure 10.4b**. Note that $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ points from the negative to the positive ends in the bar. This may seem strange at first, but we must remember that this is the field on a positive charge (electrons move in the direction opposite the electric field). The external field (such as in a circuit) will point from positive to negative as required (see **Figure 10.4c**).

Exercise 10.1 What is the induced emf and its polarity if:

(**a**) The bar in **Figure 10.4a** moves straight up at a velocity **v**.
(**b**) The bar in **Figure 10.4b** moves straight up at a velocity **v**.

Answer

(**a**) Zero. (**b**) $10\cos 30° = 8.66$ V. Polarity is shown in **Figure 10.4b**.

Example 10.2 Application: The Linear Generator A DC generator can be built as shown in **Figure 10.5a**. The bar, load, and rails form a closed circuit. The two rails are separated a distance $d = 1.5$ m and the load resistance is $R = 10\ \Omega$. Assume the rails and bar are perfectly conducting and the bar moves to the right at a constant velocity $v = 10$ m/s. The magnetic flux density is constant, directed upward, and equal to $B = 0.1$ T. Calculate:

(**a**) The electromotive force produced by this generator.
(**b**) The force required to move the bar at constant velocity.
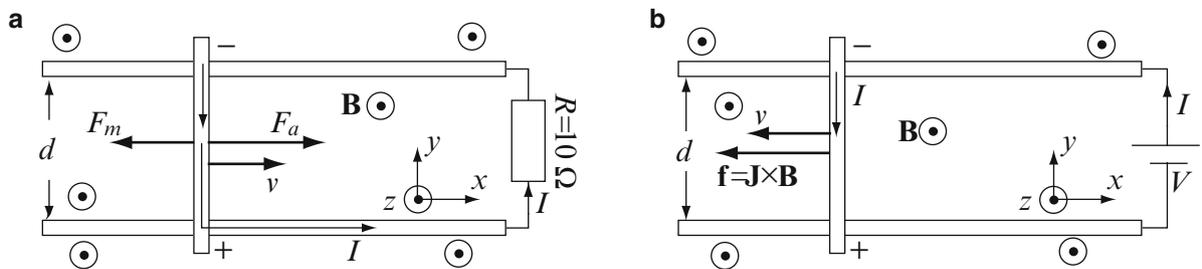(**c**) The mechanical power required for generation.

Figure 10.5 (**a**) Principle of a linear generator. (**b**) Principle of a linear motor

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Solution: A force is applied to the bar shown in **Figure 10.5a**, the bar moves, and this generates an emf on the bar. The generator creates a force $\mathbf{F}_m$ which opposes the applied force $\mathbf{F}_a$ (otherwise, the bar would accelerate). The mechanical power is calculated from velocity and force, which, in turn, is calculated from the current in the circuit and the magnetic flux density:

(**a**) To calculate the emf on the bar, we must calculate the rate of change of flux with time. As the bar moves to the right at a velocity v, after a time dt, the bar is at a new location, a distance $dx = vdt$ to the right of the original location, changing the loop area by $d(vdt)$. Since the flux density is constant, the change in flux is

$$d\Phi = B\, d(v\, dt) \quad [\text{Wb}]$$

The magnitude of the emf is

$$|\text{emf}| = \frac{d\Phi}{dt} = Bvd = 0.1 \times 1.5 \times 10 = 1.5 \quad [\text{V}].$$

The polarity of the emf is shown in **Figure 10.5a**. As the bar moves to the right, the term $\mathbf{v} \times \mathbf{B}$ points down (negative y direction). This is the direction of force on a positive charge and, therefore, the direction of current.

(b) The existence of a current in the circuit gives rise to a force $\mathbf{F}_m$. This force can be calculated from **Eq. (10.18)**:

$$\mathbf{F}_m = I \int_0^d d\mathbf{l}' \times \mathbf{B} = I \int_0^d -\hat{\mathbf{y}}dl' \times \hat{\mathbf{z}}B = -\hat{\mathbf{x}}IBd \quad [\text{N}].$$

This force is directed in the direction opposing $\mathbf{F}_a$, as shown. The induced current I is $\text{emf}/R = vBd/R$. To maintain movement at constant velocity, the two forces must balance:

$$F_m = F_a = IBd = vB^2d^2/R = 10 \times (0.1)^2 \times 1.5^2/10 = 0.0225 \quad [\text{N}].$$

(c) The mechanical power P is given by $P = \mathbf{F} \cdot \mathbf{v}$, giving the power of the applied force $\mathbf{F}_a$ as

$$P = F_a v = v^2B^2d^2/R = 10^2 \times (0.1)^2 \times 1.5^2/10 = 0.225 \quad [\text{W}].$$

This mechanical power is equal to the power dissipated through Joule's losses which, using the current $I = vBd/R$, is

$$RI^2 = v^2B^2d^2/R = 0.225 \quad [\text{W}].$$

Exercise 10.2 Suppose the rails in **Figure 10.5a** have a resistance per unit length r [Ω/m] each. The bar is perfectly conducting. Calculate the current in the resistor R at a time t if the bar starts at $t = 0$ at $x = 0$ ($x = 0$ is at the location of R) and moves at constant speed v in the negative x direction.

Answer $I(t) = \dfrac{Bvd}{R + 2rvt}$ [A]

Example 10.3 Application: The Linear Motor A linear DC motor is obtained by replacing the resistance R in **Figure 10.5a** with a source V with polarity such that the current flows in the same direction as the current in the generator (**Figure 10.5b**).

The two rails are separated a distance $d = 1.5$ m and a battery with potential $V = 12$ V is connected at one end of the rails. Assume the rails are perfectly conducting, but the bar has a resistance $R = 1\ \Omega$ and the bar is free to move. The magnetic flux density is constant, directed upward, and equal to $B = 0.1$ T:

(a) Calculate the force acting on the moving bar if the bar is held stationary.
(b) If the bar is allowed to move, find the maximum velocity of the bar.
(c) Find the equation of motion of the bar assuming the bar has mass m. Explain.

Solution: If the bar is held stationary, there is a DC current flowing through the circuit equal to V/R. The magnetic field exerts a force density on this current according to the relation $\mathbf{f} = \mathbf{J} \times \mathbf{B}$. This force is to the left (in the negative x direction according to the right-hand rule). If the bar is allowed to move, it will accelerate and its velocity will increase with time. At the same time, since the bar moves in the magnetic field, there will be an emf induced in the bar due to motion. This emf produces a current which opposes the source current. The equation of motion is found from the relation $\mathbf{F} = m\mathbf{a}$:

(a) If the bar is stationary, the current in the bar is

$$I = \frac{V}{R} = 12 \quad [\text{A}]$$

This current flows in the negative y direction as shown. The force, which is in the negative x direction, is

$$\mathbf{F}_m = I \int_0^d d\mathbf{l}' \times \mathbf{B} = I \int_0^d -\hat{\mathbf{y}} \, dl' \times \hat{\mathbf{z}} B = -\hat{\mathbf{x}} BId = -\hat{\mathbf{x}} 0.1 \times 12 \times 1.5 = -\hat{\mathbf{x}} 1.8 \quad [\text{N}].$$

(b) If the bar is allowed to move, it will move in the negative x direction under the force in part (a). An induced emf will exist in the bar which, according to **Eq. (10.12)**, is

$$\text{emf} = vBd \quad [\text{V}]$$

We do not know the velocity v, but we can still write

$$V - \text{emf} - IR = 0 \quad \rightarrow \quad I = \frac{V - \text{emf}}{R} \quad [\text{A}]$$

Substituting this into the force equation, the magnitude of $\mathbf{F}_m$ is given as

$$F_m = IBd = \frac{V - \text{emf}}{R} Bd = \frac{V - vBd}{R} Bd \quad [\text{N}]$$

This force is maximum at the start of motion when $v = 0$. As the bar moves, an emf, equal to vBd, is induced in the circuit, reducing the current in the bar. In other words, we start with a maximum force (starting force) given in (**a**), and as the bar moves, the force is reduced. In this case, there is no friction; therefore, the force must decrease to zero for the velocity to be constant. This gives

$$V - vBd = 0 \quad \rightarrow \quad v = \frac{V}{Bd} = \frac{12}{0.1 \times 1.5} = 80 \quad \left[\frac{\text{m}}{\text{s}}\right].$$

(c) The balance of forces (at any time before the acceleration diminishes to zero) requires that

$$ma = m\frac{dv}{dt} = F_m \quad [\text{N}]$$

The differential equation describing the motion is

$$m\frac{dv}{dt} = \frac{V - vBd}{R} Bd$$

In a practical linear motor, the bar, or, more often, a number of conductors, will move back and forth at a constant or varying speed, and we will also have to deal with friction forces as well as losses in the motor itself. However, this example serves to describe the principle.

10.5   Induced emf Due to Transformer Action

In describing Faraday's law, we mentioned that the emf produced by a changing flux is called a transformer action emf. The reason for this name is to distinguish it from the motional emf discussed in the previous section and because it is commonly encountered in transformers. The distinction here is between a moving conductor in the field causing a change in flux (motional emf) and a stationary loop in a changing magnetic field (transformer action emf). If the emf is connected in a closed circuit, there will also be a current in the loop and this current will produce a flux density according to Lenz's law.

In the previous section, we saw that the motional emf can be calculated either from the forces applied on the moving electrons in the conductors or as a change in flux. However, whenever we discuss stationary circuits, we must use the transformer action relation in **Eq. (10.2)**. It should also be clear that transformer action can only produce AC emfs.

Any closed circuit placed in a time-varying magnetic field experiences a current due to the induced emf. If, for example, you were to place a circuit, such as an amplifier or a computer board, in a time-dependent magnetic field, an emf will be induced in any closed circuit on the board. This type of induced emf can sometimes be detected as an annoying hum in audio amplifiers due to induced emfs from power lines.

Example 10.4 Application: The AC Generator   A uniform magnetic field is generated by a time-varying source as shown in **Figure 10.6a**. A stationary square loop is placed such that its plane is perpendicular to the magnetic flux density **B**. Assume the flux density is sinusoidal and given as $B = B_0 \sin\omega t$ [T] with $B_0 = 0.1$ T, $\omega = 100\pi$ rad/s, $a = 0.1$ m, $b = 0.1$ m:

(a) Calculate the induced emf in the loop.
(b) What is the induced emf if the loop is made of $N = 100$ turns, placed in essentially the same location?
(c) What is the solution in **(a)** and **(b)** if the loop is at an angle α to the field, as shown in **Figure 10.6c**?
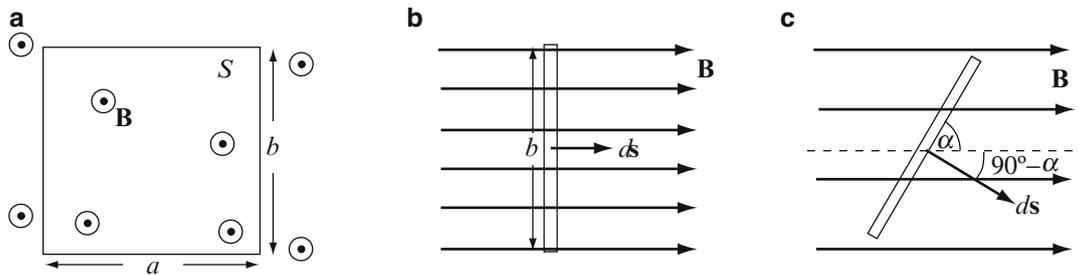
Figure 10.6 (**a**) A square loop with its plane perpendicular to a uniform, time-dependent magnetic flux density. (**b**) Side view of the figure in (**a**). (**c**) A square loop with its plane at an angle α to a uniform, time-dependent magnetic flux density

Solution: The transformer action emf is found using **Eqs. (10.1)** and **(10.7)**. The magnetic flux density is uniform; therefore, the flux is immediately available. The emf of N identical loops in the same field is N times larger than the emf of a single loop [see **Eq. (10.2)**]. In (**c**), the emf also depends on the angle the loop makes with the field because the flux depends on the angle:

(a) From **Eq. (10.7)**, the emf in the loop is

$$\text{emf} = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{s} = -\frac{dB}{dt}S = -S\omega B_0 \cos\omega t \quad [\text{V}]$$

where we used the fact that for parallel vectors **B** and $d\mathbf{s}$, $\mathbf{B} \cdot d\mathbf{s} = Bds$ (**Figure 10.6b**). With $S = ab$, the induced emf in the loop is

$$\text{emf} = -\omega ab B_0 \cos\omega t = -0.314\cos(314t) \quad [\text{V}]$$

Thus, the peak induced emf is ± 0.314 V.

(b) If there are $N = 100$ identical loops linking the same flux, the induced emf is N times larger. This can be understood from the fact that turns are connected in series:

$$\text{emf} = -\omega ab N B_0 \cos\omega t = -31.4\cos(314t) \quad [\text{V}].$$

(c) The solution now is similar, but the scalar product between **B** and $d\mathbf{s}$ is $\mathbf{B} \cdot d\mathbf{s} = Bds\cos(90 - \alpha)$ as shown in **Figure 10.6c**:

$$\text{emf} = -\omega B_0 \cos\omega t \, \cos(90 - \alpha) \int_s d\mathbf{s} = -\omega ab B_0 \sin\alpha \, \cos\omega t = -0.314\sin\alpha \, \cos 314 t \quad [\text{V}]$$

or, for N loops,

$$\text{emf} = -\omega ab N B_0 \sin\alpha \, \cos\omega t = -31.4\sin\alpha \, \cos 314 t \quad [\text{V}].$$

The induced emf is proportional to the orientation of the loop with respect to the flux density. To maximize the induced electromotive force, the loop must be perpendicular to the flux. Similarly, a loop parallel to the field ($\alpha = 0$) has zero induced emf since no flux passes through the loop. These aspects are important in design of motors, generators, and other devices.

Although this is a simple example, it indicates that the induced emf is directly proportional to frequency, number of turns, the magnetic flux density, and the area of the loop. Any one of these parameters may be used to optimize a device. As an example, if a particularly small device operating at a given frequency is needed, the flux density might be increased to decrease the area of the loop. If frequency can be increased, the same effect can be achieved without the need to increase the flux density. For example, most electric machines used in aircraft operate at 400 Hz as opposed to the more common 50 or 60 Hz devices in industry. This decreases their size and weight for the same design parameters.

10.6   Combined Motional and Transformer Action Electromotive Force

The electromotive forces in **Eqs. (10.2)** and **(10.11)** were obtained from two different situations. The first is generated in a stationary circuit by a change in flux. The second is generated due to motion in a magnetic field. The distinction indicates the source of the electromotive force and points to possible uses of the two electromotive forces. Both electromotive forces can exist together in a single circuit, as we will see shortly in an example. Therefore, the total induced electromotive force in a circuit is the sum due to motion and transformer action:

$$\text{emf} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}' - N\frac{d\Phi}{dt} \quad [\text{V}] \tag{10.20}$$

or, in a more consistent form,

$$\text{emf} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}' - N\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}' \quad [\text{V}] \tag{10.21}$$

This is a more general expression of Faraday's law in that both effects (the transformer and the motional action emfs) can exist together. The surface integration in **Eq. (10.21)** must be done over the whole area in which the change in flux takes place (usually area of a loop), whereas the line integral is around the contour (usually a conducting loop or many conducting loops) in which the emf is induced.

It should be noted that for a transformer action emf to exist, the magnetic flux density itself must be time dependent, whereas a motion action emf can be generated by motion in a time-dependent or DC magnetic flux density.

