Magnetic Materials and Properties

9

Now sing my muse, fir 'tis a weighty cause.
Explain the Magnet, why it strongly draws,
And brings rough iron to its fond embrace.
This men admire; for they have often seen
Small rings of iron, six, or eight, or ten,
Compose a subtile chain, no tye between;
But, held by this, they seem to hang in air,
One to another sticks and wantons there;
So great the Loadstone's force, so strong to bear!...

—Titus Lucretius Carus (94–50 BCE), *De Rerum Natura* (On the nature of things), T. Creech, Translation, London 1714.

9.1 Introduction

The above reference to lodestone is interesting in that it is over 2000 years old. The property of the magnetic field to attract or generate a force is universally known and is used in practical devices, probably more than most of us realize. How many applications of the permanent magnet do you recall? Did you know, for example, that many electric motors use permanent magnets or that the ignition in cars is commonly controlled by a permanent magnet and a Hall-element switch? It is therefore quite useful to identify the properties of the permanent magnet since sooner or later you will encounter it in design. Thus follows the study of magnetic properties of materials. Many materials exhibit magnetic properties, some quite surprising. The permanent magnet is only one of them. Iron, nickel, or chromium oxides on audio tapes, video tapes, or computer disks store information in the form of magnetic field variations. Solid nickel contracts when placed in a magnetic field, whereas strong magnetic fields cause atoms to tilt about their spin axes, a phenomenon that leads directly to magnetic resonance imaging (MRI). Naturally occurring materials, such as magnetite (Fe_3O_4), are found in bacteria and in brains of many animals which use this material as a biological compass for navigation in the geomagnetic field. Magnetite and hematite (Fe_2O_3) are the basis of the lodestone (a naturally occurring, magnetic stone).

Many of the magnetic properties of materials were known from antiquity, although not necessarily understood. The magnet was known at least from the times of Thales of Miletus (sixth century BCE) who spoke of the lodestone as a matter of fact (also called loadstone or leading stone, from the fact that the stone leads the mariner at sea). The name magnet was given after Magnesia, a region in Macedonia where, it is held by many, the lodestone was first found. According to a legend related by Pliny the Elder,[1] the magnet was discovered when a herdsman, apparently wearing iron-studded shoes, and using an

[1] Caius Plinius Secundus. Pliny died during the eruption of Mount Vesuvius in 79 CE, the same eruption that buried Pompeii. In his book on Natural History, published around 77 CE, he mentions both the legend of discovery of the lodestone as well as an interesting story about magnetic suspension. He goes on to describe the various types of lodestone found at the time and locations where these are found. One particularly interesting is the haematite (blood-stone, from its red color). This is essentially rust: Fe_2O_3 which is only very slightly magnetic. The common lodestone is made of magnetite (Fe_3O_4 and tends to be black). His description of the stone as the most marvelous thing there is or "lifelike" is

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iron-pointed staff, took his herd to pasture and got stuck on a hillside on Mount Ida (in Crete). Interesting, if unlikely story. What is certain is that magnets were known for ages. Some evidence points as far as 2700 BCE and definite knowledge of the lodestone traces back to 600 BCE, to the time of Thales of Miletus.[2] There are even some intriguing ancient legends of structures made of lodestone for the purpose of suspending iron statues.[3] The lodestone was also considered to have medicinal values and features in the most notable medical books of antiquity and of the Middle Ages. Magic powers were also to be found and some strange concepts of magnetic properties were prevalent. In the absence of understanding of magnetic properties, it is not surprising that magic should be associated with such a remarkable material. But perhaps the most beautiful reference is in the French name for magnet—aimant, which means loving or affectionate, or the Chinese name "loving stone" alluding to the attraction between magnets and between magnets and iron.

Systematic study of magnetism started with William Gilbert[4] who, in 1600, wrote the first serious account on magnetism. His book, *De Magnete* (originally written in Latin), is considered by many to be the beginning of electromagnetics. You may wish to read at least parts of it (in translation, if you prefer) as it gives a special flavor of ancient notions and modern experiment and, for a 400-year-old book, is surprisingly modern and readable.

Magnets are not the main topic of this chapter. However, the first true magnetic device was the permanent magnet in the form of the compass. It was known in China at least as early as 1000 CE and in Europe at least as early as 1200 CE. However, like much else about magnets, its origins are shrouded in mystery.[5] It is then not surprising that the magnet was, for a long time, the subject of inquiry. In our study, we discuss magnets because they generate forces and, therefore, have energy and work associated with them, properties we can show by simple experiment. This serves as a natural introduction to the relations between magnetic fields and force and energy and leads to the derivation of useful relations for the calculation of forces, work, and energy in the magnetic field.

In addition to magnetic properties of materials, this chapter introduces some of the most important and most useful aspects of static magnetic fields. The behavior of magnetic fields at the interface between two materials will be discussed in the form of interface conditions. The interface conditions are useful in understanding how the magnetic field behaves, say, at the contact region between a magnet and a piece of iron or in the gap between the stator and rotor of an electric motor.

Considerable space is devoted to magnetization and properties of permanent magnets both in terms of design and in terms of their general magnetic properties. Following this, we discuss inductance, energy, forces, and torque in the magnetic field as well as the concept of magnetic circuits. The latter, in particular, is a common design tool, especially in power devices.

poetic, but many attributes such as curative powers are totally absurd. According to Pliny, other common names for the stone were "live iron," "Heraclion" (a reference to either the city of Heraclea or Hercules, referring to its power over iron), and "Sideratis" (iron earth).

[2] Thales of Miletus is believed to be the earliest to mention the lodestone, although he probably learned of it during his travels in Egypt (see footnote 2 on page 96). Thales is said to have believed (according to Aristotle) that the magnet has soul since it attracted (had "sympathy" to) iron.

[3] Pliny writes that an architect by the name of Timochares by order from King Ptolemy II, Phyladelphus, began to put a vaulted roof of lodestone on the temple of Arsinoe (Ptolemy's wife) in Alexandria, so that her statue, made of iron, would be suspended in air. Another mention by Ruffinus says that in the temple of Serapis (in Memphis, Egypt), there was an iron chariot suspended by lodestones. When the stones were removed, the chariot fell and was smashed to pieces. Beda (the Venerable) says that a statue of Bellerophon's horse (Pegasus) framed of iron was placed between lodestones, with wings expanded, floating in air. These are legends, but that the idea of suspending iron by lodestone was mentioned almost 2000 years ago is remarkable in itself. Magnetic levitation seems to be not all that new, at least not in concept.

[4] William Gilbert (1540–1603) is most often mentioned as physician to Queen Elisabeth I (even though this occurred only during the last 2 years of his life), but his education included Mathematics and Physics. His main activity prior to the publication of *De Magnete* was in medicine, but he also experimented in chemistry and, certainly, in magnetics. His book became to be appreciated as one of the first to describe systematic experiment without regard to mysticism, opinions, and perpetuation of unfounded information and was the culmination of 18 years of careful (and expensive) experimental work. It is fascinating to read how he disclaims opinions of ancients and contemporaries about magnetic properties attributed to planets or constellations or that rubbing of a magnet with garlic will not cause it to lose its properties. These seemingly trivial experiments were necessary to dispute prevalent notions of the time, including that magnet will not attract iron in the presence of diamonds or that it will lose its attraction if rubbed by onions, garlic, or goat's blood. For example, in Book III, Chapter XIII, he describes an experimant in which he surrounded a lodestone by 75 "excellent" diamonds in the presence of witnesses and could observe no effect on the magnetic field or that rubbing a magnetic needle with a diamond does not affect it (one belief held was that the diamond will cause the needle to reverse its action). Gilbert died in the plague of 1603.

[5] The magnetic compass is believed to have been invented in China and perhaps later, separately, in Europe. There are also some stories about Marco Polo having brought it from China, but this cannot be true, as mention of the compass as a known instrument in Europe can be traced as far back as 1180 CE. Compasses were known to be made in Italy before 1300 CE (at Amalfi, near Naples, whereas Marco Polo returned from China in 1295). According to some accounts, the compass was known in China as early as 1100 BCE. One account gives details of a chariot, on which a figurine with outstretched arms points to the south. The figurine was pivoted and had in it lodestones to act as a compass. The time: 2637 BCE in China.

9.2 Magnetic Properties of Materials

One question that was alluded to in the previous chapter is the following: What, if any, is the effect of a magnetic field on materials? We asked a similar question about the electric field and, as a result, obtained a definition of polarization in materials and a simple definition of electric permittivity. Similar results will be obtained here for the magnetic field. The basis of discussion is to consider the atomic structure of the various materials and the effect a magnetic field can have on atoms. The effects are fascinating and lead to surprising and highly useful applications, as we will see shortly. The effects of magnetic fields on materials are varied, with some materials exhibiting little or no effect and others exhibiting very pronounced effects to the extent that we often call these materials "magnetic materials."

Consider again a permanent magnet. That it is permanently "magnetized" is the reason why it will attract another magnet or a piece of iron. Also, by experiment, we know that if we place a piece of iron on the pole of a magnet, the iron becomes "magnetized" and acquires the properties of the permanent magnet. These properties are lost after removal of the magnet. A piece of hard steel will behave similarly but will retain the magnetic property for some time. You may convince yourself of this by placing a paper clip on a magnet. When removing the clip, it loses its magnetic properties. The blade of a screwdriver, which is made of steel, becomes permanently magnetized after being placed on a magnet. On the other hand, other materials such as copper, aluminum, or plastics do not seem to be affected by the presence of the magnet.

Now for the big questions: What is magnetization? How is it imparted to materials? Why are some materials affected while others are not? We will try to answer these questions and others in the following sections, but first we must introduce some tools and, in particular, the idea of the magnetic dipole.

9.2.1 The Magnetic Dipole

The magnetic dipole is essentially a small loop, carrying a current. It is a useful concept both in itself and as a means of explaining the behavior of magnetic materials. To define the dipole and its properties, consider first a filamentary loop of any size as shown in **Figure 9.1a**. A current I flows in the loop and we wish to calculate the magnetic flux density at an arbitrary point in space $P(R,\theta,\phi)$ (in spherical coordinates). This is the field, or observation point. To calculate the field, we define an element of current of length dl' at point $P'(R',\theta',\phi')$, carrying a current I. The magnetic flux density can be calculated directly from **Eq. (8.9)** in terms of the magnetic flux density or from **Eq. (8.34)** in terms of the magnetic vector potential. The latter is easier to apply and is therefore used here. The magnetic vector potential at point P due to the loop in **Figure 9.1a** is

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\mathbf{l}'}{|\mathbf{R}-\mathbf{R}'|} \quad \left[\frac{\text{Wb}}{\text{m}}\right] \tag{9.1}$$

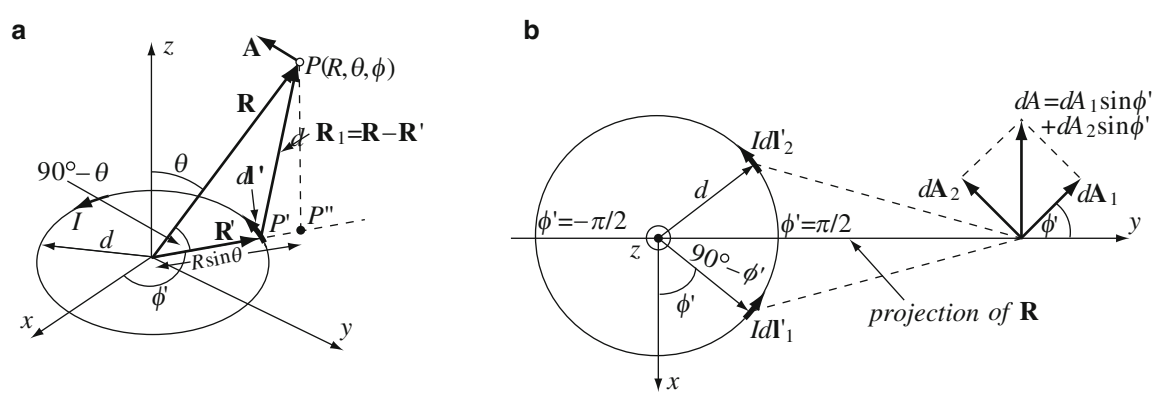


Figure 9.1 The magnetic dipole. (**a**) A loop carrying a current I and the dimensions necessary to calculate the magnetic vector potential at $P(R,\theta,\phi)$. (**b**) The magnetic vector potential at $P(R,\theta,\phi)$ is in the ϕ direction

where the magnitude of $\mathbf{R}'$ is equal to the radius of the loop and the vector $\mathbf{R}_1 = \mathbf{R} - \mathbf{R}'$ is the vector connecting the source element $d\mathbf{l}'$ to the field point P. The element of length along the loop (in spherical coordinates) is [see **Eq. (1.77)**]:

$$d\mathbf{l}' = \hat{\boldsymbol{\phi}}\, d\, d\phi' \tag{9.2}$$

and, as expected, the integration is independent of R' or θ'. This current element produces a magnetic vector potential in the ϕ direction anywhere in space. To understand the contribution of the current element to the magnetic vector potential, consider **Figure 9.1b**, which is a top view of **Figure 9.1a**. First, we note that the magnetic vector potential at point P is everywhere parallel to the current in the loop. Second, the contribution at this point is due to two symmetric elements of current $d\mathbf{l}_1'$ and $d\mathbf{l}_2'$. The tangential components add up and the normal components (those in the direction radial to the loop) cancel everywhere in space. Thus, we can write for the magnetic vector potential at point P:

$$\mathbf{A} = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin\phi'\, d\phi'}{\left|\mathbf{R} - \mathbf{R}'\right|} \quad \left[\frac{\mathrm{Wb}}{\mathrm{m}}\right] \tag{9.3}$$

The distance $\mathbf{R} - \mathbf{R}'$ from the current element to the point P is

$$\left|\mathbf{R} - \mathbf{R}'\right| = \sqrt{\left(\mathbf{R} - \mathbf{R}'\right) \cdot \left(\mathbf{R} - \mathbf{R}'\right)} = \sqrt{\mathbf{R} \cdot \mathbf{R} + \mathbf{R}' \cdot \mathbf{R}' - 2\mathbf{R} \cdot \mathbf{R}'} \tag{9.4}$$

Since $\mathbf{R} \cdot \mathbf{R} = R^2$, $\mathbf{R}' \cdot \mathbf{R}' = d^2$, and $\mathbf{R} \cdot \mathbf{R}' = Rd\cos(90 - \theta)\, \sin\phi'$, we have at P:

$$\mathbf{A} = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin\phi'\, d\phi'}{\left(R^2 + d^2 - 2Rd\cos(90 - \theta)\, \sin\phi'\right)^{1/2}} \quad \left[\frac{\mathrm{Wb}}{\mathrm{m}}\right] \tag{9.5}$$

This is an exact result. To simplify the integration, we calculate the magnetic vector potential at large distances from the loop $R \gg d$. In this case, d^2 in the above relation is negligible (but not the term involving $2Rd$, which can be much larger than d^2). With this, and using $\cos(90 - \theta) = \sin\theta$, we can write

$$\frac{1}{\left(R^2 + d^2 - 2Rd\cos(90 - \theta)\ \ \sin\phi'\right)^{1/2}} \approx \frac{1}{\left(R^2 - 2Rd\sin\theta\ \ \sin\phi'\right)^{1/2}} = \frac{\left(1 - 2(d/R)\sin\theta\ \ \sin\phi'\right)^{-1/2}}{R} \tag{9.6}$$

Using the binomial expansion approximation $(1 + x)^p \approx 1 + px$ (where all but the first two terms of the binomial expansion are neglected), with $x = -2d\sin\theta\sin\phi'/R$ and $p = -1/2$, **Eq. (9.6)** becomes

$$\frac{1}{\left(R^2 + d^2 - 2Rd\cos(90 - \theta)\, \sin\phi'\right)^{1/2}} \approx \frac{1 + (d/R)\, \sin\theta\sin\phi'}{R} \tag{9.7}$$

Substituting this approximation in **Eq. (9.5)**, we get an expression for the magnetic vector potential:

$$\begin{aligned}\mathbf{A} &\approx \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d}{2\pi} \int_{\phi'=-\pi/2}^{\phi'=\pi/2} \frac{1 + (d/R)\, \sin\theta\sin\phi'}{R}\, \sin\phi'\, d\phi' = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d}{2\pi R} \int_{\phi'=-\pi/2}^{\phi'=\pi/2} \left(\sin\phi' + \frac{d}{R}\sin\theta\sin^2\phi'\right) d\phi' \\ &= \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d}{2\pi R} \left(\left[-\cos\phi'\right]_{\phi'=-\pi/2}^{\phi'=\pi/2} + \left[\frac{\phi' d\sin\theta}{2R} - \frac{d\sin\theta\sin 2\phi'}{4R}\right]_{\phi'=-\pi/2}^{\phi'=\pi/2} \right) = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I d^2 \sin\theta}{4R^2} \quad \left[\frac{\mathrm{Wb}}{\mathrm{m}}\right]\end{aligned} \tag{9.8}$$

The magnetic flux density is now obtained using the curl of **A**:

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\mu_0 I d^2 \sin^2\theta}{4R^2}\right)\right] - \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{\partial}{\partial R}\left(\frac{\mu_0 I d^2 \sin\theta}{4R}\right)\right] \quad [\mathrm{T}] \tag{9.9}$$

In this case, **A** only has a ϕ component. Therefore, the curl of **A** has an R and a θ component. Evaluating the derivatives in **Eq. (9.9)** gives

$$\boxed{\mathbf{B} \approx \frac{\mu_0 I d^2}{4R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad [\mathrm{T}]} \tag{9.10}$$

This was a rather tedious calculation and, in the end, we only got an approximation to the solution. However, this result is rather accurate under the following two equivalent conditions:

(1) The field point is at very large distances from the loop (R large).
(2) The loop radius, d, is very small.

Both conditions are satisfied by the requirement that $R \gg d$.

This result is important because it allows the calculation of the field due to electrons orbiting around the nucleus of the atom. These will be viewed as small loops or magnetic dipoles, which produce a magnetic field outside the atom. This, in turn, will provide a model that explains the magnetic field of permanent magnets.

An interesting comparison can be made at this point with the electric dipole discussed in **Section 3.4.1.3** [see **Eq. (3.38)**]. The electric field intensity of the electric dipole and the magnetic flux density of the magnetic dipole are [after multiplying numerator and denominator in **Eq. (9.10)** by π]

$$\mathbf{E} \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{9.11}$$

$$\mathbf{B} \approx \frac{I\pi d^2}{4\pi(1/\mu_0)R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad [\mathrm{T}] \tag{9.12}$$

where, in the electric dipole, we defined p [C · m] as the magnitude of the electric dipole moment, whereas the dipole moment, which is a vector, was defined as $\mathbf{p} = q\mathbf{d}$, where q [C] is the charge of the dipole and **d** is the vector connecting the two charges as shown in **Figure 9.2a**. Using the two expressions and the analogy between the two, we can now define a magnetic dipole moment. Its magnitude equals $I\pi d^2$ and the direction, following the convention in **Figure 9.2b**, is in the z direction. In this case, $\mathbf{m} = \hat{\mathbf{z}}\,I\pi d^2$ [A · m^2], but, in general, for an arbitrary orientation of the loop, the direction of the magnetic dipole moment is in the direction perpendicular to the loop. Thus, we write, in general,

$$\mathbf{m} = \hat{\mathbf{n}}\,I\pi d^2 = \hat{\mathbf{n}}\,IS \quad [\mathrm{A}\cdot\mathrm{m}^2] \tag{9.13}$$

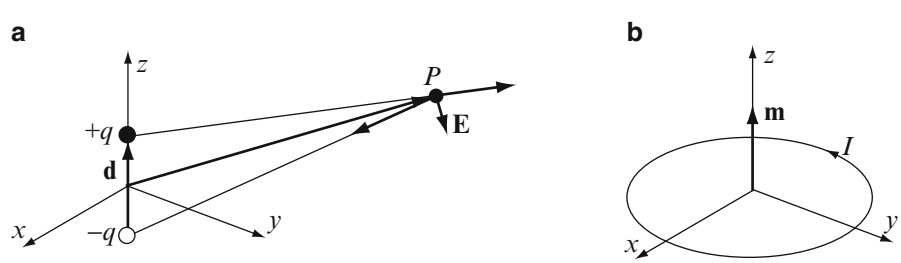


Figure 9.2 (**a**) The electric dipole. (**b**) The magnetic dipole

where S is the area of the planar loop. To simplify notation, we will assume that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ for the purposes of this section. The magnetic flux density in **Eq. (9.10)** can be written now in terms of the magnetic dipole moment **m** as

$$\boxed{\mathbf{B} \approx \frac{\mu_0 m}{4\pi R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad [\mathrm{T}]} \tag{9.14}$$

Note that the direction of the magnetic flux density depends on the location in space whereas the direction of **m** is always perpendicular to the loop. Using the magnetic dipole moment **m** and the definition of the cross product, we can write the following from **Eq. (9.8)**:

$$\mathbf{A} \approx \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I d^2 \sin\theta}{4R^2} = \frac{\mu_0\left(\hat{\boldsymbol{\phi}}\,m\sin\theta\right)}{4\pi R^2} \quad \left[\frac{\mathrm{Wb}}{\mathrm{m}}\right] \tag{9.15}$$

Since $\hat{\boldsymbol{\phi}}\,m\sin\theta = \mathbf{m} \times \hat{\mathbf{R}}$, the magnetic vector potential of a small loop at a large distance R can be written in terms of the magnetic dipole moment as

$$\boxed{\mathbf{A} \approx \frac{\mu_0 \mathbf{m} \times \hat{\mathbf{R}}}{4\pi R^2} \quad \left[\frac{\mathrm{Wb}}{\mathrm{m}}\right]} \tag{9.16}$$

Example 9.1 Dipole Moment and Equivalent Loop Current of an Atom The magnetic moment of a hydrogen atom is given as $m = 9 \times 10^{-24}$ A · m^2. Assuming that this moment is produced by a small, circular, atomic level current with radius equal to the orbit radius of the electron ($d = 5 \times 10^{-11}$ m), calculate the equivalent current produced by the atom.

Solution: The magnetic dipole moment is calculated from **Eq. (9.13)** assuming the orbiting electron to be a dipole. This assumption is fully justified for the dimensions of the atom.

From **Eq. (9.13)**,

$$|\mathbf{m}| = I\pi d^2 \quad [\mathrm{A} \cdot \mathrm{m}^2]$$

The equivalent current of the electron is

$$I = \frac{m}{\pi d^2} = \frac{9 \times 10^{-24}}{\pi \times \left(5 \times 10^{-11}\right)^2} = 1.15 \times 10^{-3} \quad [\mathrm{A}]$$

This current of 1.15 mA is a very large current for a single electron.

Example 9.2 Dipole Moment of a Square Loop A square loop with dimension $a = 1$ m on the side carries a current $I = 0.1$ A.

(a) Calculate the magnetic flux density of the loop at a height h above the center of the loop. What is the magnetic flux density at large distances?
(b) Calculate the magnetic flux density at the same location using the dipole approximation; that is, calculate the field of a dipole of identical current and area.
(c) Compare the two results at $h = 1$ m, 10 m, 100 m, and 1,000 m.

Solution: The exact flux density is calculated using the Biot–Savart law. Then, we compare this field with the field of a dipole. Consider **Figure 9.3a**. The magnetic flux density at point P equals four times the perpendicular component of the magnetic flux density of a single segment of length 1 m. The horizontal components of the four segments cancel because each two opposite segments produce fields in opposite horizontal directions. **Figure 9.3b** shows how **B** is calculated.

(a) From **Figures 9.3a** and **93.b** and assuming the current segment is in the z direction in a cylindrical system of coordinates, the magnetic flux density **B** is [from **Eq. (8.9)**]

$$\mathbf{B} = \frac{\mu I}{4\pi}\int_{z'=-a/2}^{z'=a/2} \frac{\hat{\mathbf{z}}\,dz' \times \left(\hat{\mathbf{r}}L - \hat{\mathbf{z}}z'\right)}{|\mathbf{r}|^3} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\int_{z'=-a/2}^{z'=a/2} \frac{L dz'}{\left(L^2+z'^2\right)^{3/2}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\frac{z'}{L\sqrt{L^2+z'^2}}\Bigg|_{z'=-a/2}^{z'=a/2} = \hat{\boldsymbol{\phi}}\frac{\mu_0 a I}{4\pi L\sqrt{a^2/4+L^2}} \quad [\mathrm{T}]$$

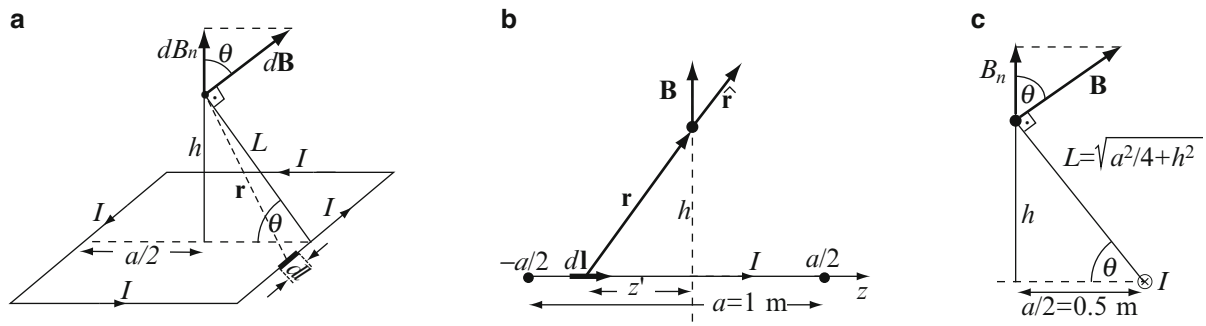


Figure 9.3 Magnetic flux density at height h [m] above the center of a square loop. (**a**) Configuration and the magnetic flux density due to an element of current. (**b**) Calculation of the flux density of one segment of the loop. (**c**) Components of **B** at height h

This flux density points in the ϕ direction (with respect to the current segment) or at an angle θ with the normal as shown in **Figure 9.3c**. Since we are only interested in the normal component, we write

$$B_n = B\cos\theta = \frac{\mu_0 a I \cos\theta}{4\pi L\sqrt{a^2/4+L^2}} \quad [\mathrm{T}]$$

From **Figures 9.3a** and **9.3c**

$$L = \sqrt{a^2/4+h^2}, \quad \cos\theta = \frac{a}{2L} = \frac{a}{2\sqrt{a^2/4+h^2}}$$

The total magnetic flux density due to all four segments is four times B_n. Substituting L and $\cos\theta$,

$$B_t = 4B_n = \frac{\mu_0 a^2 I}{2\pi\left(a^2/4+h^2\right)^{3/2}} \quad [\mathrm{T}]$$

At very large distances, $h^2 \gg a^2/4$ and we get

$$B_t \approx \frac{\mu_0 a^2 I}{2\pi h^3} \quad [\mathrm{T}].$$

(b) If we assume the loop is a dipole, the magnitude of its magnetic moment is $|\mathbf{m}| = IS = Ia^2$, where $S = a^2$ is the area of the loop. The direction of the dipole is in the direction of the axis (z direction), but here we only use the magnitude. From **Eq. (9.14)**, and setting $\theta = 0$ for the vertical component, we get

$$B_d \approx \frac{\mu_0 m}{2\pi R^3} = \frac{\mu_0 a^2 I}{2\pi h^3} \quad [\mathrm{T}].$$

(c) The magnetic flux density at 1 m, 10 m, 100 m, and 1,000 m calculated with the exact formula (B_t) and the dipole approximation (B_d) are shown in the table below:

	$h = 1$ m	$h = 10$ m	$h = 100$ m	$h = 1{,}000$ m
B_t	1.431×10^{-8} [T]	1.99×10^{-11} [T]	1.999×10^{-14} [T]	2×10^{-17} [T]
B_d	2×10^{-8} [T]	2×10^{-11} [T]	2×10^{-14} [T]	2×10^{-17} [T]

Note that at 10 m, the difference between the two results is less than 1 %. Therefore, at large distances, the dipole approximation is a good approximation to the exact flux density of the square loop.

Exercise 9.1 A rectangular loop with sides $a = 10$ mm and $b = 20$ mm is placed on the x–y plane, in free space, centered at the origin and carries a current $I = 0.5$ A.

(a) Calculate the magnetic field intensity of the loop at a general point P under the conditions used for the dipole; that is, $R \gg b$, where R is the distance from the center of the loop to point P.
(b) Calculate the magnetic flux density at any point on the x–y plane for $R \gg b$.

Answer
(a) $\mathbf{H} \approx \dfrac{7.958 \times 10^{-6}}{R^3}\left(\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta\right) \quad \left[\dfrac{\text{A}}{\text{m}}\right]$. **(b)** $\mathbf{B} \approx \hat{\boldsymbol{\theta}}\, \dfrac{1 \times 10^{-11}}{R^3} \quad [\text{T}]$.

9.2.2 Magnetization: A Model of Magnetic Properties of Materials

The simplest atomic model of a material is that of a positive nucleus and a negative cloud of electrons, each orbiting around the nucleus and each spinning about its own axis. A similar model was used in **Chapter 4** to discuss polarization of dielectrics, but, there, we neglected motion of electrons. Here, we look at a slightly modified model since we are only interested in the magnetic properties of the material. Electrons orbiting around the atom generate an equivalent current. For modeling purposes, we will assume that the electron orbits in planar circular paths and, therefore, is equivalent to a magnetic dipole as discussed in the previous section. The magnetic dipole moment of an electron is $\mathbf{m} = \hat{\mathbf{n}}\, I\pi d^2 = \hat{\mathbf{n}}\, IS$ where $\hat{\mathbf{n}}$ is the direction normal to the plane in which the electron orbits. This model of the atom is shown in **Figure 9.4a**. The equivalent model is that of a loop of radius d, carrying a current I shown in **Figure 9.4b**. This model is appropriate both for the orbiting electrons and for the electron spin in that both produce an internal magnetic field. An identical model can be obtained by replacing the loop with a very small magnet, as shown in **Figure 9.4c**, if the magnetic flux density produced by this elementary magnet is the same as that of the loop. Here, we prefer the current loop model, but the magnet model is also useful. In general, electrons in various atoms rotate in different planes and they spin around randomly oriented axes. Therefore, we can view the magnetic dipole moments of atoms in a volume as being randomly oriented, unless there is some mechanism which will force them to align in certain directions. Although we cannot actually calculate the magnetic dipole moment of any particular atom (we can certainly estimate it, as was shown in **Example 9.1**) or find its direction in space, it is reasonable to say that if there are N atoms in a volume, the total magnetic moment of the volume is the vector sum of all individual dipoles in the volume. However, since the volume dipole density can vary from point to point (depending on the local distribution of dipole moments), it is more useful to calculate this quantity in an infinitesimal volume and thus obtain a point measure of the magnetic moment. This quantity is called ***magnetization*** and is defined as

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{N} \mathbf{m}_i \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{9.17}$$

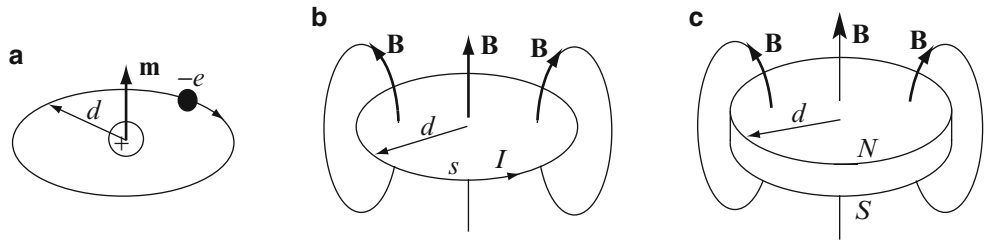


Figure 9.4 Magnetic dipole model of the atom. (**a**) Orbiting electron and its magnetic dipole moment. (**b**) The magnetic flux density due to the magnetic dipole. (**c**) Equivalent permanent magnet model

Note that since the magnetic dipole moment has units of ampere · meter^2 [A · m^2], the magnetization, which is a volume density of dipole moments, has units of ampere/meter [A/m]. The magnetization of a material can be (and often is) zero. Random orientation of dipole moments produces a zero sum vector and, therefore, zero magnetization.

What are the conditions under which nonzero magnetization can be obtained? Before we answer this question, consider what happens if all (or most) dipole moments are aligned in a given direction in space. The aligned dipoles have nonzero magnetization and therefore produce a net magnetic flux density **B**, as shown schematically in **Figure 9.5b**. In effect, we can view this dipole distribution as if it were a magnet of some known strength and shape, composed of a large number of elementary magnets as shown in **Figure 9.5c**. Thus, we conclude that a permanent magnet is any material in which the magnetic dipoles are aligned in a preferred direction and stay that way. Based on this simple model, a "stronger" permanent magnet is one in which more of the dipoles are aligned in a preferred direction. Why a material should align in a preferred direction and why it should stay that way will be discussed in the following section. The important question now is: given the magnetization of a material everywhere in its volume, what is the magnetic flux density produced by the magnetization? The reason why we pursue this path is that it provides a relation between magnetization and current density and, therefore, a very simple, physical explanation to the concept of magnetism. In this sense, the permanent magnet becomes merely an equivalent current density distribution and there is nothing mysterious about a current distribution. The model also allows evaluation of fields due to permanent magnets. To do so, we must associate the effect of the field with material volume. This is easily done from the definition of magnetization. Given a magnetization **M**, everywhere in a volume **[Eq. (9.17)]**, the magnetic dipole moment due to an element of volume dv' can be written as

$$d\mathbf{m} = \mathbf{M}dv' \tag{9.18}$$

where v' indicates only that part of the volume which is magnetized. Substituting this in **Eq. (9.16)**, the contribution to **A** due to $d\mathbf{m}$ is

$$d\mathbf{A} \approx \frac{\mu_0\, d\mathbf{m} \times \hat{\mathbf{R}}}{4\pi R^2} = \frac{\mu_0 \mathbf{M} \times \hat{\mathbf{R}}}{4\pi R^2} dv' = \frac{\mu_0 \mathbf{M} \times \mathbf{R}}{4\pi R^3} dv' \tag{9.19}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$ was used. Thus, if a volume v' is magnetized, the magnetic vector potential at a point in space is calculated using **Figure 9.6** as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{M} \times \mathbf{R}}{R^3} dv' \quad \left[\frac{\text{Wb}}{\text{m}}\right] \tag{9.20}$$

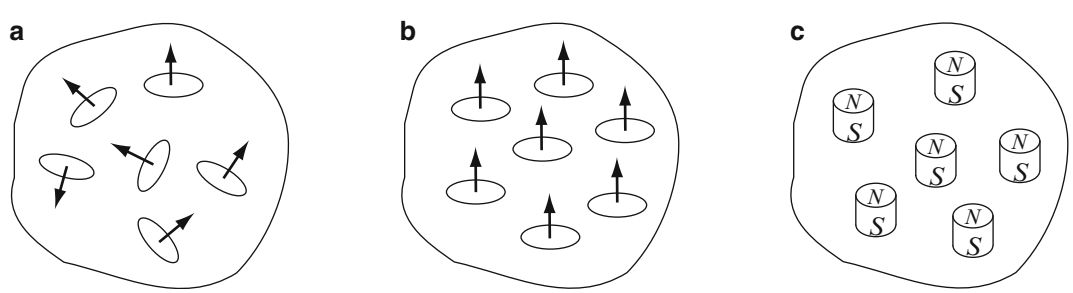


Figure 9.5 Model for magnetization in materials. (**a**) Randomly oriented dipoles. (**b**) Orientation of dipoles produces magnetization. (**c**) Permanent magnet model of magnetization

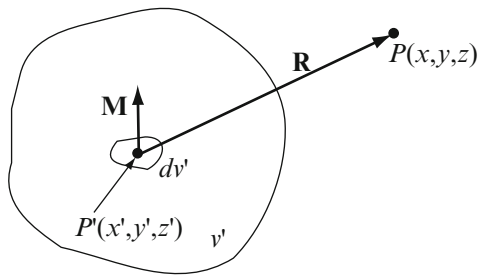


Figure 9.6 Calculation of the magnetic vector potential at point $P(x,y,z)$ due to magnetization **M**

This result can be viewed as the magnetic vector potential due to a volume of infinitesimal magnets, each with magnetization $d\mathbf{m} = \mathbf{M}dv'$. After the magnetic vector potential is known for a given volume magnetization, the magnetic flux density can be calculated directly from $\mathbf{B} = \nabla \times \mathbf{A}$.

The relation in **Eq. (9.20)** is not very useful for practical calculations, other than perhaps the fact that it indicates the direction of the magnetic vector potential and gives a general idea as to the magnitude of **A**. The magnetization is not often known, with the exception of permanent magnets. For the purpose of the calculation of the fields in permanent magnets, we seek a model in terms of equivalent currents instead of magnetization based on the model shown in **Figure 9.5b**. There are two ways to do so. One is essentially mathematical and separates the integral in **Eq. (9.20)** into a surface and a volume integral. We will not pursue it here. The method we adopt views the magnetized volume as being made of very small permanent magnets, each equivalent to a dipole. Thus, consider a very large magnetized volume as shown in **Figure 9.7a**. All magnetic dipoles point upward, and each is viewed as a small loop, with magnetic dipole moment **m**. Now, consider one stack of dipoles as shown in **Figure 9.7b**. This stack is assumed to be very long and with uniform magnetization (all dipole moments aligned in the same direction and uniformly distributed), with n dipoles per unit length of the stack. Each dipole is of radius d and carries a current I. By analogy, this stack can be viewed as a solenoid of radius d, with n turns per unit length and carrying a current I. From the result in **Example 8.10**, the magnetic flux density anywhere inside the solenoid is

$$\mathbf{B}_m = \hat{\mathbf{z}}\mu_0 nI = \hat{\mathbf{z}}\mu_0 \frac{N}{L} I \quad [\mathrm{T}] \tag{9.21}$$

where $n = N/L$ is the number of turns per unit length, written here for a finite length L, in which there are a total of N loops or dipoles. The index m indicates that this field is produced by the magnetization as opposed to applied current densities. Multiplying both numerator and denominator by the area of the dipole (the cross-sectional area of the equivalent solenoid) gives

$$\mathbf{B}_m = \hat{\mathbf{z}}\mu_0 \frac{N(\pi d^2)I}{(\pi d^2)L} \quad [\mathrm{T}] \tag{9.22}$$

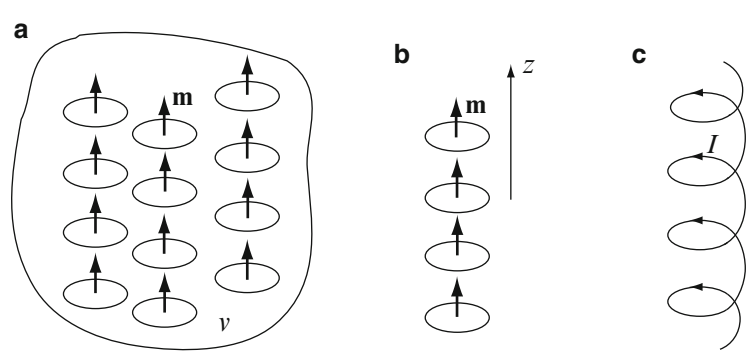


Figure 9.7 (**a**) Magnetized material viewed as distributed dipoles. (**b**) A vertical stack of dipoles within the material. (**c**) Solenoid equivalent of (**b**)

The quantity $\pi d^2 I$ is the magnitude of the dipole moment as given in **Eq. (9.13)**, whereas $\hat{\mathbf{z}}\pi d^2 I$ is the magnetic dipole moment **m**. The denominator is the volume of the solenoid of length L. Since the dipoles are uniformly distributed in the stack, we can write

$$\mathbf{B}_m = \mu_0 \frac{N\mathbf{m}}{v'} = \mu_0 \mathbf{M} \quad [\mathrm{T}] \tag{9.23}$$

This is a somewhat surprising result. Because $\mathbf{B} = \mu\mathbf{H}$, it follows that the magnetic field intensity $\mathbf{H}$ inside a permanent magnet is equal to $\mathbf{M}$. Perhaps this should have come as no surprise since we already mentioned that the units of magnetization are [A/m] and these are also the units of the magnetic field intensity. For uniform magnetization, we get

$$\mathbf{H}_m = \mathbf{M} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{9.24}$$

Applying Ampere's law ($\nabla \times \mathbf{H} = \mathbf{J}$) to **Eq. (9.24)** gives

$$\nabla \times \mathbf{H}_m = \nabla \times \mathbf{M} = \mathbf{J}_m \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right] \tag{9.25}$$

where $\mathbf{J}_m$ is the equivalent current density that causes the magnetization. Since this current occurs in the volume of the magnetized material, we call it a ***magnetization volume current density***. This definition of the magnetization volume current density is completely general, but, in our very simple example, the magnetization $\mathbf{M}$ is constant (we assumed it to be so). Thus, it immediately follows that for the conditions given in **Figure 9.7**, the magnetization volume current density $\mathbf{J}_m$ must be zero. If, however, $\mathbf{M}$ were a function of space, $\nabla \times \mathbf{M}$ would be nonzero and there would be a nonzero volume magnetization current density.

If $\mathbf{M}$ is constant, there is no magnetization volume current density, but there is an equivalent current on the surface of the magnetized material. To see how this current is produced, consider a long, cylindrical, uniformly magnetized material such as the rod magnet in **Figure 9.8a**. **Figure 9.8b** shows a slice through the magnet, with the dipoles associated with this slice. Since the dipoles are uniformly distributed, all internal dipoles cancel each other (currents in opposite directions). The only net current is that of the outer dipoles, shown in **Figure 9.8c**. Thus, the permanent magnet reduces to a solenoid with a thin sheet of current on the outer layer. This current is due to dipoles, and if we use again the idea of a stack of these currents as in **Figure 9.7b**, we can write the current per unit length of the solenoid as a surface current density $\mathbf{J}_{ms}$. This current density has units [A/m]; that is, it indicates the amount of current per unit length of the solenoid and is proportional to magnetization. Also, because of the equivalency between the solenoid and magnet, we must assume that both produce identical fields. The magnetic field of the solenoid is in the axial direction (in the direction of $\mathbf{M}$). For $\mathbf{M}$ to produce a field in this direction, the equivalent current density $\mathbf{J}_{ms}$ must be perpendicular to the magnetization as shown in **Figure 9.9a**. This condition can be satisfied if $\mathbf{J}_{ms}$ is in the direction $\mathbf{M} \times \hat{\mathbf{n}}$ on the curved surface of the cylinder. To calculate the magnitude of the current density $\mathbf{J}_{ms}$, we assume a solenoid formed by a current sheet with current density $\mathbf{J}_{ms}$, as shown in **Figure 9.9a**. The magnetic flux density inside the solenoid (it is zero outside) is calculated using Ampere's law as shown in **Figure 9.9b**. This gives

$$BL = \mu_0 J_{ms} L \quad \rightarrow \quad B = \mu_0 J_{ms} \quad [\mathrm{T}] \tag{9.26}$$

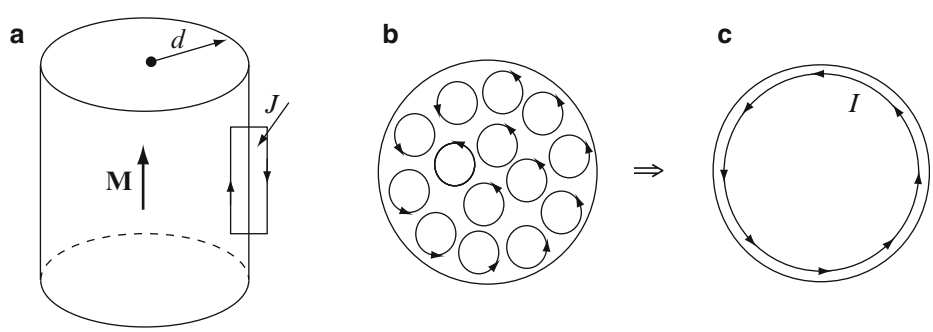


Figure 9.8 (**a**) A permanent magnet. (**b**) A slice through the magnet showing the dipoles. For constant magnetization, all internal currents cancel. (**c**) The net effect is a surface current density

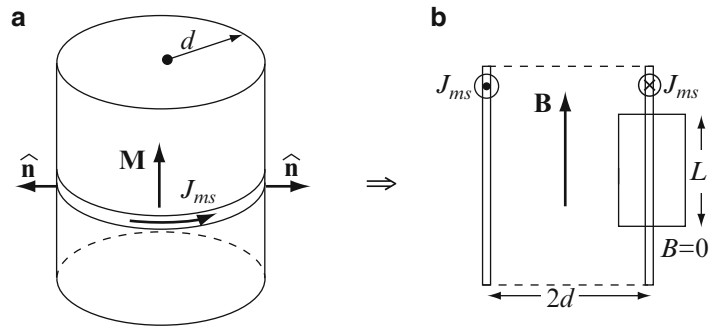


Figure 9.9 Magnetization and equivalent surface current density. (**a**) Direction of surface current density for uniform magnetization. (**b**) The solenoid equivalent model for the magnet in (**a**)

Comparing this with **Eq. (9.23)**, we see that the magnitude of $\mathbf{J}_{ms}$ must be equal to the magnitude of **M**. Thus, we can write

$$\mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{9.27}$$

The current density in **Eq. (9.27)** is called a ***magnetization surface current density***. Thus, the magnetization **M** in any material is equivalent to two current densities: a magnetization volume current density which appears whenever the magnetization within the material is nonuniform, and a magnetization surface current density which exists in uniformly or nonuniformly magnetized materials. The two current densities are

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right] \quad \text{and} \quad \mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{9.28}$$

In other words, a magnetized material can always be modeled by a surface and a volume current density. However, we must insist at this point on the fact that these currents do not exist in the sense that we cannot measure them. They are real currents on the atomic level (see **Figure 9.5**), but, normally, we will view them as fictitious currents that allow calculation in lieu of magnetization. Only the magnetization is considered to be real and measurable. To understand this and the use of equivalent currents for calculation of magnetic fields due to permanent magnets, consider the following two examples.

Example 9.3 Application: Magnetic Flux Density of a Long, Uniformly Magnetized Magnet A very long cylindrical magnet has constant magnetization everywhere inside the magnet equal to $M = 5{,}000$ A/m directed along the axis. The diameter of the magnet is 40 mm.

(a) Calculate the magnetic flux density due to this magnet everywhere in space.
(b) Design an equivalent solenoid (made of wire turns rather than a current sheet) that produces an identical magnetic flux density everywhere.

Solution: Since **M** is constant, there are no equivalent volume current densities ($\nabla \times \mathbf{M} = 0$), but there is a surface current density according to **Eq. (9.27)**. Assuming the magnetization to be in the z direction (arbitrarily), the current density is in the ϕ direction.

(a) The magnetization is perpendicular to the cross-sectional surface of the cylinder (**M** is directed along the axis of the cylinder).

$$\left|\mathbf{J}_{ms}\right| = \left|\mathbf{M} \times \hat{\mathbf{n}}\right| = M = 5000 \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The equivalent current-sheet solenoid is shown in **Figure 9.10b**. Using Ampere's law, the total current per unit length is 5,000 A/m and, from **Eq. (9.26)** law, we get inside the solenoid,

$$B = \mu_0 J_{ms} 1 = \mu_0 I = 4\pi \times 10^{-7} \times 5000 = 0.00628 \quad [\mathrm{T}]$$

The magnetic flux density outside the solenoid is zero.

Note: Permeability of most magnets is close to μ_0, although this information is not strictly required here. The equivalent solenoid is always taken to be in free space. Also, the diameter of the magnet is immaterial as long as the magnetization is constant.

(b) The equivalent solenoid in **Figure 9.10b** in the form of a current sheet is not practical. In most cases, a solenoid will be made of a single layer of wires, wound tightly. However, the total current per meter length must remain the same (5,000 A/m) or, more appropriately, 5,000 A • t/m (ampere • turns/meter). Any choice of number of turns is correct in principle. For example, we may choose 1,000 turns per meter. Each turn is 1 mm in diameter and must then carry 5 A. This is a little high for a 1 mm diameter wire but not unrealistic (current density in the 1 mm copper wire is 6.4×10^6 A/m^2 which is acceptable for copper wires). The equivalent solenoid is shown in **Figure 9.10c**.

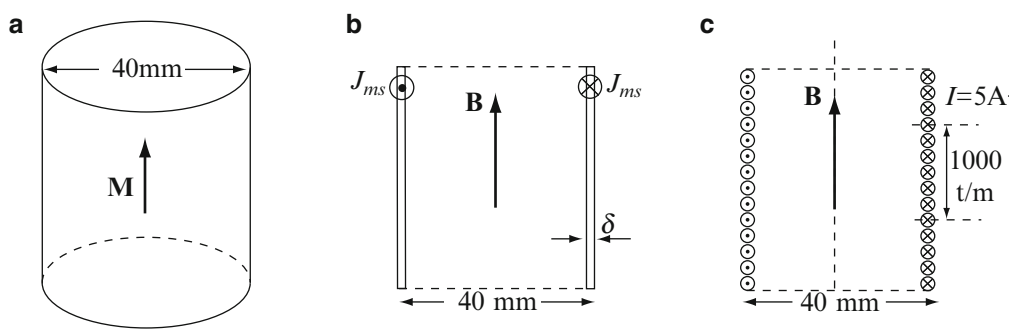


Figure 9.10 (**a**) Permanent magnet with constant magnetization **M**. (**b**) Equivalent surface current model. (**c**) Solenoid implementation of the model in (**b**)

Example 9.4 Application: Magnetic Flux Density and Equivalent Current Densities: Design of Electromagnet A very long (infinite) cylindrical magnet of radius $a = 20$ mm has magnetization $\mathbf{M}(r) = \hat{\mathbf{z}}\,500,000\,(a - r)$ A/m where r is the distance from the center of the magnet (**Figure 9.11a**).

(a) Calculate the magnetic flux density inside and outside the magnet.

(b) Design an equivalent solenoid, or combination of solenoids that will produce the same magnetic flux density everywhere.

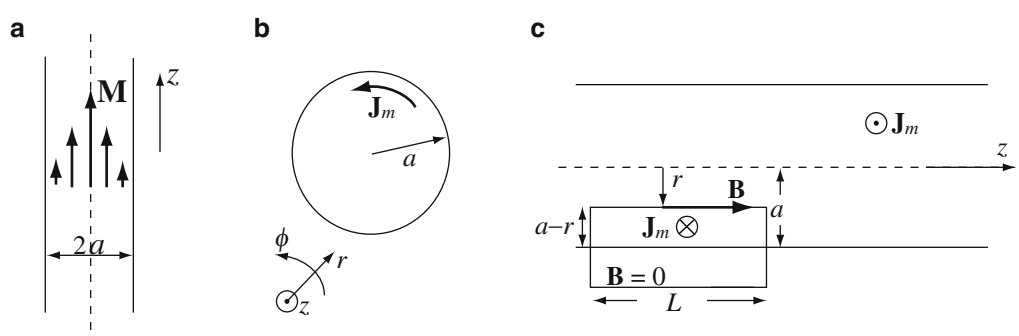


Figure 9.11 (**a**) Long magnet with magnetization **M**(r). (**b**) Equivalent volume magnetization current density. (**c**) Use of Ampere's law to calculate the magnetic flux density inside the model

Solution: We first calculate the equivalent current densities: from the current densities, the magnetic flux density everywhere in space may then be calculated, and the solenoids designed to produce the same magnetic flux density. In this particular case, the magnetization is maximum at the center of the magnet and goes down linearly until it becomes zero at $r = a$.

(a) From the curl in cylindrical coordinates, the volume magnetization current density is

$$\mathbf{J}_m = \nabla \times \mathbf{M} = -\hat{\boldsymbol{\phi}} \frac{\partial}{\partial r}(500,000(a-r)) = \hat{\boldsymbol{\phi}}\, 500,000 \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right]$$

On the surface (at $r = a$),

$$\mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{M} \times \hat{\mathbf{r}} = \hat{\mathbf{z}}\, 500,000(a-r) \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\, 500,000(a-a) = 0 \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

Because the magnetization is zero at the surface, the equivalent surface current density is also zero. This particular magnet is therefore described in terms of a volume magnetization current density alone, as shown in **Figure 9.11b**. Note that although this current density is uniform, it produces a linear magnetization, which is highest on the axis of the magnet and zero at its outer surface, as required.

Since the current density is circular, at any distance r from the center it forms a solenoid (see **Figure 9.12** for a schematic view of the magnetization current density in the volume of the magnet) and the resulting magnetic flux density outside the magnet is zero. To calculate the magnetic flux density inside the magnet, we use Ampere's law on the contour shown in **Figure 9.11c**. The current enclosed by the contour is due to the area between r and a; hence the total current is $I = J_m L(a - r)$. The magnetic flux density outside the magnet is zero; hence the magnetic flux density at a distance r from the center of the magnet is

$$BL = \mu_0(a-r)LJ_m \quad \rightarrow \quad B = \mu_0(a-r)500,000 = 0.6283(a-r) \quad [\mathrm{T}]$$

This flux density is in the positive z direction, is zero at $r = a = 0.02$ m, is equal to 0.01257 [T] at $r = 0$, and is zero outside the solenoid. The flux density increases linearly from the surface of the magnet to its center.

(b) An equivalent solenoid can be designed based on the final result. However, we cannot hope to construct a single, simple solenoid as in the previous example because the magnetic flux density inside the solenoid is r dependent; that is, because a single-layer solenoid produces a constant magnetic flux density in its interior, it cannot be used to model a varying magnetic flux density. One solution is to use a number of solenoids, one inside the other so that as we progress from the surface inward, the flux density increases. In this approach, thin layers of conductors are used (each layer is a solenoid) as shown schematically in **Figure 9.12**.

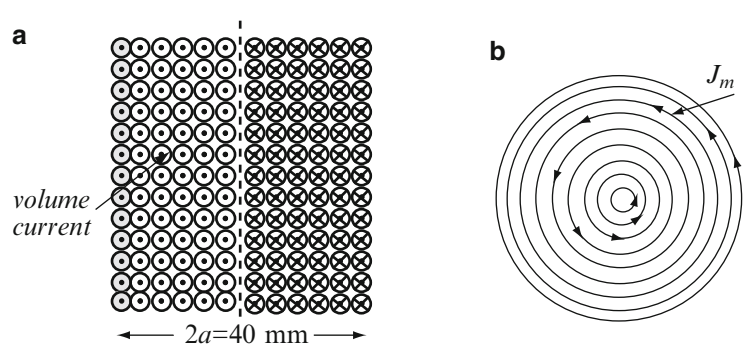


Figure 9.12 Design of a coil that models the magnet in **Figure 9.11a**. (**a**) The structure of the windings and direction of current. (**b**) Top view of **Figure 9.12a**

Now, we must choose a wire diameter that will carry enough current so that the current density is equal to J_m [A/m]. We argue as follows: if we were to wind a tight coil with a cross section of 1 m^2, the total current flowing through half the cross section would be 500,000 A. If the wires are 1 mm in diameter, there will be 10^6 wires in the cross section, and, therefore, each wire will carry 0.5 A. This is a relatively low current for this wire diameter. If this is our choice, there will

be 20 layers in the coil, each layer with 1,000 turns/meter. We could choose a 0.5 mm diameter wire. This will quadruple the number of wires and reduce the current in each wire by a factor of 4, to 0.125 A. Now there will be 40 layers representing the volume of the magnet, with 2,000 turns/meter in each layer. Either choice is valid, although the second is closer to a continuous current distribution. It should be noted that this solution is only an approximation, as the magnetic flux density does not vary continuously but rather in steps; each additional solenoid increases the flux density by 1/40th of the maximum flux density (for the 0.5 mm wire). In practical terms, the solution in **Figure 9.12** means that a multilayer coil has maximum field at the center which decreases toward the outer layer.

Note: We neglected here some important engineering considerations for the sake of simplicity. One is the question of power dissipation in the solenoids. For such a massive coil, this may be significant and, in practice, may require cooling. The second is economical: each of the designs mentioned above has implications on cost of wires, weight, power requirements, power supply design, and so on.

The main point in these examples is that magnets are not different than solenoids and can always be replaced by equivalent current densities. This has distinct advantages in that no special treatment is needed when we incorporate magnets in design. It also shows that permanent magnets are electromagnetic devices: they rely on their operation on currents, albeit currents we cannot measure. From an application point of view, a magnetic field can be produced either with a permanent magnet or an electromagnet (coil). The choice often depends on requirements, cost, and convenience, as well as mechanical properties.

Now, we must return to the question posed at the beginning of this section: why should magnetic dipoles align themselves in a specific direction in space? As with other effects, there must be a force exerted on them to align. This force is supplied by an external magnetic field. Although we have not yet discussed the question of forces in the magnetic field in any detail, we know from experiment that a magnet will align itself with another magnet if allowed to do so. Thus, following the model of small magnets, each produced by a dipole, dipoles will align with an externally produced magnetic field as shown in **Figure 9.13**.

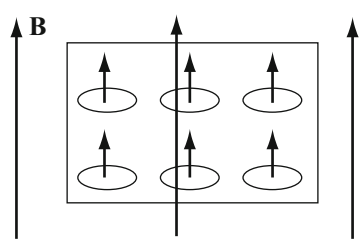


Figure 9.13 Alignment of dipole moments in an externally applied magnetic field

From this figure, it is clear that the external magnetic flux density $\mathbf{B}_e$ and the magnetic flux density produced by the magnetized material $\mathbf{B}_m$ are in the same direction and therefore add

$$\mathbf{B}_t = \mathbf{B}_e + \mathbf{B}_m \quad [\mathrm{T}] \tag{9.29}$$

where $\mathbf{B}_m$ is given in **Eq. (9.23)**. With this, the total magnetic flux density can be written as

$$\boxed{\mathbf{B}_t = \mu_0\mathbf{H}_e + \mu_0\mathbf{M} = \mu_0(\mathbf{H}_e + \mathbf{M}) \quad [\mathrm{T}]} \tag{9.30}$$

This form resembles the relations for the electric polarization in **Chapter 4 [Eq. (4.59)]**. Now, we argue that since magnetization is generated by the external field $\mathbf{H}_e$, then $\mathbf{M}$ must be proportional to $\mathbf{H}_e$. The proportionality factor is called ***magnetic susceptibility*** and indicates how susceptible the material is to magnetization. Magnetic susceptibility is a fundamental property of materials and is denoted by χ_m. The relation between the external magnetic field intensity and magnetization is written as

$$\mathbf{M} = \chi_m\mathbf{H}_e \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{9.31}$$

Substituting this back into **Eq. (9.30)** gives

$$\mathbf{B}_t = \mu_0(\mathbf{H}_e + \chi_m \mathbf{H}_e) = \mu_0(1 + \chi_m)\mathbf{H}_e \quad [\mathrm{T}] \tag{9.32}$$

Now, since the relation $\mathbf{B}_t = \mu \mathbf{H}_e$ holds in general, we can write

$$\mu = \mu_0(1 + \chi_m) \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right] \tag{9.33}$$

The term μ is called the ***magnetic permeability*** of the material. Because it depends on susceptibility, it differs from material to material. In free space, $\chi_m = 0$ and, therefore, $\mu = \mu_0$. In other materials, permeability can be larger or smaller than in free space, depending on susceptibility of the material. The range of magnetic susceptibilities is between –1 and infinity, with an equivalent permeability from zero to infinity. Most materials range in permeability between μ_0 and about 10^6 μ_0. The quantity $1 + \chi_m$ is called the ***relative permeability*** of the material since it gives the ratio between permeability μ of the given material and permeability of free space:

$$\boxed{\mu = \mu_0 \mu_r \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]} \tag{9.34}$$

This form is particularly convenient because μ_0 is a small value ($\mu_0 = 4\pi \times 10^{-7}$ H/m). Thus, the relative permeability simply indicates how much larger or smaller permeability is in relation to permeability of free space and, therefore, is a dimensionless quantity. The unit of permeability is the henry/meter [H/m], or using **Eq. (9.32)**, the tesla/(ampere/meter) or (weber/meter^2)/(ampere/meter) Wb/A · m. The quantity Wb/A is called a ***henry*** [H].[6] These units were also discussed in **Chapter 8**.

The most remarkable aspect of this definition is that it avoids the need to deal with magnetization and susceptibility which, in general, are difficult to use and, instead, the effects are lumped in a single, experimentally measurable quantity, μ. This is similar to the way we treated polarization in the electric field where we defined the permittivity to take into account polarization. With the exception of permanent magnets, most of our work will be in terms of the external magnetic field intensity **H**, permeability μ, and the magnetic flux density **B**.

9.2.3 Behavior of Magnetic Materials

In the previous section, we discussed magnetization and, more importantly, its relation with the magnetic field intensity **H**, the magnetic flux density **B**, and the equivalent current density **J** (both surface and volume current densities). Now, we wish to discuss the basic magnetic properties of materials from a macroscopic point of view; that is, we look at those parameters which are important for engineering design.

One question has been left open in the previous discussion: what is the source of the magnetization and, perhaps more significant, why should one material be capable of magnetization (for example, iron) while another material seems to be unaffected by external magnetic fields? The answer to this lies in the magnetic susceptibility or, alternatively, in the magnetic permeability of materials. Some materials are highly susceptible to magnetization while others are not. Based on our previous approach, we will view magnetic susceptibility as a fundamental, experimentally obtained property. However, it is much easier to discuss magnetic properties in terms of the relative permeability of materials. Thus, materials with high

[6] After Joseph Henry (1797–1878), Professor of Mathematics and Natural Philosophy at Albany and later at Princeton. His major contributions to electricity and magnetism are the development of the electromagnet and the study of induced currents. He was the first to show the potential uses of the magnetic force. When he developed his powerful electromagnets (the Smithsonian Institution still has one of his early electromagnets, capable of lifting up to 1.5 tons), it became evident that magnetism was of practical importance and led directly to the development of electric motors and other electromagnetic power devices. These investigations were made in parallel to those of Michael Faraday. Henry became the first secretary of the Smithsonian Institution in 1846 and served in this capacity until his death. Others of his interests led to the demonstration of telegraphy, a primitive form of which he used to communicate between his office and home. He was an enthusiastic meteorologist and is credited with influencing the establishment of the U.S. Weather Bureau.

relative permeability exhibit high magnetization when placed in a magnetic field, whereas materials with low permeability do not. In practical terms, we define three basic types of materials. These are diamagnetic, paramagnetic, and ferromagnetic materials. All materials fall under one of these groups, except vacuum, which is considered to be nonmagnetic.

9.2.3.1 Diamagnetic and Paramagnetic Materials

Diamagnetic materials are materials with relative permeabilities smaller than 1 ($\mu_r < 1$). This class includes important materials such as mercury, gold, silver, copper, lead, silicon, and water. The relative permeability of most diamagnetic materials varies between 0.9999 and 0.99999 (susceptibility varies between -10^{-5} and -10^{-4}), and for most applications, they may be assumed to be nonmagnetic (i.e., $\mu_r \sim 1$). The relative permeabilities of some diamagnetic materials are listed in **Table 9.1**.

One notable exception to the small negative susceptibility of most diamagnetic materials are superconducting materials. These materials are purely diamagnetic ($\chi_m = -1$), and, from **Eq. (9.33)**, the magnetic permeability is equal to zero ($\mu_r = 0$). Inside these materials, the magnetic flux density is always zero ($\mathbf{B} = 0$) for any external magnetic field intensity.

An interesting aspect of diamagnetism is the fact that the magnetic flux density inside the diamagnetic material is lower than the external magnetic field. If we were to place a piece of diamagnetic material over a permanent magnet, the magnet will repel the diamagnetic material, as shown in **Figure 9.14**. This can be easily explained from the equivalent magnets representation in **Figure 9.14b**. Since the magnet and the equivalent magnetic field (due to magnetization of the diamagnetic material) oppose each other, the diamagnetic material is always repelled from the magnetic field in the same way that two magnets repel each other when their magnetic flux densities oppose each other. However, this force is extremely small for all diamagnetic materials, except superconductors, in which it is very large. This repulsion is the reason why a permanent magnet floats above a superconducting material.

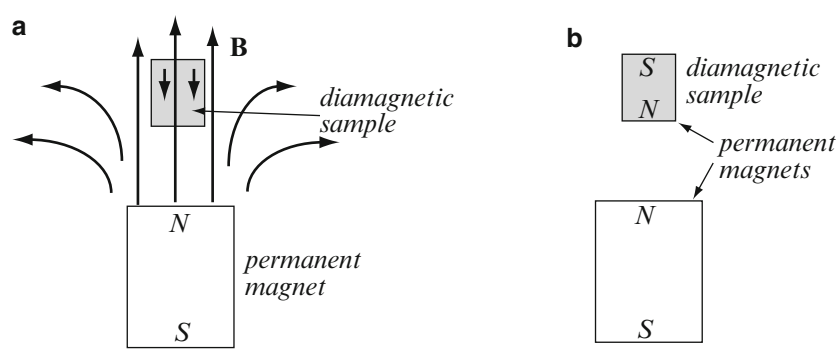


Figure 9.14 (**a**) Repulsion of a diamagnetic material by a magnetic field due to opposite internal magnetization. (**b**) A permanent magnet model of (**a**)

The behavior of diamagnetic materials can be explained based on the atomic model of materials. One model of diamagnetic materials has the magnetic moments due to orbiting electrons and electron spins cancel each other under normal conditions. Under the influence of an external field, the field due to orbiting electrons is slightly smaller than that of spins, causing a net magnetic field which opposes the external field and, thus, the lower permeability of diamagnetic materials. In fact, all materials possess this property; an external magnetic field induces a magnetization in the material which opposes the external field. However, in other types of materials, this property is obscured by larger magnetic effects, as we shall see shortly.

Paramagnetic materials are materials in which the relative permeability is slightly larger than 1. In these materials, the orbital and spin moments do not cancel and atoms have a net magnetic moment in the absence of an external magnetic field. However, since moments are oriented randomly, the net external field observed is either zero or very close to zero. Unlike diamagnetic materials, in which the net magnetic moments of atoms are induced by an external field, the magnetic moments in paramagnetic materials always exist. In the presence of an external magnetic field, these moments tend to align with the external field and increase the total field.

Relative permeability of paramagnetic materials ranges between about 1.0000001 and 1.001 (magnetic susceptibility varies between 10^{-7} and 10^{-3} and is always positive). Some common materials such as aluminum, palladium, tungsten, and air are paramagnetic. An interesting and surprising consequence is that a piece of aluminum is attracted to a magnet. However, you will need a very strong magnet to "feel" this attraction since the relative permeability of aluminum is only 1.000021. A few other materials which exhibit paramagnetism are listed in **Table 9.1**.

Table 9.1 Relative permeabilities for some diamagnetic and paramagnetic materials

Diamagnetic materials		Paramagnetic materials	
Material	Relative permeability	Material	Relative permeability
Silver	0.999974	Air	1.00000036
Water	0.9999991	Aluminum	1.000021
Copper	0.999991	Palladium	1.0008
Mercury	0.999968	Platinum	1.00029
Lead	0.999983	Tungsten	1.000068
Gold	0.999998	Magnesium	1.00000693
Graphite (Carbon)	0.999956	Manganese	1.000125
Hydrogen	0.999999998	Oxygen	1.0000019

Table 9.2 Relative permeabilities for some ferromagnetic materials

Material	μ_r	Material	μ_r
Cobalt	250	Permalloy (78.5 % Ni)	100,000
Nickel	600	Fe_3O_4 (Magnetite)	100
Iron	6,000	Ferrites	5,000
Supermalloy (5 % Mo, 79 % Ni)	10^7	Mumetal (75 % Ni, 5 % Cu, 2 % Cr)	100,000
Steel (0.9%C)	100	Permendur	5,000
Silicon iron (4 % Si)	7,000		

9.2.3.2 Ferromagnetic Materials

By far the most useful magnetic materials are the so-called ***ferromagnetic materials***. These derive their name from iron (ferrum) as the most common of the ferromagnetic materials. The relative permeability of ferromagnetic materials is much larger than 1 and can be in the thousands or higher. Some typical ferromagnetic materials are iron, cobalt, and nickel. Other materials and their relative permeabilities are given in **Table 9.2**.

Ferromagnetic materials tend to magnetize in the direction of the magnetic field and some of them retain this magnetization after the external magnetic field has been removed. When they do so, and the magnetization is permanently retained, the material becomes a permanent magnet. An additional important property of ferromagnetic materials is the dependence of magnetization on the level of the external field. Thus, magnetization in ferromagnetic materials is a nonlinear process.

The large magnetization of ferromagnetic materials cannot be explained in terms of the simple model of electron spins and orbits since these exist in all materials and most materials are not ferromagnetic. The model for ferromagnetic materials is a modified model which has been proven experimentally to be correct. Unlike other materials, the individual electron spins, instead of being randomly oriented, are oriented together in domains; that is, a number of spins in a small volume of the material are aligned in the same direction in the absence of an external magnetic field, held together by atomic coupling forces. This small volume is called a ***magnetic domain***.[7] A domain is of the order of 0.001 mm to 1.0 mm in width and has a volume between 10^{-9} mm^3 and 1 mm^3. The number of electron spins in a domain varies with the domain size, but the average is of the order of 10^{16}. Thus, although domains are small, they are very large on the atomic level. Their existence has been measured and even photographed. The domain structure of ferromagnetic materials is shown schematically in **Figure 9.15a**. Individual domains may be aligned in any direction in space. The transition region between domains is

[7] Pierre Weiss (1865–1940), French physicist and perhaps the first to initiate the study of atomic level fields, theorized in 1907 that coupling forces in domains must exist, and these forces hold the spins together. The existence of domains as well as loss of magnetization beyond a certain temperature (Curie point) were also part of Weiss' work. The domain walls are sometimes called Weiss walls or Bloch walls (after Felix Bloch, Swiss physicist and recipient of the 1952 Nobel Prize in Physics).

called a ***domain wall***. When an external magnetic field is applied, the magnetic domains that are aligned in the direction of the field remain aligned, but the domain walls move, causing them to grow at the expense of neighboring domains (**Figure 9.15b**). As the external field increases further, most of the domains will align in the direction of the magnetic field (**Figure 9.15c**). The external magnetic field intensity required to do so is relatively small whereas the resulting internal magnetic flux density is large.

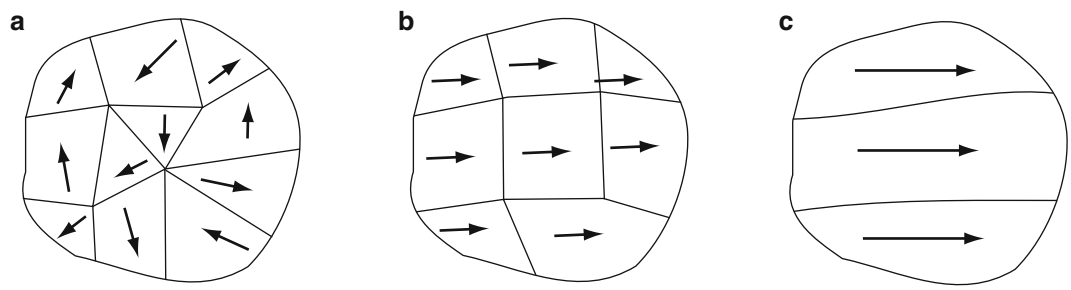


Figure 9.15 Domain model of magnetization in ferromagnetic materials. (**a**) Random orientation of domains. (**b**) External field causes aligned domains to grow. (**c**) Aligned domains occupy all or most of the volume

This aspect of magnetization is best explained using the so-called magnetization curve. The magnetization curve is an experimental plot of the internal magnetic flux density against an applied external magnetic field intensity. A typical curve for iron is shown in **Figure 9.16a**. To understand this behavior, consider first a sample of non-magnetized material, which has no internal magnetization (domains are randomly oriented) and therefore is represented by point O in **Figure 9.16a**. As the external field increases, the magnetic flux density in the sample increases along the curve shown. This is because domains start aligning with the external field. As the external magnetic field increases further, there are fewer domains left to align and, therefore, the slope of the curve decreases. At some point, all domains will be aligned with the external magnetic filed. Any increase in the external magnetic field will only increase the internal field by the increase in the external magnetic field: there is no increase in magnetization. This point is shown as point M_1 and is called a ***saturation point***. The whole curve segment between O and M_1 is called an ***initial magnetization curve***.

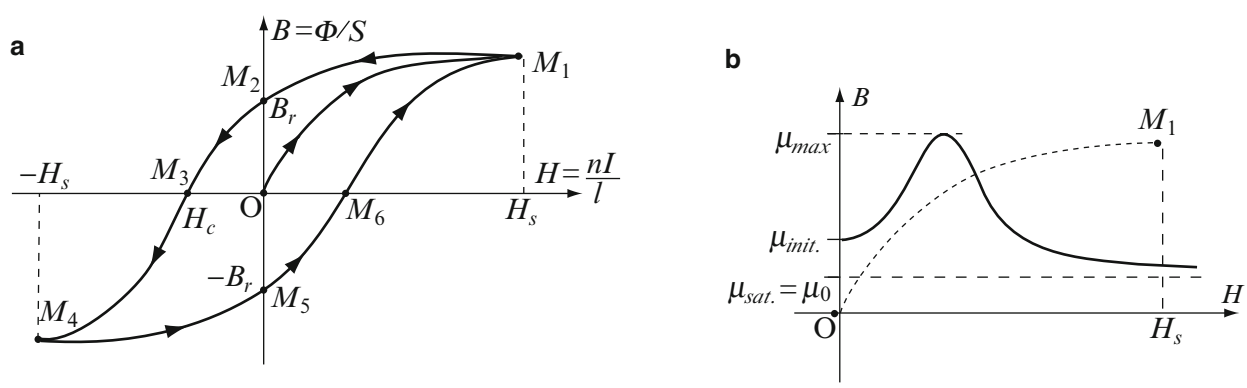


Figure 9.16 (**a**) Magnetization (**B**(**H**)) curve for iron. (**b**) Relative permeability along the initial magnetization curve

If we now decrease the external field, the magnetization curve does not follow the initial magnetization curve. Rather, the internal magnetic flux density decreases slower due to a lag in the realignment of the domains. In other words, domains tend to "retain" their magnetization. This lag in alignment gives the magnetization curve its common name: ***hysteresis curve*** ("to lag" in Greek). Reducing the field further, we eventually reach point M_2. At this point, the external magnetic field intensity is zero, but the internal magnetic flux density is B_r. B_r is called the ***remnant*** or ***residual magnetic flux density*** since it is "left

over" after the external magnetic field intensity has been removed. Thus, at this point, we have a piece of material which has an internal magnetic field in the absence of an external magnetic field.