10.6.1 The Alternating Current Generator

Consider the situation in **Figure 10.7**. The loop rotates around the axis in a constant, uniform magnetic field at angular velocity ω [rad/s]. At any given time, the normal to the loop surface makes an angle α with the magnetic field. Since the magnetic flux density is constant in time, there is no emf due to the change of flux (transformer action emf), but since the

loop is rotating, and each section of the loop moves at some velocity $\mathbf{v}$, there is an induced emf due to this motion. We calculate the induced emf in the four sections of the loop indicated in **Figure 10.7a**:

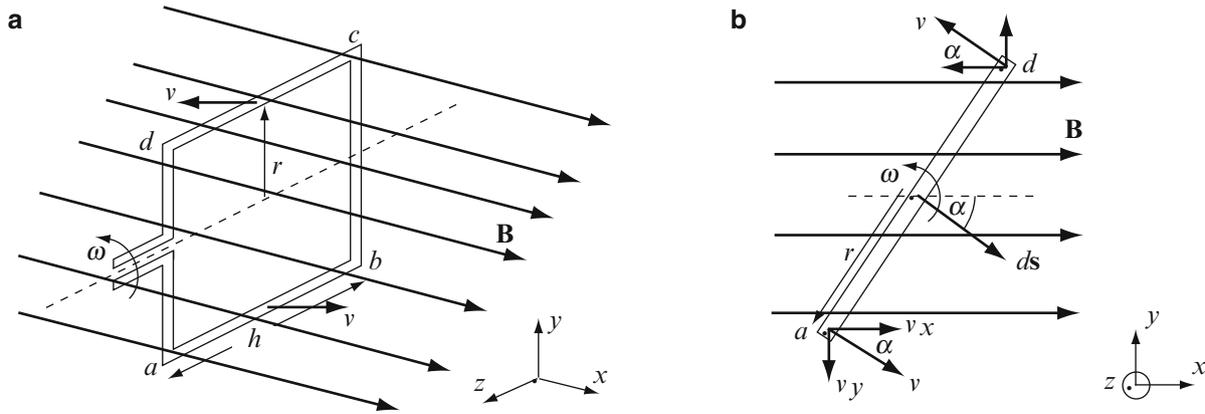$$\text{emf} = \text{emf}_{ab} + \text{emf}_{bc} + \text{emf}_{cd} + \text{emf}_{da} \quad [\text{V}] \tag{10.22}$$

Figure 10.7 The alternating current generator. (**a**) A loop rotating in a magnetic field at angular frequency ω. (**b**) The relation between the loop and magnetic field at a given instant in time

The emf in section bc and da is zero because $d\mathbf{l}$ and $\mathbf{v} \times \mathbf{B}$ are perpendicular to each other throughout the length of these two segments as can be seen in **Figure 10.7b** ($d\mathbf{l}$ is along the segments, whereas $\mathbf{v} \times \mathbf{B}$ is perpendicular to the segments, as can be verified using the right-hand rule). However, $\mathbf{v} \times \mathbf{B}$ and $d\mathbf{l}$ are in the same direction on segment ab and in opposite directions on segment cd, producing a nonzero emf. The velocity of segments ab and cd (in the system of coordinates shown in **Figure 10.7b**) is

$$\mathbf{v}_{ab} = r\omega(\hat{\mathbf{x}}\cos\alpha - \hat{\mathbf{y}}\sin\alpha) \quad \text{and} \quad \mathbf{v}_{cd} = r\omega(-\hat{\mathbf{x}}\cos\alpha + \hat{\mathbf{y}}\sin\alpha)[\text{m/s}] \tag{10.23}$$

The vector products $\mathbf{v} \times \mathbf{B}$ on segments ab and cd are

$$(\mathbf{v} \times \mathbf{B})_{ab} = r\omega(\hat{\mathbf{x}}\cos\alpha - \hat{\mathbf{y}}\sin\alpha) \times \hat{\mathbf{x}}B_0 = \hat{\mathbf{z}}B_0 r\omega\sin\alpha \tag{10.24}$$

$$(\mathbf{v} \times \mathbf{B})_{cd} = r\omega(-\hat{\mathbf{x}}\cos\alpha + \hat{\mathbf{y}}\sin\alpha) \times \hat{\mathbf{x}}B_0 = -\hat{\mathbf{z}}B_0 r\omega\sin\alpha \tag{10.25}$$

Performing the product $(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$ and integrating along segment ab and cd gives the total emf in the loop:

$$\text{emf} = B_0 2rh\omega\sin\alpha = B_0 S\omega\sin\alpha \quad [\text{V}] \tag{10.26}$$

where $S = 2rh$ is the area of the loop. N identical loops rotating together produce an emf that is N times larger:

$$\text{emf} = B_0 N2rh\omega\sin\alpha = B_0 NS\omega\sin\alpha \quad [\text{V}] \tag{10.27}$$

Also, the angle α after a time t (starting at $\alpha = 0$ at $t = 0$) is ωt. Thus the emf of the generator is

$$\text{emf} = NB_0 S\omega\sin\omega t \quad [\text{V}] \tag{10.28}$$

This device is clearly an AC generator. The emf is directly proportional to the angular velocity, the magnetic flux density, and the area of the loop. The generator can be designed as a trade-off between the various parameters. If the frequency must be constant, the loop must be rotated at a fixed angular velocity. Most generators operate on this or a very similar principle. As an example, the constant flux density may be generated by a permanent magnet or by an electromagnet and a DC source.

We also note that it is actually easier to generate an AC emf than it is to generate a DC emf. In fact, in most cases, a DC generator is an AC generator with a means of rectifying (or converting) the AC emf into a DC emf. This can be done through use of diodes or through commutators. For example, a car alternator is a three-phase AC machine supplying DC through three diodes. A commutator can be used to disconnect the loop and reverse the connection every time the output goes through zero. This is done by connecting the loop through sliding connectors on the axis of the generator. A DC generator based on commutation of the connection to a rotating loop is shown in **Figure 10.8** together with the generator waveform. The only differences between the simplified forms of the generator discussed here and practical generators are in the way the magnetic fields are generated, the magnetic paths, the arrangement of loops, and the details of mechanical construction.
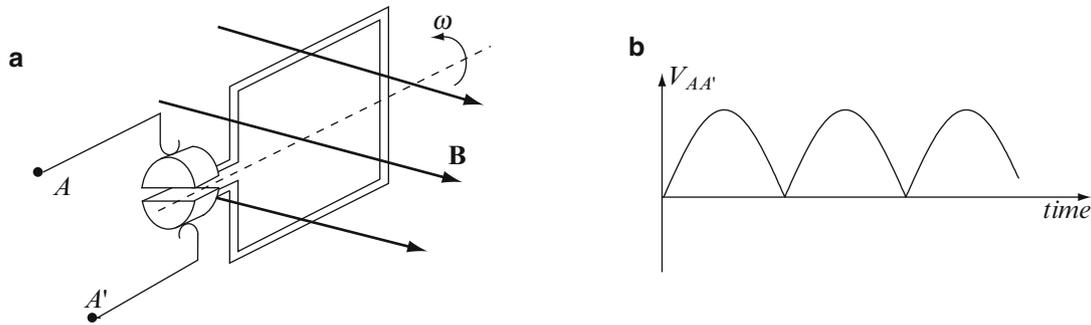
Figure 10.8 (**a**) A DC generator with commutating contacts. (**b**) The resulting output waveform

A more general situation is shown in **Figure 10.9**, where a loop rotates at an angular velocity ω_1 in a uniform, time-varying magnetic field given as $B = B_0\sin\omega_2 t$.
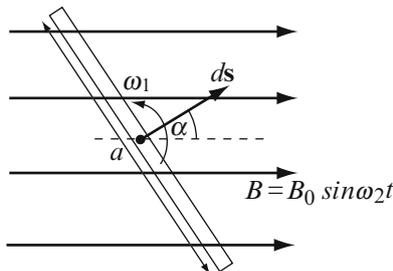
Figure 10.9 A loop rotating inside an AC magnetic field. The loop is b [m] wide (into the page)

The emf is now a superposition of induced emfs due to change of flux and due to motion of the loop. We treat each of these separately:

(a) **emf due to change of flux** (**transformer action emf**). Consider **Figure 10.9** where the loop is shown at an arbitrary fixed angle to the time-dependent magnetic flux density. The emf is

$$\text{emf}_t = -N\frac{d\Phi}{dt} = -N(S\cos\alpha)\frac{dB}{dt} = -\omega_2 SNB_0\cos\omega_2 t\ \cos\alpha \quad [\text{V}] \tag{10.29}$$

where $S = ab$ is the area of the loop and N is the number of loops.

(b) **emf due to motion**. The emf due to motion is given by **Eq. (10.27)** except that now the flux density is time dependent. Therefore:

$$\text{emf}_m = \omega_1 SNB_0\ \sin\omega_2 t\ \sin\alpha \quad [\text{V}] \tag{10.30}$$

where $S = ab$ is the area of the loop and N is the number of loops.

The total emf is

$$\text{emf} = \text{emf}_t + \text{emf}_m = -\omega_2 SNB_0 \cos\omega_2 t \cos\alpha + \omega_1 SNB_0 \sin\omega_2 t \sin\alpha \quad [\text{V}] \tag{10.31}$$

If we start with $\alpha = 0$ at $t = 0$, then $\alpha = \omega_1 t$. Thus

$$\begin{aligned}
\text{emf} &= -\omega_2 SNB_0 \cos\omega_2 t \cos\omega_1 t + \omega_1 SNB_0 \sin\omega_2 t \sin\omega_1 t \\
&= -SNB_0 [\omega_2 \cos\omega_2 t \cos\omega_1 t - \omega_1 \sin\omega_2 t \sin\omega_1 t] \quad [\text{V}]
\end{aligned} \tag{10.32}$$

If $\omega_1 = \omega_2 = \omega$, the expression can be further simplified:

$$\text{emf} = -\omega SNB_0 (\cos^2\omega t - \sin^2\omega t) = -\omega SNB_0 \cos 2\omega t \quad [\text{V}] \tag{10.33}$$

Thus, the device in **Figure 10.9** constitutes an AC generator with AC field excitation. Note however that this generator is different than the generator described in **Eq. (10.28)**. In particular, the frequency of the electromotive force now depends on both the frequency of rotation and on the frequency of the magnetic flux, whereas in **Eq. (10.28)**, the frequency depended only on the frequency of rotation. The generator in **Eq. (10.28)** is usually preferred because it provides a constant frequency. If we can regulate the rotation of the mechanical system (steam generator, water turbine, or diesel engine used to drive the loops), a constant-frequency generator is obtained. Because the amplitude also depends on frequency, it is only possible to obtain a constant amplitude if the frequency is kept constant. The magnetic field (also called excitation field) can be produced by DC sources such as a battery or permanent magnets. In large machines, such as turbogenerators or hydrogenerators, AC generators are used to generate the power required for excitation. The output from these machines is then rectified to provide DC excitation to the generators.

Example 10.5   A simple AC generator is made by inserting a loop of radius $d = 50$ mm inside a long solenoid of radius $b = 60$ mm. The number of turns per unit length of the solenoid is $n = 1,000$ turns/m and these carry a DC current $I = 1$ A. The loop is connected to the outside and is provided with an axis to rotate, as shown in **Figure 10.10**:

(a) If the loop rotates at 3,000 rpm, calculate the emf in the loop.
(b) If the loop is made of copper wire, and the wire has a diameter $D = 1$ mm, calculate the maximum current the loop can supply (shorted output). Neglect resistance of wires leading to the loop. The conductivity of copper is 5.7×10^7 S/m.
(c) How much energy must be expended in rotating the loop in one hour if the terminals of the loop are shorted?
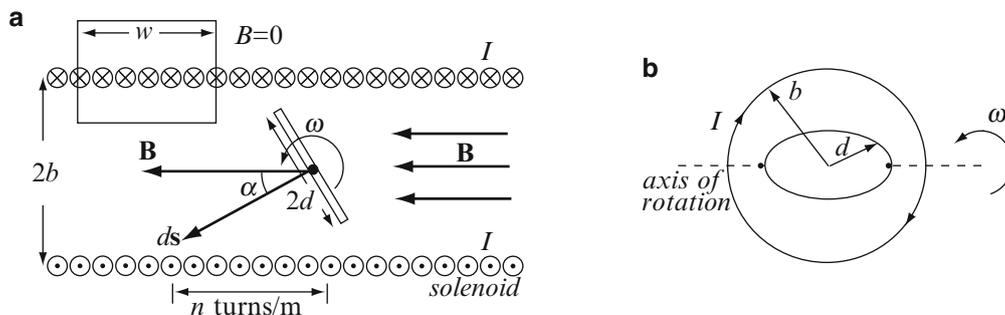

Figure 10.10   A loop rotating inside a long solenoid. (**a**) Axial cross section. (**b**) Side view

Solution: The flux density inside the solenoid is constant and may be calculated using Ampere's law. The electromotive force is calculated using the notation in **Figure 10.7b** and the transformer action emf in **Eq. (10.1)**. In **(b)**, the current is limited only by the internal resistance of the loop, whereas in **(c)**, the power dissipated multiplied by time gives the required energy:

(a) The magnetic flux density in the solenoid is calculated using a contour as shown in **Figure 10.10a**. The total current enclosed by the contour is wnI and the magnetic flux density is zero outside the solenoid:

$$Bw = \mu_0 nIw \quad \rightarrow \quad B = \mu_0 nI = 4 \times \pi \times 10^{-7} \times 1000 \times 1 = 0.0004\pi \quad [\text{T}]$$

The direction of the flux density is found from the right-hand rule and is shown in **Figure 10.10a**. The magnetic flux through the loop is

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} = \int_s B\cos\alpha \, ds = BS\cos\alpha \quad [\text{Wb}]$$

where α is the angle between $\mathbf{B}$ and $d\mathbf{s}$ as the loop rotates. Assuming zero-phase angle (i.e., $\alpha = 0$ at $t = 0$), the angle α after a time t is ωt and we get

$$\Phi = BS\cos\omega t \quad [\text{Wb}]$$

The emf is therefore

$$\text{emf} = -\frac{d\Phi}{dt} = \omega BS \sin\omega t = \left(2 \times \pi \times \frac{3000}{60}\right) \times \left(\pi \times 0.05^2\right) \times (0.0004 \times \pi) \times \sin\left(2 \times \pi \times \frac{3000}{60}t\right)$$

$$= 3.1 \times 10^{-3} \sin 314t \quad [\text{V}]$$

Note: The emf was calculated using the transformer action approach. The motional action approach gives the same result, but because the loop is circular, it is much more difficult to calculate.

(b) For a copper wire of radius r, made into a loop of radius d, the resistance is

$$R = \frac{l}{\sigma S} = \frac{2\pi d}{\sigma \pi r^2} = \frac{2d}{\sigma(D/2)^2} = \frac{2 \times 0.05}{5.7 \times 10^7 \times 0.0005^2} = 0.007 \quad [\Omega]$$

The maximum current occurs for a shorted loop at $t = 0$ (or $\omega t = \pi k$, $k = 0, 1, 2, \ldots$) and equals

$$I = \frac{\text{emf}}{R} = \frac{3.1 \times 10^{-3}}{0.007} = 0.443 \quad [\text{A}]$$

Thus, the device described here is an AC generator that can supply a peak current of 0.443 A at peak voltage (emf) of 3.1 mV, operates at 50 Hz, and has an internal resistance of 0.007 Ω.

(c) To calculate power, and therefore energy, we use the root mean square value of the current. The energy expended in 1 h is

$$W = \frac{I^2 R}{2}t = \frac{0.443^2 \times 0.007}{2} \times 3600 = 2.47 \quad [\text{J}]$$

This is 2.47 W · s or approximately 0.686 mW · h. This energy is the work required to rotate the loop against the magnetic forces on the loop (there are no other losses in this system).

Exercise 10.3 Use the dimensions and data in **Example 10.5**. Assume the loop is replaced with a very short coil with $N = 50$ turns, with the same diameter as the loop. The large solenoid is now rotated around the short coil at 3,600 rpm. Calculate:

Exercise 10.3 (continued)

(a) The emf in the short coil.
(b) The frequency and internal resistance of the generator.
(c) The peak power the generator supplies with shorted terminals.

Answer
(a) emf $= 0.186\sin120\pi t$ [V]. (b) 60 Hz, 0.35 Ω. (c) 98.8 mW.

Example 10.6 Application: The AC Generator An AC generator is made as shown in **Figure 10.11**. The coil contains $N = 500$ turns and is supplied with a sinusoidal current of amplitude $I = 10$ A and frequency 60 Hz. The relative permeability of iron is $\mu_r = 1{,}000$. The loop is 10 mm × 10 mm and rotates at 3,600 rpm. Assume the flux density in the gap is uniform and perpendicular to the iron surfaces and the $B(H)$ curve is linear:

(a) Calculate the emf of the loop.
(b) What is the waveform of the emf?
(c) Suppose you need to generate a 10 V output (peak) using this device. How many turns are required in the rotating coil?
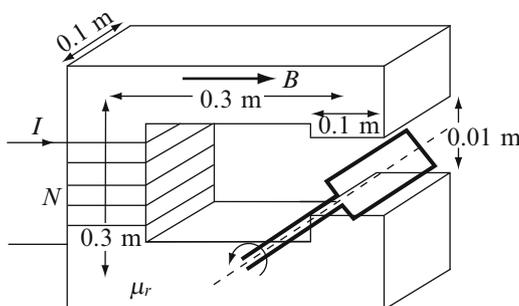
Figure 10.11 A simple AC generator

Solution: Because the coil rotates in an AC field, there are two components of the induced emf: one due to rotation and may be viewed as a motional emf. The second is due to the transformer effect and occurs even if the loop does not rotate. The emf in the coil is the sum of these two emfs. We could use the general expression in **Eq. (10.32)**, but at this stage, it is best to calculate each emf separately:

(a) **(1) emf due to change of flux**. The transformer action emf is given in **Eq. (10.29)**. However, we must first calculate the magnetic flux density in the gap of the magnetic structure. The latter is calculated using magnetic circuits assuming all flux is contained within the gap. The flux density in the gap is

$$B = \frac{\Phi}{S} = \frac{NI}{S(\mathfrak{R}_m + \mathfrak{R}_g)} = \frac{NI}{\left(\dfrac{l_m}{\mu_0\mu_r} + \dfrac{l_g}{\mu_0}\right)} = \frac{500 \times 10\sin120\pi t}{\left(\dfrac{1.19}{4 \times \pi \times 10^{-7} \times 1000} + \dfrac{0.01}{4 \times \pi \times 10^{-7}}\right)} = 0.5615\sin120\pi t \quad [\text{T}]$$

where $\mathfrak{R}_m$ is the reluctance of the magnetic path in iron, $\mathfrak{R}_g$ the reluctance in the gap, l_m is the length of the magnetic path in iron, l_g is the length of the gap, and μ_r is the relative permeability of iron. The emf in the loop is calculated from **Eq. (10.29)** using $\omega_2 = 2\pi \times 60 = 120\pi$ rad/s and $\omega_1 = 2\pi \times 3{,}600/60 = 2\pi \times 60 = 120\pi$ rad/s, but the number of turns in the loop is $N_{loop} = 1$:

$$\text{emf}_t = -\omega_2 SN_{loop}B_0 \cos \omega_2 t \cos \alpha$$

$$= -2 \times \pi \times 60 \times 0.01 \times 0.01 \times 1 \times 0.5615 \cos(2 \times \pi \times 60t) \cos\left(2 \times \pi \times \frac{3600}{60}t\right) = -0.0212 \cos^2 120\pi t \quad [\text{V}]$$

where the notation in **Figure 10.9** was used and N_{loop} was used to distinguish the number of turns in the rotating loop from those in the magnetic circuit.

(2) **emf due to motion of the loop in the magnetic field**. This is given in **Eq. (10.30)** where $\omega_1 = \omega_2 = 120\pi$ rad/s, $\alpha = \omega_1 t = 120\pi$ rad, and the number of turns in the rotating loop is again $N_{loop} = 1$:

$$\text{emf}_m = \omega_1 SN_{loop}B_0 \sin \omega_2 t \sin \alpha = 120 \times \pi \times 0.01 \times 0.01 \times 0.5615 \sin^2 120\pi t = 0.0212 \sin^2 120\pi t \quad [\text{V}]$$

The total emf is the sum of the two emfs and is also given in **Eq. (10.33)**:

$$\text{emf} = \text{emf}_t + \text{emf}_m = -0.0212(\cos^2 120\pi t - \sin^2 120\pi t) = -0.0212 \cos 240\pi t \quad [\text{V}].$$

(b) The waveform is cosinusoidal but at a frequency twice the frequency of the field, or 120 Hz. In this case the frequency of the magnetic field and that of rotation happen to be the same. If they are not, then **Eq. (10.32)** must be used instead.