To reduce the magnetic flux density to zero, we must apply a magnetic field intensity in the opposite direction. This process is called demagnetization. If the negative external magnetic field intensity is increased to H_c, the magnetic flux density is reduced to zero. This value of **H** is called the ***coercive field intensity*** (sometimes also called coercive force). Further increasing the demagnetizing field causes a negative flux density; that is, after domains have relaxed to a random pattern at H_c, they now align in the opposite direction in the sample. Eventually, all domains will be aligned in the opposite direction and a saturation point has again been reached at point M_4. Note that the magnitudes of **H** and **B** at M_1 and M_4 are the same since the domain structure is the same except for directions.

Reducing the demagnetizing field further reduces the flux density to $-B_r$, which, again, is a remnant flux density and is equal in magnitude to B_r, at point M_2. To reduce this remnant field to zero, we must increase the magnetic field intensity. At point M_6, the magnetic flux density has been reduced to zero (this is the coercive field intensity). A further increase in the magnetizing field intensity will eventually bring us to point M_1, which is the saturation point we reached before. Now, if we continue the cycle of increasing and decreasing the magnetic field intensity, we follow the outer loop in **Figure 9.16a**. The only way we can get on the initial magnetization curve again is by completely demagnetizing the material first or starting with a material that has not been subjected to magnetization. This indicates that the material has a "history": the state of the material depends on what happened to it previously.

From the magnetization curve (also called $\mathbf{B} - \mathbf{H}$ or $\mathbf{B}(\mathbf{H})$ curve), we note the following important properties:

(1) The important values, saturation point H_s, remnant flux density B_r and coercive field intensity H_c, are material dependent. Each material saturates at different field levels and has different H_c and B_r values. These values are shown in **Table 9.3** for a sample of important magnetic materials.

(2) Since the magnetization curve gives the relation between **B** and **H** ($\mathbf{B} = \mu\mathbf{H}$), the slope to the curve at any point gives the permeability of the ferromagnetic material at that field level. This slope depends on the location on the curve. Thus, permeability of ferromagnetic materials is a nonlinear function of the magnetic field intensity **H**. We usually write this as

$$\mathbf{B} = \mu(H)\mathbf{H} \quad [\mathrm{T}] \tag{9.35}$$

Also to be noted is that permeability is negative anywhere between point M_2 and M_3 because for a negative field intensity H, the magnetic flux density is positive. Similarly, between points M_5 and M_6, the magnetic field intensity is positive and the magnetic flux density is negative. Permeability is positive in the first and third quadrants of the curve. Permeability is high along the initial magnetization curve except when approaching saturation. At saturation, the relative permeability approaches 1. The permeability curve corresponding to the initial magnetization curve is shown in **Figure 9.16b**. The most important consequence of this behavior of materials is that an increase in the magnetic field intensity (which normally increases the magnetic flux density in ferromagnetic materials) may cause saturation and, therefore, a much lower increase in the magnetic flux density at higher field levels. For this reason, the magnetic flux density in magnetic devices is normally not allowed to reach saturation.

(3) The region between M_2 and M_3 is characterized by an internal magnetic flux density without an externally applied magnetic field intensity or in the presence of a demagnetizing external field. This region is where permanent magnets operate. An ideal permanent magnet will operate at point M_2, whereas most magnets operate somewhere between points M_2 and M_3.

(4) At any point on the curve, the behavior depends on the magnetization prior to reaching that point (history). The only exception is at very low magnetization levels. At the beginning of the initial magnetization curve, the external field is very low and the magnetization is reversible; that is, the magnetization does not follow a hysteresis loop but rather reverses itself along the initial magnetization curve.

(5) The surface area of the magnetization curve represents energy: the energy needed to move the domain walls and align domains. This energy is lost in the process. Thus, when magnetization is done with a periodic field (AC field), each cycle of the field traces the loop once, and during each cycle, there is a loss of energy per unit volume of the material. Although we do not calculate this energy, it is qualitatively obvious that the narrower the loop, the lower the energy loss.

(6) Narrow, tall loops represent materials with low loss per cycle, high remnant magnetic flux density, and low coercive field intensity. The low coercive field, in particular, indicates that the material can be easily magnetized and demagnetized. This is useful in applications where it is necessary to magnetize and demagnetize the material repeatedly and quickly such as in electric motors, transformers, relays, and the like. These types of materials are called

soft magnetic materials and are used in machines and transformers and other alternating current devices. Some typical materials are listed in **Table 9.3**. Note that the coercive field intensity of soft magnetic materials is low: typically less than 50 A/m.

Table 9.3 Properties of soft magnetic materials

Material	Relative permeability (max.) μ_r	Coercive field intensity H_c [A/m]	Remnant flux density B_r [T]	Saturation flux density B_s [T]
Iron (0.2 % impure)	9,000	80	0.77	2.15
Pure iron (0.05 % impure)	2×10^5	4	–	2.15
Silicon iron (3%Si)	55,000	8	0.95	2.0
Permalloy	10^6	4	0.6	1.08
Supermalloy (5 % Mo, 79 % Ni)	10^7	0.16	0.5	0.79
Permendur	5,000	160	1.4	2.45
Nickel	600	–	–	–

(7) Broad, low loops represent materials with lower remnant magnetic flux density but higher coercive field intensity. The main advantage of these materials is that they are "hard" to demagnetize once magnetized. In other words, they require a larger reverse field to reduce the remnant magnetization to zero. They are, therefore, the main candidates for production of permanent magnets. For this purpose, we actually would prefer both high remnant flux density and high coercive field intensity. Some useful hard magnetic materials and their properties are shown in **Table 9.4**. Note the very large coercive fields in some hard magnetic materials in contrast to those of soft magnetic materials.

Table 9.4 Properties of hard magnetic materials

Material	μ_r	H_c [kA/m]	B_r [T]	Curie temp. [°C]
Alnico (Aluminum–Nickel–Cobalt)	3–5	60	1.25	850
Ferrite (Barium–Iron)	1.1	240	0.38	600
Sm–Co (Sammarium–Cobalt)	1.05	700	0.9	700
Ne–Fe–B (Neodymium–Iron–Boron)	1.05	800	1.15	300

Magnetic properties of all materials are temperature dependent to a certain degree. In particular, each material has a temperature beyond which it loses its magnetization, called the ***Curie temperature.*** At this temperature, ferromagnetic materials change their magnetic behavior to that of paramagnetic materials. As a consequence, it is not possible to magnetize a ferromagnetic material above the Curie temperature. For iron, this value is approximately 770 ºC. The same happens to permanent magnets. If the permanent magnet is heated above its Curie temperature, it loses its magnetization and becomes just a piece of (paramagnetic) material. It can normally be magnetized again, but the magnetization process depends on the material and may involve more than inserting the material in a magnetic field.

9.2.3.3 Other Magnetic Materials

Although all materials are either diamagnetic, paramagnetic, or ferromagnetic, there are subclasses of materials which are sufficiently different and important in engineering to be considered as separate from the three general groups. Two of these, with important engineering applications, are the ***ferrimagnetic*** and ***superparamagnetic*** materials.

Ferrimagnetic materials, better known as ferrites, are based on ferromagnetic particles, formed and compressed together with bonding agents to form solids. The magnetic moment in ferrites is weaker than in the base ferromagnetic material from which ferrites are made (mostly iron compounds mixed with other oxides). Ferrites have low conductivity and, being made of small particles (powders), can be made into almost any shape required. Most ferrites are used at high frequencies where their low conductivity is an advantage (lower losses). For example, the antenna core of most portable radios is made of a ferrite, as are high-frequency transformers in switching power supplies.

Superparamagnetic materials are made of small ferromagnetic particles, suspended in nonconducting substrates such as resins and plastics. The most common applications are magnetic recording tapes, disks, and other media which contain iron or chromium particles, suspended in a solidified solution. Each particle is a separate magnetic entity and can be magnetized or demagnetized separately or together with its neighbors. The state of the particles is a measure of the magnetization applied and, therefore, can be used to record and retrieve data. The uses of this type of materials are vast: from magnetic tapes (audio, video, magnetic strips on credit cards) to recording media (disks). With various media, very high-density recording is possible.

Another interesting type of material in this group is the magnetic fluid. These are similar to particles used on tapes but are suspended in a liquid. The liquid may be water, oil, or a solvent (such as kerosene). Magnetic fluids are useful in the detection of magnetic fields, testing for cracks in magnetic materials, and a score of other applications in which the fluid state is more convenient than the solid. One such application, used experimentally, is the treatment of localized tumors. These are injected with a magnetic fluid and then the whole body or the specific organ is exposed to a microwave field (similar to a microwave oven but of much lower intensity). The magnetic material absorbs much more energy than the surrounding tissue, locally heating the tumor and destroying it with minimum damage to surrounding tissue. Magnetic fluids are also used in specialty loudspeakers where they also conduct heat by flowing through heat exchangers. An important application of magnetic fluids is in magnetic seals in conjunction with magnetic bearings.

9.3 Magnetic Interface Conditions

In this section we discuss the interface conditions for the static magnetic field following steps similar to those in **Section 4.6** for the electric field. Consider two materials with permeabilities μ_1 and μ_2, respectively, as in **Figure 9.17**. Because we are interested in the interface itself, we must also assume that a current may exist at the interface between the two materials. This assumption does not imply that this current exists in general; only that it may exist under certain conditions, as implied from Ampere's law ($\nabla \times \mathbf{H} = \mathbf{J}$). Thus, we assume that, in general, a current density exists on the interface between the two materials and calculate the general conditions at the interface. From these conditions, we will also be able to tell when a current density exists and calculate it.

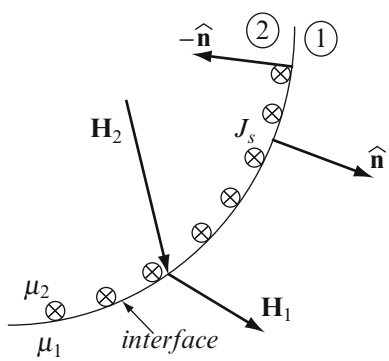


Figure 9.17 Interface between two general magnetic materials, with an assumed surface current density.

9.3.1 Interface Conditions for the Tangential and Normal Components of the Magnetic Field Intensity H

From the constitutive relation $\mathbf{B} = \mu\mathbf{H}$, we know that the magnetic field undergoes changes at the interface between two materials. We assume arbitrarily directed magnetic field intensities and magnetic flux densities in both materials and calculate the required relations between the fields at the interface. For convenience, this is done by separating the fields into the components tangential and normal to the interface and applying the magnetic field postulates to each component separately. To define the interface conditions for the magnetic field intensity **H**, consider **Figure 9.18a**. Using Ampere's law

$$\oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = I_{enclosed} \quad [\mathrm{A}] \tag{9.36}$$

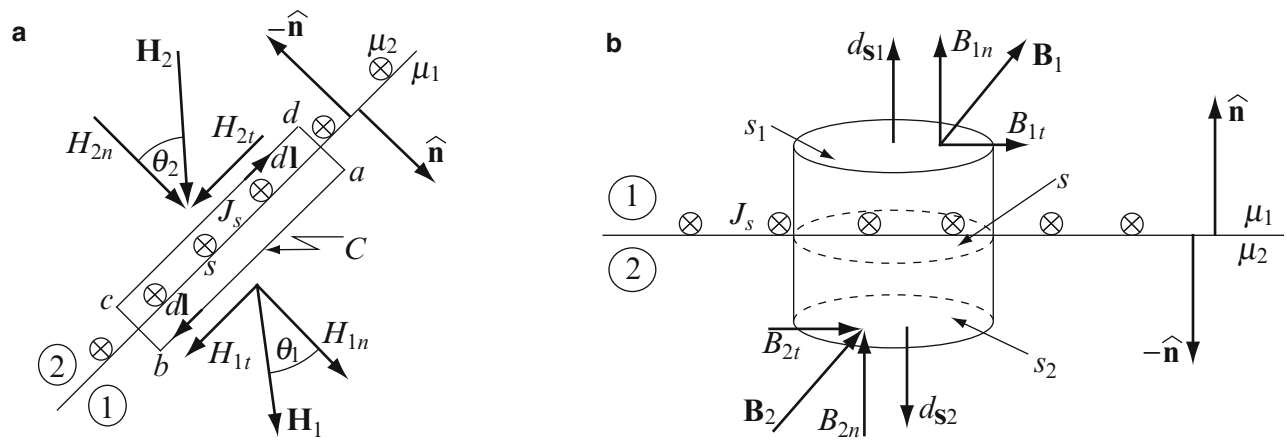


Figure 9.18 (**a**) Calculation of the relation between tangential components of the magnetic field intensity. (**b**) Calculation of the relation between normal components of the magnetic flux density

Allowing the distances bc and da to tend to zero, the total contribution due to this part of the contour is zero. Only the integration along ab and cd contributes to the left-hand side of **Eq. (9.36)**. Also, the scalar product $\mathbf{H} \cdot d\mathbf{l}$ means that only the tangential components are used. In other words, $\mathbf{H} \cdot d\mathbf{l} = H_t dl$. Denoting the surface current density by J_s [A/m] (see **Figure 9.18a**), we get

$$\int_{ab} H_{1t}\ dl_1 - \int_{cd} H_{2t}\ dl_2 = \int_{ab} J_s\ dl \tag{9.37}$$

In this expression, the vector notation was dropped since the fields are collinear with the path (H_{2t} and dl_2 are in opposite directions, hence the negative sign). The current density is perpendicular to the loop $abcda$, and since it can only depend on the width of the loop, it is integrated over the path dl. Integrating over the two segments ab and cd and setting $ab = cd$, we get

$$\boxed{H_{1t} - H_{2t} = J_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \tag{9.38}$$

This is the first condition at the interface; the discontinuity of the tangential component of the magnetic field intensity is equal to the surface current density on the interface, if such a current density exists. Note also that this current density has units of A/m indicating that it is limited to the surface and does not extend inside either medium. If there is no current on the surface, the tangential components of **H** are equal and **H** is said to be continuous at the interface:

$$H_{1t} = H_{2t} \quad \text{if} \quad J_s = 0 \tag{9.39}$$

From $\mathbf{B} = \mu\mathbf{H}$, we can also write for the general relation in **Eq. (9.38)**:

$$\boxed{\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \tag{9.40}$$

or

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} \quad \text{if} \quad J_s = 0 \tag{9.41}$$

Equations (9.38) and **(9.40)** give the basic relations between the tangential components of **H** and **B** on both sides of the interface. In particular, the tangential components of **B** are discontinuous regardless of any current density at the interface. The discontinuity in **B** is related to permeabilities of the two materials.

It must be noted here that the interface current density J_s in **Eq. (9.38)** or in **Eq. (9.40)** is perpendicular to H_{1t} and to H_{2t} as can also be seen from **Figure 9.17**. However, **Eq. (9.38)** [or **Eq. (9.40)**] only provides the magnitude of the current density.

It is essential to recall that the direction of the current density is given by the right-hand rule (see **Section 8.2** and **Figure 8.2c**). As long as the direction of the current density is properly established, these equations can be used. This is rather simple when the magnetic field intensities have only one tangential component. If $\mathbf{H}_1$ and/or $\mathbf{H}_2$ have two tangential components, **Eq. (9.38)** [or **Eq. (9.40)**] must be applied to each tangential component separately. To avoid this difficulty and to establish a more general relation, we note from **Figure 9.17** that the tangential components of the fields may be written as $\mathbf{H}_{1t} = \hat{\mathbf{n}} \times \mathbf{H}_1$ and $\mathbf{H}_{2t} = -\hat{\mathbf{n}} \times \mathbf{H}_2$ where $\hat{\mathbf{n}}$ is the normal unit vector pointing into material (1). With these relations, **Eqs. (9.38)** and **(9.40)** may be written as

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \quad \text{or} \quad \boxed{\hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = \mathbf{J}_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \tag{9.42}$$

This form of the interface conditions guarantees correct magnitude and direction for the components of $\mathbf{J}_s$ without resorting to the right-hand rule. **Equation (9.42)** is a more general form of **Eqs. (9.38)** and **(9.40)** and should be used in all instances except, perhaps, when the fields have only one tangential component.

For the normal components of the field, we use **Eq. (8.21)** and calculate the relations between the normal components of **B** on both sides of the interface by calculating the total normal flux through the interface. Because this requires calculation of the magnetic flux Φ over a closed surface, we define a cylindrical volume as shown in **Figure 9.18b**, where the normal components of **B** are perpendicular to the bases of the cylinder. Thus, B_{n1} is in the direction of ds_1, and B_{n2} is in the direction opposite ds_2. Allowing the volume of the cylinder to tend to zero (i.e., the volume of the cylinder encloses only the interface), we have

$$\oint_s \mathbf{B} \cdot d\mathbf{s} = \int_{s_1} B_{1n} ds_1 - \int_{s_2} B_{2n} ds_2 = 0 \tag{9.43}$$

With $s_1 = s_2$, the boundary condition becomes

$$\boxed{B_{1n} = B_{2n}} \tag{9.44}$$

Again, with the use of the constitutive relation $\mathbf{B} = \mu\mathbf{H}$, we can write the interface conditions for the normal component of **H** as

$$\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}} \tag{9.45}$$

The normal component of the magnetic flux density is continuous across an interface, but the normal component of the magnetic field intensity is not. The discontinuity in H_n is again related to the permeability of the two materials. The interface conditions are summarized in **Table 9.5**.

Table 9.5 Interface conditions for the static magnetic field

General conditions at an interface	Conditions at an interface without surface current density
$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$ or: $H_{1t} - H_{2t} = J_s^* \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$ $\mu_1 H_{1n} = \mu_2 H_{2n}$	$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$ or: $H_{1t} = H_{2t}$ $\mu_1 H_{1n} = \mu_2 H_{2n}$
$\hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = \mathbf{J}_s \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$ or $\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s^* \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$ $B_{1n} = B_{2n}$	$\hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = 0$ or $\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$ $B_{1n} = B_{2n}$

*This form requires the use of the right-hand rule to establish the vector relation between the tangential components of the fields and the current density

Now that the general interface conditions have been defined, it is appropriate to dwell a bit on the surface current density. Perhaps the most important question is: When do current densities exist and when not? First, since current cannot flow on the surface of nonconducting materials, at least one of the materials must be a conductor. Also, whenever the conductivities of materials are finite, currents will flow inside conductors rather than on the surface. Thus, the only practical case in which we must assume true surface current densities is at the interfaces between perfect conductors and other materials. In most applications related to steady currents (magnetostatics), the interface conditions in the second column of **Table 9.5** are used. The only exception is a situation in which the currents are constrained to flow on a surface such as a highly conducting coating on a poorly conducting base material. Whenever current density on the surface can be neglected, the tangential component of the magnetic field intensity and the normal component of the magnetic flux density are continuous across the interface.

The concept of surface current is somewhat difficult to understand since our concept of current is that of volume current densities; that is, in circuit theory, we always assumed that a conductor has finite thickness even though the thickness of the conductor was not normally used in calculation. This, of course, is still correct in the sense that only for perfect conductors ($\sigma \to \infty$) is the current limited to the surface. Since this condition is not satisfied in most practical applications and the conductivity of materials is finite, a true surface current doesn't usually occur. We defer discussion of the surface current density for materials other than perfect conductors until we discuss time-dependent fields. **Example 9.6** shows why a surface current density must exist on the surface of a superconductor.

As a direct consequence of the above interface conditions, the magnetic field (either **H** or **B**) is refracted at the interface between two materials with different permeabilities. Assuming there are no current densities on the surface, we can write for continuity of the tangential components of **H** and normal components of **B**:

$$H_{1t} = H_{2t} \quad \text{and} \quad B_{1n} = B_{2n} \tag{9.46}$$

From **Figure 9.18a**

$$\tan\theta_1 = \frac{H_{1t}}{H_{1n}} \quad \text{and} \quad \tan\theta_2 = \frac{H_{2t}}{H_{2n}} \tag{9.47}$$

and

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{H_{2n}}{H_{1n}} = \frac{B_{2n}/\mu_2}{B_{1n}/\mu_1} = \frac{\mu_1}{\mu_2} \tag{9.48}$$

where the relation $\mathbf{H} = \mathbf{B}/\mu$ and $B_{1n} = B_{2n}$ were used

Thus, if $\mu_1 > \mu_2$, $\theta_1 > \theta_2$, and if $\mu_1 < \mu_2$, $\theta_1 < \theta_2$. The refraction can be rather high because the ratio between permeabilities of different materials can be high. This means that a magnetic field almost parallel to an interface in one material may be almost perpendicular in the other material.

Example 9.5 Refraction at the Interface of a Ferromagnetic Material and Air The magnetic field intensity in an iron piece is directed at 85° to the normal to the surface as shown in **Figure 9.19**.

(a) Calculate the direction of the magnetic field intensity in air.
(b) If the magnetic flux density in iron is 1 T, what is the magnetic flux density in air?

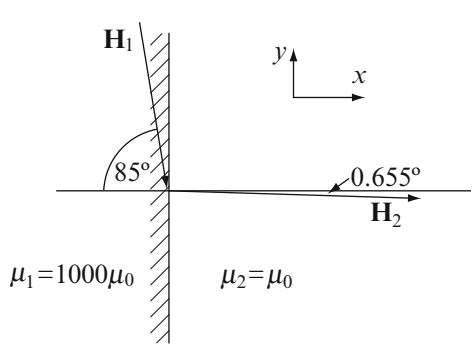


Figure 9.19 The magnetic field intensity relations at the interface between a high-permeability material and free space

Solution:

(a) The solution may be obtained by calculating the tangential and normal components of the magnetic field intensity and then applying the continuity relations for the tangential and normal components. Alternatively, we can use **Eq. (9.48)**. The latter gives

$$\tan\theta_2 = \frac{\tan 85^\circ}{1000} \quad \rightarrow \quad \theta_2 = 0.655^\circ$$

Thus, the magnetic field intensity in air is almost perpendicular to the surface. The opposite is also true: a magnetic field intensity at almost any angle (except 90°) in air will refract to an almost parallel field in the ferromagnetic material.

(b) Taking iron to be material (1) and air to be material (2), the interface conditions are

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \quad \rightarrow \quad B_{2t} = \frac{B_{1t}\mu_2}{\mu_1} \quad [\mathrm{T}]$$

$$B_{2n} = B_{1n}$$

The tangential and normal components of the magnetic flux density in iron are calculated from the magnitude of the flux density and the angle between the flux density B_1 and the normal:

$$B_{1n} = B_1 \cos 85^\circ \quad B_{1t} = B_1 \sin 85^\circ \quad [\mathrm{T}]$$

The components of the flux density in air are

$$B_{2t} = \frac{B_1\mu_2 \sin 85^\circ}{\mu_1} = \frac{B_1\mu_0 \sin 85^\circ}{1000\mu_0} = \frac{1 \times 0.9962}{1000} = 9.962 \times 10^{-4} \quad [\mathrm{T}]$$

$$B_{2n} = B_1 \cos 85^\circ = 8.716 \times 10^{-2} \quad [\mathrm{T}]$$

The magnitude of the flux density is

$$B_2 = \sqrt{B_{2t}^2 + B_{2n}^2} = 8.716 \times 10^{-2} \quad [\mathrm{T}]$$

This flux density is at an angle of 0.655° to the normal as in **Figure 9.19**.

Note: If a system of coordinates is given, the vector form can be written. For the system shown, we get

$$\mathbf{B}_2 = \hat{\mathbf{x}} B_{2n} - \hat{\mathbf{y}} B_{2t} = \hat{\mathbf{x}}\, 8.716 \times 10^{-2} - \hat{\mathbf{y}}\, 9.962 \times 10^{-4} \quad [\mathrm{T}]$$

Example 9.6 Current Density at the Surface of a Perfect Conductor A magnetic flux density of 1 T exists parallel to the surface of a large, flat superconductor. Calculate the surface current density produced by this magnetic flux density.

Solution: The magnetic field intensity in the superconducting material is zero; that is, both tangential and normal components are zero. The only way a tangential component of the magnetic field intensity can exist at the surface is if a current density exists on the surface of the superconductor.

From the general boundary conditions for **Table 9.5** and by setting the flux density in the superconductor to $B_{2t} = 0$ we get

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s \quad \rightarrow \quad J_s = \frac{B_{1t}}{\mu_0} = \frac{1}{4\pi \times 10^{-7}} = 796 \quad \left[\frac{\mathrm{kA}}{\mathrm{m}}\right]$$

The direction of the surface current density J_s is such that it cancels the external magnetic flux density inside the material; that is, the current density on the surface of the perfect conductor may be viewed as the source of an additional field which cancels the external field in the conductor. Note also that J_s is perpendicular to H_{1t} and H_{2t} as required [see **Eq. (9.42)**].

9.4 Inductance and Inductors

By now, we should have enough tools to calculate the magnetic field intensity due to currents in conductors either using Ampere's law or, if the configuration is complex, using the Biot–Savart law. From **Eq. (8.17)** or **(8.47)** we can also calculate the total magnetic flux passing through any surface. We will use these concepts to define a new and very useful quantity: the inductance.

First, consider a conducting loop C_1 in free space. This may be a bent wire or one turn of a solenoid. If a current I_1 flows in the loop, a magnetic flux density $\mathbf{B}_1$ is produced. This flux density exists everywhere in space and varies from point to point (see **Figure 9.20**). We discuss the field due to loops because they are easier to evaluate and understand and have specific applications in engineering. However, the discussion is completely general and the results apply to any current configuration, since any current must be part of a closed loop.

Now, consider a second conducting loop C_2 at some other physical location in space also shown in **Figure 9.20**. Since loop C_1 produces the magnetic flux density, all flux lines pass through loop C_1, but only some of the flux produced by loop C_1 passes through loop C_2. Assuming the flux density $\mathbf{B}$ is known (calculated using the Biot–Savart law), we can calculate the magnetic flux passing through each loop. To calculate the various fluxes we use the following notation:

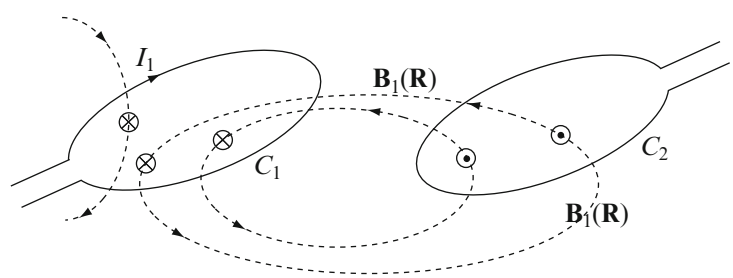


Figure 9.20 Loop (1) produces a magnetic flux, part of which passes through loop (2)

$\mathbf{B}_1$ = flux density produced by loop C_1
I_1 = current in loop C_1
Φ_{11} = magnetic flux, produced by loop C_1 that passes through loop C_1
Φ_{12} = magnetic flux, produced by loop C_1 that passes through loop C_2

In the discussion that follows, it is assumed that the medium in which the loops are placed is linear; that is, the permeability of the medium is independent of the currents in either loop.

The total flux in loop C_1 is

$$\Phi_{11} = \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{s}_1 \quad [\text{Wb}] \tag{9.49}$$

Similarly, the flux through loop C_2 is

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \quad [\text{Wb}] \tag{9.50}$$

The flux density $\mathbf{B}_1$ is not known unless the actual dimensions of C_1 and the value of I_1 are known. However, in very general terms, we recall that in the Biot–Savart law, if the current I is a constant, it may be taken outside the integral sign [see **Eq. (8.9)**]. Therefore, $\mathbf{B}_1$ is directly proportional to the current I_1:

$$\mathbf{B}_1 = I_1 \left[\frac{\mu_0}{4\pi} \oint_{C_1} \frac{d\mathbf{l}' \times \hat{\mathbf{R}}}{|\mathbf{R}|^2} \right] \quad [\text{T}] \tag{9.51}$$

where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}'$, $\mathbf{r}_1$ is the position vector of the point at which $\mathbf{B}_1$ is calculated, and $\mathbf{r}'$ is the position vector of the current element $d\mathbf{l}'$. Suppose we use this equation and calculate the magnetic flux density everywhere in space due to the loop and then substitute it in **Eqs. (9.49)** and **(9.50)** and perform the integration over s_1 and s_2. The result is the magnetic fluxes Φ_{11} and Φ_{12}, which are directly proportional to I_1:

$$\Phi_{11} = L_{11} I_1 \quad [\text{Wb}] \tag{9.52}$$

$$\Phi_{12} = L_{12} I_1 \quad [\text{Wb}] \tag{9.53}$$

The terms L_{11} and L_{12} are constants, independent of current; only the geometry of the configuration plays a role in the values of L_{11} and L_{12}. L_{11} is defined as

$$L_{11} = \frac{\Phi_{11}}{I_1} \left[\frac{\text{Wb}}{\text{A}}\right] = [\text{H}] \tag{9.54}$$

This is called the ***self-inductance*** of loop C_1. It is a self-inductance because all terms involved in its calculation relate only to itself. Loop C_2 has no effect on the self-inductance of loop C_1. Similarly,

$$L_{12} = \frac{\Phi_{12}}{I_1} \quad [\text{H}] \tag{9.55}$$

L_{12} is called the ***mutual inductance*** between loop C_1 and loop C_2. It is a mutual inductance because the flux Φ_{12} depends both on loop C_1 and on loop C_2, as is evident from **Eq. (9.50)**.

Before proceeding we note the following:

(1) Inductance (self or mutual) has units of [Wb/A]. This unit is designated as the henry [H].
(2) Inductance depends only on the geometrical configuration of the circuits and is independent of current. Although the current produces the flux, once we divide by the current, the result contains only geometrical terms (and permeability).
(3) Any device that has inductance may be called an ***inductor***. In practice, an inductor is usually a coil or a solenoid designed specifically for its inductance and often used as a circuit element.

The above results can be generalized in two ways: One is to ask ourselves what happens if instead of a single turn, we have two or more turns either in C_1 or C_2 or both. This is shown in **Figure 9.21**, where N_1 loops are placed together to form a circuit C_1 and N_2 loops are placed together to form a circuit C_2. We argue as follows: if the flux due to a single loop carrying a current I is equal to Φ, the flux due to N loops carrying the same current I is $N\Phi$ from the principle of superposition, provided that the same flux passes through all loops. Thus, the total flux in C_1 is $N_1\Phi_{11}$, and the total flux passing through C_2 is $N_2\Phi_{12}$. The equations for self- and mutual inductance can now be written as

$$\boxed{L_{11} = \frac{N_1\Phi_{11}}{I_1} \quad \text{and} \quad L_{12} = \frac{N_2\Phi_{12}}{I_1} \quad [\text{H}]} \tag{9.56}$$

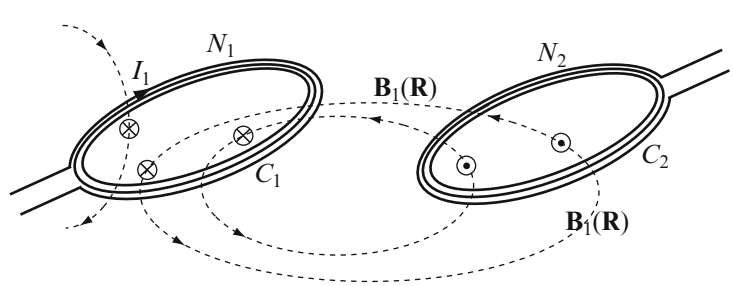


Figure 9.21 Inductance of multiple loop circuits. Circuit (1) produces a magnetic flux, part of which passes through circuit (2)

The term $N_1\Phi_{11}$ is the total flux that links circuit C_1 with itself or the ***flux linkage*** of C_1 with itself. The term $N_2\Phi_{12}$ is the total flux that links circuit C_1 with circuit C_2 or the flux linkage between circuit C_1 and C_2. These terms are denoted as

$$\Lambda_{11} = N_1\phi_1 \quad \text{and} \quad \Lambda_{12} = N_2\Phi_{12} \quad [\text{weber} \cdot \text{turns}] \tag{9.57}$$

The unit of flux linkage is the weber • turns [Wb • t]. With this notation, the self- and mutual inductances are defined as

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1} \quad \text{and} \quad L_{12} = \frac{\Lambda_{12}}{I_1} \quad [\text{H}]} \tag{9.58}$$

This definition is more general and includes in it the definition in **Eqs. (9.54)** and **(9.55)**.

The second generalization we wish to make is to assume that circuit C_2 also carries a current I_2 as shown in **Figure 9.22** and see how this affects inductance. To do so, we use superposition as follows: if the indices 1 and 2 in **Figure 9.21** are interchanged, we get an identical situation, except that now the self-inductance of circuit C_2 is calculated as L_{22} and the mutual inductance between circuit C_2 and circuit 1 is calculated as L_{21}. Thus

$$\boxed{L_{22} = \frac{N_2\Phi_{22}}{I_2} = \frac{\Lambda_{22}}{I_2} \quad \text{and} \quad L_{21} = \frac{N_1\Phi_{21}}{I_2} = \frac{\Lambda_{21}}{I_2} \quad [\text{H}]} \tag{9.59}$$

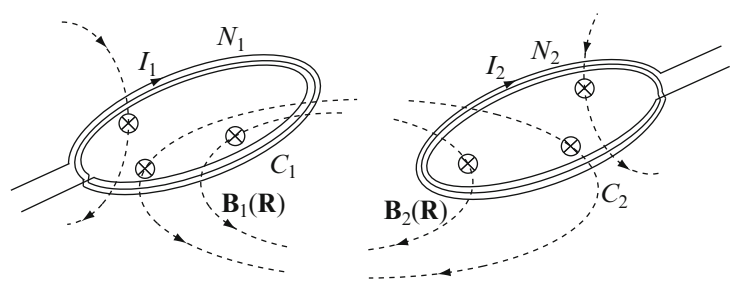


Figure 9.22 Both circuits (1) and (2) carry currents and produce flux

where the following definitions were used:

$\mathbf{B}_2$ = flux density produced by loop C_2
I_2 = current in loop C_2
Φ_{22} = magnetic flux, produced by circuit C_2 that passes through circuit C_2
Φ_{21} = magnetic flux, produced by circuit C_2 that passes through circuit C_1
Λ_{22} = magnetic flux linkage between circuit C_2 and itself
Λ_{21} = magnetic flux linkage between circuit C_2 and circuit C_1

We will see in examples that $L_{12} = L_{21}$. Although we do not prove it (it requires calculation of energy), this is a general relation and will be used in subsequent derivations.

Calculation of inductance involves the following steps:

(1) For each current-carrying conductor, calculate the flux density in space. If the conductor for which you need to calculate self-inductance does not carry a current, assume an arbitrary current. The calculation may be done using the Biot–Savart or Ampere's law.
(2) Calculate the flux passing through any circuit required. For self-inductance, only the flux in the circuit producing the flux is required.
(3) Calculate the total flux linkage linking the circuit with itself (for self-inductance) or with other circuits (for mutual inductance).
(4) Divide the appropriate flux linkage by the current that produced it to obtain the self- or mutual inductance.

It is important to remember that the relations for inductance as given here only apply for a linear medium; that is, it is assumed that the inductance is independent of current.

Example 9.7 Application: Self-inductance of a Toroidal Coil A torus with rectangular cross section and dimensions as shown in **Figure 9.23a** is wound with a coil. The coil consists of $N = 1{,}000$ turns of very fine wire wound in a uniform single layer on the surface of the torus. Permeability of the torus is μ_0. Use $b = 50$ mm, $c = 20$ mm, $d = 70$ mm, and calculate the self-inductance of the coil.

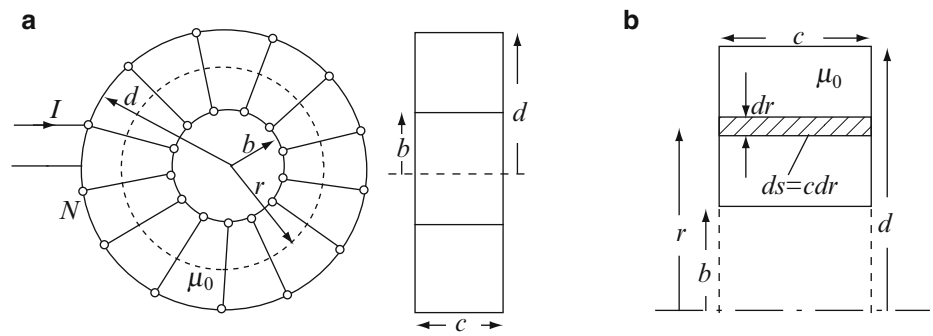


Figure 9.23 Calculation of inductance in a toroidal coil. (**a**) Dimensions. (**b**) Calculation of flux

Solution: The flux density in the torus is calculated first using Ampere's law. The flux density outside the torus is zero (see **Example 8.9**). From this, we calculate the total flux in the torus. Multiplying by the number of turns to find the flux linkage and dividing by the current in the turns gives the self-inductance.

From Ampere's law, a contour at radius $b < r < d$ inside the torus encloses N turns, each carrying a current I. The magnetic flux density is therefore

$$B = \frac{\mu_0 NI}{2\pi r} \quad [\mathrm{T}]$$

The flux density is not constant but varies with the radius r. To calculate the flux, we must integrate the flux density over the cross-sectional area of the torus. To do so, we take an elementary area $ds = cdr$ and, therefore, an element of flux $d\Phi = Bd\mathbf{s}$ as shown in **Figure 9.23b** and write

$$\Phi = \int_s Bd\mathbf{s} = \int_{r=b}^{r=d} \frac{\mu_0 NI}{2\pi r} cdr = \frac{\mu_0 NIc}{2\pi} \int_{r=b}^{r=d} \frac{dr}{r} = \frac{\mu_0 NIc}{2\pi} \ln\frac{d}{b} \quad [\mathrm{Wb}]$$

This flux passes through all windings of the coil. Therefore, the total flux linkage is

$$\Lambda = N\Phi = \frac{\mu_0 N^2 Ic}{2\pi} \ln\frac{d}{b} \quad [\mathrm{Wb \cdot t}]$$

and the self-inductance is

$$L = \frac{\Lambda}{I} = \frac{N\Phi}{I} = \frac{\mu_0 N^2 c}{2\pi} \ln\frac{d}{b} \quad [\mathrm{H}]$$

The self-inductance of the toroidal coil (not the torus) is

$$L_{11} = \frac{\mu_0 N^2 c}{2\pi} \ln\frac{d}{b} = \frac{4 \times \pi \times 10^{-7} \times 1000^2 \times 0.02}{2 \times \pi} \ln\frac{0.07}{0.05} = 1.346 \times 10^{-3} \quad [\mathrm{H}].$$

Exercise 9.2 Assume the torus in **Example 9.7** is made of iron with relative permeability of $\mu_1 = 1{,}000\mu_0$ and the coil consists of a single turn. What is the self-inductance of the coil if the flux is entirely contained within the volume of the torus?

Answer 1.346 μH.

Example 9.8 Application: Mutual Inductance Between a Wire and a Toroidal Coil—Core Memory The geometry shown in this example was used in the past as a memory device. It consisted of a small magnetic torus (usually made of ferrite) and a wire passing through the torus (see **Figure 9.24**). A current in the wire was used to magnetize the magnetic torus, which then retained this magnetization until an opposite current was used to erase it. Magnetic memories before the advent of semiconductor memories were almost exclusively used and were made of very large matrices of miniature toroids, about 1 mm in diameter. Each torus was a single memory bit. The main advantage of this type of memory was in its nonvolatility: memory could be retained after the computer was switched off. In addition to the magnetization (or write) wire, there was an erase wire and a sensing (read) wire which allowed the computer to read the memory without erasing it. Thus, each torus had three wires passing through it. The following example calculates the mutual inductance between the magnetizing wire and a coil on the torus.

A torus with mean radius r_0 [m] and a cross-sectional area as shown in **Figure 9.24b** is given. A coil with N turns is wound uniformly around the torus. A straight, long wire carrying a current I [A] passes through the center of the torus.

(a) Calculate the mutual inductance between the wire and coil if the torus is air filled. Assume $r_0 \gg b$.
(b) In an attempt to increase the mutual inductance, the torus is made of ferrite with relative permeability of 5,000. Calculate the mutual inductance between the wire and coil.

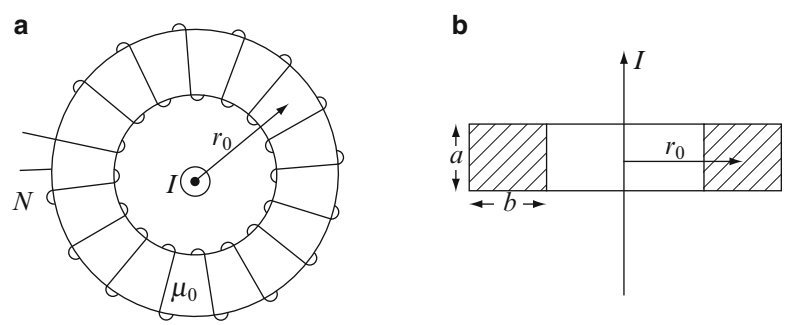


Figure 9.24 (**a**) A current-carrying wire passing through a toroidal coil. (**b**) Cross-sectional view

Solution: The wire generates a flux density in the circumferential direction. The magnetic field intensity of the wire is calculated as if the torus did not exist, since the magnetic field intensity is independent of permeability. From this, the magnetic flux density is calculated by multiplying H by the permeability μ of the torus. Because $r_0 \gg b$, we can use the flux density at r_0 as an average flux density, which is approximately uniform throughout the cross section. The total flux is the product of flux density and the cross-sectional area of the torus. The flux linkage with the N turns of the coil is found by multiplying the flux by N. Division by the current in the wire gives the mutual inductance.

(a) Using Ampere's law and a contour of radius r_0 around the wire, the field intensity is calculated as

$$H2\pi r_0 = I \quad \rightarrow \quad H = \frac{I}{2\pi r_0} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

Because $r_0 \gg b$ the flux density in the torus is uniform and equals $B = \mu_0 H$. The total flux then becomes

$$\Phi = Bab = \mu_0 Hab = \frac{\mu_0 abI}{2\pi r_0} \quad [\text{Wb}]$$

This flux links all N turns of the coil. The total flux linkage is

$$\Lambda_{12} = N\Phi = \frac{\mu_0 abNI}{2\pi r_0} \quad [\text{Wb} \cdot \text{t}]$$

and the mutual inductance is

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu_0 abN}{2\pi r_0} \quad [\text{H}]$$

The mutual inductances are equal: $L_{21} = L_{12}$. This fact can be used to our advantage. For example, in the above example, it is relatively easy to calculate the mutual inductance L_{12}. The mutual inductance L_{21} is much more difficult to define since, now, we must calculate the magnetic flux density due to the torus inside the loop created by the wire and its return at infinity. Because the two inductances are the same, we can choose to calculate that which is easiest.

(b) If the torus is made of ferrite, the only change is in the permeability of ferrite. The permeability is 5,000 times larger and so is the mutual inductance:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{5000\mu_0 abN}{2\pi r_0} \quad [\text{H}].$$

Example 9.9 Mutual Inductance Between Straight Wire and Loop A straight, long wire is placed in a plane. A rectangular loop is also in the same plane as shown in **Figure 9.25**. Calculate the mutual inductance between wire and loop.

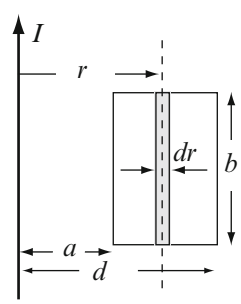


Figure 9.25 Calculation of flux linking a straight wire and a loop; calculation of mutual inductance

Solution: To calculate the mutual inductance, we first calculate the magnetic flux density of the wire everywhere inside the loop. Then, the flux is found by integrating the flux density over the loop. Division by current gives the mutual inductance.

The magnetic flux density of a wire is (see **Example 9.8**):

$$B = \frac{\mu_0 I}{2\pi r} \quad [\text{T}]$$

Taking a differential area of length b and width dr (see **Figure 9.25**) gives the magnetic flux as

$$\Phi = \int_s B ds = \int_{r=a}^{r=d} \frac{\mu_0 Ib}{2\pi r} dr = \frac{\mu_0 Ib}{2\pi} \ln r \Big|_{r=a}^{r=d} = \frac{\mu_0 Ib}{2\pi} \ln \frac{d}{a} \quad [\text{Wb}]$$

The mutual inductance between the wire and loop is

$$L_{12} = \frac{\Phi}{I} = \frac{\mu_0 b}{2\pi} \ln \frac{d}{a} \quad [\text{H}]$$

As we discussed in **Example 9.8**, the calculation of L_{12} is relatively simple whereas the direct calculation of L_{21} (mutual inductance between loop and wire) is much more involved. Because $L_{12} = L_{21}$, we chose to calculate L_{12}.

Example 9.10 Self- and Mutual Inductances in Multiple Coils Three coils are wound on a toroidal core with properties and dimensions as shown in **Figure 9.26**. Assume $a \gg (b - a)$ and that all three coils are uniformly wound around the coil (one on top of the other) and calculate:

(a) The self-inductances of each of the three coils.
(b) The mutual inductances between coils (1) and (2), between coils (2) and (3), and between coils (1) and (3).

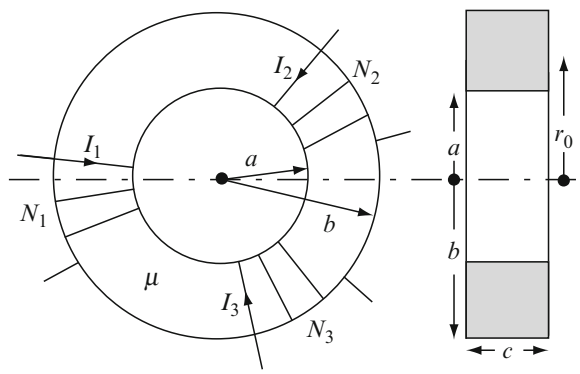


Figure 9.26 Three coils wound on a common torus. Each coil is uniformly wound around the core

Solution: To calculate the self-inductance, we assume each coil, separately, carries an arbitrary current, calculate the flux, then the flux linkage, and then divide by current in the coil to find the self-inductance of the coil. The mutual inductance is calculated similarly by taking one coil at a time, assuming a current through the coil, calculating the flux in the core, and then calculating the flux that links each pair of coils. Division by the current generating the flux gives the mutual inductance.

(a) We start with coil (1). Assuming a current I_1, the flux density in the coil is (see **Example 9.8**):

$$B_1 = \frac{\mu N_1 I_1}{2\pi r_0} \quad [\mathrm{T}]$$

where $r_0 = (a + b)/2$ is the average radius (since $a \gg (b - a)$).
The flux generated by coil (1) is

$$\Phi_1 = B_1 S = \frac{\mu N_1 I_1 (b-a)c}{\pi(b+a)} \quad [\mathrm{Wb}]$$

Because, in a torus, all flux is contained within the core of the torus, this flux links all N_1 turns of coil (1). Thus, the flux linkage is

$$\Lambda_{11} = N_1 \Phi_1 = N_1 B_1 S = \frac{\mu N_1^2 I_1 (b-a)c}{\pi(b+a)} \quad [\mathrm{Wb \cdot t}]$$

The self-inductance of coil (1) is

$$L_{11} = \frac{\Lambda_{11}}{I_1} = \frac{\mu N_1^2 (b-a)c}{\pi(b+a)} \quad [\mathrm{H}]$$

Repeating the process for each of the remaining two coils, we get

$$L_{22} = \frac{\Lambda_{22}}{I_2} = \frac{\mu N_2^2 (b-a)c}{\pi(b+a)}, \quad L_{33} = \frac{\Lambda_{33}}{I_3} = \frac{\mu N_3^2 (b-a)c}{\pi(b+a)} \quad [\mathrm{H}]$$

Note: The self-inductance may also be written as

$$L_{ii} = \frac{\mu N_i^2 S}{l} \quad [\mathrm{H}]$$

where N_i is the number of turns in coil i, S is the cross-sectional area of the core, and l is the average length of the core.

(b) To calculate the mutual inductance, say, between coils (1) and (3), we first calculate the magnetic flux density and magnetic flux Φ_1 produced by coil (1) as in **(a)**. All of this flux links with coil (3). Thus, the flux linkage between coils (1) and (3) is

$$\Lambda_{13} = \Phi_1 N_3 = \frac{\mu N_1 N_3 I_1 (b-a)c}{2\pi r_0} = \frac{\mu N_1 N_3 I_1 (b-a)c}{\pi(b+a)} \quad [\mathrm{Wb \cdot t}]$$

The mutual inductance between coil (1) and coil (3) is

$$L_{13} = \frac{\Lambda_{13}}{I_1} = \frac{\mu N_1 N_3 (b-a)c}{\pi(b+a)} \quad [\mathrm{H}]$$

Before continuing, we note that if we were to calculate the flux due to coil (3) rather than coil (1), we would obtain

$$\Phi_3 = B_3 S = \frac{\mu N_3 I_3 (b-a)c}{\pi(b+a)} \quad [\mathrm{Wb}]$$

The flux linkage between coils (3) and (1) is

$$\Lambda_{31} = \Phi_3 N_1 = \frac{\mu N_3 N_1 I_3 (b-a)c}{\pi(b+a)} \quad [\mathrm{Wb \cdot t}]$$

and the mutual inductance between coil (3) and coil (1) is

$$L_{31} = \frac{\Lambda_{31}}{I_3} = \frac{\mu N_3 N_1 (b-a)c}{\pi(b+a)} \quad [\mathrm{H}]$$

Clearly, $L_{13} = L_{31}$. Repeating the process, we obtain for the other two pairs of mutual inductances:

$$L_{12} = L_{21} = \frac{\mu N_1 N_2 (b-a)c}{\pi(b+a)}, \quad L_{23} = L_{32} = \frac{\mu N_2 N_3 (b-a)c}{\pi(b+a)} \quad [\mathrm{H}]$$

As for the self-inductance, the mutual inductance may also be written in general for the mutual inductance between coils (i) and (j) on the same core:

$$L_{ij} = \frac{\mu N_i N_j S}{l} \quad [\mathrm{H}].$$

Exercise 9.3 Consider **Example 9.9**. Assume that the wire passes over the loop such that the loop is symmetric about the wire (see **Figure 9.27**). What is the mutual inductance between wire and loop?

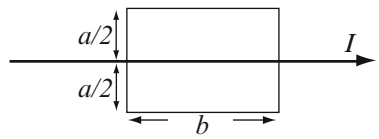


Figure 9.27 A current-carrying wire passing over the center of a loop

Answer $L_{12} = 0$, because the total net flux linking the wire and loop is zero.

9.4.1 Inductance per Unit Length

The self- and mutual inductances defined in **Eqs. (9.58)** and **(9.59)** are the total self- or mutual inductances of the corresponding loops because the flux linkage was calculated based on the total flux through the loops. However, if the loops are very large in size as is the case of long cables, the flux linkage becomes very large (or infinite) and, therefore, the total inductance is very large (or infinite). In such cases it is more useful to define the ***inductance per unit length*** of the structure. All results obtained so far apply except that we calculate the flux linkage of a 1 m section of the structure instead of the total flux linkage. Naturally, the unit of inductance now becomes the [H/m]. **Examples 9.11, 9.12,** and **Exercise 9.4** discuss some of the details involved in the calculation of inductance per unit length.

Example 9.11 Application: Self-inductance of a Long Solenoid—Inductance per Unit Length Find the self-inductance per unit length of a long solenoid with $n = 100$ turns per unit length and a diameter of $d = 50$ mm.

Solution: We assume a current I in the turns and use Ampere's law to calculate the magnetic flux density inside the solenoid. Since the flux density is constant in the solenoid, the total flux is obtained by multiplying the magnetic flux density by the cross-sectional area and then the flux linkage is found by multiplying the total flux by the number of turns per unit length. Division of the flux linkage per unit length by the current gives the inductance per unit length. Note that only inductance per unit length of a solenoid has practical meaning. The total inductance is infinite.

To calculate the flux density, we use Ampere's law as in **Example 8.10**, using contour a as shown in **Figure 9.28**. The length of the contour is arbitrary. The flux density outside the solenoid is zero (see contour b, which includes both sides of the solenoid and, therefore, the total net current is zero). Thus,

$$B = \frac{\mu_0 nIL}{L} = \mu_0 nI \quad [\mathrm{T}]$$

This is the same result we obtained in **Example 8.10**. The total flux is

$$\Phi = BS = \mu_0 nI\pi\frac{d^2}{4} \quad [\mathrm{Wb}]$$

where d is the diameter of the solenoid. The flux linkage per unit length of the solenoid is

$$\Lambda = \Phi n = \mu_0 n^2 I\pi\frac{d^2}{4} \quad \left[\frac{\mathrm{Wb}\cdot\mathrm{t}}{\mathrm{m}}\right]$$

since the flux passes through all turns of the solenoid. The self-inductance per unit length is

$$L_{11} = \frac{\Lambda}{I} = \mu_0 n^2\pi\frac{d^2}{4} \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

Thus, whereas the flux density is independent of the diameter of the solenoid, its inductance depends on the diameter. For the values given, the inductance per unit length is

$$L_{11} = 4 \times \pi \times 10^{-7} \times 100^2 \times \pi \times \frac{0.05^2}{4} = 2.47 \times 10^{-5} \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

or in more standard notation $L_{11} = 24.7$ μH/m.

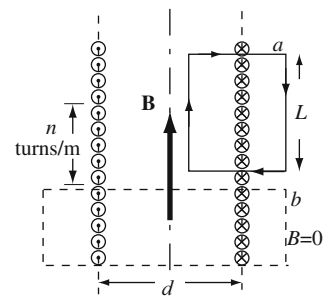


Figure 9.28 Calculation of flux density and inductance per unit length of a solenoid

9.4.2 External and Internal Inductance

So far we have defined and calculated the self- and mutual inductance of a number of configurations, including the inductance per unit length of a long solenoid. In all of these configurations we assumed the current-carrying conductors to be thin and that the flux used for computation of the flux linkage and inductance was external to the conductors—the flux inside the conductors, if any, was neglected. The inductance calculated from the flux linkage external to the conductors is also called the ***external inductance***. If the current-carrying conductors are of finite dimensions, part of the flux produced will actually exist inside the conductors and produce a flux linkage of its own. As a result, part of the inductance is due to the flux linkage interior to the conductors. This inductance is called the ***internal inductance***. The total inductance is the sum of the external and internal inductances. In many instances, the internal inductance is small compared to the external inductance and is therefore neglected. For this reason, what is normally called inductance refers more often to external inductance. **Example 9.12** and **Exercise 9.4** explore the concepts of internal, external, and total inductance per unit length.

Example 9.12 Application: Inductance per Unit Length of Coaxial Cables A coaxial cable is made of an inner solid conductor of radius b [m] and an outer thin, flexible conductor of radius a [m] and negligible thickness, separated by a dielectric. Assume all materials have permeability μ_0 [H/m]. Calculate the self-inductance of the coaxial cable per unit length of the cable.

Solution: We assume a current I flows into the outer conductor and out of the inner conductor as shown in **Figure 9.29a** and that the current is uniformly distributed in the inner conductor (uniform current density). For ease of calculation, we divide the conductor into three domains: One is the inner conductor ($0 < r < b$). The second is the domain between the two conductors ($b \leq r \leq a$). The third is the exterior of the outer conductor ($r > a$). The flux in each of the domains is calculated first. Then, we calculate the flux linkage and, finally, divide the total flux linkage by the current to obtain the inductance. Unlike calculation of inductance in solenoids, where the flux linkage is clearly defined, here we will have to define the flux linkage for a differential element of current and calculate the total flux linkage by integration.

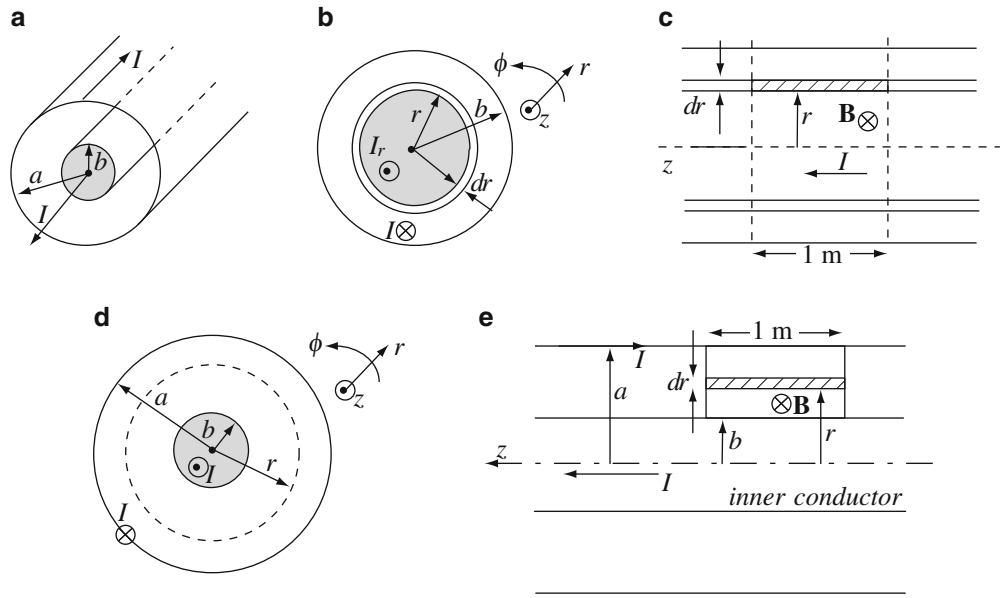


Figure 9.29 Calculation of internal and external self-inductance in a coaxial cable. **(a)** Geometry and dimensions. **(b)** Calculation of flux density in the inner conductor. **(c)** Calculation of flux linkage in the inner conductor. (**d**) Calculation of flux density between conductors. (**e**) Calculation of flux linkage between conductors

(1) Inside the inner conductor ($0 < r < b$). We draw a contour at radius $0 < r < b$. The total current enclosed by the contour is the area enclosed by the contour (**Figure 9.29b**) multiplied by the current density in the conductor. The current density in the inner conductor is

$$J_i = \frac{I}{\pi b^2} \quad \left[\frac{\text{A}}{\text{m}^2}\right]$$

The total current enclosed by the contour of radius $r < b$ is

$$I_r = \pi r^2 J_i = \frac{\pi r^2 I}{\pi b^2} \quad [\text{A}]$$

The magnetic flux density at radius r is found by applying Ampere's law along the contour of radius r:

$$2\pi r B_r = \mu_0 I_r = \frac{\mu_0 r^2 I}{b^2} \quad \rightarrow \quad B_r = \frac{\mu_0 r I}{2\pi b^2} \quad [\text{T}], \quad (0 < r < b)$$

Now, suppose we define an element of current in the form of a ring of radius r, thickness dr, and width 1 m as shown in **Figures 9.29b** and **9.29c**. The flux in the ring (i.e., the flux that passes through the area of the ring $ds = 1 \times dr$) perpendicular to B is

$$d\Phi_1 = \mathbf{B}_r \cdot d\mathbf{s} = \frac{\mu_0 I r}{2\pi b^2} \times 1 \times dr = \frac{\mu_0 I r dr}{2\pi b^2} \quad [\text{Wb}]$$

This flux is produced by the current enclosed by the ring (current outside the ring produces zero flux density at the location of the ring). Thus, the flux linking the ring equals the flux $d\Phi_1$ multiplied by the ratio of the current enclosed and the total current in the conductor (**Figure 9.29b**):

$$d\Lambda_1 = d\Phi_1 \frac{I_r}{I} = d\Phi_1 \frac{\pi r^2}{\pi b^2} = \frac{\mu_0 I r^3 dr}{2\pi b^4} \quad [\text{Wb} \cdot \text{t}]$$

where the ratio between the currents equals the ratio between the cross-sectional areas occupied by the current enclosed by the ring and the total cross-sectional area of the conductor because the current is uniformly distributed. The total flux linkage is found by integrating over r from $r = 0$ to $r = b$:

$$\Lambda_1 = \int_{r=0}^{r=b} \frac{\mu_0 I r^3 dr}{2\pi b^4} = \left.\frac{\mu_0 I r^4}{8\pi b^4}\right|_{r=0}^{r=b} = \frac{\mu_0 I}{8\pi} \quad [\text{Wb} \cdot \text{t}]$$

Note that this flux linkage is independent of the radius of the conductor. Because it is entirely due to flux linkages within the conductor's volume this is an internal inductance. The internal inductance due to the inner conductor is therefore

$$L_1 = \frac{\mu_0}{8\pi} \quad \left[\frac{\text{H}}{\text{m}}\right].$$

(2) The second domain is between the two conductors $b \leq r \leq a$. On the contour shown in **Figure 9.29d**, using Ampere's law (and the right-hand rule),

$$B 2\pi r = \mu_0 I \quad \rightarrow \quad B = \frac{\mu_0 I}{2\pi r} \quad \text{or} \quad \mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \quad [\text{T}]$$

Using **Figure 9.29e**, the element of area is $1dr$ and the flux per unit length through the area shown is

$$\Phi = \int_{r=b}^{r=a} \mathbf{B} \cdot d\mathbf{s} = \int_{r=b}^{r=a} \frac{\mu_0 I}{2\pi r} 1dr = \frac{\mu_0 I}{2\pi} \ln \frac{a}{b} \quad [\text{Wb}]$$

The flux linkage equals Φ ($N = 1$) and the inductance per unit length is

$$L_{11} = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ln \frac{a}{b} \quad \left[\frac{\text{H}}{\text{m}}\right]$$

Note, again, that the current I is irrelevant. It is required for formal computation but disappears from the final result and inductance depends only on physical dimensions and permeability. This inductance is due to flux linkages external to the conductors and, therefore, is the external inductance per unit length of the cable.

For $r > a$, the flux density is zero (the total net current enclosed by any contour of radius $r > a$ is zero). Thus, there is no contribution to inductance due to this domain.

The total inductance per unit length of the coaxial cable is the sum of the internal and external inductances:

$$L = L_1 + L_{11} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{a}{b} \quad \left[\frac{\text{H}}{\text{m}}\right].$$

Exercise 9.4 A coaxial cable as used in TV antennas has an inner conductor which is 0.5 mm in diameter and an outer conductor 5 mm in diameter. The outer conductor is a very thin shell.

(a) Calculate the total inductance per unit length of the cable.
(b) What are the internal and external inductances per unit length?
(c) What is the total inductance per unit length if you assume the inner conductor to be a very thin shell with outer diameter 0.5 mm instead of a solid conductor?

Answer

(a) $L = 5.1 \times 10^{-7}$ [H/m].
(b) External inductance: $L = 4.6 \times 10^{-7}$ H/m. Internal inductance: $L = 0.5 \times 10^{-7}$ H/m.
(c) $L = 4.6 \times 10^{-7}$ H/m.

From these examples, it is apparent that self- and mutual inductances exist even in cases when these are not self-evident. In particular, the inductance of straight wires and infinitely long structures requires us to "look" for a generalized loop. However, if there is a current in a conductor, this conductor must be part of a closed circuit, perhaps closing at infinity, and a loop always exists. For infinite structures, the loop is somewhat artificial since we only calculate the inductance per unit length.

We defined inductance as the ratio between flux linkage and current, but the basic question of why we do so remains; that is, what does inductance do that cannot be done from field relations directly? Part of the answer lies in the fact that inductance is geometry dependent rather than current dependent. Thus, the calculation of flux, for example, can be carried out by first calculating the inductance and then the flux linkage from **Eqs. (9.56)** and **(9.57)**. The second part of the answer will be given in detail in the following section, where we discuss energy. However, at this point, you might wish to recall that in electrostatics, we defined capacitance as the ratio between charge and potential. Inductance has equivalent use in magnetics as capacitance has in electrostatics, including storage of energy.

9.5 Energy Stored in the Magnetic Field

The normal process of defining energy in a field is to start with the force in the field, calculate work performed by the field (negative work) or against the field (positive work), that is, start with $dW = \mathbf{F} \cdot d\mathbf{l}$, and then find the energy density associated with the field. We used this approach in the calculation of electrostatic energy (see **Section 4.8**). In the magnetic field, this approach cannot be followed. The reason is that a proper definition of energy in the magnetic field must start with the time-dependent field. Since we have not discussed this aspect of fields yet (and will not do so until **Chapter 10**), we must either accept the equations for energy as given, define them by analogy to the relations in electrostatics, or not define them at all at this stage. A compromise seems to be appropriate here: we will use known relations from circuit theory to find the energy stored in the magnetic field of an inductor. Later, in **Chapter 12**, we will devote considerable time to power and energy in the general electromagnetic field (time dependent as well as time harmonic). The relations we define here for energy will then become simplifications of the more general energy relations.

Consider the *RL* circuit in **Figure 9.30**. From Kirchhoff's circuital law, the voltage across the *RL* circuit is

$$V = V_R + V_L = RI + L\frac{dI}{dt} \quad [\text{V}] \tag{9.60}$$

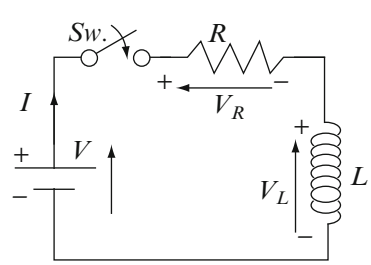


Figure 9.30 *RL* circuit used to define energy stored in the inductor *L*

Here, we are only interested in the second term of **Eq. (9.60)** (the inductive term). The instantaneous power in the inductor is

$$P_L = V_L I = LI\frac{dI}{dt} \quad [\text{W}] \tag{9.61}$$

and the magnetic energy stored in the inductor is the integrated power over time:

$$\boxed{W_m = \int_0^t P_L dt = \int_0^I LI\frac{dI}{dt}dt = \int_0^I LIdI = \frac{1}{2}LI^2 \quad [\mathrm{J}]} \tag{9.62}$$

It is useful to compare this to the electrostatic energy stored in a capacitor, which is given as $CV^2/2$, where C is capacitance and V the voltage on the capacitor. This comparison shows that the role of C is taken by L, whereas that of V is taken by I. It also indicates that if storage of energy in the capacitor is characterized by the potential across the plates, storage of energy in an inductor is characterized by the current in the inductor. Any change in this current changes the stored energy in the inductor.

This relation for magnetic energy is convenient for the calculation of energy stored in an inductor, such as the one shown in **Figure 9.30**. However, in other configurations, this is not so. Consider, for example, the two coils in **Figure 9.22**. Each circuit can be viewed as an inductor with self-inductances L_{11} and L_{22}. The energy stored in each of these inductors is properly given by **Eq. (9.62)**. However, there are also two mutual inductances L_{12} and L_{21}. How do we calculate the energy stored in these mutual inductances and what is the total energy stored in a system? Furthermore, suppose we have N coupled circuits. Can we find a simple way of calculating the energy in the system? To find some answers to these questions, we start with **Eq. (9.62)** but rewrite it in terms of the magnetic flux to obtain a more general statement of energy in the magnetic field.

Consider, first, the two circuits in **Figure 9.22**. Each circuit produces a flux, some of which links with the other circuit. These are as follows:

Φ_{11} = Flux that links C_1 alone (produced by C_1 and passing through C_1)
Φ_{22} = Flux that links C_2 alone (produced by C_2 and passing through C_2)
Φ_{12} = Flux that links C_1 and C_2 and is produced by C_1 (dashed lines)
Φ_{21} = Flux that links C_2 and C_1 and is produced by C_2 (dashed lines)

Now, we can calculate the total flux in circuit C_1 as the flux produced by the circuit itself (Φ_{11}) and that portion of the flux produced by C_2 that links with C_1 (i.e., Φ_{21}). Depending on the directions of currents in C_1 and C_2, these fluxes may oppose each other or may be in the same direction (the fluxes in **Figure 9.22** are shown as opposing each other, but inverting the current in one of the circuits will make both fluxes in the same direction). Thus, we can write for the total flux in each circuit:

Flux in circuit C_1:

$$\Phi_1 = \Phi_{11} \pm \Phi_{21} \quad [\mathrm{Wb}] \tag{9.63}$$

Flux in circuit C_2:

$$\Phi_2 = \Phi_{22} \pm \Phi_{12} \quad [\mathrm{Wb}] \tag{9.64}$$

The flux linkage in each circuit is the flux through the circuit multiplied by the number of loops in the circuit:

$$\Lambda_1 = N_1\Phi_1 = N_1\Phi_{11} \pm N_1\Phi_{21} \quad [\mathrm{Wb \cdot t}] \tag{9.65}$$

$$\Lambda_2 = N_2\Phi_2 = N_2\Phi_{22} \pm N_2\Phi_{12} \quad [\mathrm{Wb \cdot t}] \tag{9.66}$$

We can now define the total inductance of each circuit by dividing the flux linkage by its current. These inductances are denoted L_1 and L_2 and include both the self-inductance and the mutual inductance:

$$L_1 = \frac{\Lambda_1}{I_1} = \frac{N_1\Phi_1}{I_1} = \frac{N_1\Phi_{11}}{I_1} \pm \frac{N_1\Phi_{21}}{I_1} \quad [\mathrm{H}] \tag{9.67}$$

$$L_2 = \frac{\Lambda_2}{I_2} = \frac{N_2\Phi_2}{I_2} = \frac{N_2\Phi_{22}}{I_2} \pm \frac{N_2\Phi_{12}}{I_2} \quad [\mathrm{H}] \tag{9.68}$$

Using the definition of energy in **Eq. (9.62)**, we can write for the total energy in the two circuits:

$$W_m = \frac{1}{2}LI^2 = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 \quad [\mathrm{J}] \tag{9.69}$$

Substituting **Eqs. (9.67)** and **(9.68)** into this relation gives

$$W_m = \frac{1}{2}\left(\frac{N_1\Phi_{11}}{I_1} \pm \frac{N_1\Phi_{21}}{I_1}\right)I_1^2 + \frac{1}{2}\left(\frac{N_2\Phi_{22}}{I_2} \pm \frac{N_2\Phi_{12}}{I_2}\right)I_2^2 \quad [\mathrm{J}] \tag{9.70}$$

Thus, we get a relation in terms of the total fluxes Φ_1 and Φ_2:

$$W_m = \frac{1}{2}N_1(\Phi_{11} \pm \Phi_{21})I_1 + \frac{1}{2}N_2(\Phi_{22} \pm \Phi_{12})\ I_2 = \frac{1}{2}N_1\Phi_1 I_1 + \frac{1}{2}N_2\Phi_2 I_2 \quad [\mathrm{J}] \tag{9.71}$$

This can be written as a general expression for any number of circuits n:

$$\boxed{W_m = \frac{1}{2}\sum_{i=1}^{n} N_i\Phi_i I_i \quad [\mathrm{J}]} \tag{9.72}$$

where Φ_i is the total flux, that is, the flux in loop i due to all loops in the system.
From the definition of inductance, we can also write, in general **[**see, for example, **Eq. (9.56)]**:

$$LI \ = \ N\Phi \tag{9.73}$$

Thus, we get

$$N_1\Phi_{11} = L_{11}I_1, \quad N_1\Phi_{21} = L_{21}I_2, \quad N_2\Phi_{22} = L_{22}I_2, \quad N_2\Phi_{12} = L_{12}I_1 \tag{9.74}$$

Substituting these in **Eq. (9.71)** gives

$$W_m = \frac{1}{2}\ L_{11}I_1^2 \pm \frac{1}{2}L_{21}I_1I_2 + \frac{1}{2}L_{22}I_2^2 \pm \frac{1}{2}L_{12}I_2I_1 \quad [\mathrm{J}] \tag{9.75}$$

Collecting terms, using the relation $L_{12} = L_{21}$, and because both mutual fluxes are positive or both are negative, gives

$$W_m = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 \pm L_{12}I_1I_2 \quad [\mathrm{J}] \tag{9.76}$$

We can again generalize this relation for any number of inductors, each with its own inductance and mutual inductances between each two inductors by properly defining the sign of the energy term due to mutual inductance. To do so, we define mutual inductance L_{ij} to be positive if the current in L_i produces a flux which adds to the flux in the loop L_j with which it couples. Otherwise, it is negative. With this assumption, the energy in a system of n inductors is

$$\boxed{W_m = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(L_{ij}I_iI_j\right) \quad [\mathrm{J}]} \tag{9.77}$$

Note that in this expression, if one current is negative, any term I_iI_j, $i \neq j$, in the expression is negative, whereas terms I_iI_j, $i = j$, are always positive.