(c) Since the amplitude of the emf is directly proportional to the number of turns and the emf above was generated in a single turn, we can write the emf per turn as

$$\text{emf}_0 = 0.0212 \qquad [\text{V/turn}]$$

A 10 V peak (20 V peak-to-peak) output requires

$$\frac{10}{\text{emf}_0} = \frac{10}{0.0212} = 471.7 \quad [\text{turns}]$$

The rotating coil should contain 472 turns (the number of turns is usually given in integer numbers).

Exercise 10.4 In **Example 10.6**, (a) calculate the emf if the coil is supplied by a DC source. (b) How many turns are required for a 10 V peak output?

Answer

(a) $\text{emf} = -0.0212\cos 120\pi t$ [V]. (b) 472 [turns].

Example 10.7 A coil is wound on a torus as shown in **Figures 10.14a** and **10.14b**. The coil consists of N turns, carries a current $I = I_0 \sin \omega t$ [A], and has an average radius r [m] $(r \gg b)$ as shown:

(a) Calculate the induced emf in the coil. What is the meaning of this emf?
(b) Show that the induced emf is proportional to the inductance of the coil.
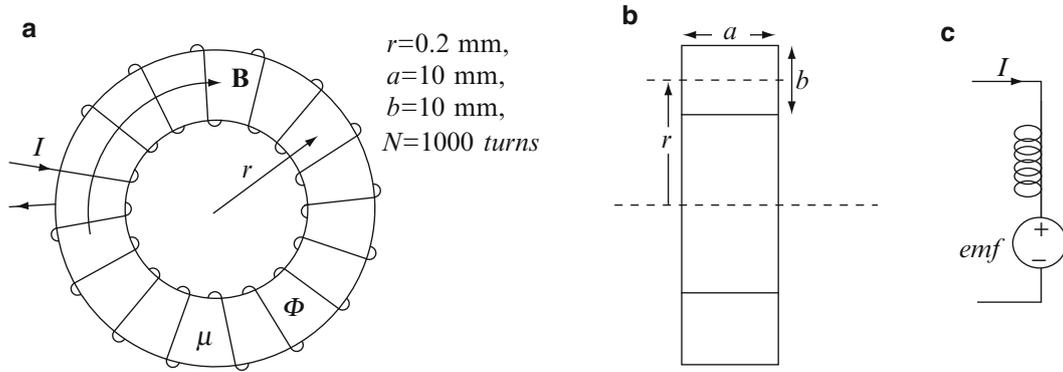(c) Suppose the permeability of the torus is very large. What is the induced emf in the coil?

Figure 10.12 (**a**) A wound torus. (**b**) Cross section showing dimensions. (**c**) Equivalent circuit of the inductor

Solution: The induced emf in the coil is due to the change in flux (transformer action) in the coil itself. That is, because the coil is fed with an AC source, it produces an AC flux which in return induces an emf in itself. After calculating the flux in the core of the torus, the emf is calculated from Faraday's law. In (**c**), the induced emf is also very large because the inductance of the coil is directly proportional to permeability:

(**a**) Since $r \gg b$, we can use the flux density at r as the average, uniform flux density throughout the core. The flux density is found from Ampere's law (see **Example 8.9** or **9.10**):

$$2\pi r B = \mu N I \quad \rightarrow \quad B = \frac{\mu N I_0 \sin\omega t}{2\pi r} \quad [\text{T}]$$

and the direction of the flux density is as shown in **Figure 10.12a**. The flux in the core is

$$\Phi = BS = \frac{\mu N a b I_0 \sin\omega t}{2\pi r} \quad [\text{Wb}]$$

The induced emf in the coil due to change of flux through the core is

$$\text{emf} = -N\frac{d\Phi}{dt} = -\frac{\mu N^2 a b \omega I_0 \cos\omega t}{2\pi r} \quad [\text{V}]$$

The induced emf is shown in **Figure 10.12c** as a source in series with the ideal coil. This source opposes the current as required by Lenz's law and is the AC voltage measured on the coil. Without this emf (such as if the coil is driven by a DC source), the voltage on the coil would be zero except for any voltage drop that might exist because of the resistance of the coil.

(**b**) The inductance of any device is the flux linkage divided by current. In this case, the self-inductance of the coil is

$$L = \frac{N\Phi}{I} = \frac{\mu N^2 a b}{2\pi r} \quad [\text{H}]$$

Thus, the emf in the coil is

$$\text{emf} = -L\frac{dI}{dt} = -\frac{\mu N^2 a b \omega I_0 \cos\omega t}{2\pi r} \quad [\text{V}]$$

This expression is a direct result of Lenz's law and is extremely important in AC analysis of circuits. We will use it in the following section.

(c) If the permeability is large, so is the inductance of the coil. Therefore, the emf is also very large. If permeability tends to infinity, so does the induced emf. The same effect can be obtained with finite permeability by increasing the magnitude of the time derivative of the current (in this case, by increasing the frequency). This effect is responsible for large pulses that occur when the current in inductive circuits is changed quickly (such as when connecting or disconnecting a circuit). In electronic circuits, it is often required to protect devices, such as output power stages in amplifiers, motor and relay drivers, and the like, from being damaged due to inductive pulses.

Exercise 10.5   A small coil has a self-inductance of 10 μH. A sinusoidal current of amplitude 0.1 A and frequency 1 kHz passes through the coil. **(a)** Calculate the emf measured on the coil. **(b)** What is the voltage if frequency changes to 100 kHz? **(c)** Calculate the emf at 1 kHz for a 10 mH coil.

Answer

(a) $-6.283 \times 10^{-3} \cos 2000\pi t$ [V]. **(b)** $-0.6283 \cos 200000\pi t$ [V]. **(c)** $-6.283 \cos 2000\pi t$ [V].

10.7   The Transformer

The transformer is a device designed to transform voltages and currents (and, therefore, impedances). It is an AC device and operates on the principles of Faraday's law. The transformer consists of two or more coils and a magnetic path that links the coils. There is a variety of transformers with different types of paths, but they all operate on the same principles. Power transformers are designed primarily for voltage transformation and operate at relatively high currents. The magnetic path is made of a ferromagnetic material like iron to produce a low-reluctance magnetic path (see **Figure 10.13**). Typically, the iron core of the transformer is laminated to reduce induction of currents in the core which contribute to losses. Impedance-matching transformers are normally designed for low-power applications. There are other types of transformers, some with iron cores, some without a core (air-core transformers), and still others with ferrite cores. There are also transformers which do not look like transformers but act as such. For a device to be considered a transformer, it must have two or more coils, coupled together by a common flux, whatever the physical construction of the device.
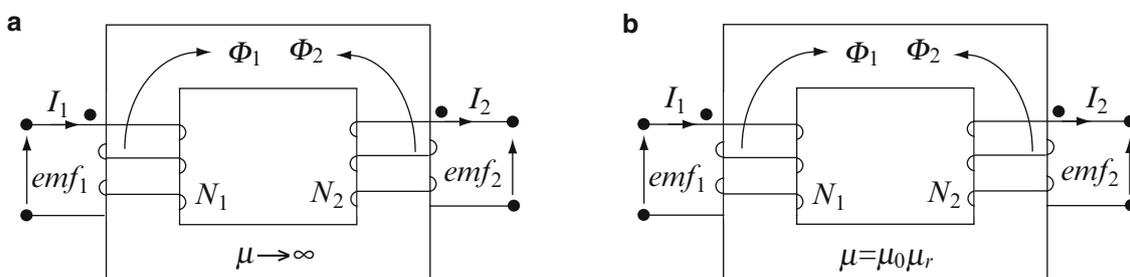

Figure 10.13 The transformer. emf$_2$ and I_2 are induced quantities. (**a**) The ideal transformer has a core with infinite permeability. (**b**) Core with finite permeability

10.7.1 The Ideal Transformer

An ideal transformer is one in which all flux links the coils of the transformer (i.e., flux does not leak out of the magnetic path). This implies that the permeability of the magnetic path is high (ideally, it should be infinite) and the path is closed. In addition, we assume there are no losses in the transformer.

The flux in a magnetic circuit was calculated in **Eq. (9.102)**:

$$\Phi = \frac{\sum_{i=1}^{n} N_i I_i}{\sum_{j=1}^{k} \mathfrak{R}_j} \quad \text{[Wb]} \tag{10.34}$$

where N_i is the number of turns of coil i, I_i is the current in this coil, and $\mathfrak{R}_j$ is the magnetic reluctance of the jth segment of the path. The reluctance of the magnetic path was given in **Eq. (9.99)**:

$$\mathfrak{R} = \frac{l_m}{\mu S} \quad \left[\frac{1}{\text{H}}\right] \tag{10.35}$$

where l_m is the length of the magnetic path, μ its permeability, and S the cross-sectional area of the path. Assuming that there are no losses in the core or in the coils of the transformer in **Figure 10.13**, the flux in **Eq. (10.34)** becomes

$$\Phi = \frac{N_1 I_1 - N_2 I_2}{\mathfrak{R}} \quad \text{[Wb]} \tag{10.36}$$

where the negative sign in front of $N_2 I_2$ is due to the fact that I_2 is an induced current and Lenz's law stipulates that the induced current must produce a flux opposing the flux that produces it. Because the core is made of iron with high permeability, we may assume $\mu \to \infty$ and, therefore, $\mathfrak{R} \to 0$ and we can write

$$N_1 I_1 - N_2 I_2 = \Phi \mathfrak{R} \approx 0 \tag{10.37}$$

This, however, is only an approximation. In many cases, this approximation is very good and gives results that are very close to the exact values. From this relation, we get

$$N_1 I_1 = N_2 I_2 \tag{10.38}$$

The total flux in the core is the same in both coils and the emfs across the two coils are

$$V_1 = \text{emf}_1 = -N_1 \frac{d\Phi}{dt}, \quad V_2 = \text{emf}_2 = -N_2 \frac{d\Phi}{dt} \quad \text{[V]} \tag{10.39}$$

From these, the voltage ratio between primary and secondary is

$$\boxed{\frac{\text{emf}_1}{\text{emf}_2} = \frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2} = a} \quad \text{[dimensionless]} \tag{10.40}$$

where a denotes the turn ratio, also called the ***transformer ratio***.

Note: The emf in each coil is opposite in sign to the applied voltage on the coil, if any, so that Kirchhoff's voltage law gives zero in the primary and secondary circuits. This is the same as saying that the total flux in an ideal transformer is zero. In a real transformer, Kirchhoff's voltage law in each circuit results in a small voltage difference ($V_1 > \text{emf}_1, V_2 < \text{emf}_2$). The difference is due to losses in the transformer and the transformer ratio changes accordingly.

While most transformers are designed either to transform currents or voltages, they also change the impedance of the circuit. The impedance of the primary circuit is given by the ratio of emf_1 and I_1:

$$Z_1 = \frac{\text{emf}_1}{I_1} = \frac{a\,\text{emf}_2}{I_2/a} = a^2 Z_2 \quad [\Omega] \tag{10.41}$$

or, if Z_L is a load impedance,

$$\boxed{\frac{Z_1}{Z_L} = a^2}$$

(10.42)

The impedance Z_1 is, in fact, the effective load impedance seen by the source. Impedance matching is sometimes the primary function of the transformer. However, regardless of the function, the impedance seen from the primary or the effective impedance in the primary circuit depends on the turn ratio squared and the impedance of the secondary.

Although an actual transformer includes losses due to resistance of the conductors, induced currents in the core, and currents needed to magnetize the core (as well as capacitive losses), the main approximation used to define an ideal transformer was the assumption that the permeability is infinite and, therefore, that the reluctance of the magnetic path is zero. In practical applications, this is never the case, and in some transformers, like air-core transformers, the above approximations cannot be used at all. In many transformers, the losses are relatively small (sometimes less than 1%) and the above approximations are quite good. However, in low-power transformers, losses may be high relative to the total power capacity of the transformer.

10.7.2 The Real Transformer: Finite Permeability

The transformer in **Figure 10.13b** consists of a core with relatively high reluctance (low permeability); therefore, the approximation of infinite permeability cannot be used. If we wish to calculate the ratio between primary and secondary, we must calculate the flux in the magnetic circuit. We first assume that all flux links both coils of the transformer (no flux leakage), but permeability is finite. The net flux in the magnetic circuit linking both coils in **Figure 10.13b** is

$$\Phi = \frac{\mu S}{l}(N_1 I_1 - N_2 I_2) \quad [\text{Wb}]$$

(10.43)

Now, using Faraday's law, the emf in each coil is calculated as

$$\text{emf}_1 = N_1 \frac{d}{dt}\left(\frac{\mu S}{l}(N_1 I_1 - N_2 I_2)\right),$$

$$\text{emf}_2 = N_2 \frac{d}{dt}\left(\frac{\mu S}{l}(N_1 I_1 - N_2 I_2)\right) \quad [\text{V}]$$

(10.44)

Rearranging the terms, we get

$$\boxed{\text{emf}_1 = \frac{\mu S}{l}N_1^2 \frac{dI_1}{dt} - \frac{\mu S}{l}N_1 N_2 \frac{dI_2}{dt} \quad [\text{V}]}$$

(10.45)

$$\boxed{\text{emf}_2 = \frac{\mu S}{l}N_1 N_2 \frac{dI_1}{dt} - \frac{\mu S}{l}N_2^2 \frac{dI_2}{dt} \quad [\text{V}]}$$

(10.46)

In **Example 9.10**, we calculated the self-inductances and mutual inductances of three coils on a closed magnetic core of finite permeability μ. Using those results for the first two coils, we have

$$L_{11} = \frac{\mu S}{l}N_1^2, \quad L_{12} = L_{21} = \frac{\mu S}{l}N_1 N_2, \quad L_{22} = \frac{\mu S}{l}N_2^2 \quad [\text{H}]$$

(10.47)

Using these relations, **Eqs. (10.45)** and **(10.46)** become

$$\boxed{\mathrm{emf}_1 = L_{11}\frac{dI_1}{dt} - L_{12}\frac{dI_2}{dt} \quad [\mathrm{V}]}$$

(10.48)

$$\boxed{\mathrm{emf}_2 = L_{21}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} \quad [\mathrm{V}]}$$

(10.49)

Here, we have used the fact that the current in the secondary is due to induction; that is, it only exists if the current in the primary exists. According to Lenz's law, the flux produced by this current is always in opposition to the flux due to the primary. Therefore, the flux in the core is small (it is zero for an ideal transformer and for a nonideal transformer with zero losses).

To more easily identify the emfs induced in various coils, the so-called dot convention is used. A dot is placed on the terminal of the coil which, when a current flows into the dot, produces a flux in the direction of the net flux in the core. In the case of transformers, this means that when the current increases on a dotted terminal, all dotted terminals experience an increase in emf. A current flowing into a dot produces a positive emf and a current flowing away from a dot produces a negative emf. In **Figure 10.13b**, I_1 flows into the dot and I_2 flows away from the dot. The emfs in **Eqs. (10.48)** and **(10.49)** that are associated with I_1 are positive whereas those associated with I_2 are negative.

The induced emfs in **Eqs. (10.48)** and **(10.49)** may also be understood in terms of impedances. In particular, in the frequency domain, d/dt is replaced with $j\omega$ and the emfs in **Eqs. (10.48)** and **(10.49)** become

$$\mathrm{emf}_1 = j\omega L_{11}I_1 - j\omega L_{12}I_2 \quad [\mathrm{V}]$$

(10.50)

$$\mathrm{emf}_2 = j\omega L_{21}I_1 - j\omega L_{22}I_2 \quad [\mathrm{V}]$$

(10.51)

where I_1 and I_2 are now phasors. Note also that the emf and current in each coil are 90° out of phase.

Although the relations in **Eqs. (10.48)** and **(10.49)** may look very different than those for the ideal transformer, they are, in fact, very similar. In particular, because all flux is contained within the core and there are no losses, the ratio between the voltage (emf) in the primary and secondary remains the same as for the ideal transformer in **Eq. (10.40)**. Therefore, any transformer in which all flux is contained within the core and which has no losses behaves as an ideal transformer regardless of the permeability of the core. This can be seen most easily from **Eq. (10.44)**.

10.7.3 The Real Transformer: Finite Permeability and Flux Leakage

In the previous two sections, we assumed that all flux produced by a coil is contained within the core. This is not always the case, as we have seen in **Section 9.4**. There are conditions under which some of the flux closes outside the core or there is no core to begin with. Consider two coils in air. In this case, we do not know how much of the flux connects the two coils, but we can assume that a fraction of the flux produced by one coil links the second coil. Suppose this fraction is k. Because of that, the mutual inductances also change by this fraction; that is, if all flux links both coils, the inductances in **Eq. (10.47)** are obtained. If only a fraction k links coil 1 and coil 2, we get

$$L_{11} = \frac{\mu S}{l}N_1^2, \quad L_{12} = L_{21} = k\frac{\mu S}{l}N_1N_2, \quad L_{22} = \frac{\mu S}{l}N_2^2 \quad [\mathrm{H}]$$

(10.52)

We note that

$$L_{11}L_{22} = \frac{\mu S}{l}N_1^2\frac{\mu S}{l}N_2^2 \;\rightarrow\; \sqrt{L_{11}L_{22}} = \frac{\mu S}{l}N_1N_2$$

(10.53)

or

$$L_{12} = L_{21} = k\sqrt{L_{11}L_{22}} \quad [\mathrm{H}]$$

(10.54)

The constant k is called a ***coupling coefficient*** and indicates how much of the flux produced by one coil links the other. For the ideal transformer in **Section 10.7.1** and for the transformer in **Section 10.7.2**, the coupling coefficient is equal to 1 (all flux links both coils). In air-core transformers, the coupling coefficient is almost always smaller than 1. If the coupling coefficient is known, the emfs in each coil can be calculated from the self-inductances of the two coils and the coupling coefficient. For the transformer in **Figure 10.13b**, but now assuming that $k \leq 1$, the emfs are

$$\boxed{\text{emf}_1 = L_{11}\frac{dI_1}{dt} - k\sqrt{L_{11}L_{22}}\frac{dI_2}{dt} \quad [\text{V}]} \tag{10.55}$$

$$\boxed{\text{emf}_2 = k\sqrt{L_{11}L_{22}}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} \quad [\text{V}]} \tag{10.56}$$

Thus, the smaller the coupling coefficient, the closer the emf is to that of a simple coil, and the closer k is to 1, the closer the behavior is to that of an ideal transformer.