A simple consequence of **Eq. (9.76)** is a formula for the calculation of inductances in series. If L_1 and L_2 are connected in series, then $I_2 = I_1$ and we can write

$$W_m = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_1^2 \pm L_{12}I_1^2 = \frac{L_{eq}I_1^2}{2} \quad [\mathrm{J}] \quad \rightarrow \quad L_{eq} = L_{11} + L_{22} \pm 2L_{12} \quad [\mathrm{H}] \tag{9.78}$$

This expression can be easily extended to any number of series inductances (see **Example 9.14**). The sign of the mutual inductance term is determined as above, based on the direction of fluxes produced by the currents in each inductor.

Example 9.13 Application: Superconducting Magnetic Energy Storage (SMES) One method of storing large amounts of energy is to use a superconducting coil. The total energy stored in the coil is given by **Eq. (9.62)**. To store considerable amounts of energy, the coil must have a large inductance and negligible resistance (therefore the need for superconducting coils). Proposed systems include underground coils that can be used during peak power consumption. To do so, special switches connect the coil to the grid whenever necessary.

A proposed superconducting storage ring is made as a toroidal coil. The cross-sectional radius is $d = 1$ m and the radius of the torus is $r = 1$ km. The torus is wound with $N = 150{,}000$ turns and can carry a current $I = 100{,}000$ A. The geometry is shown in **Figure 9.31**.

(a) Calculate the magnetic flux density in the torus.
(b) What is the total amount of energy stored in this torus?
(c) A city requires 100 MW of power. How long can a storage ring of this type power the city in case of a blackout in power generation? Assume there are no losses in the conversion of energy from stored DC energy to AC energy required by standard grids.

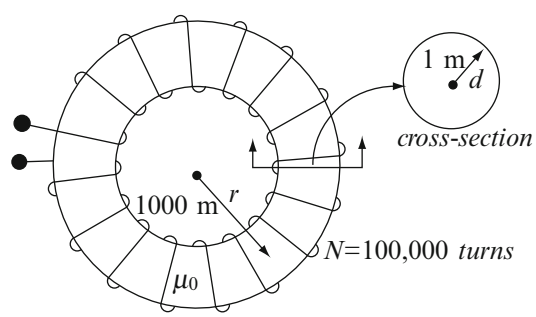


Figure 9.31 A large superconducting ring for energy storage

Solution: The flux density in a torus has been calculated in **Example 9.7** by assuming it is uniform in the torus. This assumption is based on the fact that the radius of the ring (1,000 m) is large compared to the radius of the cross section (1 m). The inductance of the torus is calculated from the flux density by multiplying the flux density by the cross-sectional area S and the number of turns to obtain the flux linkage and then dividing by the current in the turns of the torus.

(a) For the dimensions, properties, and current given here, the flux density in the torus is (see **Example 9.7**):

$$B = \frac{\mu_0 NI}{2\pi r} = \frac{4 \times \pi \times 10^{-7} \times 150{,}000 \times 100{,}000}{2 \times \pi \times 1000} = 3 \quad [\mathrm{T}].$$

(b) To calculate the total energy stored in the inductor, we calculate first the inductance and then use **Eq. (9.62)** or **Eq. (9.77)** with $n = 1$. The total flux in the torus is

$$\Phi = BS = \frac{\pi d^2 \mu_0 NI}{2\pi r} = \frac{d^2 \mu_0 NI}{2r} \quad [\mathrm{Wb}]$$

The total flux linkage is $N\Phi$:

$$\Lambda = N\Phi = \frac{d^2 \mu_0 N^2 I}{2r} \quad [\mathrm{Wb \cdot t}]$$

The self-inductance of the storage ring is

$$L = \frac{\Lambda}{I} \quad \rightarrow \quad L = \frac{d^2 \mu_0 N^2}{2r} \quad [\mathrm{H}]$$

The stored energy in the ring is

$$W = \frac{LI^2}{2} = \frac{d^2\mu_0 N^2 I^2}{4r} = \frac{1^2 \times 4 \times \pi \times 10^{-7} \times 2.25 \times 10^{10} \times 10^{10}}{4 \times 1000} = 7.07 \times 10^{10} \quad [\text{J}].$$

(c) Since energy is power integrated over time and the city requires 100 MW, the energy needs of the city may be met for

$$t = \frac{W}{P} = \frac{7.07 \times 10^{10}}{1 \times 10^8} = 707 \quad [\text{s}]$$

This is approximately 11 min, 47 s.

Example 9.14 Connection of Coils in Series An inductance of 150 mH is required in an electric circuit. However, because of a shortage of this type of inductor, an engineer decides to use three inductors, each 50 mH, in series as shown in **Figure 9.32a**. The inductors, when connected, have a mutual inductance $L_{12} = L_{23} = 10$ mH and $L_{13} = 5$ mH. The three inductors are identical and identical currents produce identical flux in each inductor.

(a) Calculate the inductance of the circuit and the total stored energy for a current $I = 1$ A.
(b) What is the maximum inductance that can be obtained with the three inductors?
(c) What is the minimum inductance possible with the three inductors?
(d) Is it possible to obtain an inductance of 150 mH using the three inductors given? How?

Solution: (a) Although the inductors are identical, L_3 produces a flux which opposes the flux in coils (1) and (2). Therefore, the mutual inductance terms between coil (3) and coils (1) and (2) are negative. The mutual inductance between coils (1) and (2) is positive because their fluxes are in the same direction. **(b)** For the inductance to be maximum, all mutual inductance terms must be positive [see **Eq. (9.78)**]. Similarly, in **(c)**, as many of the mutual inductance terms as possible must be negative for minimum inductance.

(a) Because L_{13}, L_{23}, L_{31}, and L_{32} are negative, the total inductance is

$$L_{eq} = L_{11} + L_{22} + L_{33} + 2L_{12} - 2L_{23} - 2L_{13} = 150 + 20 - 20 - 10 = 140 \quad [\text{mH}]$$

The energy stored is

$$W_t = \frac{L_{11}I^2}{2} + \frac{L_{22}I^2}{2} + \frac{L_{33}I^2}{2} + \frac{2L_{12}I^2}{2} - \frac{2L_{23}I^2}{2} - \frac{2L_{13}I^2}{2} = \frac{L_{eq}I^2}{2} = 70 \quad [\text{mJ}]$$

The total energy stored in the three 50 mH inductors is 70 mJ.

(b) Maximum inductance is obtained by flipping the connections on L_3 (**Figure 9.32b**). The result is

$$L_{\max} = L_{11} + L_{22} + L_{33} + 2L_{12} + 2L_{23} + 2L_{13} = 200 \quad [\text{mH}]$$

(c) To obtain minimum inductance, the connections must be such that as many of the largest mutual inductance terms are negative. We cannot make them all negative because when we flip another coil, those terms connected to this coil that were initially negative become positive. To minimize inductance, we flip the coil (or coils) that produces the largest negative mutual inductances. In this case, either coil (1) or (2) will do. Choosing to reverse coil (2) (see **Figure 9.32c**), we get

$$L_{\min} = L_{11} + L_{22} + L_{33} - 2L_{12} - 2L_{23} + 2L_{13} = 120 \quad [\text{mH}].$$

(d) The only way three 50 mH coils can produce a series inductance of 150 mH is if their mutual inductances are zero. This can be done if the fluxes of the three coils are noninteracting; that is, if the flux in one coil does not couple into any other coil. This means the coils are shielded or their fluxes are enclosed. For example, if each coil is made in the form of a torus, the flux of each coil is contained in the torus and there will be no flux linkage between the coils.

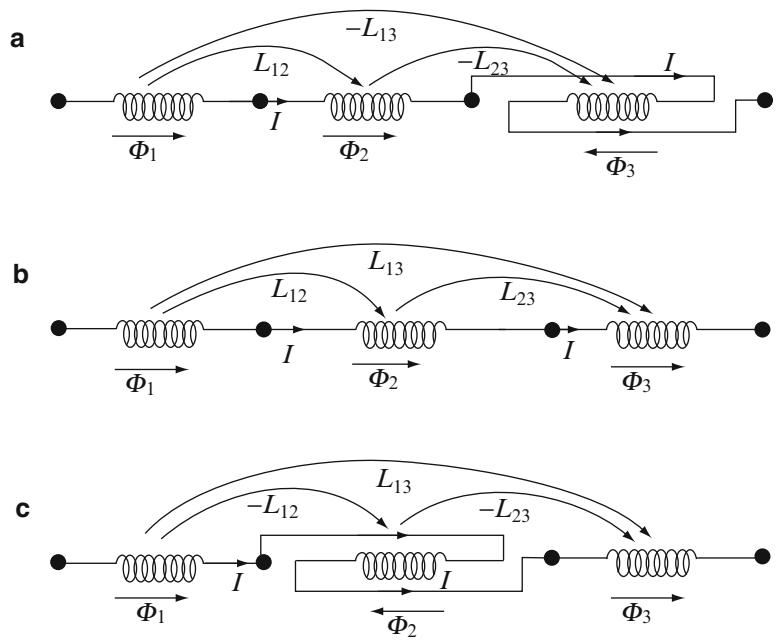


Figure 9.32 Connection of inductors in series. (**a**) Φ_3 opposes the fluxes Φ_1 and Φ_2. (**b**) All three fluxes are in the same direction. (**c**) Φ_2 opposesΦ_1 and Φ_3

9.5.1 Magnetostatic Energy in Terms of Fields

In the previous section, we found the energy in a system of inductors to be a simple summation process. This, while useful, has the distinct disadvantage that it only allows the calculation of energy in a system of inductors. What do we do if, for example, we need to calculate the energy from known field quantities such as **B** or **H**? A good example is the calculation of energy in the field of a permanent magnet. In this case, the inductance is not as useful and, yet, the energy stored in the magnetic field must often be calculated. Thus, it is important that we generalize the expressions above further, to include all physical situations, not only discrete inductors.

This generalization starts with **Eq. (9.72)**. However, now we argue as follows: If any system of currents is given, these can always be divided into any number of loops or current segments. For example, a solenoid can be viewed as an infinite number of loops, a straight, solid conductor can be viewed as an infinite number of filamentary currents, and a general conductor can be viewed as a collection of current segments of some defined shape. Thus, we may view the above sum as an infinite sum over single loops or current segments (i.e., $N_i = 1$). With this assumption, Φ_i is the flux that links with the elementary segment or loop. Since the sum is infinite, we will replace the sum by an integral, but before we do so, recall that the flux can be calculated by integrating the flux density over the surface through which the flux flows.

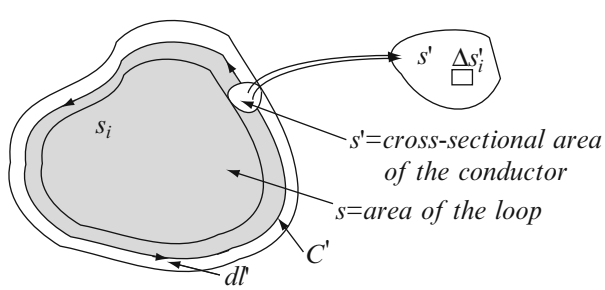


Figure 9.33 A general closed contour in a conductor of cross-sectional area s' used to calculate energy stored in the magnetic field

Now, consider **Figure 9.33**. A thick, closed conductor of arbitrary shape carries a current I. The cross section of the conductor is shown separately and is denoted by s' to indicate that the current in the conductor flows through this area. We wish to calculate the magnetic energy stored in the field produced by this conductor. The method is as follows: Take an

elementary conductor with cross-sectional area $\Delta s'$. Now, we can calculate the flux passing through the loop formed by this elementary closed conductor as

$$\Phi_i = \int_{s_i} \mathbf{B} \cdot \hat{\mathbf{n}}\, ds_i \quad [\text{Wb}] \tag{9.79}$$

where the surface enclosed by the conductor is shaded in **Figure 9.33** and we assume that the flux density **B** can be evaluated using the Biot–Savart law. Note that the area s_i is the area of the loop, not that of the cross section of the conductor! For each elementary closed conductor of this type, we can write a similar equation. The current in the elementary conductor shown is ΔI_i

$$\Delta I_i = \frac{I}{s'} \Delta s_i' \quad [\text{A}] \tag{9.80}$$

Now, we can write the total energy stored in the magnetic field using **Eq. (9.72)** as

$$W_m = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{s'} \Delta s_i' \int_{si} \mathbf{B} \cdot \hat{\mathbf{n}}\, ds_i \quad [\text{J}] \tag{9.81}$$

where I/s' is the current density J in the conductor. The magnetic flux density can be written in terms of the magnetic vector potential **A** as $\mathbf{B} = \nabla \times \mathbf{A}$:

$$W_m = \frac{1}{2} \sum_{i=1}^{n} J \Delta s_i' \int_{si} (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}}\, ds_i \quad [\text{J}] \tag{9.82}$$

Using Stokes' theorem to convert the surface integral to a closed contour integral around the surface, substituting this in **Eq. (9.82)**, and noting that **J** is in the direction of $d\mathbf{l}'$ (as defined in **Chapter 8**), we can write for the energy

$$W_m = \frac{1}{2} \sum_{i=1}^{n} \left[J \Delta s_i' \oint_{C'} \mathbf{A} \cdot d\mathbf{l}' \right] = \frac{1}{2} \sum_{i=1}^{n} \left[\Delta s_i' \oint_{C'} \mathbf{A} \cdot \mathbf{J} dl' \right] \quad [\text{J}] \tag{9.83}$$

As we allow $\Delta s_i'$ to tend to zero, the sum becomes an integral:

$$\boxed{W_m = \frac{1}{2} \int_{s'} \left[\oint_{C'} \mathbf{A} \cdot \mathbf{J} dl' \right] ds' = \frac{1}{2} \int_{v'} \mathbf{A} \cdot \mathbf{J} dv' \quad [\text{J}]} \tag{9.84}$$

where s' is the cross-sectional area of the conductor and C' is the circumference of the loop. Thus, the above integration is over the volume of the conductor, v'. This must be so because outside the conductor **J** is zero and the contribution to this integral is zero. Note, also, that we started with the flux Φ, which exists both outside and inside the conductor itself. However, the use of Stokes' theorem allowed us to convert this surface integral to a contour integral and this contour is entirely within the volume of the conductor.

This relation is important because it allows the calculation of energy from the source: from the current density and the resulting magnetic vector potential. Also, recalling the Biot–Savart law, the magnetic vector potential is directly proportional to the current I (or current density J) and, therefore, energy is related to I^2 or J^2. We know this to be true from Joule's law in **Section 7.5**. Also, this relation shows that energy is volume related.

The energy in **Eq. (9.84)** seems to be stored in the conducting volume itself since it is obtained by integration over the volume of the conductor. However, we may argue that energy can also be associated with the field **B** or **H**, regardless of location of the current that produces the fields. The same can be said about the field of a permanent magnet. Otherwise, a magnet will have no energy associated with it (no current). To show that this is the case, we can proceed in two directions. From Ampere's law we can replace **J** in **Eq. (9.84)** by $\nabla \times \mathbf{H}$ and obtain a relation for **B**. It is, however, easier to start with the general energy relation in **Eq. (9.72)** and the definition of flux in **Eq. (9.79)**.

Consider a simple current-carrying conductor such as the wire shown in **Figure 9.34**. The conductor can be thin or thick. We wish to calculate the energy associated with an element of volume due to a conductor. **Equation (9.72)** is now

$$W_m = \frac{1}{2}\sum_{i=1}^{1} \Phi_i I_i = \frac{1}{2}\Phi_1 I_1 \quad [\mathrm{J}] \tag{9.85}$$

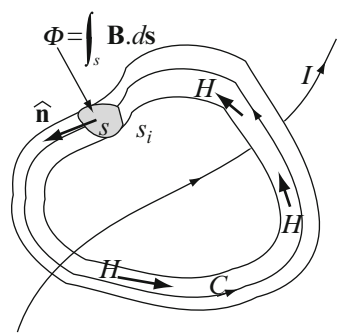


Figure 9.34 An arbitrary volume in the form of a general ring around the current I

where we assumed $N = 1$. This does not diminish from the generality of the derivation. If there are multiple currents, the energy due to each current can be calculated separately. The flux generated by the current can now be calculated by first using Ampere's law to calculate the magnetic field intensity: The surface s over which the flux is evaluated is arbitrary and can be as small as necessary:

$$I = \oint_C \mathbf{H} \cdot d\mathbf{l} \quad \text{for single loops} \quad \text{or}: \quad NI = \oint_C \mathbf{H} \cdot d\mathbf{l} \quad \text{for multiple loops} \tag{9.86}$$

The contour C is again arbitrary. However, we will choose a convenient contour such that $\mathbf{H}$ and $d\mathbf{l}$ are in the same direction to simplify the evaluation of the scalar product. Referring to **Figure 9.34**, we created a volume, in the form of a general tube, with cross section s in which the flux density is constant so that the flux Φ is independent of the location in the cross section along the tube. With these considerations in mind, we can write

$$W_m = \frac{1}{2}\int_s \mathbf{B} \cdot \hat{\mathbf{n}}\, ds \oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{1}{2}\oint_C \left(\int_s \mathbf{B} \cdot \hat{\mathbf{n}}\, ds\right) \mathbf{H} \cdot d\mathbf{l} \quad [\mathrm{J}] \tag{9.87}$$

Note that the surface integral representing flux was inserted into the line integral because the flux is assumed to be constant along the contour C. Since $\mathbf{B} \cdot \hat{\mathbf{n}}$ is the scalar component of $\mathbf{B}$ in the direction normal to s (see **Figure 9.34**), we can write

$$W_m = \frac{1}{2}\oint_C \left(\int_s \mathbf{B} \cdot \hat{\mathbf{n}}\, ds\right) \mathbf{H} \cdot d\mathbf{l} = \frac{1}{2}\oint_C \left(\int_s \mathbf{B} \cdot \mathbf{H} ds\right) dl = \frac{1}{2}\int_v \mathbf{B} \cdot \mathbf{H} dv \quad [\mathrm{J}] \tag{9.88}$$

This magnetic energy represents the total energy enclosed in the (arbitrary) volume of the tube in **Figure 9.34**. This tube is entirely outside any current or current density distribution and, therefore, its energy represents energy in space (or any other material that may be present in the volume). Because of this, the integrand of **Eq. (9.88)** may be viewed as a volume energy density in space. Using the relation $\mathbf{B} = \mu\mathbf{H}$, we can rewrite this relation in a number of useful forms:

$$\boxed{W_m = \frac{1}{2}\int_v \mathbf{B} \cdot \mathbf{H} dv = \frac{1}{2}\int_v \mu\mathbf{H} \cdot \mathbf{H} dv = \frac{1}{2}\int_v \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} dv \quad [\mathrm{J}]} \tag{9.89}$$

Any of these relations is appropriate to use depending on which is more convenient. Note, also, that the integration may be done over any volume, including the whole of space. The result is always the energy stored in the chosen volume. If the

energy is over the whole space, it represents the total energy in the system. If permeability μ is a constant, then **B** and **H** are always in the same direction and we can write the scalar products as $\mathbf{B}\cdot\mathbf{H} = BH$, $\mathbf{H}\cdot\mathbf{H} = H^2$, and $\mathbf{B}\cdot\mathbf{B} = B^2$. Viewing the integrand in **Eq. (9.89)** as an energy density, w_m, we can write

$$\boxed{w_m = \frac{\mathbf{B}\cdot\mathbf{H}}{2} = \frac{BH}{2} = \frac{\mu H^2}{2} = \frac{B^2}{2\mu} \quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right]} \tag{9.90}$$

Similarly, the integrand in **Eq. (9.84)** is an energy density in terms of the magnetic vector potential **A** and current density **J** and we can write

$$\boxed{w_m = \frac{\mathbf{A}\cdot\mathbf{J}}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right]} \tag{9.91}$$

In practical use it is common to first calculate the magnetic energy density and then integrate over the volume of a device, or over some part of space to calculate the total magnetic energy stored in a given volume.

We conclude this section by returning to the initial result in **Eq. (9.62)**. If the energy stored in the inductor is known, it can be used to calculate the inductance as

$$W_m = \frac{1}{2}LI^2 \quad \rightarrow \quad L = \frac{2W_m}{I^2} \quad [\mathrm{H}] \tag{9.92}$$

Since it is sometimes easier to calculate the energy due to a current-carrying conductor or a system of conductors, it is often easier to calculate the energy in the system using any of the above equations [such as **Eq. (9.89)**] and calculate the inductance of the system from energy. For example, using the general result for energy 1

$$L = \frac{1}{I^2}\int_v \mathbf{B}\cdot\mathbf{H}dv \quad [\mathrm{H}] \tag{9.93}$$

However, we should be careful since **Eq. (9.93)** gives the total inductance and it is not always easy to differentiate between self- and mutual inductances. This relation should only be used if only a self-inductance exists or if the total inductance of the system is needed.

Example 9.15 Stored Magnetic Energy An infinitely long solenoid with radius a [m] and n turns per unit length is given. The turns carry a current I [A]. A long piece of iron, of radius b [m], is placed in the solenoid, as shown in **Figure 9.35**. The relative permeability of iron is μ_r and that of free space is 1. Calculate the total work per unit length of the solenoid necessary to pull the iron completely out of the solenoid. Assume iron does not saturate and the **B**(**H**) curve of iron is linear. Does this work increase or decrease the potential energy of the system? Explain.

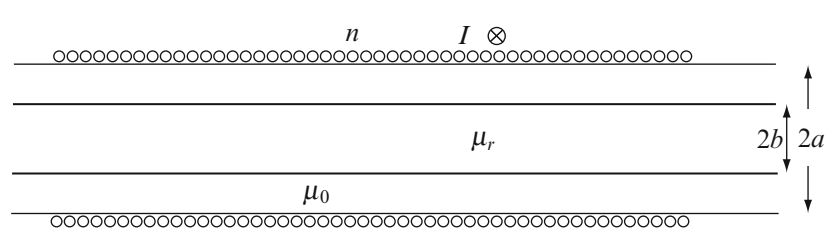


Figure 9.35 Long solenoid with an iron core filling part of the volume

Solution: The energy per unit length of the solenoid with the iron and, separately, with air is calculated. The difference between the two energies is the work per unit length necessary to take out the iron.

Applying Ampere's law inside the iron:

$$B = \mu_0 \mu_r n I \quad [\mathrm{T}]$$

In air, outside the iron:

$$B = \mu_0 n I \quad [\mathrm{T}]$$

The energy per unit length with the iron is

$$W_m = \int_v \frac{B^2}{2\mu} dv = \frac{B^2 \pi b^2}{2\mu} + \frac{B^2 \pi \left(a^2 - b^2\right)}{2\mu_0} = \frac{\mu n^2 I^2 \pi b^2}{2} + \frac{\mu_0 n^2 I^2 \pi \left(a^2 - b^2\right)}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right]$$

After removing the iron, the flux density is that in free space and the energy in the solenoid is

$$W_0 = \int_v \frac{B^2}{2\mu_0} dv = \frac{B^2 \pi a^2}{2\mu_0} = \frac{\mu_0 n^2 I^2 \pi a^2}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right]$$

The work per unit length required to remove the iron is the difference between the final energy (after removing the iron) and initial energy (with iron in):

$$W = W_0 - W_m = \frac{\mu_0 n^2 I^2 \pi a^2}{2} - \frac{\mu_0 n^2 I^2 \pi b^2}{2} - \frac{\mu_0 n^2 I^2 \pi \left(a^2 - b^2\right)}{2} = -\frac{n^2 I^2 \pi b^2 (\mu - \mu_0)}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right]$$

This work is negative ($\mu > \mu_0$); that is, removing the iron decreases the potential energy in the system.

Example 9.16 Total Inductance Two solenoids are placed one inside the other (see **Figure 9.36**). Each carries a current I as shown and has n turns per unit length. Calculate the total inductance per unit length of the system using the energy method.

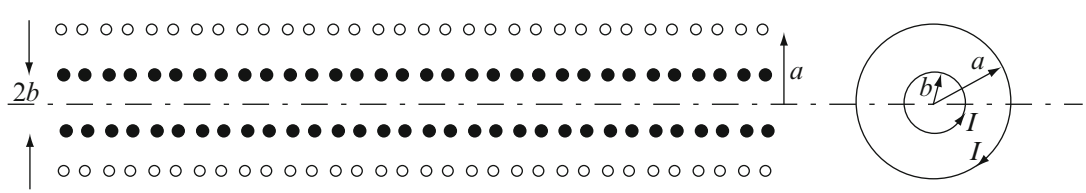


Figure 9.36 Two solenoids, one inside the other, shown in axial cut and in cross section

Solution: We first calculate the flux density everywhere in space due to each solenoid separately. Inside the smaller solenoid, the flux density is zero since the flux of the two solenoids is equal and opposite to each other. In the area between the smaller and larger solenoids, the flux density due to the outer solenoid is nonzero whereas outside the outer solenoid the flux density is again zero.

The flux density between the two solenoids ($b < r < a$) is

$$B = \mu_0 n I \qquad [\mathrm{T}]$$

Because this is constant throughout the area, the stored magnetic energy per unit length of the solenoids is

$$W = \frac{B^2}{2\mu_0} \pi \left(a^2 - b^2\right) = \frac{\mu_0 n^2 I^2 \pi \left(a^2 - b^2\right)}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right]$$

This must also be equal to:

$$W = L\frac{I^2}{2} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right] \quad \rightarrow \quad L = \frac{2W}{I^2} \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

The total inductance per unit length is H

$$L_t = \mu_0 n^2 \pi \left(a^2 - b^2\right) \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

The same result may be obtained by evaluating L_{11}, L_{22}, L_{12}, and L_{21}, but the energy calculation is simpler (see **Exercise 9.5**).

Exercise 9.5 In **Example 9.16**, evaluate the self-inductances per unit length (L_{11}, L_{22}) of the two solenoids and the mutual inductances between them (L_{12}, L_{21}) and show that the total inductance is the same as that found in **Example 9.16**.

Answer

$$\begin{aligned} &L_{11} = \mu_0 n^2 \pi b^2, \quad L_{22} = \mu_0 n^2 \pi a^2, \quad L_{12} = L_{21} = \mu_0 n^2 \pi b^2 \quad [\mathrm{H/m}] \\ &L_t = L_{11} + L_{22} - L_{12} - L_{21} = \mu_0 n^2 \pi \left(a^2 - b^2\right) \quad [\mathrm{H/m}] \end{aligned}.$$

9.6 Magnetic Circuits

Magnetostatics.m

A useful and relatively simple tool in design of magnetic devices is the idea of the magnetic circuit. It is based on an analogy between the fundamental equations and properties of the static magnetic field and the static electric field. This analogy is based on the following relations:

Electric field	Magnetic field	
$\mathbf{E} = \frac{1}{\sigma}\mathbf{J} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$	$\mathbf{H} = \frac{1}{\mu}\mathbf{B} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$	(9.94)
$V = \int_C \mathbf{E} \cdot d\mathbf{l} \quad [\mathrm{V}]$	$NI = \oint_C \mathbf{H} \cdot d\mathbf{l} \quad [\mathrm{A} \cdot \mathrm{t}]$	(9.95)
$I = \int_s \mathbf{J} \cdot d\mathbf{s} \quad [\mathrm{A}]$	$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} \quad [\mathrm{Wb}]$	(9.96)

The "current" in the magnetic circuit is represented by the magnetic flux, whereas "voltage" is represented by the term NI. The latter, defined in Ampere's law [for example, **Eq. (9.86)**], is often called the ***magnetomotive force***. The voltage V is an ***electromotive force*** supplied by a source such as a battery. We must note, however, a number of points:

(1) The magnetic circuit is only an analogy: flux is not a "magnetic current" and the magnetomotive force is not a "magnetic voltage."

(2) This analogy between circuits only applies if the conditions for a circuit are satisfied. In particular, an electric circuit requires that current flows in closed circuits in conductors. Similarly, the flux must flow in closed circuits in "magnetic paths." These concepts must be carefully defined and understood before we can properly use the analogy between electric and magnetic circuits.

(3) The magnetomotive force in a magnetic circuit is supplied either by a coil with N turns and current I or by an equivalent permanent magnet, as shown in **Figure 9.37**.

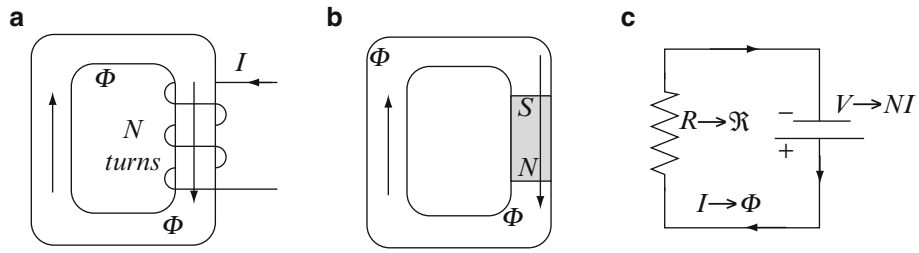


Figure 9.37 (**a**) Magnetomotive force in a magnetic circuit generated by a coil with N turns and current I. (**b**) A magnetomotive force generated by a permanent magnet. (**c**) The equivalent circuit for (**a**) or (**b**)

In the equivalent circuit in **Figure 9.37c**, the meaning of the equivalent resistance $\mathfrak{R}$ has not yet been defined. To do so, we use the simple magnetic circuit in **Figure 9.38**. The torus is used here because it satisfies the basic condition of a magnetic circuit; namely, all flux is contained within the magnetic circuit (magnetic core). The magnetic field intensity for the torus was calculated in **Example 9.7**. Its magnitude is

$$H = \frac{NI}{2\pi r_a} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{9.97}$$

where r_a is the average radius of the torus. The field intensity is uniform in the cross-sectional area of the torus (approximately) and, therefore, the flux is given by

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} = \int_s \mu \mathbf{H} \cdot d\mathbf{s} = \mu H S = \mu \frac{NI}{2\pi r} S \quad [\text{Wb}] \tag{9.98}$$

where μ is the permeability of the material and S is the cross-sectional area of the magnetic circuit. The magnetomotive force (NI) and the flux (Φ) are known. From the equivalent circuit in **Figure 9.37c**, the equivalent "resistance" in the circuit is

$$\mathfrak{R} = \frac{NI}{\Phi} = \frac{NI}{\mu \dfrac{NI}{2\pi r_a} S} = \frac{2\pi r_a}{\mu S} = \frac{l}{\mu S} \quad \left[\frac{1}{\text{H}}\right] \tag{9.99}$$

where $l = 2\pi r_a$ is the average length of the magnetic path. $\mathfrak{R}$ is called ***magnetic reluctance*** (often shortened to ***reluctance***) and is analogous to resistance in an electric circuit. Note that in this sense, μ is viewed as a "magnetic conductivity" and $1/\mu$ as a "magnetic resistivity." The term $1/\mu$ is also called ***reluctivity*** of the magnetic material in analogy to the *resistivity* of conductors.

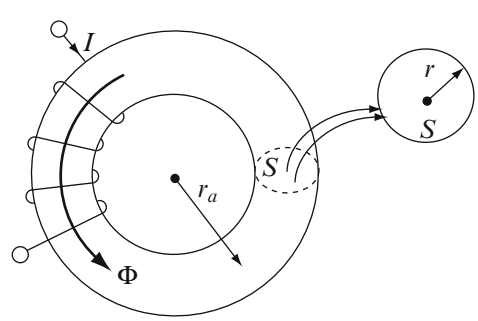


Figure 9.38 A closed magnetic circuit used to define reluctance

The calculation of reluctance of a circuit is straightforward: It is equal to the length of the magnetic circuit divided by the cross-sectional area and magnetic permeability. It depends on the physical size of the device and on its permeability.

The use of a torus in the above derivation was arbitrary: Any closed magnetic circuit would do. In fact, we may consider a magnetic circuit made of a number of materials with perhaps many branches. As long as all flux is contained within the magnetic circuit, the circuit can be analyzed using this method. In particular, consider the magnetic circuit in **Figure 9.39,** which is made of two sections, one with permeability μ_1 and the other μ_2.

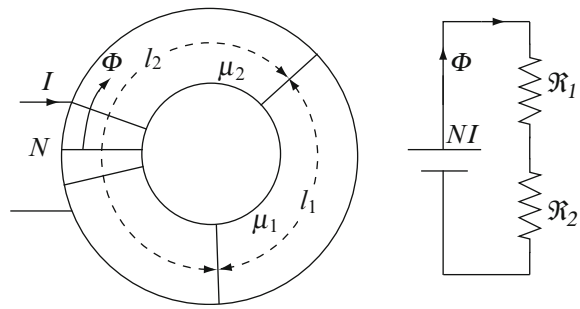


Figure 9.39 Magnetic circuit made of two magnetic materials and equivalent circuit

The reluctance of each section is calculated from **Eq. (9.99)**. In this case, there are two reluctances, connected in series:

$$\Re_1 = \frac{l_1}{\mu_1 S}, \quad \Re_2 = \frac{l_2}{\mu_2 S} \qquad \left[\frac{1}{\mathrm{H}}\right] \tag{9.100}$$

The flux in the magnetic circuit is

$$\Phi = \frac{NI}{\Re} = \frac{NI}{\Re_1 + \Re_2} = \frac{NI}{l_1/\mu_1 S + l_2/\mu_2 S} \qquad [\mathrm{Wb}] \tag{9.101}$$

Note also that any material can be included in the circuit, as long as the conditions of the circuit are satisfied. If, however, a very high reluctance material is included (low μ), such as free space, the length of this material must be kept to a minimum; otherwise the flux will spread out and the circuit is not a true circuit any more. In other words, when large air gaps are included in the magnetic circuit, the analysis of the magnetic circuit using this method may not be valid or a large error may be introduced.

The expression in **Eq. (9.101)** may be generalized for any number of magnetomotive forces and reluctances in a closed circuit as

$$\boxed{\Phi = \frac{\sum_{i=1}^{n} N_i I_i}{\sum_{j=1}^{k} \Re_j} \qquad [\mathrm{Wb}]} \tag{9.102}$$

Example 9.17 Magnetic Circuit with a Gap The magnetic circuit in **Figure 9.40** is given. Calculate the magnetic field intensity H in the gap.

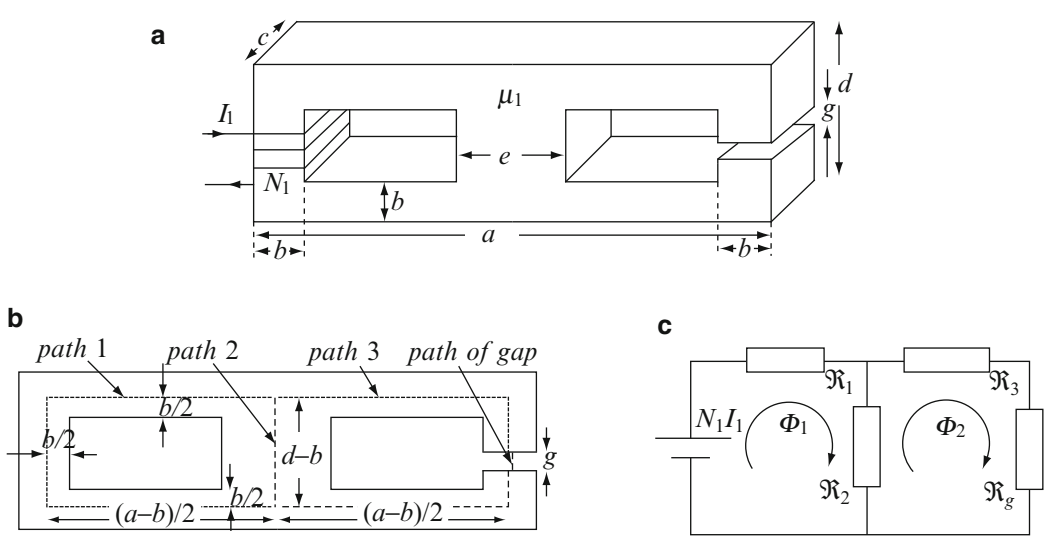


Figure 9.40 A magnetic circuit. (**a**) Dimensions. (**b**) Definition of average magnetic paths. (**c**) Equivalent circuit

Solution: The equivalent circuit with a source equal to $N_1 I_1$ and reluctances in each path is shown in **Figure 9.40c**. The three reluctances are calculated using the average paths shown in **Figure 9.40b**. Note that the cross-sectional area everywhere is bc except in the central leg, where it is ec. Since the flux in the gap is needed, it is best to use two loops as shown in **Figure 9.40c**, but other methods can be used.