Example 10.8 Application: The Toroidal Transformer A widely used transformer in high-quality audio and test equipment is built around a toroidal core, as in **Figure 10.14**. The transformer shown is designed to supply 48 V at 2 A for the output stage of an amplifier. The toroidal core has a cross-sectional area of $S = 400$ mm^2, a mean magnetic path of $l = 200$ mm, and the relative permeability of the core is infinite. The primary operates at 240 V, 60 Hz. If the primary coil must have 800 turns to generate the required flux in the core, calculate:

(a) The number of turns in the secondary and current in the primary.
(b) Show that the result in **(a)** remains unchanged if the permeability of the core is finite, as long as all flux remains contained within the core.
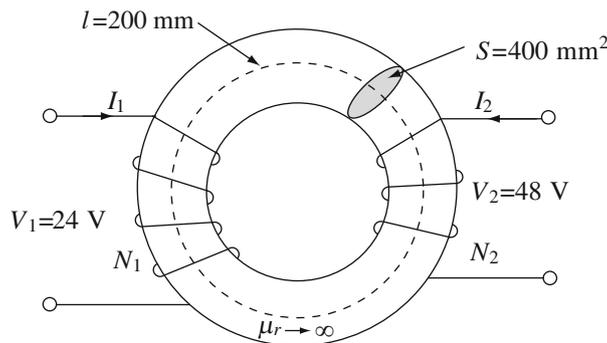
Figure 10.14 A toroidal power transformer

Solution: In **(a)**, we may use the expressions for the ideal transformer since the reluctivity of the magnetic path is zero. In **(b)** we use the expressions in **Eqs. (10.45)** and **(10.46)** and calculate the ratio between the emfs:

(a) From **Eq. (10.40)**

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \quad \rightarrow \quad N_2 = \frac{N_1 V_2}{V_1} = \frac{800 \times 48}{240} = 160 \quad [\text{turns}]$$

where V_1 is the voltage on the primary and V_2 is the voltage on the secondary. The current in the primary is also calculated from **Eq. (10.40)**:

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} \quad \rightarrow \quad I_1 = \frac{N_2 I_2}{N_1} = \frac{160 \times 2}{800} = 0.4 \quad [\text{A}]$$

(b) The ratio between emf$_1$ and emf$_2$ from **Eqs. (10.45)** and **(10.46)** is

$$\frac{\text{emf}_1}{\text{emf}_2} = \frac{\dfrac{\mu S}{l}N_1^2\dfrac{dI_1}{dt} - \dfrac{\mu S}{l}N_1N_2\dfrac{dI_2}{dt}}{\dfrac{\mu S}{l}N_1N_2\dfrac{dI_1}{dt} - \dfrac{\mu S}{l}N_2^2\dfrac{dI_2}{dt}} = \frac{N_1\left(N_1\dfrac{dI_1}{dt} - N_2\dfrac{dI_2}{dt}\right)}{N_2\left(N_1\dfrac{dI_1}{dt} - N_2\dfrac{dI_2}{dt}\right)} = \frac{N_1}{N_2}$$

This is the same as in (a); therefore, the ratio remains unchanged for any value of μ. However, if μ is low, the flux will tend to leak, invalidating the assumptions used to obtain **Eqs. (10.45)** and **(10.46)**. If this happens, **Eqs. (10.55)** and **(10.56)** must be used. In practical design, if μ is large (but finite), it is safe to use the assumptions for the ideal transformer.

Notes:

(1) Because $N_1I_1 = N_2I_2$ and since I_2 produces a flux which opposes that due to I_1, the net flux in the core is zero. In reality, there will be a small flux due to losses.

(2) Because this is an ideal transformer, the dimensions of the core and frequency of the source are not important: They do not figure in the calculation. In practical transformers, the dimensions define the maximum flux density allowable without the core reaching saturation.

Toroidal transformers are favored for audio applications because they are inherently low-leakage transformers, even at low core permeability. Typically, they offer the shortest magnetic path and, therefore, the lowest reluctance in addition to being economical in both winding and core materials. However, since the winding of coils is usually done after the core is assembled (made of stacked-up laminations or of strips of the lamination material wound in the form of a torus), it is complicated and requires special winding equipment. Toroidal transformers are also very useful in switching and high-frequency applications.

> **Example 10.9** A high-frequency transformer is made in the form of two coils on a nonmagnetic form ($\mu = \mu_0$), as in **Figure 10.15**. The self-inductance of coil (1) is 10 μH and of coil (2) 20 μH. The current in the primary is $I_0\sin\omega t$ where $f = 1$ MHz, $I_0 = 0.1$ A, and the secondary is open. The coupling coefficient between the two coils is 0.2. Calculate:
>
> (a) The voltage (emf) required in the primary to sustain the given current
> (b) The voltage (emf) in the secondary
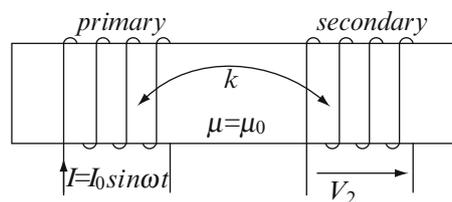
Figure 10.15 A high-frequency transformer; $k \leq 1$

Solution: This transformer must be treated as a real transformer, using **Eqs. (10.55)** and **(10.56)**. In addition, because the secondary coil is open, $I_2 = 0$:

(a) From **Eq. (10.55)**, the induced emf in the primary, induced by itself, is

$$\text{emf}_1 = L_{11}\frac{dI_1}{dt} = L_{11}\frac{d(I_0\sin\omega t)}{dt} = L_{11}\omega I_0\cos\omega t$$

$$= 10 \times 10^{-6} \times 2 \times \pi \times 10^6 \times 0.1 \cos(2 \times \pi \times 10^6 t) = 6.283\cos(2 \times \pi \times 10^6 t) \qquad [\text{V}]$$

(b) From **Eq. (10.56)**, the emf induced in the secondary by the primary is

$$\text{emf}_2 = k\sqrt{L_{11}L_{22}}\frac{dI_1}{dt} = k\sqrt{L_{11}L_{22}}\omega I_0\cos\omega t$$

$$= 0.2 \times \sqrt{10 \times 10^{-6} \times 20 \times 10^{-6}} \times 2 \times \pi \times 10^6 \times 0.1\cos(2 \times \pi \times 10^6 t) = 1.78\cos(2\pi \times 10^6 t) \qquad [\text{V}]$$

This emf is the open circuit voltage on the secondary coil.

Example 10.10 Application: The Current Transformer Although **Eqs. (10.55)** and **(10.56)** define the general transformer, the current transformer is unique in that its primary coil is connected in series with the circuit in which it operates. In many cases, the primary coil is part of the circuit and the transformer core surrounds it. Three examples of current transformers are shown in **Figure 10.16**. The first, in **Figure 10.16a**, is a simple transformer that, in principle, can also be used as a voltage transformer. What makes it unique is the low number of turns in the coils, especially in the primary coil. This is necessary since it is connected in series with the circuit and it should have low impedance. The transformer in **Figure 10.16b** is similar except that the primary is a single turn passing through the core. In this case, the turn ratio a is $1/N_2$. This particular arrangement is often used because it does not require connections into the circuit and is particularly useful for measuring purposes (i.e., $aI_1 = I_2$). If appropriate, or necessary, the primary may be made of two or more turns by passing the wire through the core two or more times. A common measuring device based on this principle is shown in **Figure 10.16c**. This is a clamping ampere meter. It is essentially a current transformer without the primary coil. The secondary coil is connected to a measuring device such as a digital meter or bridge. The core is split and hinged such that it can be opened and closed around the wire in which we wish to measure the current. With N_2 turns in the secondary coil and since the primary in this case has a single turn, the measured current in the primary is equal to $1/a$ times the current in the secondary, which is measured directly. The advantage of this device is that it measures current without the need to cut the circuit, but, as you might expect, it is only accurate at relatively high currents.
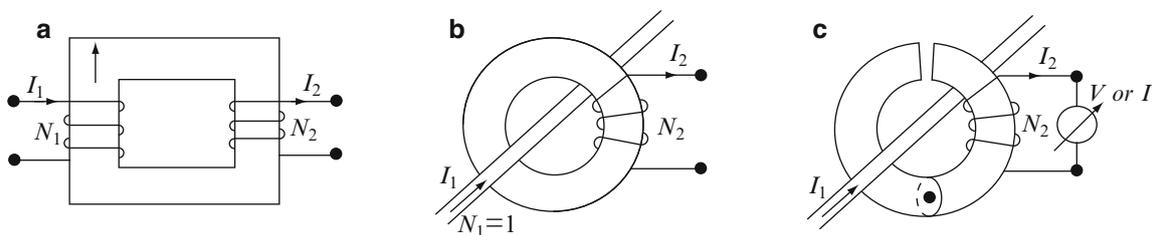
Figure 10.16 (**a**) A current transformer. (**b**) A single-turn transformer. (**c**) A clamping ampere meter

One point of interest with current transformers is that the voltage across the secondary can be very high. For this reason, the secondary should always be shorted unless a low-impedance load or low-impedance measuring device is connected to the secondary.

It is required to design a current transformer that will continuously measure a sinusoidal current supplied to an installation by placing the transformer over one of the wires leading to the installation. The peak current expected is 100 A at 60 Hz.

A toroidal core, made of iron with average radius of $a = 30$ mm, is available. The cross section of the torus is circular, with a radius $b = 10$ mm. Relative permeability of the iron is $\mu_r = 200$. The torus is inserted over the wire as shown in **Figure 10.17** and the secondary coil is connected to a voltmeter. The voltmeter can measure between $V = 0$ and $V = 1$ V (peak):

(a) Calculate the number of turns in the secondary of the current transformer for full-scale reading at 100 A.
(b) Suppose you do not wish to use an iron core for the solenoid because of induced currents in the iron. Can you use an air-core torus? If so, what is the number of turns required if the torus is made of plastic, with the same dimensions as before and for the same reading?

Figure 10.17 A toroidal current transformer used to measure the current in a conductor. (**a**) General view. (**b**) Cross section with dimensions
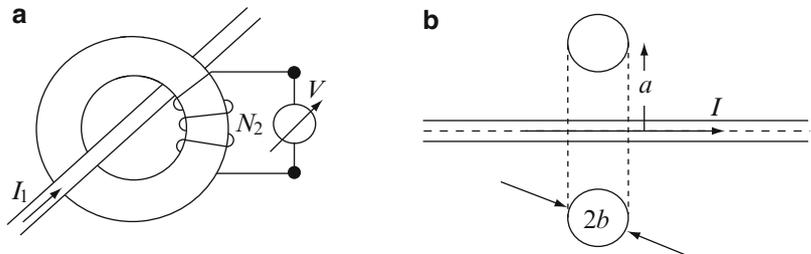
Solution: The magnetic flux density inside the torus is calculated as for any infinitely long wire carrying a current I. The flux in the torus is then calculated, and from the flux, the emf is calculated using Faraday's law for a single turn. The number of turns is the ratio between the full-scale reading and the emf of a single turn:

(a) The magnetic flux density at a distance a from the wire at full-scale current is (see **Example 8.6**)

$$B = \frac{\mu_0 \mu_r I \sin\omega t}{2\pi a} = \frac{4 \times \pi \times 10^{-7} \times 200 \times 100\sin120\pi t}{2 \times \pi \times 0.03} = 0.1334 \sin(120\pi t) \quad [\text{T}]$$

Assuming this to be uniform in the cross-sectional area of the torus, the flux in the core is the flux density multiplied by the cross-sectional area:

$$\Phi = BS = B\pi b^2 = 0.133 \times \pi \times 0.01^2 \sin(120\pi t) = 4.189 \times 10^{-5} \sin(120\pi t) \quad [\text{Wb}]$$

The induced emf in a loop is

$$\text{emf}_0 = -\frac{d\Phi}{dt} = -\omega B\pi b^2 = -2 \times \pi \times 60 \times 4.189 \times 10^{-5}\cos(120\pi t) = -0.01579 \cos(120\pi t) \quad [\text{V}]$$

This is the emf per turn. The peak emf per turn is 0.01579V. Thus, the number of turns required in the secondary coil is

$$N_2 = \frac{V_2}{\text{emf}_0} = \frac{1}{0.01579} = 63.3 \quad [\text{turns}] \quad \rightarrow \quad N_2 = 63 \quad [\text{turns}].$$

(b) As long as the torus is centered with the wire and the turns on the torus are uniform around its circumference, an air-filled toroidal coil may be used just as well. However, the magnetic flux density and magnetic flux in the torus are 200 times smaller since, now, $\mu_r = 1$. The emf per turn will also be 200 times smaller and the number of required turns is 200 times larger or 12,665 turns. The iron core is therefore a better solution. Lamination of the core can reduce losses and heating in the core to a minimum.

Note: The current transformer discussed here is an ideal transformer because we assumed there are no losses and all flux in the core remains contained, in spite of the relatively low permeability of the core.

10.8   Eddy Currents

Up to this point, we assumed that an induced emf (induced voltage) can be generated in a loop, or any conducting wire, regardless of shape. If the loop is closed in a circuit, the induced emf produces an induced current. However, Faraday's law as written in **Eq. (10.1)** does not require the existence of a physical loop: Induction of flux, and therefore electromotive force, exists even if an actual loop is not obvious. To see this, consider a time-dependent, uniform magnetic flux density as in **Figure 10.18a**. If we place a loop in this flux density, an induced emf is generated in the loop. Now, consider the situation in **Figure 10.18b**, where a cylindrical conductor is placed in the changing magnetic flux density. We do not have a loop per se, but we can view the cylinder as being composed of thin short-circuited cylindrical loops, as in **Figure 10.18c**. Each one of these loops will have an emf that produces an induced current in each loop. Now, considering again the cylinder in **Figure 10.18b**, it is obvious that the magnetic flux density induces currents in the volume of the cylinder. These currents are called induced currents, eddy currents, or Foucault currents[3]. Normally, we view eddy currents as undesirable because they dissipate power in the volume of materials and therefore generate heat (losses) in the material. This is certainly the case in transformers and in some machines. However, as we will see in examples that follow, there are important applications of eddy currents including levitation, heating and melting of materials, nondestructive testing for material integrity, and induction machines, where eddy currents are essential.
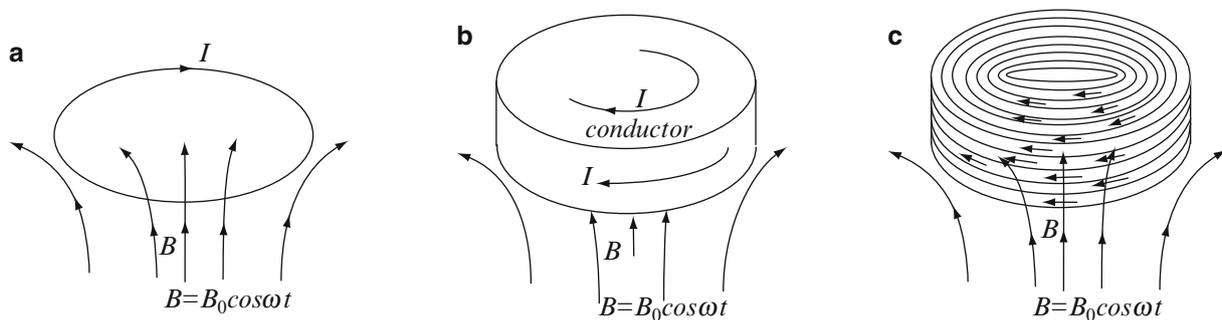
Figure 10.18 (**a**) A time-dependent flux density generates an induced current in a loop. (**b**) A time-dependent magnetic flux density generates induced currents in a conducting volume. (**c**) The conducting volume is seen as being made of short-circuited conducting loops similar to the loop in (**a**)

Example 10.11 Application: Losses in Conducting Materials   A circular disk of radius $d = 100$ mm and thickness $c = 1$ mm is placed in a uniform, AC magnetic field as in **Figure 10.19a**. The magnetic flux density varies as $B = B_0 \sin \omega t$ [T] and is directed perpendicular to the disk. The conductivity of the disk is $\sigma = 10^7$ S/m, $f = 50$ Hz, and the amplitude of the magnetic flux density is 0.2 T:

(**a**) Calculate the instantaneous power dissipated in the disk due to induced (eddy) currents. Assume the magnetic field is not modified by the induced currents and the field remains constant throughout the disk.
(**b**) What is the peak power loss at 100 Hz, under the same assumptions?

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[3] *Induced currents* is the generic name associated with currents in the bulk of conducting materials. The term *eddy currents* is the common name used to distinguish induced currents occurring in the bulk of materials with induced currents in thin wire loops. The name *Foucault currents* is commonly used in France and is named after Jean Bernard Leon Foucault (1819–1868) as a tribute to his extensive contribution to many areas of science, most notably to optics and electromagnetics. Foucault is best remembered for his pendulum, which measured, for the first time, the rotation of the Earth (1851), but he also invented the gyroscope (1852) and a method of photographing stars (1845) and also showed that heat has wave properties. Many other techniques, including the modern method of making mirrors, are due to him.