The two loop equations are

$$N_1 I_1 = \phi_1(\mathfrak{R}_1 + \mathfrak{R}_2) - \Phi_2 \mathfrak{R}_2$$
$$\phi_2(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g) - \Phi_1 \mathfrak{R}_2 = 0$$

The fluxes are calculated as

$$\Phi_1 = \frac{N_1 I_1(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g)}{(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g)(\mathfrak{R}_1 + \mathfrak{R}_2) - \mathfrak{R}_2^2}, \quad \Phi_2 = \frac{N_1 I_1 \mathfrak{R}_2}{(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g)(\mathfrak{R}_1 + \mathfrak{R}_2) - \mathfrak{R}_2^2} \quad [\text{Wb}]$$

For the leg in which the gap is, the flux density in iron is

$$B_2 = B_g = \frac{\Phi_2}{bc} = \frac{N_1 I_1 \mathfrak{R}_2}{bc\{(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g)(\mathfrak{R}_1 + \mathfrak{R}_2) - \mathfrak{R}_2^2\}} \quad [\text{T}]$$

and the magnetic field intensity in the gap is

$$H_g = \frac{B_g}{\mu_0} = \frac{N_1 I_1 \mathfrak{R}_2}{\mu_0 bc\{(\mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_g)(\mathfrak{R}_1 + \mathfrak{R}_2) - \mathfrak{R}_2^2\}} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

The reluctances needed are calculated using the average path lengths in **Figure 9.40b**, as follows:

$$\mathfrak{R}_1 = \frac{a + d - 2b}{\mu_1 bc}, \quad \mathfrak{R}_2 = \frac{d - b}{\mu_1 ec}, \quad \mathfrak{R}_3 = \frac{a + d - 2b - g}{\mu_1 bc}, \quad \mathfrak{R}_g = \frac{g}{\mu_0 bc} \quad \left[\frac{1}{\text{H}}\right]$$

Thus, the magnetic field intensity in the gap is

$$H_g = \frac{N_1 I_1 \dfrac{d - b}{\mu_1 ec}}{\mu_0 bc\left\{\left(\dfrac{d - b}{\mu_1 ec} + \dfrac{a + d - 2b - g}{\mu_1 bc} + \dfrac{g}{\mu_0 bc}\right)\left(\dfrac{a + d - 2b}{\mu_1 bc} + \dfrac{d - b}{\mu_1 ec}\right) - \left(\dfrac{d - b}{\mu_1 ec}\right)^2\right\}} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

Writing $\mu_1 = \mu_0 \mu_{r1}$ and simplifying the expression gives

$$H_g = \frac{\mu_{r1} N_1 I_1 (d - b) b}{[b(d - b) + e(a + d - 2b - g) + \mu_{r1} eg][e(a + d - 2b) + b(d - b)] - b^2 (d - b)^2} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

Example 9.18 Application: Use of Permanent Magnets in Magnetic Circuits A cylindrical magnet of length a [m], radius b [m], and uniform magnetization **M** [A/m] is inserted in a magnetic path as shown in **Figure 9.41**. The cross section of the magnetic path is constant and equal to that of the magnet. Two small gaps, each of length d [m], are also present. Other dimensions and properties are given in the figure. Assume permeabilities of the path are infinite, except in gaps and magnet, where the permeability is μ_0 [H/m]. Calculate the magnetic field intensity H in each of the gaps.

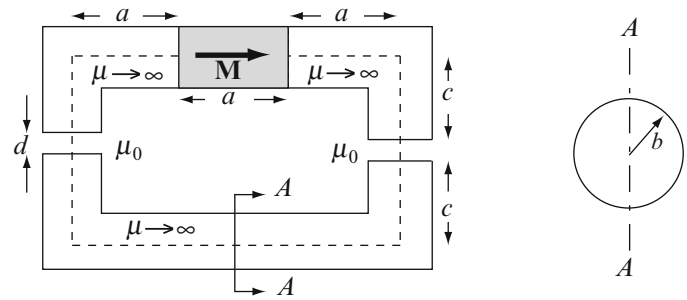


Figure 9.41 Use of a permanent magnet to generate the flux in a magnetic circuit

Solution: To calculate the flux, it is first necessary to find the equivalent magnetomotive force that produces the magnetization in the magnet as an equivalent solenoid problem. The flux in the magnetic circuit is calculated assuming there is no flux leakage in the gaps. The equivalent circuit includes the two gap reluctances as well as the reluctance of the magnet itself. The flux and flux density in the gaps are now found from the equivalent circuit.

For a uniformly magnetized magnet, the equivalent current density (current per unit length of the magnet) is [see **Eq.** (9.27)]:

$$\mathbf{M} \times \hat{\mathbf{n}} = \mathbf{J} = \hat{\boldsymbol{\phi}} M \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

This current density is on the surface of the magnet, is directed circumferentially, and produces a flux density in the same direction as **M**. The total equivalent current producing the flux density is equal to Ma. Thus, the magnetomotive force is equal to Ma. The reluctance of the path includes the two gaps and the length of the magnet since, now, the magnet has been replaced by a solenoid, the volume occupied by the magnet having permeability of free space:

$$\mathfrak{R} = \frac{2d + a}{\mu_0 \pi b^2} \qquad \left[\frac{1}{\text{H}}\right]$$

Because the reluctance in iron is zero ($\mu \to \infty$), the flux in the circuit is

$$\Phi = \frac{Ma}{\mathfrak{R}} = \frac{\mu_0 \pi b^2 Ma}{2d + a} \qquad [\text{Wb}]$$

Dividing by the cross-sectional area πb^2 gives the magnetic flux density:

$$B = \frac{\mu_0 Ma}{2d + a} \quad [T] \quad \to \quad H = \frac{B}{\mu_0} = \frac{Ma}{2d + a} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

Note: We assumed that the magnet can be replaced by a solenoid and that by doing so, the permeability of the magnet equals that of free space. In practice, this approximation is good for long magnets and for magnets in a closed magnetic path, as is the case here. The relation for B shows that in effect the gap has increased by the length of the magnet (because the magnet has low permeability). Nevertheless, the flux is contained within the magnet and thus the magnetic circuit method is valid. Good magnets have a relative permeability between 1 and 3. If the actual permeability of the permanent magnet is known, it should be used in the design.

Exercise 9.6 Calculate the magnetic field intensity in the gaps of **Example 9.18** for finite permeability of iron. Assume the relative permeability of iron is μ_r.

Answer

$$H = \frac{\mu_r M a}{\mu_r(2d + a) + (5a + 4c - 2d)} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

It is useful to generalize these results for more complex circuits, including circuits with multiple loops. To do so, we write Kirchhoff's laws for magnetic circuits based on the discussion in **Chapter 7**. By analogy between currents and fluxes and between voltages and magnetomotive forces, we can write directly:

Kirchhoff's current law

$$\sum_i \Phi_i = 0 \quad \left(\text{from } \sum_i I_i = 0\right) \tag{9.103}$$

Kirchhoff's voltage law

$$\sum_i N_i I_i = \sum_j \mathfrak{R}_j \Phi_j \quad \left(\text{from } \sum_i V_i = \sum_j R_j I_j\right) \tag{9.104}$$

In addition, the connection of reluctances in series and parallel follows the same rules as the connection of resistances in an electric circuit.

In summary, the solution to a magnetic problem using magnetic circuits follows the following steps:

(1) Determine if the flux generated in the circuit is contained within the magnetic circuit. Look in particular if permeabilities of materials in the magnetic paths are high, since flux tends to follow high-permeability paths (i.e., low-reluctivity paths). Large air gaps and low-permeability materials will tend to allow flux to "leak" out of the circuit, invalidating the assumptions of a circuit.

(2) Determine the average path lengths for each material or section of material and find the reluctances of each material using **Eq. (9.99)**.

(3) Locate and calculate the magnetomotive forces in the circuit using **Eq. (9.95)** These are either the ampere turns of coils or the equivalent ampere turns of permanent magnets.

(4) Draw an equivalent circuit in terms of voltages (magnetomotive forces), currents (fluxes), and resistances (reluctances).

(5) Use **Eqs. (9.103)** and **(9.104)** to find the unknowns. These are usually the fluxes in various parts of the magnetic circuit. From flux, other magnetic circuit parameters can be obtained.

Once an equivalent circuit has been obtained, you can use any analysis tool you wish. For complex magnetic circuits, you may even wish to use a DC circuit analysis computer program.

9.7 Forces in the Magnetic Field

We started **Chapter 8** with a short discussion on force in the magnetic field. The initial purpose was only to show that the magnetic field is related to force in a way similar to the relation of the electric field to force. Now, it is time to revisit force, quantify it, and see what the differences between forces in the magnetic and electric field are.

The magnetic field exerts a force on a moving charge which is directly proportional to the velocity and magnitude of the charge. How do we know that? Why, by experiment of course! Measuring both the force in a magnetic field in the absence of an electric field and the force in an electric field in the absence of a magnetic field, we obtain a force relation which is both general and distinguishes between electric and magnetic forces. The relation is called the Lorentz force equation (also called the Coulomb–Lorentz equation) and is written as

$$\boxed{\mathbf{F}_{total} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad [\text{N}]} \tag{9.105}$$

where $\mathbf{F}_e$ is the force due to the electric field and $\mathbf{F}_m$ is the force due to the magnetic field. If the electric field is zero, only a magnetic force is present. The Lorentz force equation for the magnetic field gives the magnetic force as

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \quad [\mathrm{N}] \tag{9.106}$$

The importance of the Lorentz force equation is that it is fundamental in electromagnetics; that is, it serves the same purpose as any of the postulates we have introduced for the electric or magnetic fields. When we come to the point of unifying the theory of electromagnetics, **Eq. (9.105)** will become one of the seven basic relations needed to completely define the theory.

We must note here the following:

(1) From a dimensional point of view, the term $\mathbf{v} \times \mathbf{B}$ must have units of electric field intensity [V/m]. This means that, in fact, the force can be viewed as an electric force, proportional to the magnetic flux density. More on this connection between the electric and magnetic field will be said in **Chapter 10**.
(2) The magnetic force term can only exist if the velocity of charges is nonzero. If all charges are stationary, only an electric force exists.
(3) The magnetic force is directed perpendicular to both direction of motion (velocity) and direction of the magnetic flux density **B**. This is a consequence of the vector product between **v** and **B**.
(4) A velocity perpendicular to the magnetic field will cause the charge or charged body to move in a circular motion, with the plane of the circle perpendicular to the direction of the magnetic field (**Figure 9.42a**).
(5) A charge moving in the direction of the magnetic flux density experiences no force (**Figure 9.42b**) because the vector product of two parallel vectors is zero.
(6) A charge with velocity components perpendicular and in the direction of the field experiences a helical motion, with the normal component causing the circular motion and the tangential component causing the translation of the path of motion (**Figure 9.42c**).

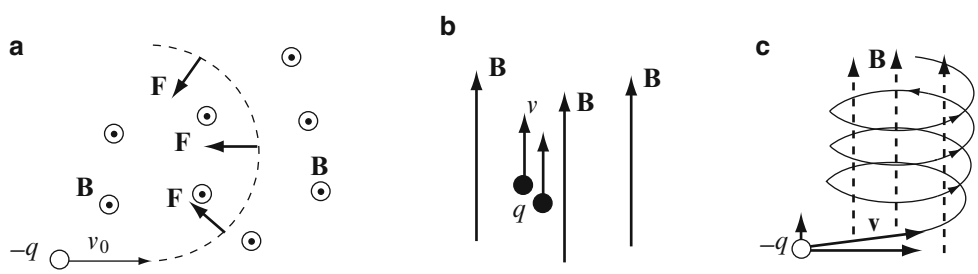


Figure 9.42 (**a**) A charge moving perpendicular to the magnetic field experiences a radial force causing circular motion. (**b**) Motion parallel to the field generates no force. (**c**) Motion with components perpendicular and parallel to the field causes the charge to move in a helical path

Example 9.19 Path of Electrons in a Magnetic Field An electron is injected into a uniform magnetic field at right angle to the flux density.

(a) If the velocity of the electron is constant at $v = 100{,}000$ m/s and the magnetic flux density is $\mathbf{B} = \hat{\mathbf{z}}\,0.5\,\mathrm{T}$, calculate the path of the electron. Assume velocity is perpendicular to **B**.
(b) Describe qualitatively the path of the electron if, in addition to the magnetic flux density in **(a)**, there is also an electric field intensity of magnitude $E = 10^5$ V/m, in the direction opposite **B**.

Solution: The magnetic and electric forces on a charged particle are defined by the Lorentz force equation. The magnetic force is perpendicular to the magnetic field and direction of motion (velocity) and the electric force is in the direction of the electric field.

(a) The magnetic force is given as

$$\mathbf{F_m} = -q\mathbf{v} \times \mathbf{B} \quad \rightarrow \quad F = \left|q\mathbf{v} \times \mathbf{B}\right| = 1.6 \times 10^{-19} \times 10^5 \times 0.5 = 0.8 \times 10^{-14} \quad [\mathrm{N}]$$

The direction of the force is perpendicular to the plane formed by **v** and **B** (r direction) and hence is perpendicular to both the flux density **B** and velocity **v** as shown in **Figure 9.43a**. Because of this, the electron will move in a circle, counterclockwise (see also **Figure 9.42a**). The radius of the circle is defined by the centrifugal force on the electron and the latter depends on its mass. The radius is calculated by equating the magnitudes of the two forces:

$$\frac{mv^2}{r} = F_m \quad \rightarrow \quad r = \frac{mv^2}{F_m} = \frac{9.1 \times 10^{-31} \times 10^{10}}{0.8 \times 10^{-14}} = 1.138 \times 10^{-6} \quad [\text{m}]$$

The radius of motion of the electron is 1.138 μm.

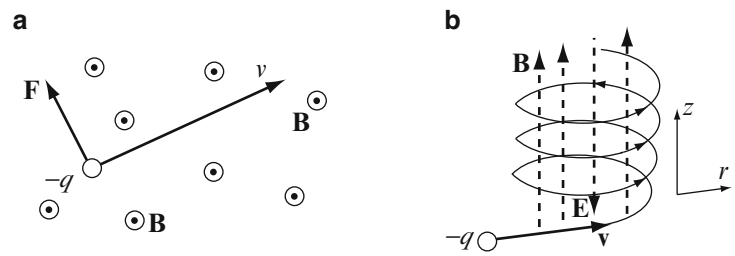


Figure 9.43 (**a**) Force on a moving electron in the presence of a magnetic field. (**b**) Path of an electron in the presence of an electric, and a magnetic field

(b) If an electric field intensity **E** exists in the direction opposite that of the magnetic field, the electron also experiences a force due to the electric field. The magnitude of the electric force is

$$|F_e| = qE = 1.6 \times 10^{-19} \times 10^5 = 1.6 \times 10^{-4} \quad [\text{N}]$$

This force is in the direction of **B** because the charge is negative. The combined effect of the magnetic and electric field causes the electron to move in an upward, counterclockwise spiral path, as shown in **Figure 9.43b**. The radius is defined by **B**, the pitch by **E**. The radius is as above (1.138 μm). The electric force is constant, causing the electron to accelerate in the z direction. The acceleration at low speeds is $a_z = F_e/m = 1.6 \times 10^{-14}/9.1 \times 10^{-31} = 1.758 \times 10^{16}$ m/s^2, but as the speed increases, the electron becomes relativistic and the rest mass cannot be used. The electron will eventually reach a speed close to the speed of light, provided the electric field can be maintained over the path of the electron.

The magnetic force given in **Eq. (9.106)** is not particularly convenient to apply since it relies on forces on individual charges. It is usually not possible to account for individual charges (say electrons) and we certainly do not know the velocities of individual charged particles. Instead, it is more convenient to develop relations for forces on currents or current densities in the magnetic field. This is not done only for convenience's sake: In practical applications, the forces on current-carrying conductors are extremely important and there is a real benefit to be gained from knowledge of these forces.

To find appropriate relations, we recall the definition of current density in terms of charges in a conductor. Consider **Figure 9.44a**, where a volume $dv' = ds'dl'$ is given. The volume contains N charges per unit volume. The number of charges in the differential volume is Ndv', all moving at an average velocity in the direction of the conductor in response to an externally applied electric field intensity $\mathbf{v} \times \mathbf{B}$. The force on the differential volume can, therefore, be written as

$$d\mathbf{F}_m = Ndv' q\mathbf{v} \times \mathbf{B} \quad [\text{N}] \tag{9.107}$$

where q is the charge of an individual particle (such as an electron). The use of $d\mathbf{F}_m$ indicates the force due to the volume dv'. Following the discussion in **Section 7.3.2 [Eq. (7.6)]**, the current density in this segment of conductor is given as $\mathbf{J} = Nq\mathbf{v}$ [A/m^2]. Thus, the force in **Eq. (9.107)** can be written as

$$d\mathbf{F}_m = \mathbf{J} \times \mathbf{B}dv' \quad [\text{N}] \tag{9.108}$$

where the product $\mathbf{J} \times \mathbf{B}$ may be viewed as a volumetric force density **f**:

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} \quad \left[\frac{\text{N}}{\text{m}^3}\right] \tag{9.109}$$

The direction of force is given by the right-hand rule and is shown in **Figure 9.44b**. For a given volume, the total force is an integration of this relation over the volume:

$$\boxed{\mathbf{F}_m = \int_{v'} \mathbf{J} \times \mathbf{B} dv' \quad [\mathrm{N}]} \tag{9.110}$$

This is a common statement of the magnetic (Lorentz) force and, in many ways, is a more useful relation since it gives the force on distributed currents. From this, we can now find the force on a current-carrying conductor such as a thin wire or a current loop (closed circuit).

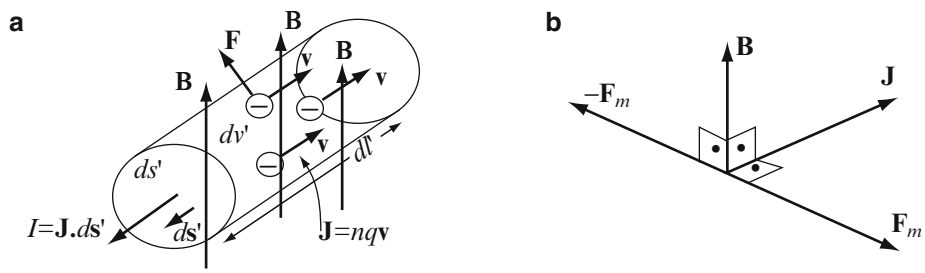


Figure 9.44 Forces on currents in the magnetic field. (**a**) Force on a differential volume carrying current density **J**. (**b**) Direction of force on a current-carrying conductor in the magnetic field

The force on a current-carrying conductor can be written from **Eq. (9.110)** and **Figure 9.44a** by noting that the total current through the conductor is equal to the current density multiplied by the cross-sectional area of the conductor:

$$I = \mathbf{J} \cdot d\mathbf{s}' \quad [\mathrm{A}] \quad \text{or} \quad I d\mathbf{l}' = \mathbf{J} dv' \quad [\mathrm{A} \cdot \mathrm{m}] \tag{9.111}$$

Substituting this in **Eq. (9.108)** we obtain the force on an element of current as[8]

$$\boxed{d\mathbf{F}_m = I d\mathbf{l}' \times \mathbf{B} \quad [\mathrm{N}]} \tag{9.112}$$

where the direction of $d\mathbf{l}'$ is in the direction of the current I in the conductor. For a conducting segment of length L' and carrying a current I, the total force is

$$\mathbf{F}_m = \int_{L'} I d\mathbf{l}' \times \mathbf{B} \quad [\mathrm{N}] \tag{9.113}$$

Similarly, if the force on a closed circuit is needed, a closed contour integration is required:

$$\mathbf{F}_m = \oint_c I d\mathbf{l}' \times \mathbf{B} \quad [\mathrm{N}] \tag{9.114}$$

The fundamental relation for force in the present approach is that of **Eq. (9.112)** or **Eq. (9.108)**. It gives the force on an element of current in a magnetic field. Now, we can also calculate the force between two current elements by noting that the magnetic flux density can be generated by an element of a second current. Referring to **Figure 9.45**, an element of current $d\mathbf{l}_1'$ produces a magnetic flux density $d\mathbf{B}_{12}$ at the location of a second element of current $d\mathbf{l}_2'$. This second element will experience a force $d\mathbf{F}_{12}$ caused by the action of the field of current segment $d\mathbf{l}_1'$ on current segment $d\mathbf{l}_2'$. The opposite is also true: current element $d\mathbf{l}_2'$ produces a magnetic flux density at the location of $d\mathbf{l}_1'$ and this produces a force $d\mathbf{F}_{21}$ on element $d\mathbf{l}_1'$ due to the field of $d\mathbf{l}_2'$. Let's take this in three steps:

[8] This force is called the Laplace force although we derived it from the Lorentz force. The distinction between the two forces is in the fact that the Laplace force is a macroscopic force on the ensemble of moving charges whereas the Lorentz force is the microscopic force on a moving charged particle.

(1) The magnetic flux density due to current element $d\mathbf{l}_1'$ at a distance R_{12} in space is given by the Biot–Savart law as

$$d\mathbf{B}_{12} = \frac{\mu_0 I_1}{4\pi} \frac{d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}}{R_{12}^2} \tag{9.115}$$

where $\mathbf{R}_{12}$ is the vector connecting $d\mathbf{l}_1'$ and $d\mathbf{l}_2'$, as shown in **Figure 9.45**.

(2) Calculate the force acting on $d\mathbf{l}_2'$ using **Eq. (9.112)**:

$$d\mathbf{F}_{12} = I_2 d\mathbf{l}_2' \times d\mathbf{B}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} d\mathbf{l}_2' \times \left[\frac{d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}}{R_{12}^2}\right] \quad [\mathrm{N}]. \tag{9.116}$$

(3) Calculate the total force on any segment of C_2 (P_3 to P_4) due to a segment of the circuit C_1 (P_1 to P_2) by integrating along C_2 and then along C_1:

$$\boxed{\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{p_3}^{p_4} \int_{p_1}^{p_2} \frac{d\mathbf{l}_2' \times \left(d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}\right)}{R_{12}^2} \quad [\mathrm{N}]} \tag{9.117}$$

To find this relation, we assumed that a circuit exists, but only part of each circuit contributes to the force. This is useful whenever we have current segments or when a force per unit length is required. If, instead, we need to calculate the force due to an entire closed circuit on a second closed circuit, the integration must be carried out as closed contour integrals over each circuit. Using the notation of **Figure 9.45**, we can write:

$$\boxed{\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_2' \times \left(d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}\right)}{R_{12}^2} \quad [\mathrm{N}]} \tag{9.118}$$

What about the force $\mathbf{F}_{21}$ (force on circuit C_1 due to the magnetic field produced by circuit C_2)? We could derive the force in the same manner as above. However, from Newton's third law, $\mathbf{F}_{21}$ is the reaction force to $\mathbf{F}_{12}$. Therefore, $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Equation (9.118) is known as the Ampere force law. This force was obtained for two thin current-carrying conductors (the thin current assumption is required for Biot–Savart's law to apply). However, superposition of filamentary currents can be used to calculate the forces between thick conductors. Note that the cross products must be taken properly (the vector product in parentheses is calculated first) and the direction of the vector $\mathbf{R}_{12}$ is from $d\mathbf{l}_1'$ to $d\mathbf{l}_2'$. Note also that I_1 and I_2 are always positive numbers, whereas $d\mathbf{l}_1'$ and $d\mathbf{l}_2'$ can be either positive or negative, depending on their directions in space. Taking the contours as positive in the directions of the currents ensures the correct sign for the force.

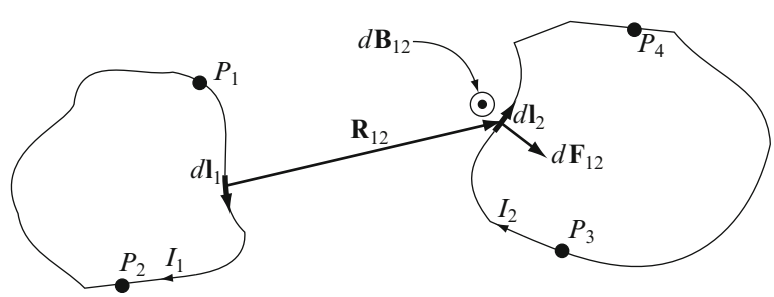


Figure 9.45 Relation between a current element in loop (1) and the force it exerts on a current element in loop (2)

Example 9.20 Application: Force Exerted by the Geomagnetic Field on Power Distribution Lines A 500 km long DC power distribution line carries a current of 1,000 A.

(a) Assuming that the perpendicular component of the terrestrial magnetic flux density is 50 μT (typical), calculate the total force exerted on one conductor of the distribution line.

(b) Find the direction of the force for the two-conductor line carrying current as above.

Solution: The force may be calculated using **Eq. (9.114)** directly since the magnetic field can be assumed to be constant everywhere on the line.

(a) The force is

$$\mathbf{F}_m = \oint_C I d\mathbf{l}' \times \mathbf{B} \quad [\mathrm{N}]$$

The magnitude of the force is calculated as

$$F_m = IB \int_C dl' = LIB \quad [\mathrm{N}]$$

Note the use of the open integration. The force for a segment of the circuit is calculated, but the implicit assumption that the circuit closes must be made for the current to flow. The total force on a conductor is

$$F_m = 500,000 \times 1000 \times 0.00005 = 25,000 \quad [\mathrm{N}]$$

This is a significant force, but it is only 0.05 N per meter length of the cable. This is negligible compared to other forces, including the weight of the cable, wind and snow loads, tension due to temperature variations, and forces between conductors due to their currents (see **Example 9.21**).

(b) Using the right-hand rule for the force, the forces on each conductor are such that the conductors are moved toward each other, as shown in **Figure 9.46**.

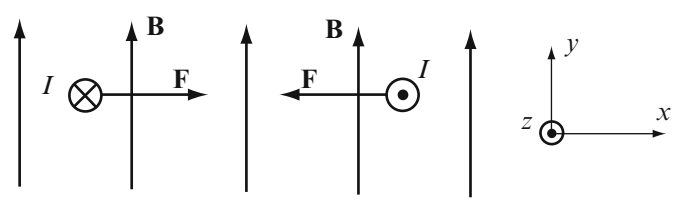


Figure 9.46 Forces on overhead current-carrying conductors due to the terrestrial magnetic field

Example 9.21 Application: Force Between Two Overhead Transmission Lines Carrying Direct Currents Two currents cause a force on each other. In overhead transmission lines, the currents may be quite high. Because of this force, when the current is switched on under load or a short circuit exists on the line, the cables may swing violently from side to side. The distance between the cables must be such that under the most severe swing, either due to faults on the line or wind, the cables do not touch.

A long power transmission line carries a current $I = 1{,}200$ A. The distance between two towers is $L = 100$ m and the distance between the two conductors is $d = 3$ m. Calculate the total force between the two cables between each two towers. What is the direction of this force?

Solution: The force may be calculated using Ampere's force law or, alternatively, using **Eq. (9.112)** directly. The latter is possible because the force is constant along the line, and because it is much simpler, we use it here. The configuration is shown in **Figure 9.47**.

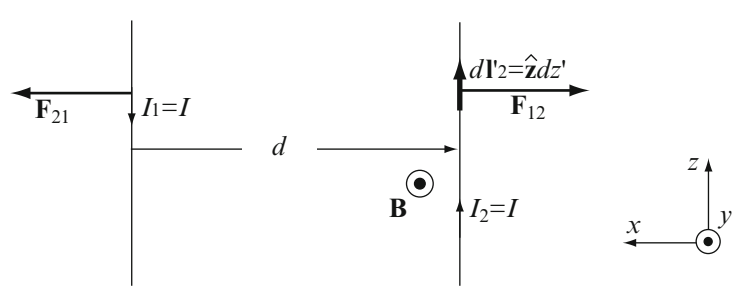


Figure 9.47 Forces on power lines due to the currents in the lines

The magnetic flux density at a distance d from the current I in **Figure 9.47** at the location of the second conductor [current (2)] is (see **Example 9.9**):

$$\mathbf{B} = \hat{\mathbf{y}}\,\frac{\mu_0 I}{2\pi d} \quad [\mathrm{T}]$$

From **Eq. (9.112)** and taking $d\mathbf{l}_2 = \hat{\mathbf{z}}dz$ we get

$$d\mathbf{F}_m = I\,d\mathbf{l}_2' \times \mathbf{B} = I\hat{\mathbf{z}}\,dz' \times \hat{\mathbf{y}}\,\frac{\mu_0 I}{2\pi d} = -\hat{\mathbf{x}}\,\frac{\mu_0 I^2 dz'}{2\pi d}$$

This is the force exerted on segment (2) by segment (1). Since the force is independent of y, the force on a section of length $L = 100$ m is simply 100 times larger:

$$\mathbf{F}_{12} = -\hat{\mathbf{x}}\,\frac{\mu_0 I^2 L}{2\pi d} = -\hat{\mathbf{x}}\,\frac{4 \times \pi \times 10^{-7} \times 1200^2 \times 100}{2 \times \pi \times 3} = -\hat{\mathbf{x}}\,9.6 \quad [\mathrm{N}]$$

The force on the left cable (1) is to the left, and on the right cable ($\mathbf{F}_{12} = -\mathbf{F}_{21}$) is to the right and equal in magnitude. This force tends to separate the wires.

Example 9.22 A very long, thin conductor passes midway between two square loops as shown in **Figure 9.48**. The wire carries a current $I_0 = 5$ A, loop (1) carries a current $I_1 = 2$A, and loop (2) carries a current $I_2 = 4$ A as shown in **Figure 9.48**. Calculate the total force on the wire due to the two loops.

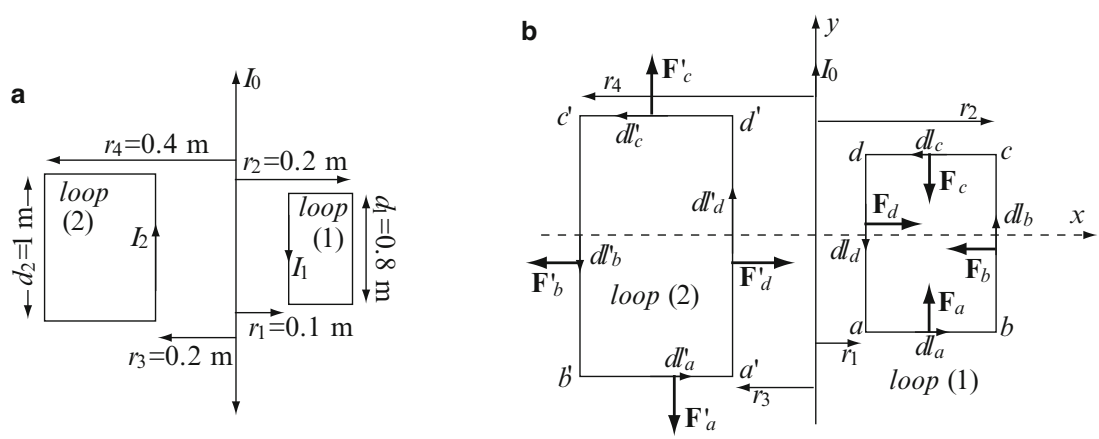


Figure 9.48 Forces between two loops and a current-carrying wire

Solution: There are two ways this problem may be solved. One is to use the general expression in **Eq. (9.118)**. Doing so requires treating the force on the wire due to each loop separately. The force due to loop (1) is first evaluated integrating over the two closed paths: loop (1) and the wire from $-\infty$ to $+\infty$. Then, the process is repeated for loop (2) and the two forces are summed. A simpler way, one we follow here, is to calculate the field due to the wire at the two loops and then calculate the force the wire exerts on each of the segments of the loops using **Eq. (9.112)** or **Eq. (9.114)**. Since the force on the wire is then the negative of the force on the loops, the force on the wire is also immediately available.

The magnetic flux density at any location in space due to the infinitely long wire is given as $\mu_0 I/2\pi r$. In this case, the direction of the magnetic flux density is in the positive z direction to the left of the wire and in the negative z direction to the right of the wire:

$$\mathbf{B}_2 = \hat{\mathbf{z}}\,\frac{\mu_0 I}{2\pi r} \quad [T] \quad \text{for} \quad x < 0, \quad \mathbf{B}_1 = -\hat{\mathbf{z}}\,\frac{\mu_0 I}{2\pi r} \quad [\mathrm{T}] \quad \text{for} \quad x > 0$$

With the magnetic flux densities above, the forces on the segments of loop (1) are **[see Eq. (9.112)]**

$$\mathbf{F}_1 = \oint_{C_1} I d\mathbf{l} \times \mathbf{B}_1 = \int_a^b I d\mathbf{l}_a \times \mathbf{B}_1 + \int_b^c I d\mathbf{l}_b \times \mathbf{B}_1 + \int_c^d I d\mathbf{l}_c \times \mathbf{B}_1 + \int_d^a I d\mathbf{l}_d \times \mathbf{B}_1 \quad [\mathrm{N}]$$

On loop (2), we get

$$\mathbf{F}_2 = \oint_{C_2} I d\mathbf{l}' \times \mathbf{B}_2 = \int_{b'}^{a'} I d\mathbf{l}'_a \times \mathbf{B}_2 + \int_{a'}^{d'} I d\mathbf{l}'_b \times \mathbf{B}_2 + \int_{d'}^{c'} I d\mathbf{l}'_c \times \mathbf{B}_2 + \int_{c'}^{b'} I d\mathbf{l}'_d \times \mathbf{B}_2 \quad [\mathrm{N}]$$

These forces are shown in **Figure 9.48b**. The first thing to note is that $\mathbf{F}_a$ and $\mathbf{F}_c$ cancel each other as do $\mathbf{F}'_a$ and $\mathbf{F}'_c$ Second, the forces on opposite sides of the loop are in opposite directions, but $\mathbf{F}_d > \mathbf{F}_b$ and $\mathbf{F}'_d > \mathbf{F}'_b$. Thus, the total force on the loops, exerted by the central wire, is

$$\mathbf{F}_l = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_d + \mathbf{F}_b + \mathbf{F}'_b + \mathbf{F}'_d = \int_d^a I d\mathbf{l}_d \times \mathbf{B}_1 + \int_b^c I d\mathbf{l}_b \times \mathbf{B}_1 + \int_{c'}^{b'} I d\mathbf{l}'_b \times \mathbf{B}_2 + \int_{a'}^{d'} I d\mathbf{l}'_d \times \mathbf{B}_2 \quad [\mathrm{N}]$$

Now, we must evaluate the magnetic flux density at the locations of the segments of the loops. The directions of the various elements of length are given in **Figure 9.48b**. With these, the force on the loops is

$$\begin{aligned}\mathbf{F}_l = &\int_d^a I_1(-\hat{\mathbf{y}}dy) \times \left(-\hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi r_1}\right) + \int_b^c I_1(\hat{\mathbf{y}}dy) \times \left(-\hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi r_2}\right) \\ &+ \int_{c'}^{b'} I_2(-\hat{\mathbf{y}}dy) \times \left(\hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi r_4}\right) + \int_{a'}^{d'} I_2(\hat{\mathbf{y}}dy) \times \left(\hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi r_3}\right) \quad [\mathrm{N}]\end{aligned}$$

Note that the direction of the field is defined by the vector product rather than the direction of integration. Evaluating the four integrals gives

$$\mathbf{F}_l = \hat{\mathbf{x}}\,\frac{\mu_0 I_0 I_1 d_1}{2\pi r_1} - \hat{\mathbf{x}}\,\frac{\mu_0 I_0 I_1 d_1}{2\pi r_2} + \hat{\mathbf{x}}\,\frac{\mu_0 I_0 I_2 d_2}{2\pi r_3} - \hat{\mathbf{x}}\,\frac{\mu_0 I_0 I_2 d_2}{2\pi r_4} \quad [\mathrm{N}]$$

With the values given, the total force is

$$\begin{aligned}\mathbf{F}_l &= \hat{\mathbf{x}}\,\frac{\mu_0 I_0}{2\pi}\left[\frac{I_1 d_1}{r_1} - \frac{I_1 d_1}{r_2} + \frac{I_2 d_2}{r_3} - \frac{I_2 d_2}{r_4}\right] \\ &= \hat{\mathbf{x}}\,\frac{4 \times \pi \times 10^{-7} \times 5}{2 \times \pi}\left[\frac{2 \times 0.8}{0.1} - \frac{2 \times 0.8}{0.2} + \frac{4 \times 1}{0.2} - \frac{4 \times 1}{0.4}\right] = -\hat{\mathbf{x}}\,1.8 \times 10^{-5} \quad [\mathrm{N}]\end{aligned}$$

The force on the wire ($\mathbf{F}_w = -\mathbf{F}_l$) is $\hat{\mathbf{x}}\,1.8 \times 10^{-5}$ [N].

Exercise 9.7 Application: Experimental Definition of the Ampere One method of defining the ampere is as follows: A 1 m length of thin wire is placed parallel to and a distance 1 m from a very long straight wire as shown in **Figure 9.49**. Each wire carries a current I. The force between the segment and the infinite wire is measured. If the current in both wires is 1 A, what is the measured force between the segment and the long wire? (The standard for the ampere is a modified form of this arrangement which uses coils rather than straight wires to increase the measured force and, therefore, reduce the error involved in the measurement of very small forces).

Answer 2×10^{-7} [N].

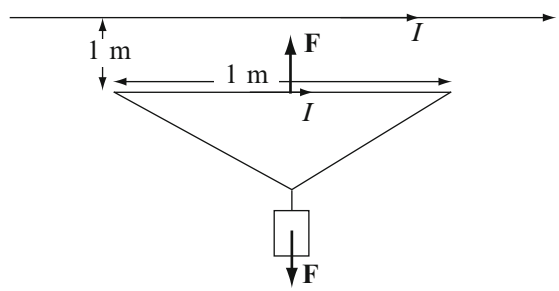


Figure 9.49 Method of defining the ampere in terms of measured forces

9.7.1 Principle of Virtual Work: Energy in a Gap

The calculation of force using Ampere's force law in **Eq. (9.118)** is particularly suited for simple configurations such as segments of wires or long conductors. There are, however, a large number of applications in which this method is next to impossible to apply. For example, the force between two magnets cannot be calculated using this method. Another method, which is sometimes easier to apply, is the method of ***virtual work*** or ***virtual displacement.*** We discussed the principle in **Section 4.8.2**. It relies on the basic method of measuring force; when measuring force (as, for example, in weighing), we allow the force to move the body (on which the force operates) a small distance such that a mechanism or sensor may be activated. This distance is then a measure of the force. If a force **F** exists between two pieces of material as shown in **Figure 9.50a**, energy is also associated with the system. If we now allow the two pieces to move a distance dl, the energy in the gap will be reduced by an amount $\mathbf{F} \cdot d\mathbf{l}$. If allowed, the two pieces will move closer together reducing the gap volume (**Figure 9.50b**). Thus, assuming the potential magnetic energy of the system to be W before the motion, the change in the potential energy of the system is

$$\mathbf{F} \cdot d\mathbf{l} = -dW \quad [\mathrm{N}] \tag{9.119}$$

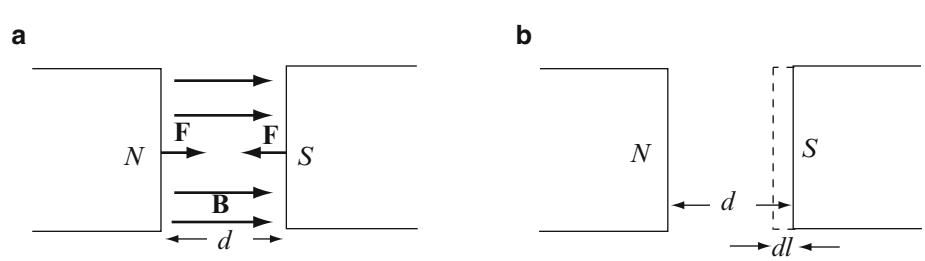


Figure 9.50 Method of virtual work. (**a**) Force between two magnets. (**b**) Calculation of force through virtual work

Since the total derivative dW can be written in terms of the gradient (see **Sections 2.3.1** and **4.8.2**) as

$$dW = (\nabla W) \cdot d\mathbf{l} \tag{9.120}$$

The force is

$$\boxed{F = -\nabla W \quad [\mathrm{N}]} \tag{9.121}$$

Thus, the magnetic force is the negative of the gradient in potential energy. In expanded form, we can write in Cartesian coordinates

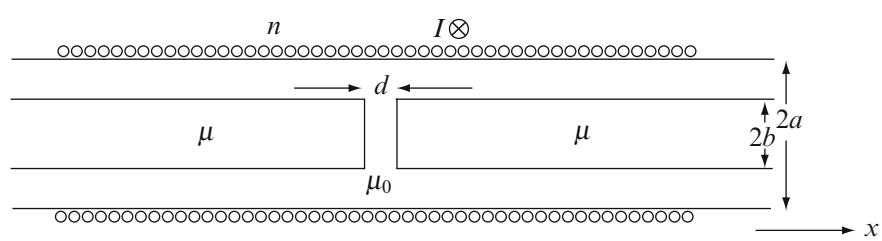
$$\mathbf{F} = \hat{\mathbf{x}} F_x + \hat{\mathbf{y}} F_y + \hat{\mathbf{z}} F_z = -\hat{\mathbf{x}} \frac{\partial W}{\partial x} - \hat{\mathbf{y}} \frac{\partial W}{\partial y} - \hat{\mathbf{z}} \frac{\partial W}{\partial z} \quad [\mathrm{N}] \tag{9.122}$$

As expected, the force may have three components in space provided that energy changes as the body moves in a particular direction. If it does not, the force in that particular direction is zero.

Note: It is implicit that the virtual displacement dl does not change the energy density or total energy in the gap since in reality, nothing moves.

Example 9.23 Application: Forces on Iron Pieces in a Uniform Magnetic Field—The Solenoid Valve An infinitely long solenoid of radius a [m] has n turns per unit length and carries a current I [A]. Two very long pieces of iron, each of radius a [m], are placed in the solenoid and are separated by a very small gap of length d [m], as shown in **Figure 9.51**. Permeability of iron is μ [H/m] and that of free space is μ_0 [H/m]. Calculate the force between the two pieces of iron. What is the direction of this force?

Figure 9.51



Solution: The magnetic field intensity in the solenoid is calculated, and from it, the magnetic flux density in the gap. Now, allowing a virtual displacement of the iron pieces, we calculate the change in energy due to this displacement, from which we obtain the force.

The magnetic flux density is in the negative x direction with magnitude

$$B = \mu H = \mu n I \quad [\mathrm{T}]$$

Since the gap is small, we may assume that the same flux density exists in the gap and that the flux density is constant throughout the gap. The total energy in the gap is [see **Eq. (9.89)**]

$$W_m = \frac{B^2}{2\mu_0} v_{gap} = \frac{B^2 \pi b^2 d}{2\mu_0} = \frac{\mu^2 n^2 I^2 \pi b^2 d}{2\mu_0} \quad [\mathrm{J}]$$

To calculate the force, we replace d (width of gap) by dl to get an expression for dW and calculate the total force on the poles using **Eq. (9.122)**. This is the same as to calculate the change in energy due to the change in volume caused by the displacement dl. We assume dl is positive and, therefore, increases the potential energy in the system; that is, dW as well as dl are positive. If the right pole moves a distance dl to the right, the force due to this positive displacement is

$$F = -\hat{\mathbf{x}}\,\frac{dW_m}{dl} = -\hat{\mathbf{x}}\,\frac{\mu^2 n^2 I^2 \pi b^2}{2\mu_0} \quad [\mathrm{N}]$$

We could also assume dl to be negative (the right pole moves a distance dl to the left). This reduces the gap, and, therefore, both dW and dl are negative. The resulting force is again in the negative x direction. In other words, it does not matter what direction we assume for dl as long as the change in energy dW is calculated accordingly. This force acts on the right pole and tends to close the gap because this reduces the energy in the system. The force on the left pole is in the positive x direction (from Newton's law) and it also tends to close the gap. If allowed, the pieces would move toward each other.

Example 9.24 Application: Force in a Magnetic Circuit—The Airless Sprayer An airless sprayer is a device used to spray fluids such as paints or pesticides without the use of compressed air. The device uses a simple piston action to push the fluid through an orifice at high velocity. The pump used in most sprayers is a hinged electromagnet with a gap, as shown in **Figure 9.52a**. When the current is switched on, the gap closes, moving the piston. When the current is switched off, the piston returns to its original position. This motion is sufficient to pump fluid in and to expel it at high velocity. Commercial sprayers use an AC source, causing the piston to move 100 or 120 times per second, depending on the frequency of the electrical grid. The same idea is used in small air pumps such as those used in fish tanks. The most characteristic aspect of these devices is relatively high noise levels and heat produced in the magnetic core. This type of device is also used extensively to open and close valves and other types of linear actuators.

A pump used to spray paint is made as shown in **Figure 9.52b**. The coil contains $N = 5000$ turns and carries a current $I = 0.1$ A. Permeability of iron is $\mu = 1{,}000\mu_0$ and the gaps are $d = 5$ mm long. Assume all flux is contained in the gaps (no leakage of flux). Other dimensions are given in the figure.

(a) Calculate the force exerted by the moving piece on the piston.
(b) What is the force if the gaps are reduced to 1 mm?

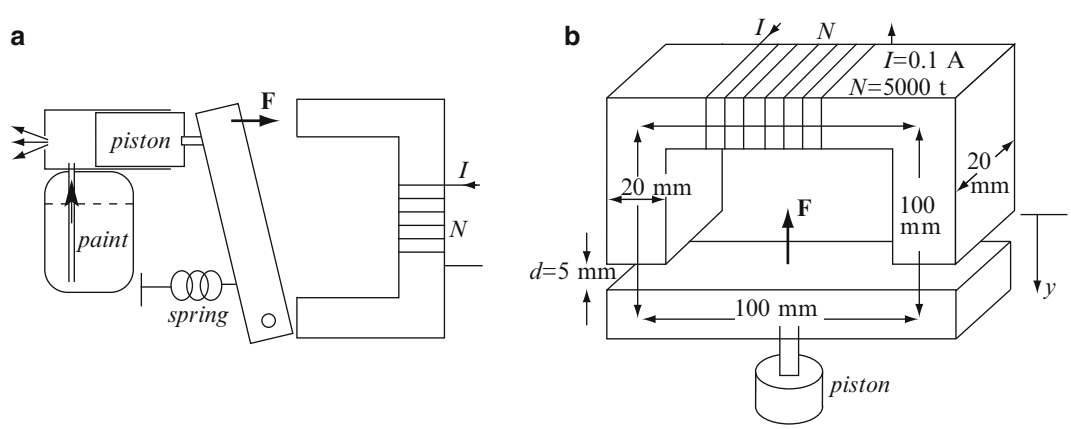


Figure 9.52 (**a**) A hinged electromagnet used in an airless sprayer. (**b**) A modified form of the electromagnet in (**a**) used for calculation of forces

Solution: In this case, there are two gaps. The energy in the two gaps is calculated and then we assume the lower piece moves up a distance dl, thereby reducing the gaps and reducing the potential energy stored in the gaps. Thus, both dl and dW are negative.

(a) The magnetic flux density in the gap is calculated using the magnetic circuit shown in **Figure 9.52b**. The reluctances of the iron and gaps are

$$\mathfrak{R}_i = \frac{l_{iron}}{\mu_{iron} S} = \frac{0.39}{1000 \times 4 \times \pi \times 10^{-7} \times 0.02^2} = 7.759 \times 10^5 \quad \left[\frac{1}{\mathrm{H}}\right]$$

$$\mathfrak{R}_g = \frac{l_{gap}}{\mu_0 S} = \frac{0.01}{4 \times \pi \times 10^{-7} \times 0.02^2} = 1.99 \times 10^7 \quad \left[\frac{1}{\mathrm{H}}\right]$$

where S is the cross-sectional area of the magnetic circuit. The flux density in iron and, therefore, in the gap is

$$B_g = \frac{\Phi_g}{S} = \frac{NI}{S\left(\mathfrak{R}_i + \mathfrak{R}_g\right)} = \frac{5000 \times 0.1}{0.02^2 \times \left(7.759 \times 10^5 + 1.99 \times 10^7\right)} = 0.06 \quad [\mathrm{T}]$$

Since the flux density is constant in the gaps, the total energy stored in the two gaps is

$$W_m = \frac{B^2}{\mu_0} v_{gap} = \frac{B^2 S d}{\mu_0} \quad \rightarrow \quad dW = \frac{B^2 S\, dl}{\mu_0} \quad [\mathrm{J}]$$

For a length $-dl$ (which reduces the gap), dW is also negative. From **Eq. (9.122)**, we get

$$\mathbf{F} = -\hat{\mathbf{y}}\, \frac{-dW_m}{-dl} = -\hat{\mathbf{y}}\, \frac{B^2 S}{\mu_0} = -\hat{\mathbf{y}}\, \frac{0.06^2 \times 0.02^2}{4 \times \pi \times 10^{-7}} = -\hat{\mathbf{y}}\, 1.146 \quad [\mathrm{N}].$$

(b) The reluctance in iron remains the same, but the reluctance in the gap has decreased by a factor of 5:

$$\mathfrak{R}_g = \frac{l_{gap}}{\mu_0 S} = \frac{0.002}{4 \times \pi \times 10^{-7} \times 0.02^2} = 3.979 \times 10^6 \quad \left[\frac{1}{\mathrm{H}}\right]$$

The flux density is therefore

$$B_g = \frac{NI}{S\left(\mathfrak{R}_i + \mathfrak{R}_g\right)} = \frac{5000 \times 0.1}{0.02^2 \times \left(7.759 \times 10^5 + 3.979 \times 10^6\right)} = 0.263 \quad [\mathrm{T}]$$

The force is

$$\mathbf{F} = -\hat{\mathbf{y}}\, \frac{B^2 S}{\mu_0} = -\hat{\mathbf{y}}\, \frac{0.263^2 \times 0.02^2}{4 \times \pi \times 10^{-7}} = -\hat{\mathbf{y}}\, 22.02 \quad [\mathrm{N}]$$

The force increases as the gap decreases because the flux density is higher. In practical designs of this type, the gap is kept at a minimum and the motion of the moving piece is also small to keep the force more or less constant. If, for example, the moving piece were to move 4 mm the gap would change from 5 mm to 1 mm and the force would vary from 1.146 N to 22.02 N.

9.8 Torque

Torque is the product of force **F** and length of arm d about an axis, as shown in **Figure 9.53**. In our case, the force is a magnetic force. Torque is an important aspect of all rotating machinery. It is responsible for the rotation of the magnetic needle in a compass, as well as rotation of the rotor in an electric motor.

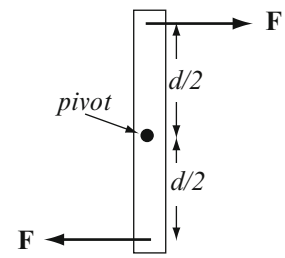


Figure 9.53 Relation between force and torque

To define torque, consider a small, square loop, w by d meters in size, placed in a uniform magnetic flux density $\mathbf{B}$, as shown in **Figure 9.54a**. The loop is constrained to rotate about an axis. A current flows in the loop as shown. Using **Eq. (9.113)**, we can calculate the forces on each of the four sides of the loop. On side ab, the current is in the positive z direction, and $\mathbf{B}$ is in the positive x direction:

$$\mathbf{F}_{ab} = \int_a^b I\,d\mathbf{l}' \times \mathbf{B} = \int_a^b I\left(\hat{\mathbf{z}}dl'\right) \times \hat{\mathbf{x}}B = \hat{\mathbf{y}}\,wIB \quad [\mathrm{N}] \tag{9.123}$$

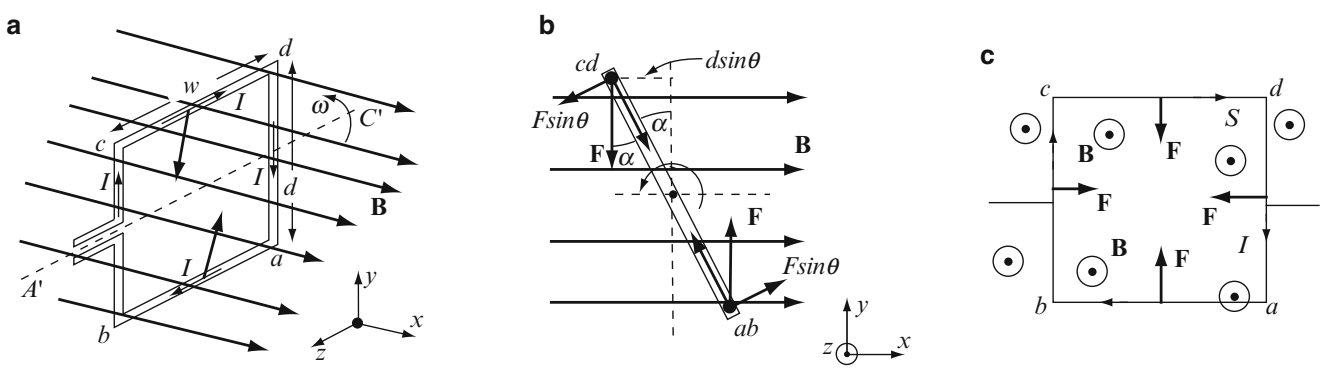


Figure 9.54 Forces and torque on a loop in a magnetic field. (**a**) Configuration and dimensions. (**b**) Plane of the loop at an angle α to the normal. (**c**) Forces in the plane of the loop do not contribute to torque

On side cd, the current is in the opposite direction and therefore

$$\mathbf{F}_{cd} = -\hat{\mathbf{y}}\,wIB \quad [\mathrm{N}] \tag{9.124}$$

The forces on sides ad and bc are in the direction of the axis (z axis; see **Figure 9.54c**) and, therefore, cannot contribute to torque. Thus, only the forces on ab and cd contribute to torque. The magnitude of torque is therefore

$$T = 2F\frac{d\sin\alpha}{2} \quad [\mathrm{N \cdot m}] \tag{9.125}$$

where $(d/2)sin\alpha$ is the armlength on which the force operates and F is the magnitude of $\mathbf{F}_{cd}$ (or $\mathbf{F}_{ab}$) (**Figure 9.54b**). The factor of 2 is due to contribution of the two forces to torque. Thus, the torque is

$$T = wdIB\sin\alpha \quad [\mathrm{N \cdot m}] \tag{9.126}$$

This is the general expression for the magnitude of torque. We can also write this in a different way by noting the following:

(a) The product $wdI = m$ is the magnitude of the magnetic dipole moment of the loop $\mathbf{m} = \hat{\mathbf{n}}\,wdI$.
(b) The term $wdIB\sin\alpha = mB\sin\alpha$ can be written as the magnitude of the vector product of the vectors $\mathbf{m}$ and $\mathbf{B}$, since torque is perpendicular to $\mathbf{B}$ and to $\mathbf{m}$.

Thus,

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad [\mathrm{N \cdot m}] \tag{9.127}$$

This expression has the advantage that the direction of torque becomes evident. Because torque is perpendicular to both **m** and **B**, it must be in the direction of the axis (**m** is perpendicular to the surface of the loop; see also **Section 9.2.1**).

Example 9.25 Application: Torque on a Square Coil and the Principle of Electric Motors A square coil consists of $N = 100$ turns, tightly packed together. The coil is placed in a uniform magnetic flux density $B = 0.2$ T as shown in **Figure 9.55**. The coil is $a = 200$ mm on the side and carries a current $I = 10$ A. Calculate the maximum torque on the coil.

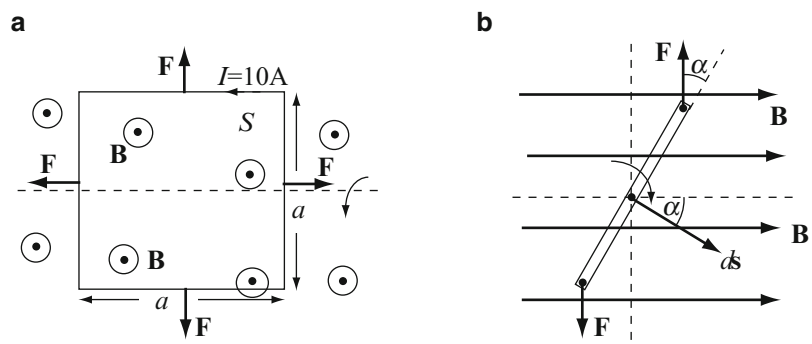


Figure 9.55 Square coil in a uniform magnetic field. (**a**) Top view. (**b**) Side view along the axis of rotation

Solution: The torque of the square loop is given in **Eq. (9.126)**. Since all 100 turns of the coil are close together, we may assume they are at the same angle with respect to the field and, therefore, the torque of the coil is N times larger.

With $w = d = a$, the torque is

$$T = Na^2IB\sin\alpha = 100 \times 0.2^2 \times 10 \times 0.2\sin\alpha = 8\sin\alpha \quad [\mathrm{N \cdot m}]$$

The maximum torque is 8 N·m and occurs when the plane of the coil (loop) is aligned with the magnetic flux density ($\alpha = 90°$). This torque is responsible for rotation in electric motors. The use of coils, rather than single loops, allows a significant increase in torque. In practical motors, when the coil has rotated one half-turn, the direction of the current is reversed so that the coil continues to rotate.

Example 9.26 Torque on an Electron The electron, as it rotates around its axis, produces a current as was shown in **Example 9.1**. This current is in effect a small loop. Suppose an electron rotates around the hydrogen atom in a circle of radius $r = 5 \times 10^{-11}$ m and produces the equivalent of 1 mA current. Calculate the magnitude and direction (with respect to the axis of rotation) of the torque if the electron is placed in the field of a magnetic resonance imaging magnet of 3 T. Assume the magnetic flux density and the axis of the electron are at an arbitrary angle as shown in **Figure 9.56**.

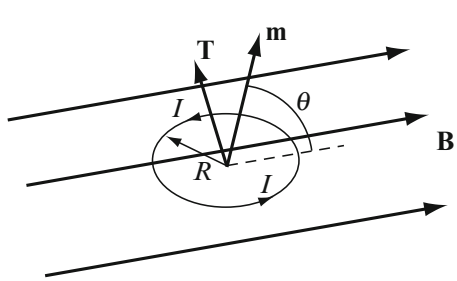


Figure 9.56 An electron spinning in the field of a magnetic resonance imaging device

Solution: After calculating the magnetic dipole moment of the electron, we use **Eq. (9.127)** to calculate the torque.

The magnitude of the magnetic moment of the loop is

$$m = \pi r^2 I = \pi\left(5 \times 10^{-11}\right)^2 \times 10^{-3} = 7.854 \times 10^{-24} \quad \left[\mathrm{A \cdot m^2}\right]$$

First, we note that the torque is in the plane of the loop, perpendicular to the plane formed by **B** and the axis of the loop. The magnitude of the torque is

$$T = \left|\mathbf{m} \times \mathbf{B}\right| = mB\sin\theta = 7.854 \times 10^{-24} \times 3\ \sin\theta = 2.356 \times 10^{-23}\ \sin\theta \quad [\mathrm{N \cdot m}]$$

The torque is maximum when the axis of rotation of the electron and the magnetic flux density are perpendicular to each other, and zero when they are parallel. The torque is always in the plane of the loop created by the rotating electron, and perpendicular to the flux density.

9.9 Applications

Application: Flispping of Magnetization in Lava as an Indication of Flipping of Earth's Magnetic Field An interesting manifestation of magnetization and the Curie temperature can be found in lava layers that have solidified while under the influence of the terrestrial magnetic field. There is evidence that the magnetic field of the Earth flips direction every 30,000 to 50,000 years or so. Lava deposits on the bottom of oceans, in particular in the Atlantic Ocean near Iceland and in the south Pacific, show a pattern of very long strips, alternately magnetized north to south and south to north. This gave rise to the theory that the magnetic field of the Earth reverses regularly. The explanation of the regular magnetization features on the seabed is that molten rock, as it surged from the Atlantic ridge, has solidified in the presence of the geomagnetic field. As the temperature rose above the Curie temperature, the magnetic particles in the rock gained the preferred direction of the field and remained this way after solidification, permanently recording the magnetic field at the time. Subsequent lava flow pushed these strips further from the ridge, forming new strips, again forming a record of the geomagnetic field. This is an interesting aspect of magnetization because it indicates how permanent a magnetic recording can be. The magnetic record remained intact over millions of years and, in fact, this magnetization should remain unchanged until the rock melts again. Thus, we have the greatest magnetic recorder of all times: nature itself. The fact that the geomagnetic field can reverse does not have a ready explanation, especially since such reversals seem to be quick and most likely accompanied by severe disturbances to the balance of life on the planet. The phenomenon of magnetized strips was discovered when sensitive magnetic field measuring devices (magnetometers) were used to detect submarines by detecting the disturbance their iron hull causes in the surrounding geomagnetic field.

In old manuals for production of magnetic needles for compasses, the recipe calls for the iron needle to be red hot and held in the north–south direction. In this position, the needle is dipped into water. In this process, the iron becomes hard steel and the domains remain aligned in the direction north–south (lengthwise in the needle). Thus, a magnetic needle is produced in the same way that magnetization in the Earth's crust has been produced on the ocean floor.

Application: Magneto-optic Recording A common method of recording data on high-density disks is the magneto-optic method, which is another simple application of the Curie temperature. The magnetic medium is a disk, coated with a magnetic material which, like any other magnetic material, has a Curie temperature beyond which it loses its magnetization. In this application, a laser beam is used to heat the material beyond the Curie temperature, at the point at which we wish to record information. Then, the recording magnetic field is applied, and as the point cools below the Curie temperature, the information magnetization is retained. Reading of the information is done as with any other recording device. The advantage of this method over other magnetic recording methods is that the beam can be very small in diameter, allowing high recording densities. The heating is local and fast, as is the cooling and the required magnetic field is low.

Application: Ferrite Cores for Coils and Transformers We will only discuss transformers in **Chapter 10**, but it is easy to see that if we need a very complicated shape for a core, it is difficult to prepare these shapes from solid materials such as iron. On the other hand, ferrite materials are made of powdered ferromagnetic materials, mixed together, molded into the desired

shape, and then sintered (or "burned"). In principle, any ferromagnetic material can be made into a ferrite. Iron oxides form the basis of most ferrites (for example, Fe_3O_4) mixed with bivalent compounds such as Nickel oxide (NiO), Barium oxide (BaO), Manganese oxide (MnO), or others of the same type. The final product can have almost any shape and size and the properties of the solid material depend both on the materials used and the processing of the materials. Because of the sintering process used to bind the compounds together, the final material is very hard, is brittle, and can only be worked by grinding. Conductivity is low (normally below 10^{-5} S/m), but some materials have a conductivity of the order of 1 S/m. Ferrite cores are used extensively in coils and transformers, particularly where operation at high frequencies is required. As a ferromagnetic material, ferrites have a relatively low coercive field intensity (usually below 100 A/m), relative permeability between 10 and 10,000, remnant flux density below about 0.5 T, and a rather square magnetization curve.

Application: Magnetic Recording; Magnetic Tapes, Strips, and Disks A very useful application of magnetic materials is in recording of signals. Although the properties of various magnetic recording media vary in composition and quality, the principles are the same. In its simplest form, we may think of a magnetic medium as containing a layer of magnetic particles (not unlike very small permanent magnets) on a substrate and a means of orienting these particles by external fields. A magnetic recording medium, either tape (or strip) or disk, is made by one of two methods: one consists of coating a base material with ferromagnetic particles in a binding material. Each particle is independently suspended and its domains can be oriented by an external magnetic field. The second method is deposition of a ferromagnetic alloy in the form of a thin layer on a nonferromagnetic base material such as aluminum. The materials used are various iron and cobalt alloys. This method is normally used for disks.

In particulate media, the particles are diluted in a binder to about 20 to 50 % volume and coated onto a substrate film made of polyethylene or on rigid aluminum disks. The binder contains substances that protect the film and the recording head from physical damage. The particles may be any number of available ferromagnetic materials, including cobalt, nickel, iron, their oxides, and others. Most tapes use iron oxides and in particular Fe_2O_3. A simple recording system is shown in **Figure 9.57**. The recording head is a closed magnetic circuit in which a very thin gap (as small as 1 μm) has been cut. The flux produced by the coil passes through the gap and part of it leaks around the gap. This leakage field is proportional to the current in the coil. As it interacts with the particles on the tape, it magnetizes them at levels that are proportional to the signal being recorded. The magnetization in the particles is the recorded signal. When required, the signal can be read by a process that will be described in **Chapter 10**. Although magnetic tapes were used in the past most current applications are restricted to magnetic strips on credit cards and access cards and to magnetic hard drives.

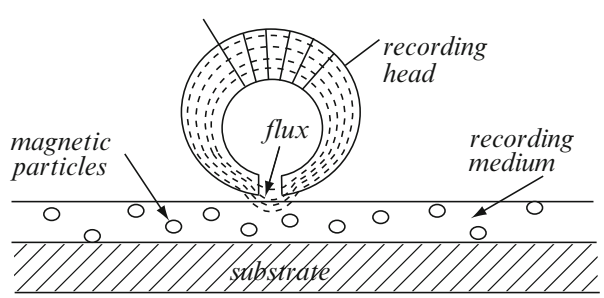


Figure 9.57 A simple recording mechanism showing a recording head and magnetic tape

Application: Magnetic Particle Testing A common method of nondestructive testing of ferromagnetic products is the magnetic particle testing. The idea is simplicity itself. Any ferromagnetic material when subjected to an external magnetic field is magnetized to a certain extent. If the material is uniform, the magnetic field is confined to the interior of the ferromagnetic material. If, however, there is a discontinuity in the material such as cracks, some of the magnetic flux "leaks" out. This field can then interact with external agents such as magnetic particles. If we now sprinkle ferromagnetic particles on the object, these will be attracted to the location of the flaw clearly indicating it. To many, this process is simply known as "magnafluxing" (Magnaflux is a trademark, not a method). Magnetic particles can be as simple as iron filings but can be much finer, with antioxidants and wetting materials added. Some processes call for dry powders, and others for wet solutions. Magnetic particles are sometimes dyed to facilitate detection; some are fluorescent to be observed under special

(black) lighting. There are many particle sizes used in industry ranging from 1 to 25 μm in size for wet methods and from 100 to 1,000 μm for dry methods, depending on application. Magnetic particles are normally spherical or elongated and are mostly ferromagnetic oxides. Pure ferromagnetic materials can also be used, but these are not stable. Oxides, on the other hand, are stable and can be stored and used without their properties changing in the process.

Application: Magnetite as a Guiding Mechanisms Used by Bacteria and Animals An intriguing use of magnetic materials by microorganisms and animals is in guidance in the terrestrial magnetic field. Magnetite is a naturally occurring substance and traces of the material have been found in some organisms as well as birds and other animals, including humans. There is a strong indication that many animals are, or were in the past, using this magnetic material as a natural compass. Pigeons almost certainly use terrestrial magnetism as part of their guiding mechanism. Migrating birds also use this compass, in addition to other methods. Although this sense is not developed in humans, the occurrence of magnetite in our brains suggests that at some time in our past, we may have had at least a partial ability to navigate using the terrestrial magnetic field. However, there is no definite proof of this and the magnetite in our brains may well be leftover particles with no specific function since their occurrence is at very low densities.

Some microbes use magnetite to orient themselves along the magnetic field. In these bacteria, called ***magnetotactic bacteria***, a string of single-domain magnetite (Fe_3O_4) particles (easily visible under microscopes) forms a compass that keeps the bacteria oriented along the geomagnetic or any other local field. The magnetic moment of the cells is fixed and the bacteria have no control over it. In other animals and insects, there is experimental evidence of active use of the magnetic field, indicating that these animals have magnetic sensors. Experimental data have been collected on pigeons whereby small magnets attached to their heads have shown to disorient them. Bees seem to use the magnetic field, as do amphibians such as salamanders. Some experiments have even focused on training animals and insects to detect low-level magnetic fields and local anomalies in the terrestrial magnetic field.

Application: Magnetic Shielding—Passive Magnetic Shielding of MRI Equipment It is often required to shield instruments from the effects of naturally or artificially produced magnetic fields. For example, we might want to measure very low magnetic fields at levels much below the geomagnetic field. In other cases, we might need to calibrate a magnetic device, and the effects of external fields must be eliminated. In still other cases, the very large magnetic fields produced by electromagnets must be contained so they do not interfere with instruments in their vicinity. Examples of this type of work abound. In a magnetic resonance imaging device, the fields produced by the magnet are of the order of 1 to 3 T. Yet, within a few meters, we need to operate computers, testing equipment, and place operators who might not appreciate being exposed to high magnetic fields continuously. The threat of such high magnetic fields to equipment is real. For example, the magnetic field required to erase a computer disk is about 0.008 T. Clearly, any disk brought close to a magnetic resonance imaging device will be affected. A similar magnetic flux density is capable of resetting a cardiac pacemaker. Other examples are the need to measure the magnetic field produced by the brain. This field is orders of magnitude smaller than the geomagnetic field. The solution to these problems is magnetic shielding. In its simplest form, magnetic shielding provides a low-reluctance path which the magnetic field takes, thus steering the field away from the shielded area. An example is shown in **Figure 9.58**. The coil at the center of the shielded structure produces a magnetic field. By placing a box of iron around the coil, most of the field is contained in the iron because it has a high relative permeability and, therefore, low reluctivity. This method is equivalent to shorting an electric circuit to prevent current from flowing into a certain part of the circuit. Magnetic shields can be rather big structures. For example, in an MRI installation, the shield may be inserted in the walls of the room. For a room $4 \times 4 \times 2.5$ m (not a very large room) with an iron shield, 50 mm thick in all four walls, ceiling, and floor, the total volume of iron is 3.6 m^3. At a weight of approximately 7.8 tons/m^3, the total weight of the shield is 28 tons. This is not a trifling weight, if nothing else, then for design of foundations for the building. In the above example, shielding is used to contain the magnetic field in a given area. If the source of the magnetic field is outside the shield, the magnetic field is excluded from the area inside the shield.

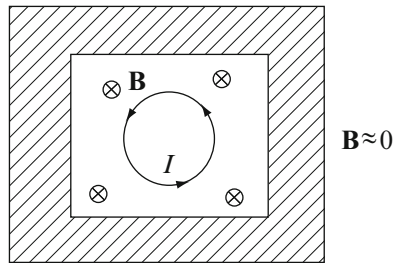


Figure 9.58 A simple magnetic shield used to contain the field inside the structure. The same structure will shield the interior from external magnetic fields

9.10 Experiments

Experiment 1 (Demonstrates: Ferromagnetic attraction in a magnetic field; high-permeability materials). A useful tool in the detection of ferromagnetic materials is the magnet. Take a simple magnet and move it on the wall. You can detect any ferromagnetic materials such as nails, screws, pipes, or reinforcing rods in concrete if these are close enough to the surface, the magnet is strong, and they are made of ferromagnetic materials. This method is used in simple "nail detectors" which consist of a small magnet, pivoted on a small frame. The frame is moved on the wall until the magnet moves. This movement indicates a force and, therefore, a ferromagnetic material in its vicinity. Pivoted magnets are useful tools in detecting and characterizing magnetic fields. The compass is, in fact, a very small pivoted magnet.

Experiment 2 (Demonstrates: Permanent magnets; attraction, repulsion, field patterns, magnetic poles). The concept of field pattern and even field lines can be seen by placing a permanent magnet under a piece of cardboard and sprinkling ferromagnetic particles on the cardboard. The particles will align with the magnetic field and will show the field pattern. Try to move the magnet and use different magnet shapes to see the pattern. You can obtain suitable powder in the form of iron particles by filling a piece of iron (hard work) or by scraping some rust from a rusted iron piece (much easier) or by grinding a piece of badly rusted sheet iron (such as a piece of tailpipe) to fine particles with a hammer (easiest and most satisfying).

Experiment 3 (Demonstrates: Magnetic shielding). Using an iron box, the magnetic field of a permanent magnet can be reduced outside the box. A magnetic path between the two poles reduces the field to almost zero. In the case of a horseshoe magnet, it is enough to "short" the poles with a high-permeability material to prevent most of the field from existing outside the volume of the magnet.

Place a magnet on a table. Measure the magnetic flux density. If you do not have a magnetic field measuring device (called a magnetometer), you may estimate it from the motion of a compass. For example, you may move the compass from far away toward the magnet. The further from the magnet the compass reacts, the stronger the magnetic field. Now, place the magnet in an iron box such that the magnet is somewhere in the middle of the box. Any suitable box will do. You may even invert a steel drawer or an iron pan over the magnet. Measure the field again. You should get a much smaller field because some of the flux lines close through the shield.

Experiment 4 (Demonstrates: Magnetization of ferromagnetic materials). Take a screwdriver or any other steel tool. Rub it over a strong magnet (you can obtain a good magnet from an old loudspeaker, although you will have to do some dismantling to get to the magnet). See that the tip is magnetized. If you do not succeed, you may wish to try another screwdriver. If it is made of good quality steel, it should magnetize well (this may be a good method to check if the promise of the salesperson of a good quality tool holds water). Lift a paper clip with the tip of the screwdriver. Try to lift another paper clip with the first such that you have two clips hanging. Remove the two clips. Is any of them magnetic? Why? Explain why a second clip can be lifted with the first clip.

Experiment 5 (Demonstrates: Magnetization below and above the Curie temperature). Take a small nail or needle. Cut the nail's head. With a suitable set of pliers heat the nail with a gas burner until the nail is red hot. Make sure the nail points north to south. After the nail is hot, without changing its direction, dip it in a cup of water to cool it. Now, you have a

magnetic needle. Place the needle on a small piece of cork or straw to float it on water. It should show the direction north–south. Check with another nail to see if, in fact, the nail has been magnetized. Reheating the nail and cooling it slowly, perhaps in the east–west direction, will eliminate the magnetization.

9.11 Summary

The current chapter applies the Biot–Savart and Ampere's laws and introduces most of the concepts in magnetostatics including inductance, energy, and interface conditions. We start with the magnetic flux density and the dipole moment of a small loop of radius d carrying current I, at large distances ($R \gg d$) (see **Figure 9.1**).

$$\mathbf{B} \approx \frac{\mu_0 m}{4\pi R^3}\left(\hat{\mathbf{R}}\,2\cos\theta + \hat{\theta}\sin\theta\right) \quad [\mathrm{T}] \tag{9.14}$$

$\mathbf{m} = \hat{\mathbf{n}}\,I\pi d^2$ [A · m^2] is the ***magnetic dipole moment*** ($\hat{\mathbf{n}}$ is the normal to the loop).