Figure 10.19 Induced currents due to change in flux. (**a**) Geometry. (**b**) A ring of radius r and differential width used to calculate the flux
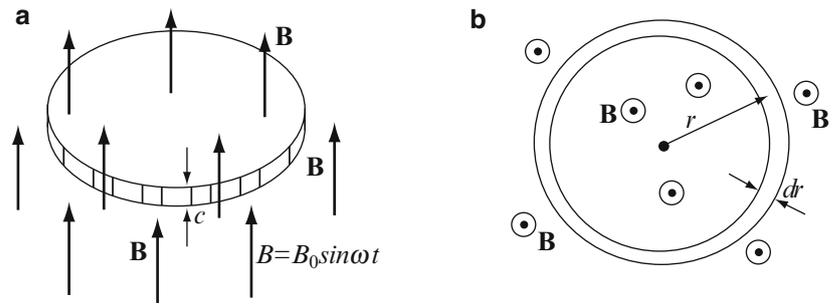
Solution: To calculate the current, an infinitesimal ring is "cut" out of the disk and viewed as a loop. Now, we can calculate the emf induced in this ring and its resistance. From these, we obtain the power dissipated in the infinitesimal ring. To find the total power dissipated in the disk, we integrate the power over all rings that make up the disk:

(**a**) Consider **Figure 10.19b**. The total flux enclosed in the ring of radius r is

$$\Phi = \pi r^2 B = \pi r^2 B_0 \sin\omega t \quad [\text{Wb}]$$

The emf (neglecting the sign since only the power is needed, not the direction of current) is

$$\left|\frac{d\Phi}{dt}\right| = \omega\pi r^2 B_0 \cos\omega t \quad [\text{V}]$$

To calculate power, we need the resistance of the ring. This is calculated for a ring of length $2\pi r$ and cross-sectional area equal to cdr as

$$R = \frac{l}{\sigma S} = \frac{2\pi r}{\sigma cdr} \quad [\Omega]$$

The instantaneous power dissipated in this infinitesimal ring is

$$dP(t) = \frac{V^2}{R} = \frac{\text{emf}^2}{R} = \frac{(\omega\pi r^2 B_0 \cos\omega t)^2}{2\pi r/\sigma cdr} = \frac{\omega^2 \pi r^3 B_0^2 \sigma c(\cos\omega t)^2 dr}{2} \quad [\text{W}]$$

Since the disk is made of an infinite number of rings varying in radius from zero to d, we integrate this expression over r and get

$$P(t) = \int_{r=0}^{r=d} \frac{\omega^2 \pi r^3 B_0^2 \sigma c(\cos\omega t)^2 dr}{2} = \frac{\omega^2 \pi d^4 B_0^2 \sigma c(\cos\omega t)^2}{8} \quad [\text{W}]$$

For the values given above, this power is

$$P(t) = \frac{(2 \times \pi \times 50)^2 \times \pi \times 0.1^4 \times 0.2^2 \times 10^7 \times 10^{-3} \times \cos^2(2 \times \pi \times 50t)}{8} = 1550.3\cos^2(314.16t) \quad [\text{W}]$$

The peak power dissipated is 1,550.3 W.

(b) Since the power dissipated is proportional to the square of the frequency and all other parameters remain unchanged, the peak power dissipated at 100 Hz is four times larger or 6,201.2 W. This power is very large considering the small volume involved. The result is quick heating of the material or even melting. This method of heating metals is commonly used in both melting (induction melting) and heat treatment of conducting materials. Perhaps the most common method of surface hardening (such as on bearing surfaces and rotating shafts) is the use of induction heating coils to locally heat the surface that needs to be hardened followed by quenching in oil. Because coils can be made to fit rather awkward surfaces, the method is versatile, and because heating is quick, it is fast and efficient.

In practice, the magnetic flux density does change in the material (we shall see in **Chapter 12** why and how) and the power dissipated is smaller than that found here. Also, because of the change in the magnetic flux density in the material, more power is dissipated on the surface of the conductor than in its interior. This property is often used to produce localized surface heating such as in hardening of surfaces of rotating shafts.

Example 10.12 Induced Currents Due to Change in Flux Consider the thin conducting ring in **Figure 10.20a**. The flux density $\mathbf{B} = \mathbf{B}_0$ [T] is constant and uniform throughout its cross section. At a given time $t = 0$, the flux density $\mathbf{B}$ starts to increase as

$$\mathbf{B}(t) = \mathbf{B}_0(1 + kt) \qquad [\text{T}]$$

where k is a constant. Calculate the induced current in the ring. Assume the ring is thin and the induced currents do not affect the magnetic field. Numerical values are: cross-sectional area of the ring $s = 1$ mm^2, $r = 10$ mm, $\sigma = 10^7$ S/m, $B_0 = 1$ T, $k = 60$ T/s.

Figure 10.20 Induced currents due to change in flux. (**a**) A conducting ring in a magnetic field. (**b**) Relation between magnetic and electric fields. (**c**) Equivalent circuit showing the induced emf in the ring
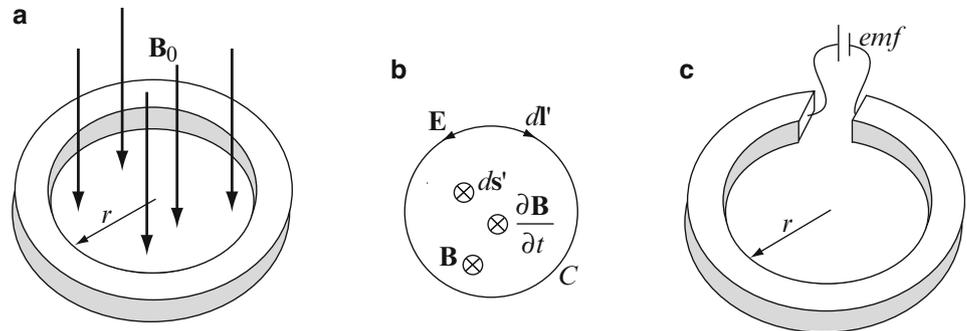
Solution: There are two methods to solve this problem: (1) The increase in the magnetic flux density causes an induced electric field intensity in the closed loop, which may be calculated using **Eq. (10.5)**. This electric field generates a current density in the material of the loop equal to $\sigma\mathbf{E}$. Assuming the current density is uniform in the conductor, the current is found by multiplying the current density by the cross-sectional area of the conducting ring. (2) The increase in the magnetic flux density induces an emf in the ring. This emf produces a current equal to the emf divided by the resistance of the loop. We show both methods.

Method (1) The change in flux density produces an electric field intensity $\mathbf{E}$ in the direction shown in **Figure 10.20b**, assuming that the flux density increases as indicated. The induced electric field intensity, induced current density, and induced current can now be calculated using **Eq. (10.5)**:

$$\oint_C \mathbf{E} \cdot d\mathbf{l}' = \int_{s'} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}'$$

where s' is the surface defined by the circular ring and C is the circumference of the ring. Noting that $\mathbf{B}$ depends only on time and that the pairs of vectors $\mathbf{E}$, $d\mathbf{l}'$, and $\partial \mathbf{B}/\partial t$, $d\mathbf{s}'$ are collinear gives:

$$E2\pi r = -\pi r^2 \frac{\partial B}{\partial t} \quad \rightarrow \quad |E| = \frac{r}{2}\frac{\partial}{\partial t}(B_0 + B_0 kt) = \frac{krB_0}{2} \quad \left[\frac{V}{m}\right]$$

From this, the current density $J = \sigma E$ and current are

$$J = \frac{\sigma k B_0 r}{2} \quad \left[\frac{A}{m^2}\right] \quad \text{and} \quad I = \frac{\sigma k B_0 rs}{2} \quad [A]$$

Method (2). The emf in the closed loop equals

$$|\text{emf}| = \left|\int_{s'} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}'\right| = \pi r^2 B_0 k \quad [V]$$

The emf can be viewed as a voltage source in the ring, as shown in **Figure 10.20c**. Viewing the loop as a circuit, the emf is

$$\text{emf} = RI \quad [V]$$

where R is the resistance of the loop and I the current in the loop. The resistance of the ring is

$$R = \frac{l}{\sigma s} = \frac{2\pi r}{\sigma s} \quad [\Omega]$$

where s is the cross-sectional area. Combining the last three relations, we get

$$\pi r^2 B_0 k = \frac{2\pi r}{\sigma s} I \quad \rightarrow \quad I = \frac{\sigma k B_0 sr}{2} \quad [A]$$

This is identical to the result obtained in method (1). With the given numerical values, the current in the loop is 3 A.

10.9   Applications

Application: The Magnetic Brake An interesting and very useful application of induced currents is the magnetic brake. To outline the principle involved, consider **Figure 10.21a**. An electromagnet generates a flux density **B** in the gap. This field is assumed to be constant. A pendulum-like flat piece, made of a conducting material, is placed such that it can move into the gap. If the current in the electromagnet, I, is zero, the oscillation of the pendulum is not affected by the structure. If there is a current in the coil, the movement of the conducting plate into the magnetic field **(Figure 10.21b)** generates induced currents in the plate itself due to the motion of the conductor in the magnetic field. The flux of the induced currents is such that it opposes the field **B**. According to Lenz's law, the induced currents tend to maintain this condition by opposing the flux. **Figure 10.21b** gives the direction of the fields. The electric field intensity due to the induced currents is given as $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ and we get

$$\mathbf{J} = \sigma \mathbf{E} = \sigma \mathbf{v} \times \mathbf{B} \quad \left[A/m^2\right] \tag{10.57}$$
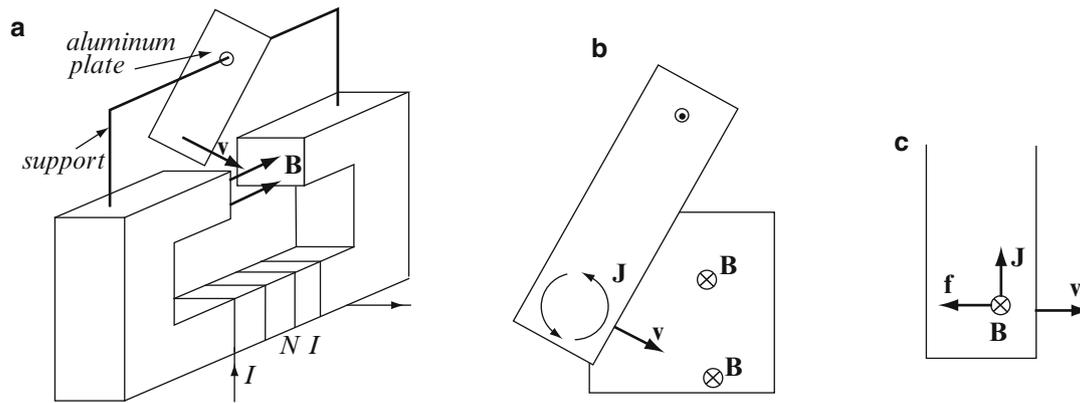
Figure 10.21 (**a**) The magnetic brake. (**b**) Direction of fields in the magnetic brake. (**c**) Direction of induced currents in the plate

The velocity at which the plate penetrates into the gap is responsible for the magnitude of the induced currents. The relation between the current density **J**, magnetic flux density **B**, and force density **f** at a point is shown in **Figure 10.21c**.

Using **Eq. (9.109)** we get the volumetric force density **f**:

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} = \sigma(\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \quad [\text{N/m}^2] \tag{10.58}$$

If all vectors are mutually orthogonal, as is the case in this example, the total force is

$$F = \sigma v B^2 V_0 l \quad [\text{N}] \tag{10.59}$$

and its direction, given by the cross-product $\mathbf{J} \times \mathbf{B}$, opposes the direction of **v**. This has the effect of damping the movement of the plate into the gap. If the conductivity σ of the plate were infinite, the plate would be repelled from the gap. In reality, σ is finite and the plate is decelerated as the power due to induced currents is dissipated in the plate. The plate penetrates into the gap, decelerating, and, eventually, reaches a state of static equilibrium at the lowest point of its oscillation. In the case this does not happen immediately; the plate may continue to oscillate in a highly damped motion until the pendulum has completely stopped. This principle is used extensively on locomotives and trucks. Conducting disks are installed on the axles of the vehicle and electromagnets are placed around them such that the disks move in the gap of the electromagnets, as in **Figure 10.22**.

Figure 10.22 A practical magnetic brake. Braking takes place by the interaction of the electromagnet and eddy currents in the disk
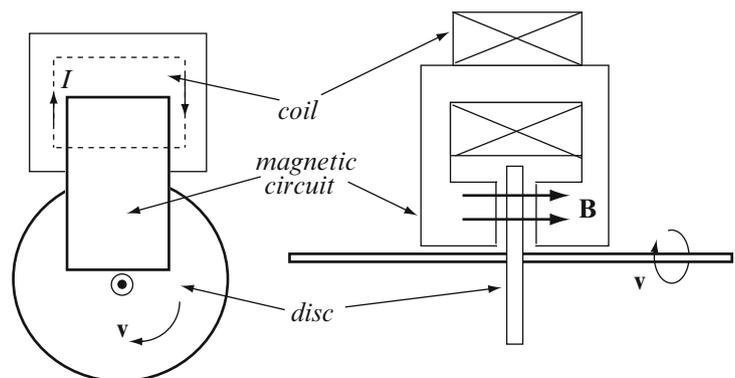
When the mechanical brakes are applied, a current is also applied to the electromagnet and the braking effects of the mechanical and magnetic brakes are added together. We note, however, that the braking effect assumes a velocity **v**. For this reason, electromagnetic brakes cannot be used to completely stop a vehicle, only to slow it down. The magnetic brake is therefore more appropriately called a magnetic retarder or damper. Electric brakes have many advantages. First, they brake better at high speeds and are natural antilocking brakes, since locking of the wheels will immediately release the brakes.

Similarly, dragging and binding are not possible because they are noncontact devices. On the other hand, they dissipate large amounts of energy, need considerable electric power, and must be supplemented by mechanical brakes.

Application: The Acyclic[4] (Homopolar) Generator and Motor—Faraday's Disk The idea of moving a bar in a magnetic field is fundamental to all generators in one way or another, as was amply shown in the previous sections. One particularly simple method is to rotate the bar in a magnetic field rather than translate it. By doing so, the motion is greatly simplified. The basic idea is shown in **Figure 10.23**. It consists of a bar, pivoted at one end and rotated in the magnetic field. Two connections are made: one at the pivot (axis) and one at the moving end. An emf is generated in the bar which is proportional to the speed of motion. In this case, the output is proportional to the frequency of rotation, but it remains DC. A more common implementation of the same idea, one that simplifies the connections, is a disk on a shaft rotating in the magnetic field, as shown in **Figure 10.24**. The rotating disk acts the same as the rotating bar, but in practical terms, the connections are easier to make and the device is balanced. This method of generation is one of only a small number of methods that allow direct generation of DC power.
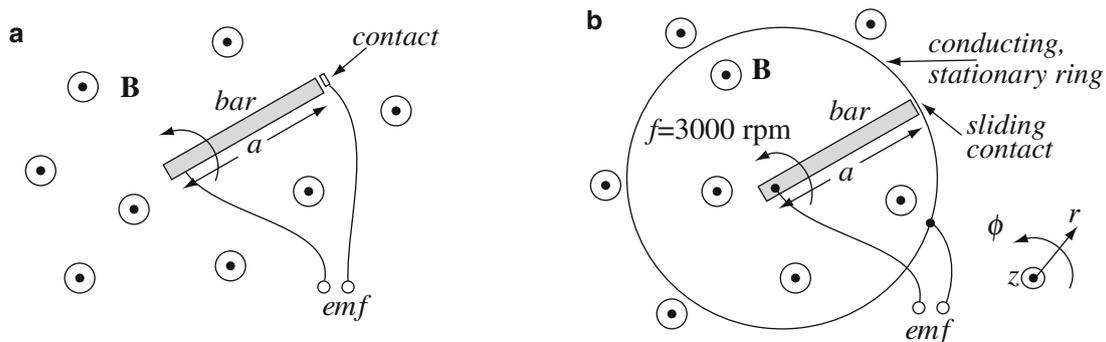
Figure 10.23 The principle of the acyclic generator. (**a**) A rotating bar perpendicular to a magnetic field. (**b**) The bar in (**a**) with a more practical sliding ring connection
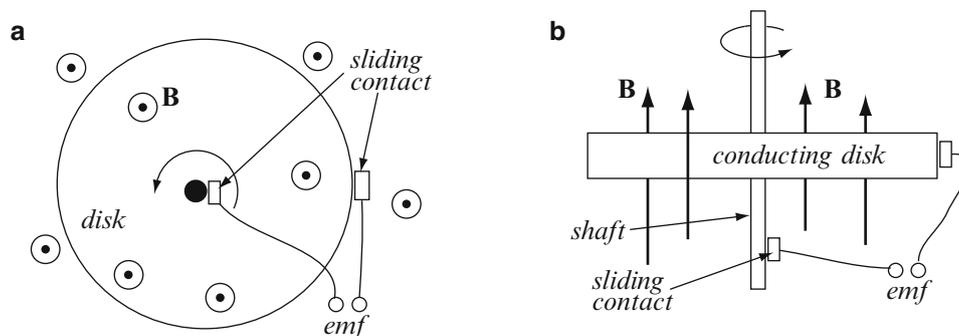
Figure 10.24 A practical acyclic generator. (**a**) A conducting disk rotates in a magnetic field. Connections are made through sliding contacts. (**b**) Side view of the generator

Application: The Acyclic (Homopolar) Motor Consider now the opposite problem: The contacts on the disk or the pivoted bar of the previous application are connected to an external source as shown in **Figure 10.25**. The disk now rotates as a motor (see **Problem 10.14**). The homopolar motor is particularly suitable for applications where low-voltage, high-current sources are available (for example, submarines) or for applications that require high torque.

[4] An acyclic or homopolar generator is a machine in which the emf induced in the moving conductors maintains the same polarity with respect to the conductors as the conductors move.
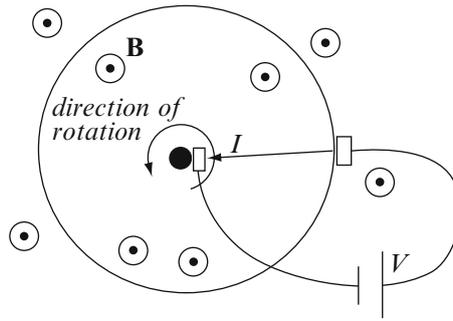
Figure 10.25 The acyclic motor

Application: The Watt-Hour Meter A useful device based on the interaction of induced eddy currents in a conductor is the common watt-hour meter found in many homes. The meter is built of an aluminum disk on a spindle. The disk is placed between the poles of a magnetic yoke, as shown in **Figure 10.26**. Three coils are wound on the yoke. The upper, center coil is called a voltage coil since the current in the coil and, therefore, the field it generates depends on the line voltage. The two lower coils are connected in series with the load. These coils generate a field that is proportional to the current in the load. Both the current and voltage coils generate eddy currents in the conducting disk. However, either one, by itself, produces no torque in the disk. When both are present, the interaction of the current and voltage fields produces a torque, proportional to the product of current and voltage. The speed of the disk is therefore proportional to power. In addition, the meter employs permanent magnets as retarders or braking devices whose braking force is proportional to the speed of the disk. This is required to avoid acceleration of the disk and, therefore, inaccurate measurements. The shaft of the disk is geared and turns indicator dials that register the energy consumption. Although there are more modern instruments that use direct measurement of current and voltage (as well as phase angle) and display power consumption digitally, the common watt-hour meter is still used extensively because it is an accurate, reliable, and inexpensive instrument. The rotating disk in watt-hour meters is normally set such that it can be seen through a window in front of the instrument, giving a simple indication of its operation.
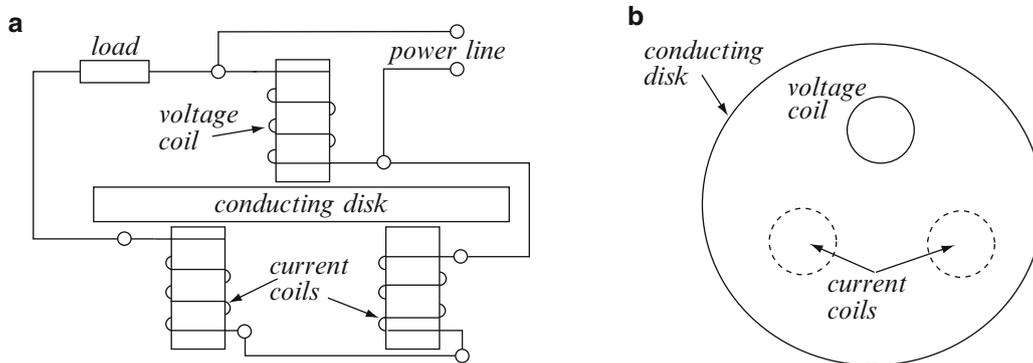
Figure 10.26 The watt-hour meter. (**a**) Side view. (**b**) Top view

Application: The Rail Launcher One application of the magnetic force exerted on currents is the rail launcher or rail gun. In its simplest form, it consists of two rails with a conducting projectile that shorts the rails as shown in **Figure 10.27a**. The current in the rails generates a flux density between the rails and the interaction of the current in the projectile, and this

field will force the projectile out. This method has been used to demonstrate the possibility of firing projectiles at velocities much higher than those possible with explosives for both peaceful and military applications. One possible application that has been proposed is to use this device to fire satellites into orbit, since the initial velocities that can be achieved are high enough to permit such applications. This means that escape velocities are possible or that much greater damage can be done when the gun is used for military purposes. The rail gun also finds applications in the acceleration of very small masses in particle research.
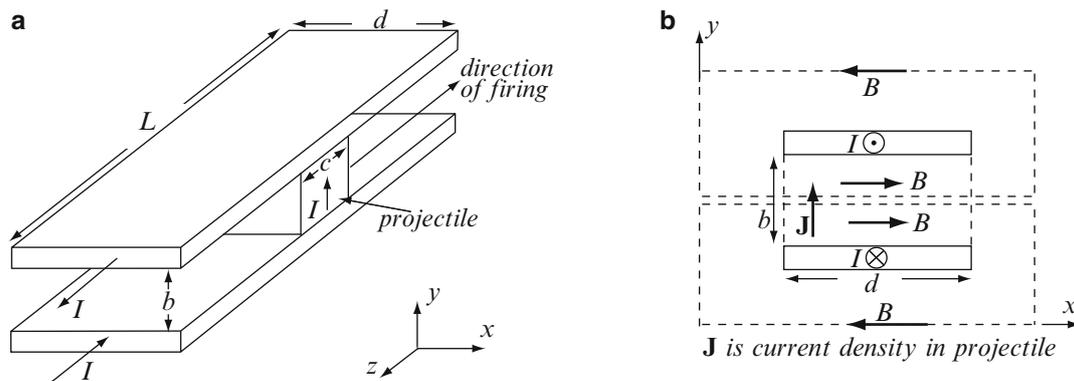
Figure 10.27 The rail launcher. (**a**) Configuration. (**b**) The magnetic flux density between the rails and its relation to current density in the projectile

Application: Eddy Current Testing of Materials   One of the most common methods of testing conducting materials for flaws is the eddy current method. It consists essentially of a coil, connected to a constant current or constant voltage AC source. A current flows in the coil and produces a magnetic field in the vicinity of the coil as shown in **Figure 10.28a**. If the conditions in the space around the coil do not change, the coil inductance, and therefore its AC impedance, remains constant. Now, suppose we bring the coil near a conducting material as in **Figure 10.28b**. There are now induced currents in the conducting material due to the induced emf. This causes the impedance of the coil to decrease (i.e., more power must be provided by the source). The current in the coil changes (for constant voltage supply) or the voltage on the coil changes (for constant current supply). This establishes the reference reading (current, voltage, or impedance). If, however, there is a flaw in the material, such as a crack or inclusion, the induced currents in the conducting material change and so does the impedance of the coil. Monitoring the coil impedance (measuring the current for constant voltage sources or the voltage for constant current sources) gives a direct reading of the condition of the material. Any variation from the constant reading obtained with the "good" material is an indication of some change in the material, either material condition (cracks, inclusions, corrosion) or material properties (changes in conductivity or permeability of the material). This configuration is shown in **Figure 10.28c**. A test and the voltage on the coil (for constant current supply) is shown in **Figure 10.29**. This method of testing is commonly employed for testing of conductors, including airframes in aircraft, aluminum skins on wings, and other critical parts, and in testing of tubing in air-conditioning units and power plants. In testing tubes, two coils are inserted inside the tubes and the two coils connected in series (**Figure 10.30a**) to provide a differential output as in **Figure 10.30b**. As long as both coils are in the vicinity of sound material, the output is zero. If one coil approaches a defect, such as a hole, it will show a different impedance than the other coil and the output will be nonzero, as shown in **Figure 10.30b**.
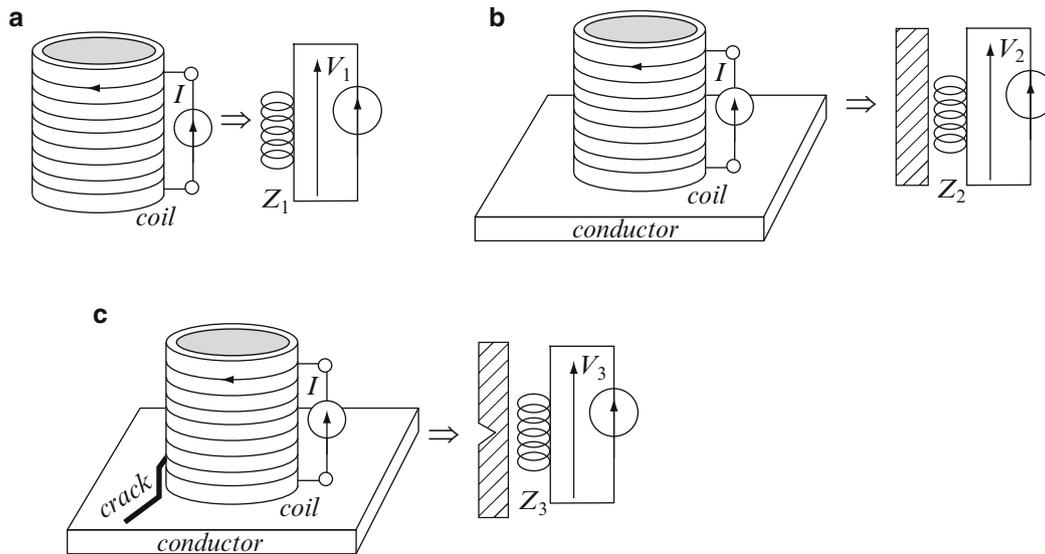
Figure 10.28 Eddy current testing of materials. (**a**) The coil and its equivalent circuit. (**b**) Testing of an intact material produces an output V_2. (**c**) A flaw produces a different output V_3
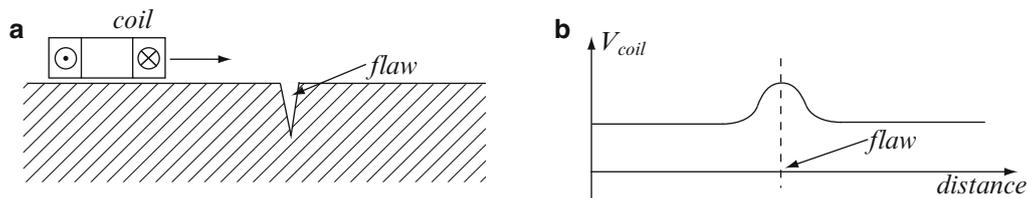

Figure 10.29 Eddy current testing for surface flaws. (**a**) A simple coil passes over the flaw. (**b**) The output signal due to a flaw
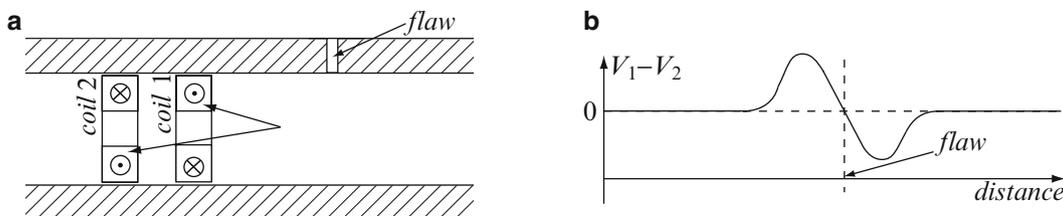

Figure 10.30 Eddy current testing of tubulars. (**a**) Two coils connected in opposition pass through the tube. (**b**) The output signal due to a flaw

Application: Electromagnetic Propulsion of Ships and Submarines The force in the rail gun above can be used in a somewhat different way. Assume that the rails are submerged in seawater. Since seawater is conductive (approximately 4 S/m), the whole space between the rails is filled by a conductor. A current passing through the seawater will generate a force on the water which is then pushed out of the space, provided a magnetic field exists in this space. By enclosing the rails with nonconducting surfaces (i.e., creating a tube), the water will be expelled as a jet. The reaction force moves the water craft forward in much the same way as in a water-jet boat. This device is a ***magnetohydrodynamic*** pump and has been proposed as a propulsion mechanism for submarines. The main advantage would be reduced noise (one of the main means of detection of submarines is the noise they generate by their propellers, a noise which travels quite far in water). However, the currents required are large and the efficiency of the system is very low, mainly because of the very low conductivity of

seawater. To increase efficiency, it is necessary to use very large magnetic fields which, in turn, require cryogenically cooled magnets (or superconducting magnets) with all the associated cooling equipment, weight, and energy.

The same method can be used for pumping molten metals and, in particular, molten sodium as used in some nuclear power plants. Since these metals have high conductivities, it is a very efficient method in particular since it acts as a pump without moving parts. In the movement of corrosive materials, or molten metals at high temperatures, this is an overriding requirement.

Application: The Magnetohydrodynamic (MHD) DC Generator One method of generating electricity is the magneto-hydrodynamic method. The principle is that of a moving bar in a magnetic field, not unlike that used in **Example 10.1**. However, instead of the moving bar, a conducting fluid moves between two conducting electrodes, as shown in **Figure 10.31**. Two magnets generate a very high magnetic field in a channel between them, called a *magnetohydrodynamic* (MHD) *channel*. Two conducting electrodes are placed at right angles to the field, insulated from the magnets and each other. A conducting fluid is now forced through the channel. The magnetic field and velocity product $\mathbf{v} \times \mathbf{B}$ are as shown in the figure which indicates that the left plate becomes negative whereas the right plate is positive. The conducting fluid can be any fluid. For example, we may pump seawater through the channel. Most experimental generators use highly ionized exhaust gases from the burning of fuels. A very simple *MHD* generator is made of a channel as above, connected to the exhaust of a jet engine. To increase conductivity of the gases, these are seeded with conducting ions such as alkali metal vapors. In coal-fired *MHD* generators, potassium carbonate is used as seed to increase conductivity. The most attractive feature of these generators is the fact that they are stationary and produce DC directly. Most experimental *MHD* generators have low efficiency (below 15 %), but because the system can be fully contained and operating in closed circuits, it has considerable promise, especially for solar power generation. In spite of its simplicity, the engineering challenges of *MHD* are quite difficult to handle. One is the need for very large magnetic fields which can only be obtained at superconducting temperatures. The other is the very high speeds and pressures, as well as temperatures in the channel. See **Problem 10.25** for a sample calculation.
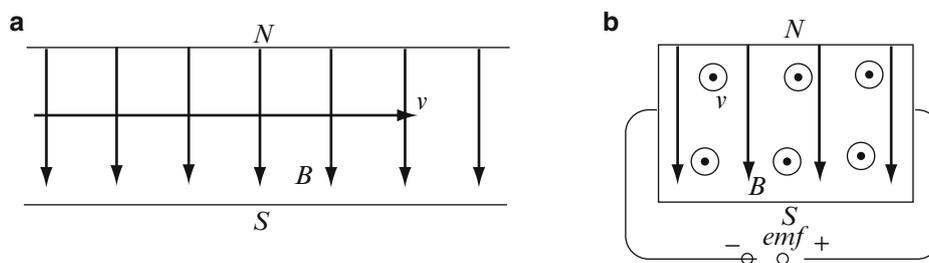

Figure 10.31 The magnetohydrodynamic generator. (**a**) Relation between fluid velocity and magnetic field. (**b**) Cross-sectional view showing the emf

10.10 Experiments

**Experiment 1 (The Electromagnetic Brake. Demonstrates: Induced Currents Due to Motion, Electromotive Force, Lenz's Law, Faraday's Law of Induction).** The principle of the electromagnetic brake can be demonstrated with a strong magnet and a thick conducting plate. A suitable magnet is a magnet with a small gap between its poles. Two large loudspeaker magnets placed so that the opposite poles are located on two sides of the gap can be used. **Figures 10.32a** and **10.32b** show two possible configurations.

The conducting plate can be a simple aluminum plate. It should be at least 5 mm thick (the thicker the better) and 200 mm wide. A hole is drilled in the plate so that it can be suspended freely. A pencil may be passed through the hole to create a pivot.

Lift the plate and allow it to swing through the gap between the two magnets. Notice the retarding effect of the magnet. If the magnet is strong, and the plate thick, the plate should come to an almost complete stop within the gap. Otherwise, it will oscillate somewhat and then stop. An alternative is to simply drop the aluminum plate between the poles and observe its motion. The same

effect can be demonstrated by allowing a magnet to slide down a slanted, conducting, but nonmagnetic plate. Use a thick plate and a strong magnet. A flat disk magnet is most convenient in this case. The motion of the magnet down the slope is retarded and it will move slowly downhill. A similar slope on a nonconducting surface should demonstrate the difference.

Figure 10.32 Two methods of demonstrating magnetic braking. (**a**) Use of two magnets. (**b**) Use of a single magnet with a gap
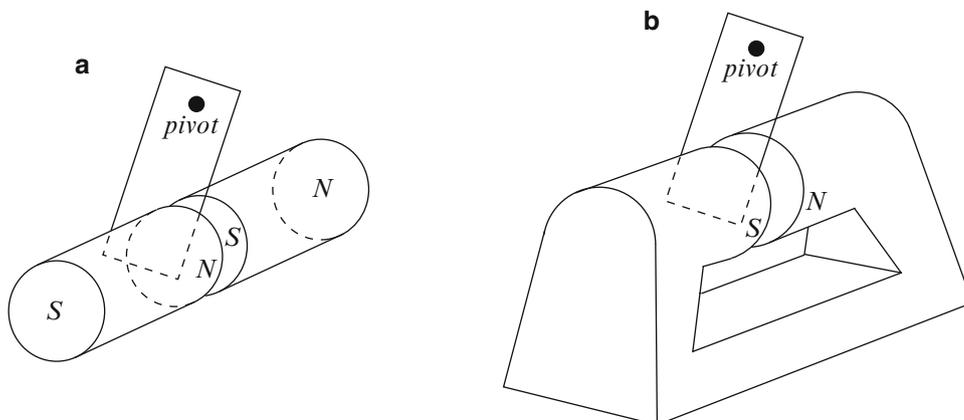
Experiment 2 (The generator. Demonstrates: Electromotive Force, Induction, AC Generation, Lenz's Law, Faraday's Law). The principle of induction can be demonstrated by using a permanent magnet as the source of the magnetic field and a coil. A loudspeaker magnet is suitable for this purpose. Use a coil with an iron core and a large number of turns. A low-voltage relay coil is suitable. Connect two light-emitting diodes (LEDs) across the leads of the coil as shown in **Figure 10.33**. The diodes are connected in opposition, such that one diode lights up when the electromotive force is of one polarity, the other when it has an opposite polarity. Use of two different color diodes will allow you to distinguish between polarities of the electromotive force induced in the coil. Move the coil across the magnet, as close as possible to the magnet. As the coil is moved in one direction, one diode lights up. Move the coil toward the magnet and away from the magnet. The diodes will light up alternately. If the magnet is strong, the iron core will tend to stick. You can reduce this by placing a piece of cardboard on the magnet and sliding the coil on the cardboard or sliding the magnet beneath the cardboard.

Figure 10.33 Demonstration of generator action. (**a**) Construction. (**b**) Operation
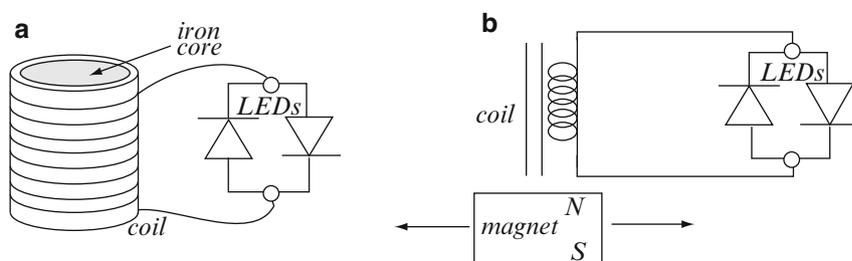
Experiment 3 (Eddy Current Levitation. Demonstrates: Lenz's Law, Forces Between Currents, Eddy Currents). Levitation of conducting bodies due to induced currents can be demonstrated as follows: A small disk of conducting material (i.e., aluminum) will be suspended in a magnetic field due to the force that exists between the inducing and the induced field (Lenz's law). **Figure 10.34** shows the basic principle. A simple device can be built from a transformer by changing it from a closed magnetic circuit to an open magnetic circuit. To do so, it will be necessary to remove the top laminations by sawing off the laminations as shown in **Figure 10.35a**. When doing so, care must be taken not to damage the coils. A transformer with low-voltage coils should be used for safety reasons. Drill a small hole in the center of the central leg of the transformer and insert a nonconducting peg in the hole (a toothpick is suitable). Make a small aluminum disk (25–75 mm in diameter) or a small square aluminum plate and drill a small hole at the center. Insert the disk over the peg as shown in **Figure 10.35b**. Now, the disk is resting on the laminations. Connect the primary coil of the transformer to an AC source. The disk will lift and balance at a certain distance depending on the magnetic field and the weight of the disk.
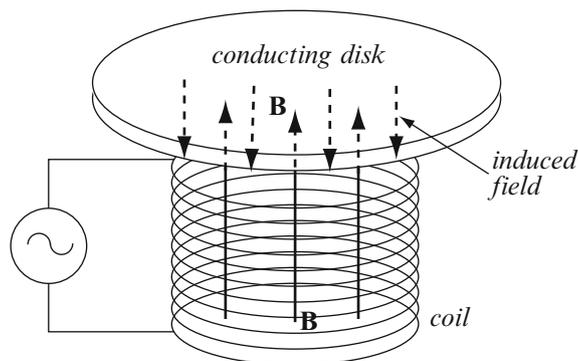
Figure 10.34 Eddy current levitation. A conducting disk is suspended above a coil due to reaction force on the induced currents in the disk

Caution The transformer now operates with an open magnetic path. The losses in the transformer are high and the core will get hot quickly. Do not leave the supply connected for more than a few seconds at a time.

One useful application of this method of levitation is for assembly line moving of devices. In particular, the method is very attractive for the fast moving of silicon wafers in the production of semiconductors. The devices are levitated high enough to eliminate friction and then, by properly sequencing the levitating coils, moved in any direction using the forces generated between the coils and wafers.
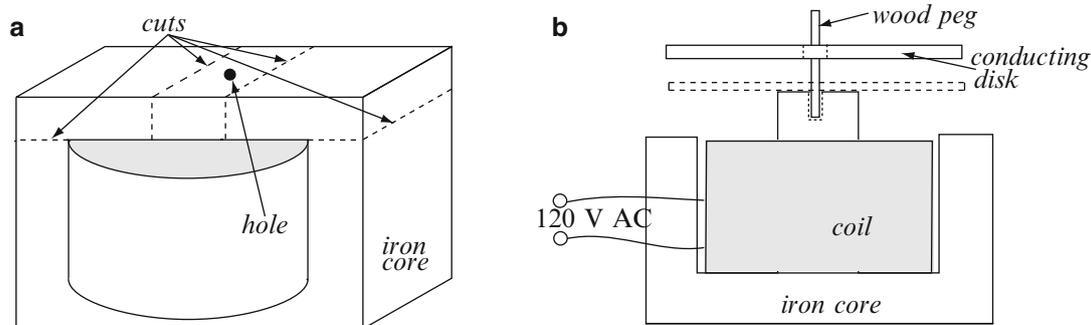
Figure 10.35 Construction of an eddy current levitator. (**a**) Location of cuts. (**b**) Side view with the disk shown suspended

Experiment 4 (Stored Energy in Inductors. Demonstrates: Inductance, Stored Energy, Faraday's Law, Lenz's Law). Take a small power transformer and identify the primary and secondary leads. Connect the secondary to an LED through a small resistor as shown in **Figure 10.36**. Now connect a 1.5 to 6 V battery to the two secondaries by closing the switch S for a few seconds and then open the switch. The LED will light momentarily. As the current is disconnected, the flux through the coil of the transformer is reduced from some value to zero in a short time. The result is an induced emf which can be many times larger than the voltage of the battery. The polarity of this emf is opposite that of the battery and hence the LED lights up and dissipates the stored energy in the coil.

Warning Do not perform the experiment on the primary coil and make sure nothing else is connected to the transformer.
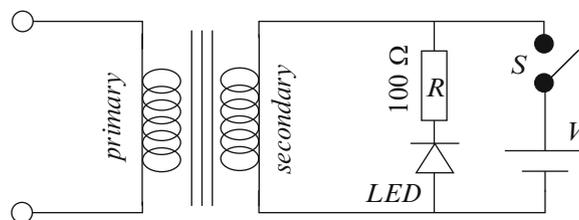
Figure 10.36 Demonstration of Lenz's law and of stored magnetic energy

Experiment 5 (Permanent Magnet Levitation. Demonstrates: Magnetic Forces). You can build a simple levitation system using permanent magnets. On a square piece of thin plywood, glue four magnets. Ring magnets such as those used to hold down notes on magnetic boards are suitable. Make sure all four magnets are glued such that identical poles point in the same direction by testing with a fifth magnet. If all four attract or all four repel the fifth magnet, they are properly aligned. On a second, identical piece of wood, glue four additional magnets in exactly the same fashion, with the same polarity facing away from the wood plate. Now, drill two holes through both plates and pass wooden pegs through them. These pegs will prevent the plates from rotating. Place one plate on a table and slide the second over the pegs so that the four magnets of one plate face the four magnets of the other plate. The upper plate will be suspended above the lower at some distance depending on the strength and weight of the magnets and the weight of the plate.

10.11 Summary

Following the study of electrostatics and magnetostatics, we now look into time-dependent phenomena, starting with **Faraday's law** of induction. Faraday's law was originally observed as an induced voltage (or **electromotive force (emf)**) in a loop due to motion of a magnet in its vicinity. For a single loop or for N loops in the same location, it takes the forms:

$$emf = -\frac{d\Phi}{dt} \quad [\text{V}] \tag{10.1}$$

$$\text{or } emf = -N\frac{d\Phi}{dt} \quad [\text{V}] \tag{10.2}$$

This observation modifies the first postulate of the electric field (curl equation):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.8}$$

$$\text{or} \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t} \tag{10.5}$$

Lentz's law accompanies Faraday's law and gives meaning to the negative sign. It states: "The direction of the *emf* is such that the flux generated by the induced current opposes the change in flux."

An emf may be viewed as being generated by motion or by inherent time dependency of the field.

Motion action emf is produced by motion of a conductor in a magnetic field:

$$emf = \int_a^b (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \quad [\text{V}] \tag{10.12}$$

where a conductor extending from a to b moves at a velocity $\mathbf{v}$ in a magnetic flux density $\mathbf{B}$.

Transformer action emf requires that the magnetic flux density be time dependent:

$$emf = \oint_C \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{V}] \tag{10.6}$$

The **transformer** is a device that relies on its operation on induced emfs. In an ideal transformer, there are no losses and the magnetic path has low reluctance. For a two-coil closed path transformer with path reluctance $\Re$ (**Figure 10.13b**), the flux along the path is

$$\Phi = \frac{N_1 I_1 - N_2 I_2}{\Re} \tag{10.36}$$

The terminal voltages, currents, and impedances of the primary (1) and secondary (2) coils are related by the transformer ratio a

$$\frac{emf_1}{emf_2} = \frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2} = a \tag{10.40}$$

and
$$\frac{Z_1}{Z_2} = a^2 \tag{10.42}$$

In the real transformer, the reluctance is not necessarily very low but we still assume a closed magnetic path. The emfs in the primary (1) and secondary (2) are now given in terms of self and mutual inductances of the two coils (see **Figure 10.13b**):

$$emf_1 = L_{11}\frac{dI_1}{dt} - L_{12}\frac{dI_2}{dt} \quad [\text{V}] \tag{10.48}$$

$$emf_2 = L_{21}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} \quad [\text{V}] \tag{10.49}$$

If the magnetic path is not closed, the coupling between the coils is weaker and we define a coupling coefficient $0 < k < 1$. Since $L_{12}=L_{21}$ we write

$$emf_1 = L_{11}\frac{dI_1}{dt} - k\sqrt{L_{11}L_{22}}\frac{dI_2}{dt} \quad [\text{V}] \tag{10.55}$$

$$emf_2 = k\sqrt{L_{11}L_{22}}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} \quad [\text{V}] \tag{10.56}$$

Problems

Motional emf

10.1 Motional emf. Two trains approach each other at velocities v_1 and v_2 as shown in **Figure 10.37**. Assume the trains' axles are good conductors and the rails have a resistance r [Ω/km]. Use $d = 2$ m. The trains are $P = 10$ km apart. Also given: $v_1 = 100$ km/h, $v_2 = 120$ km/h, $r = 0.1$ Ω/km. The vertical component of the terrestrial magnetic flux density is $B_0 = 0.05$ mT. Calculate:

(a) The current in the rails produced by the motion of the trains in the terrestrial magnetic flux density at the instant shown in **Figure 10.37**.
(b) The current in the rails as a function of time taking $t = 0$ at the instance shown in **Figure 10.37**.
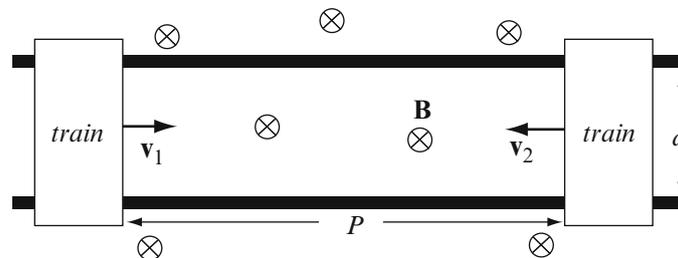
Figure 10.37

10.2 Application: Motional emf as a Motion Detector Mechanism. Suppose you want to know if a train is moving on a rail but it is too far to see. Assume the rails are a distance d [m] apart and are insulated from each other. To check for motion of the train, you measure the potential difference between the two rails:

(a) Calculate the velocity of the train assuming that the magnetic flux density of the Earth is equal to B_0 [T] and is perpendicular to the surface of the Earth, pointing downward. The potential difference measured is V [V] with its positive pole on the top rail in **Figure 10.38**.

(b) In which direction is the train moving?

(c) If the resistance per meter length of a rail equals r [Ω/m], calculate the distance P from the measurement point to the train. How can this distance be found by a simple measurement at the point where the observer is located?

(d) Can these measurements be used to avoid collision of trains?
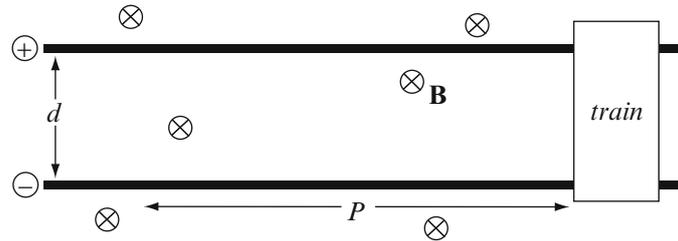
Figure 10.38

Induced emf

10.3 Forces Due to Induction. Two very long, parallel rails are placed in a uniform, time-dependent magnetic field. The resistance of the rails is negligible. On the left side, the rails are shorted with a wire of resistance R [Ω]. The wire is fixed and not allowed to move. On the right, a bar with zero resistance is placed on the rails **(Figure 10.39)** so that it is free to move. The magnetic flux density is given as $\mathbf{B} = -\hat{\mathbf{z}} B_0 \cos(\omega t)$ [T]:

(a) Calculate the magnitude and direction of the force acting on this bar in the instance shown in **Figure 10.39**.

(b) What is the force at $t = 0$?

(c) Suppose now that the bar is free to move. Describe qualitatively the motion of the bar.
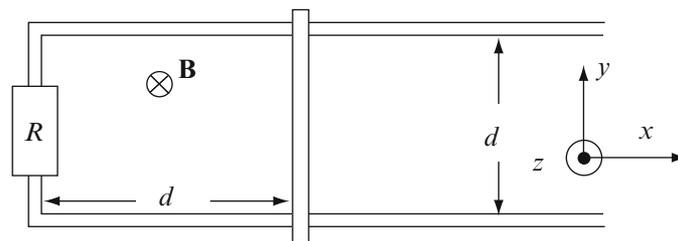
Figure 10.39

10.4 Motional emf. An infinitely long, straight wire of radius r_1 [m] carries a uniform current density J [A/m^2]. The direction of the current is as shown in **Figure 10.40**. Another infinite, thin wire parallel to the first wire moves to the right at a velocity v_0 [m/s]. Calculate the induced voltage (emf) per unit length in the moving wire, when the moving wire is at a distance r [m] from the thick wire. The wires are shown in cross-section.
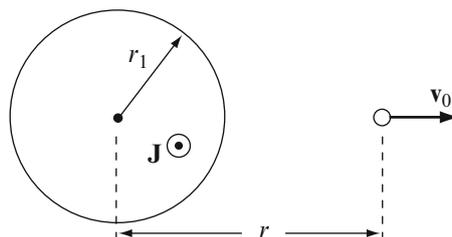

Figure 10.40

10.5 **Induced emf in a Plate**. An infinitely long strip of conducting material of width d [m] moves in a uniform, sinusoidal magnetic flux density $B = B_0 \cos\omega t$ [T] at a constant velocity v [m/s]. The field makes a $45°$ angle with the direction of movement of the strip. Calculate the induced emf between any two points on opposite sides (a, b) of the strip (**Figure 10.41**). Show polarity.
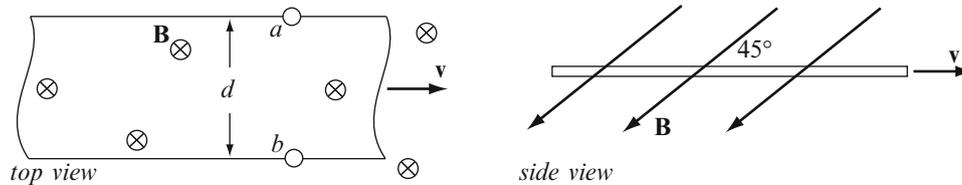

Figure 10.41

10.6 **Induced emf in a Rotating Bar**. Two conducting hoops, each of radius r_1 [m], are connected at one point by a straight conducting bar. The two hoops are kept parallel to each other and rotate at 600 rpm in a constant magnetic field as shown in **Figure 10.42**. The magnetic field is parallel to the hoops and perpendicular to the bar. Calculate the potential difference (emf) between the two hoops.
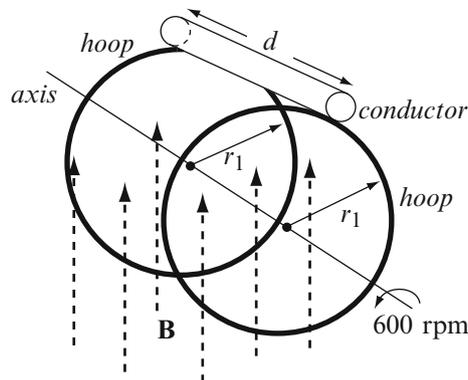
Figure 10.42

10.7 **Motion and Transformer Action emf in Loops**. A square loop and a round loop move at a constant velocity v_0 [m/s] in the x direction ($\mathbf{v} = \hat{\mathbf{x}} v_0$), as shown in **Figure 10.43**. The magnetic field intensity is given as $\mathbf{H} = \hat{\mathbf{z}} H_0 \cos(\omega t)$ [A/m]. Calculate the ratio between the induced voltages (emf) in the two loops.
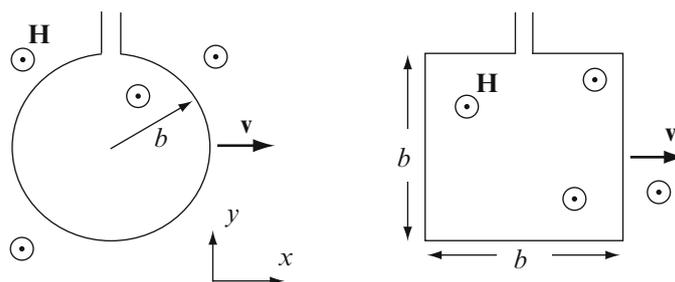
Figure 10.43

10.8 **Application: Generation of Very High emf/Current: Explosive Method.** A long solenoid has a total of N turns of radius d. The solenoid is placed in the magnetic flux density of the Earth, $\mathbf{B}$, such that $\mathbf{B}$ is parallel to its axis, and the two ends of the solenoid are shorted. The total resistance of the wires of the solenoid is R. Using an explosive charge, the solenoid is collapsed (i.e., it is flattened sideways as shown in **Figure 10.44**). Calculate the current generated in the solenoid (average) during the collapsing of the solenoid assuming that this happens in a time Δt. Use $\Delta t = 1 \mu s$, $B = 0.00005$ T, $N = 1{,}000$, $R = 1 \Omega$, $d = 1$ m. **Note.** Explosive methods are sometimes used to generate very high magnetic fields of short duration for research purposes. This problem looks at the emf and short circuit currents of the collapsing coil.
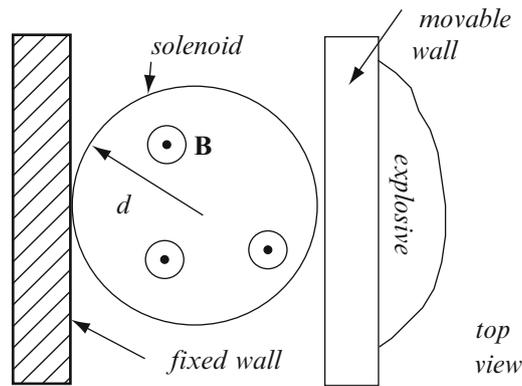
Figure 10.44

Generator emf

10.9 emf in a Generator. A piece of wire is bent and rotated in a uniform, magnetic field **B** [T] as shown in **Figure 10.45**. The frequency of rotation is f (rotations per second). Calculate the induced emf between the terminals a and b. Show the polarity of the induced voltage (emf).
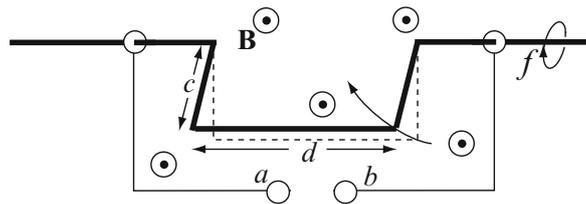

Figure 10.45

10.10 emf in Rotating Bar. A conducting bar of length l [m] is pivoted at one end and rotates at an angular velocity ω [rad/s] in a magnetic field **B** [T] (z directed). Find the induced emf on the bar if it rotates perpendicular to the flux density in the counterclockwise direction for:

(a) $\mathbf{B} = \hat{\mathbf{z}} B_0$.
(b) $\mathbf{B} = \hat{\mathbf{z}} B_0 e^{-r}$, $0 \leq r \leq l$.

10.11 Homopolar Generator. A spoked, conducting wheel rotates in an AC magnetic field as shown in **Figure 10.46**. The frequency of rotation is ω [rad/s]. The magnetic field intensity is $H = 10^6 \cos(\omega t)$ [A/m]. A conducting hoop is provided on which the outer wire slides freely:

(a) Calculate the induced voltage (emf) between the center of the wheel and its outer perimeter.
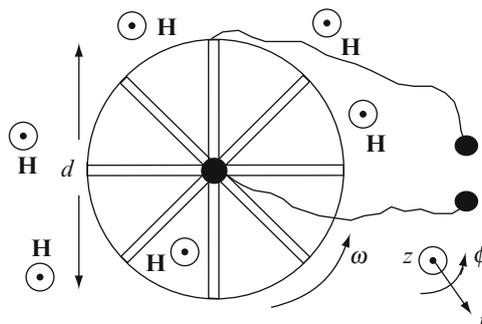(b) How does the solution change if the wheel is a solid conducting wheel?

Figure 10.46

10.12 **Application: Homopolar Generator.** A disk of radius 100 mm rotates in a uniform magnetic flux density at 6,000 rpm as shown in **Figure 10.47**. The magnetic flux density is sinusoidal, with amplitude 0.1 T and frequency 400 Hz. A wire is connected to the center of the disk and one is sliding on its edge, making good contact. Calculate the voltage (motional emf) between points A and B.
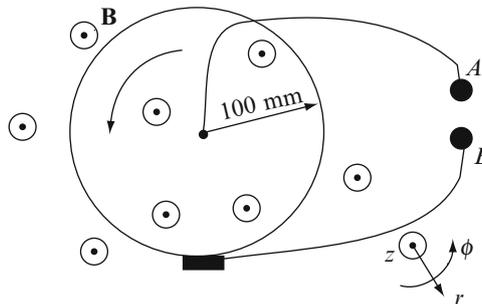

Figure 10.47

10.13 **Power Line Current Sensor.** A sensor designed to detect and estimate the current in a power line remotely is proposed as follows: A small coil of area S is placed at ground level at the center between the two conductors of an AC power line (see **Figure 10.48**). The line carries a current I_0 (RMS) at frequency f and is sinusoidal. The surface of the loop is parallel to the ground. If the height of the lines and their separation is known, find a relation between the current in the line and the emf produced in the coil. Assume the coil has area S [m^2] and N turns. All materials involved have permeability of free space.
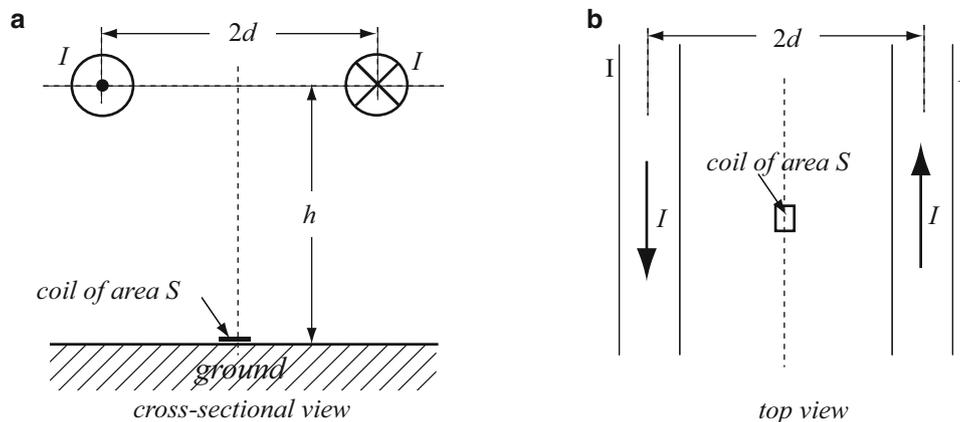

Figure 10.48

10.14 **Torque in Acyclic Motor.** A disk of radius 100 mm is placed in a uniform DC magnetic flux density of 0.2 T. A wire is connected to the center of the disk, and one is sliding on its edge, making good contact. The disk is connected to a DC source that supplies a current of 100 A through the disk (**Figure 10.49**). Calculate the torque generated by the device.
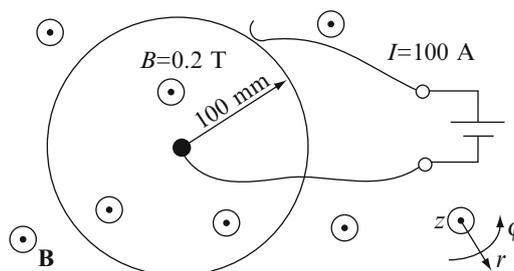

Figure 10.49

Transformers

10.15 Ideal Transformer. An ideal transformer has a primary to secondary turn ratio of 10. If the transformer's primary coil is connected to 110 V and transfers 150 W, calculate:

(a) The voltage and current in the secondary.
(b) The current in the primary.
(c) The impedance of the primary and secondary coils. What is the ratio between the impedance in the primary and secondary?

10.16 Current Transformer. An infinitely long, thin wire, carrying a current I, is located at the center of a torus as shown in **Figure 10.50**. Calculate the induced voltage (emf) in an N turn coil, uniformly distributed around the torus. Use $a = 20$ mm, $b = 30$ mm, $d = 10$ mm, $f = 100$ Hz, $I_0 = 1$ A, $\mu_r = 100$, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $N = 200$.
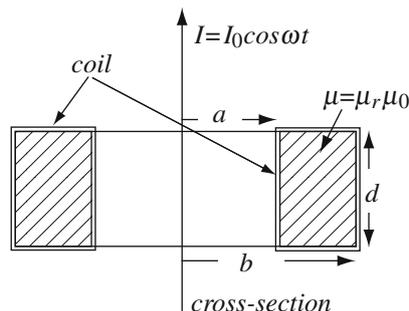

Figure 10.50

10.17 Application: High-Power Ideal Transformer. A transformer is used in a hydroelectric power plant to step up the output from a 720 MW generator. The generator operates at 60 Hz, 18 kV. The output voltage required for transmission is 750 kV:

(a) If an ideal transformer is used, calculate the turn ratio and input and output currents at full load.
(b) Ideal transformers do not exist. Suppose the transformer has 1 % losses (typical for these transformers). Calculate the required turn ratio to maintain an output voltage of 750 kV at full load. Assume all power loss occurs in the iron core of the transformer.
(c) Calculate the output current and the flux in the magnetic core for rated input power with the turn ratio in (b). Use a relative permeability of 1,000, length of the magnetic path of 4 m, and cross-sectional area of 0.2 m². Assume that the number of turns in the primary is given as $N_1 = 10$.

10.18 Application: Design of a Current Transformer. You have an AC digital voltmeter with its basic range 199.9 mV rms and are requested to monitor the input power to a house by measuring the input current. The maximum current expected is 150 A (rms) at 60 Hz. You choose to build a simple current transformer by cutting an iron ring (so that the ring can be slipped over the wire) and winding a coil on the ring. The emf in this coil is your reading. The ring is of average radius of 20 mm, a cross-sectional area of 100 mm², and relative permeability of 200:

(a) How many turns are required to produce full-scale reading for maximum input?
(b) Suppose you wish now to use the same device to measure currents up to 500 A (rms). How many turns are required for this purpose?

10.19 Application: Transformer with Multiple Windings. The magnetic circuit in **Figure 10.51** is given. Coil 1 has $N_1 = 100$ turns and carries a current $I_1 = I_0\cos(\omega t)$ and coil 2 has $N_2 = 50$ turns and carries a current $I_2 = I_0\cos(\omega t)$ where $I_0 = 1$ A, $\omega = 314$ rad/s. The currents in the two coils are such that the magnetic flux density is in the direction shown in the figure. Coil 3 has $N_3 = 1,000$ turns. Dimensions are average lengths and are given in the figure. The cross-sectional area of the core is 100 cm². Relative permeability of the core is 100:

(a) Calculate V_1 and V_2 (the voltages across the terminals of coils No. 1 and No. 2).
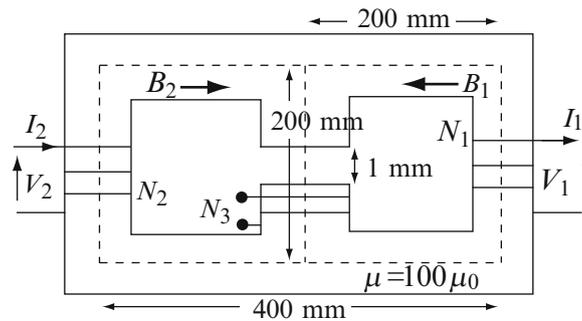(b) Calculate the induced voltage in coil No. 3.

Figure 10.51

10.20 Application: Loosely Coupled Transformer. Two coils, each with inductance of 10 mH, are placed next to each other to form a transformer. It was determined that the coefficient of coupling between the coils is 0.1 (i.e., 10 % of the flux produced by each coil passes through the second coil). One coil, designated as the primary, is connected to a sinusoidal current with amplitude 1 A and frequency of 100 kHz. Calculate:

(a) The emf in the primary and secondary with the secondary open circuited.
(b) The emf in the primary and secondary if a current of amplitude 0.05 A is drawn from the secondary.

10.21 Induced Current in a Loop: A Nonconventional Transformer. Determine the induced voltage (emf) $v(t)$ in the rectangular loop shown next to a very long wire carrying a time-varying current $I(t) = I_0\sin\omega t$ [A] (**Figure 10.52**).
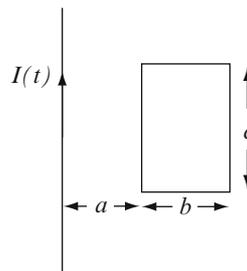

Figure 10.52

10.22 Current Transformer with Air Core. Calculate the induced voltage (emf) in an air-filled, uniformly wound toroidal coil of N turns and cross-sectional area as shown in **Figure 10.53**. The central wire is very long and carries a current $I(t) = I_0\sin\omega t$ [A].
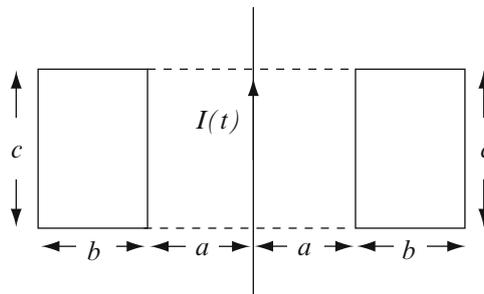

Figure 10.53

10.23 Application: Power Line Power Scavenging for Sensing. As part of a smart grid, one proposes to place sensors on power lines to sense a variety of parameters such as temperature, current, corrosion, vibrations, and others and transmit these parameters wirelessly to a central location. To do so, the sensors are placed on the power line itself by attaching them to one conductor of the power line. To power the sensors and the wireless transmitter, one can use the magnetic field produced by the line by designing an appropriate transformer. A solution is to place a toroidal coil

around the conductor as shown in **Figure 10.54**. Suppose a power line carries a sinusoidal current $I = 500$ A (RMS) at a frequency $f = 60$ Hz. The toroidal core has an inner radius $a = 30$ mm, outer radius $b = 50$ mm, and thickness $c = 20$ mm. Its relative permeability is 100:

(a) Calculate the number of turns needed on the torus to produce an RMS voltage of 6 V.
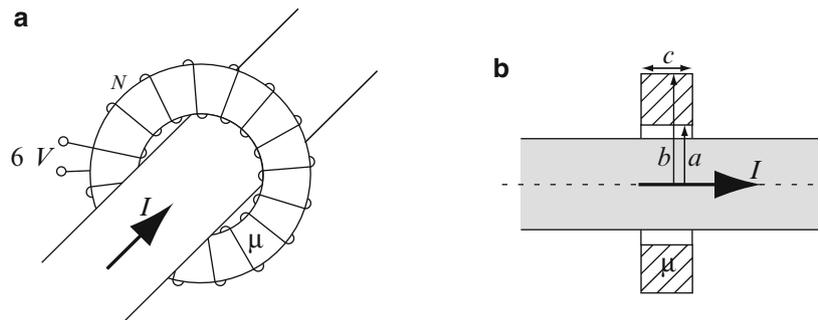(b) What is the maximum theoretical power available for use in the sensors?

Figure 10.54 Structure of power line power scavenging. (**a**) General view, (**b**) axial cross section

10.24 **Ground Fault Circuit Interrupt** (**GFCI**). An important safety device is the GFCI (also called a residual current device (RCD)). It is intended to disconnect electrical power if current flows outside of the intended circuit, usually to ground, such as in the case when a person is electrocuted. The schematic in **Figure 10.55** shows the concept. The two conductors supplying power to an electrical socket or an appliance pass through the center of a toroidal coil. Normally the currents in the two conductors are the same and the net induced emf due to the two conductors cancel each other, producing a net zero output in the current sensor. If there is a fault and current flows to ground, say a current I_g, the return wire will carry a smaller current and the current sensor produces an output proportional to the ground current I_g. If that current exceeds a set value (typically 30 mA), the induced emf causes the circuit to disconnect. These devices are common in many locations and are required by code in any location in close proximity to water (bathrooms, kitchens, etc.).

Consider the GFCI shown schematically in **Figure 10.55**. The device is designed to operate in a 50 Hz installation and trip when the output voltage is 100 μV RMS. For a toroidal coil with average diameter $a = 30$ mm and a cross-sectional diameter of $b = 10$ mm:

(a) Calculate the number of turns needed if an air-filled toroidal coil is used and the device must trip at a ground current of 6 mA.
(b) Calculate the number of turns needed to trip at a current of 6 mA if a ferromagnetic torus with relative permeability of 2,000 is used as a core for the coil.
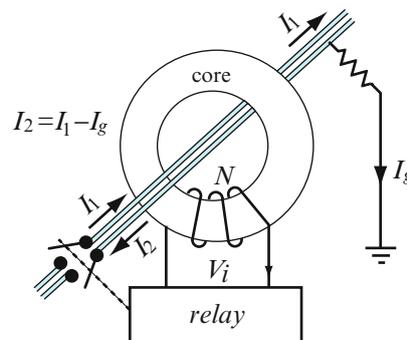

Figure 10.55 Principle of a GFCI sensor

10.25 **Application: The MHD Generator.** A small magnetohydrodynamic (MHD) generator is proposed as part of the exhaust system of a jet engine. The MHD channel is 200 mm by 200 mm in cross section and is 1 m long. A magnetic field of 1 T is generated between two surfaces as shown in **Figure 10.56**. The exhaust moves at 200 m/s and is seeded to produce a conductivity of 50 S/m in the channel. Calculate:

(a) The output voltage (emf) of the generator. Show polarity.
(b) The internal resistance of the generator.
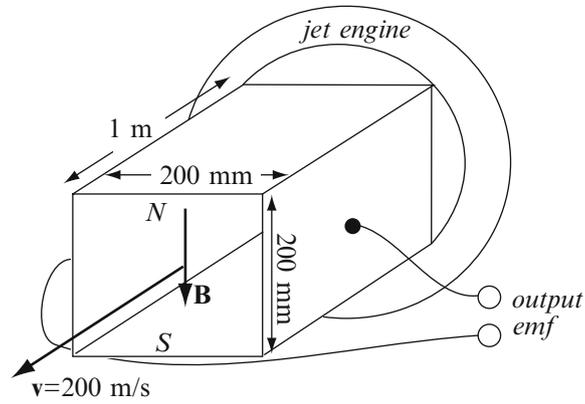(c) Maximum power output to a load under maximum power conditions.

Figure 10.56