The dipole moment leads to the definition of ***magnetization*** and ***magnetization current density***. The magnetization $\mathbf{M}$ is due to a ***magnetization volume current density*** $\mathbf{J}_m$ and a ***magnetization surface current density*** $\mathbf{J}_{ms}$:

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right], \quad \mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{9.28}$$

The magnetization manifests itself in the permeability μ of the material:

$$\mathbf{B} = \mu_0\mathbf{H}_e + \mu_0\mathbf{M} = \mu_0\mu_r\mathbf{H}_e \quad [\mathrm{T}] \tag{9.30–9.34}$$

where $\mu = \mu_0\mu_r$ [H/m] is the magnetic permeability of the material, μ_r [dimensionless] is its relative permeability, and $\mathbf{H}_e$ is the magnetic field intensity external to the medium. The higher the magnetization, the higher the permeability of the medium.

Magnetic Materials—Properties

Diamagnetic materials are materials with relative permeability slightly smaller than 1.

Paramagnetic materials have relative permeability slightly higher than 1.

Ferromagnetic materials are characterized by very high permeability ($\mu_r \gg 1$).

Ferromagnetic materials exhibit ***hysteresis***—a nonlinear effect due to magnetic domains whereby the relation between the magnetic flux density and magnetic field intensity under AC conditions follows a closed path **(Figure 9.16).**

Hysteresis is responsible for losses but also for the existence of permanent magnets—magnetized materials that retain their magnetization.

Soft magnetic materials are those materials that can be easily demagnetized.

Hard magnetic materials are "hard" to demagnetize and are used for production of permanent magnets.

Interface conditions for the magnetic field define the behavior at interfaces. The magnetic interface conditions between two materials are [see **Eqs. (9.38)** through **(9.45)** and **Table 9.5**]:

$$\hat{\mathbf{n}} \times (\mathbf{H}_{1t} - \mathbf{H}_{2t}) = \mathbf{J}_s, \quad \hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_{1t}}{\mu_1} - \frac{\mathbf{B}_{2t}}{\mu_2}\right) = \mathbf{J}_s \quad \text{and} \quad B_{1n} = B_{2n}, \quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

$\hat{\mathbf{n}}$ points into material (1) and a surface current density $\mathbf{J}_s$ [A/m] may exist at the interface between conductors and nonconductors (see **Figure 9.17**).

Inductance is the ratio of flux linkage and the current that produces it. It is independent of current and only depends on physical dimensions and permeability. Given two circuits made of N_1 and N_2 loops and carrying currents I_1 and I_2, respectively, we define ***self-inductances*** L_{11}, L_{22} and ***mutual inductances*** L_{12}, L_{21} as **[**see **Eqs. (9.56)** through **(9.59)]**:

$$L_{11} = \frac{N_1\Phi_{11}}{I_1} = \frac{\Lambda_{11}}{I_1}, \quad L_{22} = \frac{N_2\Phi_{22}}{I_2} = \frac{\Lambda_{22}}{I_1}, \quad L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}, \quad L_{21} = \frac{N_1\Phi_{21}}{I_2} = \frac{\Lambda_{21}}{I_1} \quad [\mathrm{H}]$$

Φ_{11} is the flux produced by circuit (1) linking all turns of circuit (1), Φ_{12} is the flux produced by circuit (1) linking all turns of circuit (2), and so on. $\Lambda = N\Phi$ is called ***flux linkage***.

Inductance entails assumption of a current in a circuit, calculation of the flux density, calculation of flux, and flux linkage followed by division by the current that generated the flux. The inductance is independent of the assumed current. In infinite structures a more useful relation is self- and mutual inductance per unit length of the device.

External inductance—inductance due to flux outside conductors
Internal inductance—inductance due to flux within the conductor's volume.

Energy stored in the magnetic field is closely related to inductance even where inductors cannot be clearly identified. The basic definition starts with the energy stored in an inductor L due to passage of current I:

$$W_m = \frac{LI^2}{2} \quad [\mathrm{J}] \tag{9.62}$$

In a system of n loops or coils, each with N_i turns and current I_i, the magnetic energy is

$$W_m = \frac{1}{2}\sum_{i=1}^{n} N_i\Phi_i I_i \quad [\mathrm{J}] \tag{9.72}$$

where Φ_i is the total flux in loop i due to all current-carrying loops in the system. Alternatively it may be written in terms of inductance as

$$W_m = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}I_iI_j \quad [\mathrm{J}] \tag{9.77}$$

where L_{ij} is the self-inductance ($i = j$) or mutual inductance ($i \neq j$) between circuits i,j, and the current I_i, I_j are the currents in the circuits. If the currents I_i, I_j produce fluxes in the same direction, L_{ij} is considered positive, if not, negative.

A more general approach is in terms of fields. This is particularly useful when inductances cannot be identified and calculated such as in space. The magnetic energy can be calculated from various field quantities as follows:

$$W_m = \frac{1}{2}\int_{v'} \mathbf{A}\cdot\mathbf{J}dv' = \frac{1}{2}\int_{v} \mathbf{B}\cdot\mathbf{H}dv = \frac{1}{2}\int_{v} \mu H^2 dv = \frac{1}{2}\int_{v} \frac{B^2}{\mu}dv \quad [\mathrm{J}] \tag{9.84, 9.89}$$

v' here is the volume in which the current density $\mathbf{J}$ exists whereas v is the volume in which the magnetic field is nonzero. The first of these is useful where $\mathbf{J}$ is easily identified such as in conducting media. In space, where currents may not exist, calculation in terms of the magnetic flux density and field intensity is used.

The integrand in the energy expressions is the energy density:

$$w_m = \frac{\mathbf{A}\cdot\mathbf{J}}{2} = \frac{\mathbf{B}\cdot\mathbf{H}}{2} = \frac{BH}{2} = \frac{\mu H^2}{2} = \frac{B^2}{2\mu} \quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right] \tag{9.84, 9.91}$$

Magnetic circuits are based on the equivalence between currents in closed electric circuits and flux in closed magnetic paths. The requirements are for permeability to be high and any gaps in the circuits to be as small as possible to avoid flux leakage around the gaps. Under these conditions the flux in any closed path within the magnetic circuit is

$$\Phi = \frac{\sum_{i=1}^{n} N_iI_i}{\sum_{j=1}^{k} \mathfrak{R}_j} \tag{9.102}$$

With

$$\mathfrak{R}_j = \frac{l_j}{\mu_j S_j} \quad \left[\frac{1}{\mathrm{H}}\right] \tag{9.99}$$

where N_i is the number of turns in coil i, I_i its current, and $\mathfrak{R}_j$ is the magnetic reluctance of segment j of the magnetic path with l_j the length of the segment, μ_j its permeability, and S_j its cross-sectional area (see **Figure 9.37**). Magnetic circuits allow simple calculation of fluxes and magnetic fields in devices that satisfy the basic requirements of a magnetic circuit.

Forces in the magnetic field are defined based on the Lorentz force equation ($\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$) in **Eq. (9.106)**, which governs forces on moving charges. Since moving charges constitute currents, **Eq. (9.106)** can be developed into more useful relations. For currents in a volume we have

$$\mathbf{F}_m = \int_{v'} \mathbf{J} \times \mathbf{B} dv' \quad [\mathrm{N}] \tag{9.110}$$

where $\mathbf{J} \times \mathbf{B}$ is a volume force density [N/m^3]. In a thin wire carrying current I in a magnetic field:

$$\mathbf{F}_m = \int_{L'} I d\mathbf{l}' \times \mathbf{B} \quad [\mathrm{N}] \tag{9.113}$$

In all cases, the flux density $\mathbf{B}$ is due to sources other than I (I cannot cause a force on itself).

Equation (9.113) is the basis of Neumann's formula, which defines a force between any two current-carrying wire segments that are part of closed circuits (see **Figure 9.45**):

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{p_3}^{p_4} \int_{p_1}^{p_2} \frac{d\mathbf{l}_2' \times \left(d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}\right)}{R_{12}^2} \quad [\mathrm{N}] \tag{9.117}$$

Here one segment extends between p_1 and p_2, the second between p_3 and p_4, and the vector $\mathbf{R}_{12}$ connects $d\mathbf{l}_1'$ and $d\mathbf{l}_2'$. Both **Eqs. (9.113)** and **(9.117)** can be extended for closed loops [see **Eqs. (9.114)** and **(9.118)**].

Principle of Virtual Work Given the magnetic energy in a system, a force, such as between two faces of a gap, may be calculated by allowing a virtual displacement of one of the surfaces. The change in energy is related to force (**Section 9.7.1**):

$$F = -\nabla W \quad [\mathrm{N}] \tag{9.121}$$

If there is a force on a system, there may also be a torque. ***Torque*** is simply the force multiplied by armlength. In the case of a loop with magnetic dipole moment $\mathbf{m}$, the torque may be written as

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad [\mathrm{N} \cdot \mathrm{m}] \tag{9.127}$$

Problems

Magnetic Dipoles and Magnetization

9.1 Application: Magnetic Dipole. A small circular loop of radius a [m] and a small, square loop, $a \times a$ [m^2] are given.

- **(a)** Calculate the magnetic dipole moment of the circular loop.
- **(b)** Justify based on physical considerations the following statement: "At large distances, the magnetic dipole of the square loop is the same as that of a circular loop of identical area."
- **(c)** Calculate the magnetic flux density at a general point in space due to each loop for the conditions in **(b)**.

9.2 Magnetic Dipole. A loop of radius a [m] carries a current I [A]. A second, small loop of radius b [m] is placed at point P_1 (on the axis of the first loop, at height h [m] above the loop) or at point P_2 (at a horizontal distance h [m], on the plane of

the first loop) so that the planes of the two loops are parallel in either case **(Figure 9.59)**. In both cases, $h \gg b$, $a > b$ and $h \gg a$. Calculate the total flux through the small loop produced by the large loop when:

(a) The small loop is at point P_1.
(b) The small loop is at point P_2.

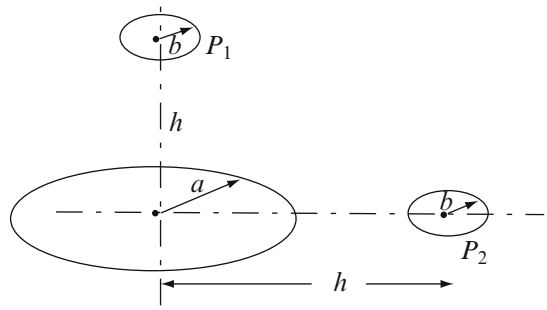


Figure 9.59

9.3 Magnetization. A very long cylindrical magnet has constant magnetization **M** [A/m], directed as in **Figure 9.60**. A solenoid made of thin wire is wound tightly around the magnet, with n turns per unit length of the coil. What must be the current (magnitude and direction) in the coil to cancel the magnetization of the magnet?

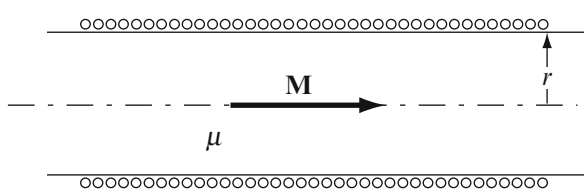


Figure 9.60

9.4 Application: Magnetization and Magnetic Field in a Permanent Magnet. A very long cylindrical magnet has magnetization **M** [A/m] as shown in **Figure 9.60**. The magnet is made of a material with permittivity μ [H/m]. Calculate the magnetic field intensity inside the magnet.

9.5 Flux Density in a Magnet. For a permanent magnet, the flux density inside the magnet can be written as $B_i = \mu_i H_i$. Show that in this case, the permeability must have a negative sign. Explain why this must be so and the practical meaning of the relation.

Magnetic Interface Conditions

9.6 Magnetic Interface Conditions. A magnetic field intensity **H** is given at the interface between materials (1) and (2) as shown in **Figure 9.61**. Calculate the direction and magnitude of the magnetic flux density in materials (2) and (3). Assume there are no surface currents on the interfaces.

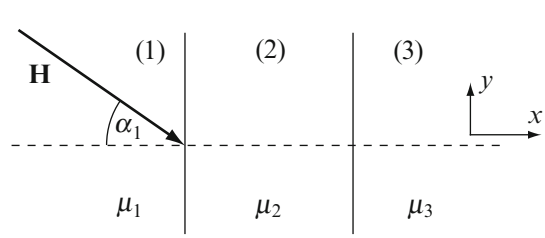


Figure 9.61

9.7 Interface Conditions for Ferromagnetic Media. The relative permeability of a large, flat piece of ferromagnetic material is 100. If the magnetic field intensity in the iron must be at 60° to the surface, what must be the direction of the magnetic field intensity at the surface of the iron piece in air? Assume there are no currents on the interfaces.

9.8 Interface Conditions and Flux Density. A two-layer magnetic sheet is made as shown in **Figure 9.62**. Each sheet is d [m] thick. Permeabilities are $\mu_1 = \mu_0$, $\mu_2 = 200\mu_0$, $\mu_3 = 50\mu_0$, and $\mu_4 = \mu_0$. A magnetic flux density in material (1) is given at 30° to the normal and of magnitude $B = 0.01$ T. Calculate the magnetic flux density (magnitude and direction) in materials (2), (3) and (4).

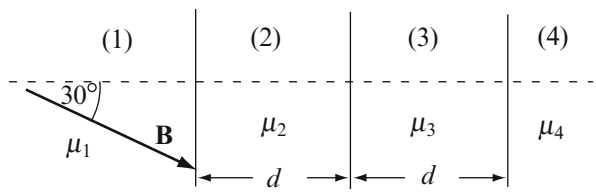


Figure 9.62

Inductance

9.9 Application: Self- and Mutual Inductances of Coils. A coil is wound uniformly in the form of a torus (see **Figure 9.63**). A long solenoid, of radius $a < b$ [m] and n turns per unit length, is inserted in the central hole of the torus. Calculate:

(a) The self-inductance of the toroidal coil.
(b) The self-inductance per unit length of the solenoid.
(c) The mutual inductance between the toroidal coil and the solenoid.

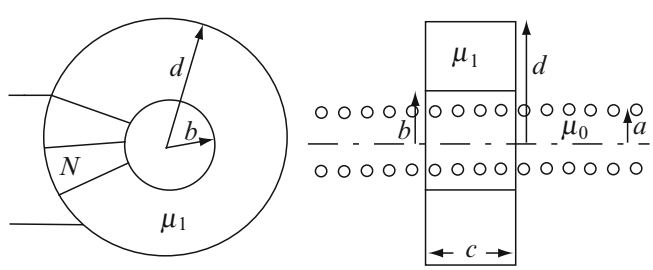


Figure 9.63

9.10 Application: Self- and Mutual Inductances of Coils. Three coils are wound on a toroidal core with properties and dimensions as shown in **Figure 9.64**. Assume $b - a \ll a$ and $b > a$ and calculate:

(a) The self-inductance of coils (1), (2), and (3).
(b) The mutual inductances between coils (1) and (2), between coils (2) and (3), and between coils (1) and (3).

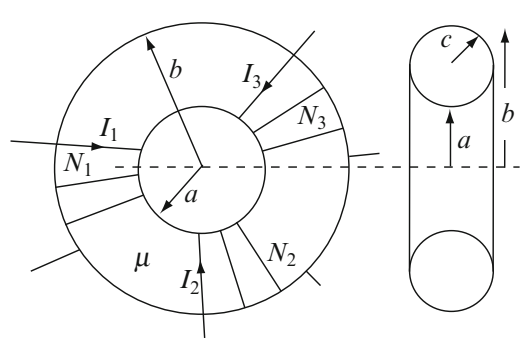


Figure 9.64

9.11 Application: Self-inductance of Long Solenoid. A long solenoid is wound on a hollow cylinder made of iron. Dimensions are given in **Figure 9.65**. Permeability of iron is $\mu_1 > \mu_0$ and that of free space is μ_0 [H/m]. The solenoid carries a current I [A] and has n turns per unit length. Calculate the self-inductance per unit length of the solenoid.

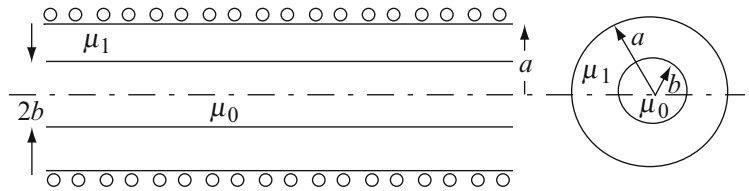


Figure 9.65

9.12 Inductance of a Double Cylinder. A thin sheet of copper is bent into an infinitely long double cylinder as shown in cross section in **Figure 9.66**. Calculate the inductance per unit length of the double cylinder. For purposes of calculation, assume $d \to 0$.

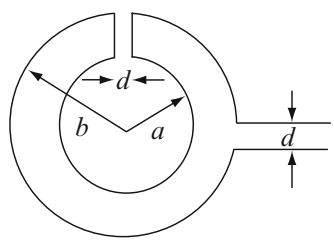


Figure 9.66

9.13 Mutual Inductance Between Wire and Solenoid. A very long wire (infinite for practical purposes) is located at a distance d [m] from the center of an infinite solenoid (**Figure 9.67**). The solenoid has a radius b [m] ($b < d$) and n turns per unit length. The solenoid and the wire are at 90° to each other, as shown. Calculate the mutual inductance between solenoid and wire.

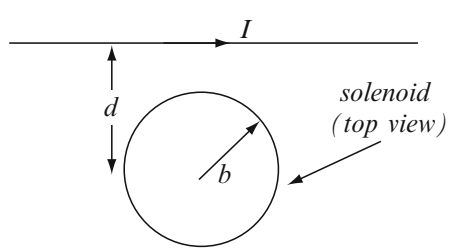


Figure 9.67

9.14 Mutual Inductance Between Coil and Solenoid. A very long (infinite) solenoid with N turns per unit length and radius a [m] carries a current $I = I_0$ [A]. A coil with two turns and radius b [m] ($b > a$) is located as shown in **Figure 9.68**. The coil and solenoid do not touch.

(a) Calculate the mutual inductance between the coil and solenoid.

(b) Suppose the coil is now twisted so that it makes a 45° angle with the axis of the solenoid. What is now the mutual inductance between the coil and solenoid? (The solenoid is entirely enclosed within the coil.)

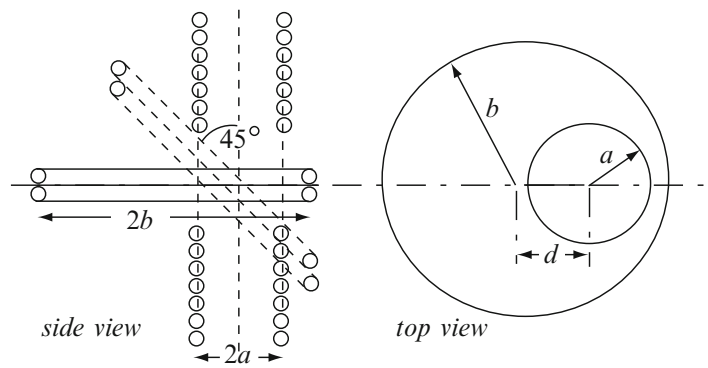


Figure 9.68

9.15 Application: Mutual Inductance Between Loops. The geometry and conditions in **Problem 9.2** are given. Calculate:

(a) The mutual inductances L_{12} and L_{21} when the small loop is at point P_1 in **Figure 9.59** and parallel to the large loop. Write the necessary assumptions.
(b) The mutual inductances L_{12} and L_{21} when the small loop is at point P_2 in **Figure 9.59** and parallel to the large loop. Write the necessary assumptions.
(c) Based on the calculation in **(a)**, does the relation $L_{12} = L_{21}$ hold? Explain.

9.16 Inductors in Series and Parallel. Three identical inductors are connected as shown in **Figure 9.69**. The inductors experience no mutual inductance.

(a) Calculate the total inductance for the series connection in **Figure 9.69a**.
(b) Calculate the total inductance for the parallel connection in **Figure 9.69b**.
(c) Calculate the total inductance for the connection in **Figure 9.69c**.

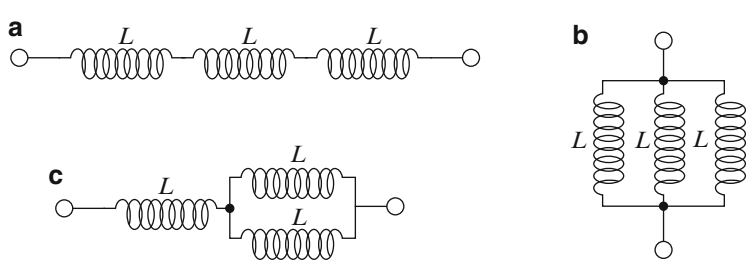


Figure 9.69

9.17 Application: Inductance of Coaxial Cables. A coaxial cable is made of an inner shell of radius b [m] and an outer, concentric shell of radius a [m], separated by a dielectric with permeability μ_0 [H/m]. Calculate the inductance per unit length of the coaxial cable. Assume the shells are very thin.

9.18 Application: Internal and External Inductances. A coaxial cable is made of a solid inner conductor of radius b [m] and a concentric, outer solid conductor of inner radius a [m] and outer radius c [m]. The conductors are separated by a material of permeability μ_0 [H/m]. Assume any current in the conductors is uniformly distributed throughout the conductor's cross-sectional area and the permeability of the conductors is μ_0 [m]. Calculate:

(a) The internal inductance per unit length due to the inner conductor.
(b) The internal inductance per unit length due to the outer conductor.
(c) The external inductance per unit length of the cable.
(d) The total inductance per unit length of the cable. Compare with the result in **Problem 9.17**.

9.19 Application: Inductance Per Unit Length of Cables. Two two-wire cables carry a current I_1 [A] and I_2 [A] as shown in **Figure 9.70**. Both cables can be assumed to be very long (infinite) and both are placed flat on a plane. Calculate:

(a) The external self-inductance per unit length of each cable.
(b) The mutual inductance per unit length between the two cables.

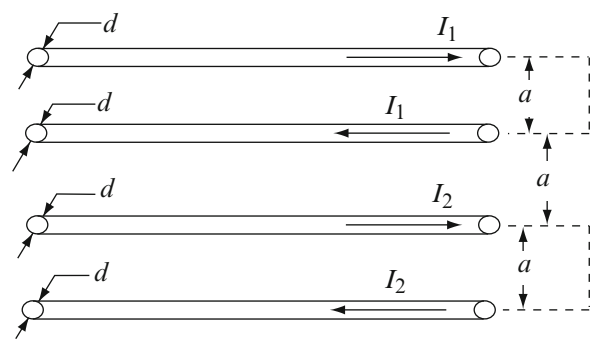


Figure 9.70

9.20 Self- and Mutual Inductance in Printed Circuit Boards. The configuration shown in **Figure 9.71** represents part of a printed circuit board (PCB). The two outer conductors represent the power supply traces and may be assumed to be long, whereas the loop in the center is part of the functional circuit and is small. The outer traces are of width $2t$. The permeability of the PCB is μ_0 [H/m] and the dimensions are as follows: $a = 25$ mm, $b = 20$ mm, c $= 4$ mm, $d = 6$ mm, and $t = 1$ mm. Calculate:

(a) The external self-inductance per unit length of the outer circuit.
(b) The total self-inductance per unit length of the outer circuit.
(c) The mutual inductance between the loop and the external circuit. Neglect the thickness of the traces of the small loop for the purpose of this calculation.

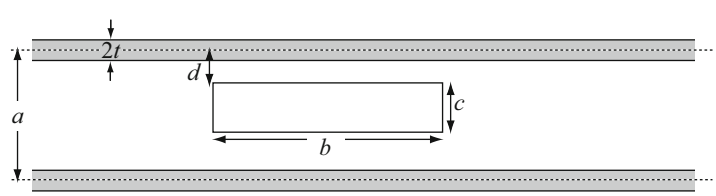


Figure 9.71

Energy

9.21 Stored Energy in a System of Inductors. An infinite solenoid with n turns per unit length and radius a [m] carries a current I [A]. A shorter solenoid, of length d [m] and radius b [m], is placed over the first as shown in **Figure 9.72**a. The two solenoids do not actually touch. **Figure 9.72b** shows the relation of the two solenoids from an axial view. The second solenoid also carries a current I [A] and has a total of N turns. Assume both solenoids to be long (i.e., infinite for calculation purposes) and that the flux densities of the solenoids are in the same direction. Calculate the change in the energy in the system (made of the two solenoids) when the short solenoid is removed.

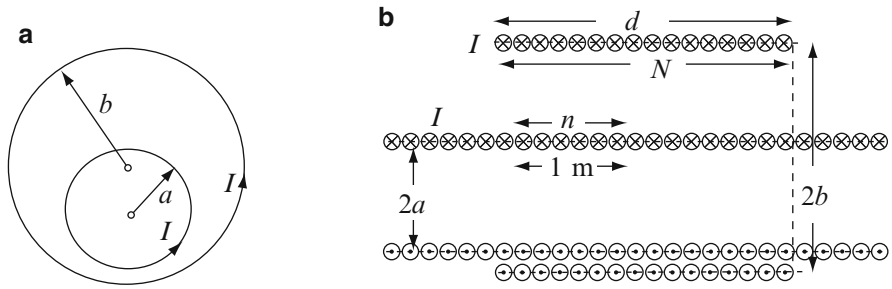


Figure 9.72

9.22 Change in Stored Energy Due to Change in Material properties. A toroidal coil of inner radius r_1 [m], outer radius r_2 [m], and square cross section is shown in **Figure 9.73**. The material of the torus has permeability $\mu_1 > \mu_0$, the coil has N turns and carries a current I [A]. Calculate the change in the stored magnetic energy if the material inside the torus is removed (the core of the torus is in effect replaced with free space).

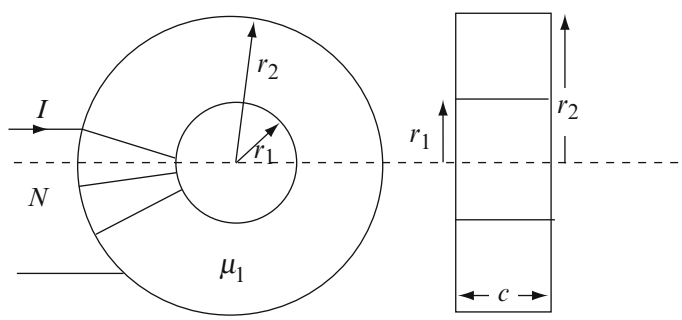


Figure 9.73

9.23 Stored Energy in a Gap. A toroidal coil of inner radius r_1 [m], outer radius r_2 [m], and square cross section is given as shown in **Figure 9.73**. The material of the torus has permeability $\mu_1 > \mu_0$, the coil has N turns and carries a current I [A]. Calculate the change in the stored magnetic energy if a gap of length l_g [m] is cut in the torus. Assume the gap does not create flux leakage and the average radius is $(r_2 + r_1)/2$ [m].

9.24 Stored Energy in Series-Connected Inductors. Consider the torus and configuration given in **Figure 9.64**. For the given values and with $N_3 = 2N_2 = 2N_1$ turns calculate:

(a) The total energy stored in the torus, if all three fluxes produced by the coils are in the same direction.
(b) Now, all three coils are connected in series to a current I [A] ($I_1 = I_2 = I_3 = I$). It is required that the torus store the minimum possible energy. Show how the coils must be connected and calculate the energy stored and the equivalent inductance under this condition.
(c) The configuration now is as in **(b)**, but it is required that the torus store maximum energy. Show how the connections must be made to accomplish this and calculate the stored energy and the equivalent inductance.

9.25 Application: Energy Stored in a Magnet. A cylindrical permanent magnet of radius a [m] with constant magnetization $\mathbf{M} = \hat{\mathbf{z}} M_0$ [A/m] and infinite in length is given. Calculate the energy stored per unit volume of the magnet. Assume permeability of the magnet is μ_0 [A/m].

9.26 Work Necessary to Modify a Magnetic Core. An infinitely long solenoid of radius a [m] has n turns per unit length. The turns carry a current I [A]. A long piece of iron of radius b [m] is located in the solenoid, as shown in **Figure 9.74**. The relative permeability of iron is μ_r and that of free space is μ_0 [H/m]. Calculate the total work per unit length of the solenoid necessary to pull the iron completely out of the solenoid. Does this work increase or decrease the potential energy of the system? Explain.

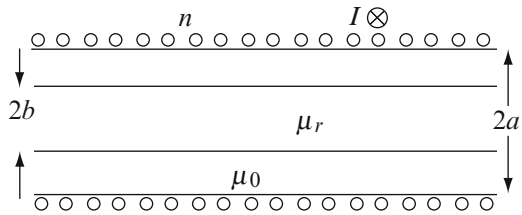


Figure 9.74

9.27 Change in Energy Due to Change in Mutual Inductance. A square loop is located outside a very long solenoid with a total of N turns. Both the solenoid and the loop carry a current I [A] as shown in **Figure 9.75a**. Assume that the solenoid is d [m] long but that its field is identical to an infinite solenoid of the same radius. The loop is now inserted inside the solenoid such that the plane of the loop is perpendicular to the axis of the solenoid **(Figure 9.75b)**. Calculate the change in energy in the system (loop inside solenoid) due to this action.

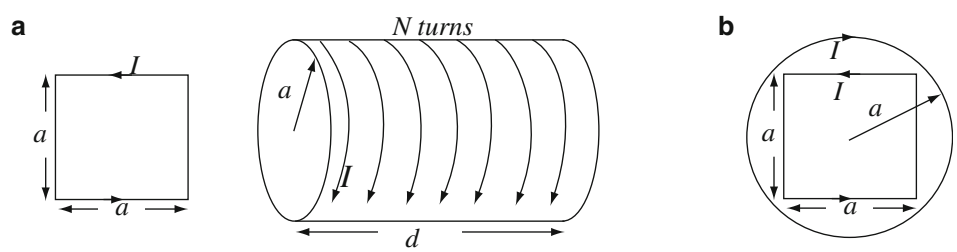


Figure 9.75

Magnetic Circuits

9.28 Application: Flux in a Toroidal Magnetic Circuit. A straight, long wire passes at the center of a torus as shown in **Figure 9.76** and carries a current I [A]. The torus has permeability μ [H/m].

(a) Calculate the total flux in the torus due to the current in the wire.
(b) What is the flux if a gap of length l_g [m] is cut in the torus (dotted lines)? Assume $(d - b) \ll b$.

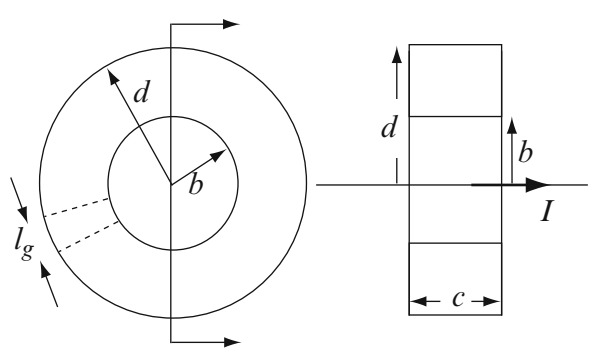


Figure 9.76

9.29 Application: Magnetic Circuits. Calculation of Flux and Field Intensity. A magnetic circuit is given in **Figure 9.77**. A single turn of wire carrying a current I [A] is placed in the gap. Assume all flux is contained within the magnetic circuit and the magnetic path length is the average length of the corresponding section. Assume $e > b$ and calculate:

(a) The minimum and maximum magnetic field intensity (H) in the circuit. Where do these occur?
(b) The flux in the central leg of the magnetic circuit.

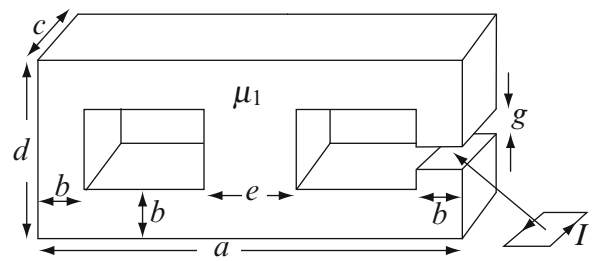


Figure 9.77

9.30 Application: Mutual Inductance in a Magnetic Circuit. A torus with average radius r_0 [m] and cross-sectional area as shown in **Figure 9.78** is given. A coil with N turns is wound around the torus. A small gap of length l_g [m] is cut in the torus and a loop of radius a [m] is inserted in the gap, such that the loop is centered in the gap. Calculate the mutual inductance between the loop and the coil. Assume $r_0 \gg b$.

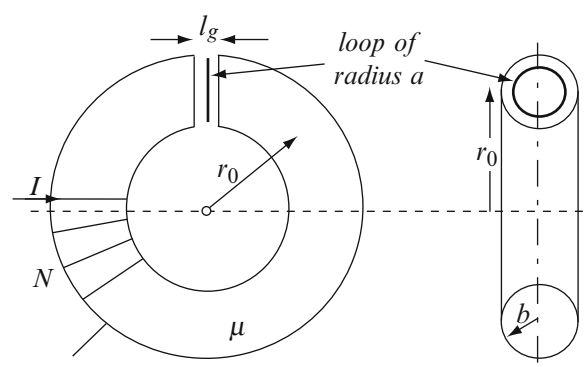


Figure 9.78

9.31 Magnetic Circuit with Different Materials. The magnetic circuit in **Figure 9.79** is given. The two halves are made of different materials with different permeabilities. The length of the gap is l_g [m]. The cross-sectional area of the core is the same everywhere. Calculate the magnetic field intensity in the gap. Assume $a \gg l_g$ and $d \gg l_g$.

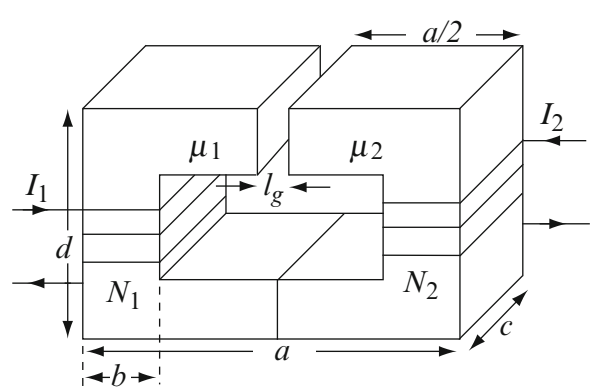


Figure 9.79

9.32 Magnetic Circuit with Different Materials. A torus is made of two types of materials with four small gaps as shown in **Figure 9.80**. Assume the gaps have properties of free space. The coil has N turns and carries a current I [A]. Calculate the flux density in the gaps assuming there is no flux leakage at the edges of the gaps.

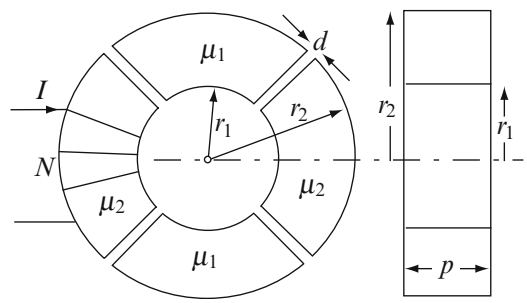


Figure 9.80

9.33 Application: Stored Energy and Magnetic Flux Density in a Magnetic Circuit. The magnetic circuit in **Figure 9.81** is given. The two halves are made of different materials with different permeabilities. The length of each gap is l_g [m]. The cross-sectional area of the core is constant. Take the magnetic path as the average length of the core:

(a) If I_1 and I_2 are known, calculate the total energy stored in the magnetic field.
(b) If I_1 is known, calculate I_2 such that the magnetic field intensity in the gap is zero. Show its direction.

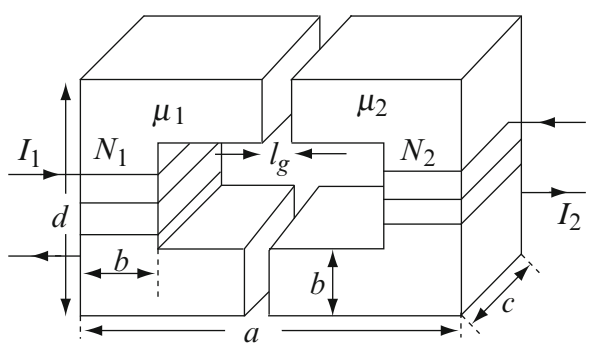


Figure 9.81

9.34 Stored Energy in a Magnetic Device. The core shown in **Figure 9.82** is made of two pieces. Two gaps exist between the two pieces as shown. The cross-sectional area of the two pieces is the same and is constant. The relative permeability of the upper and lower piece tends to infinity. Currents, number of turns, and dimensions are as shown. Calculate the total magnetic energy stored in the magnetic field of this device.

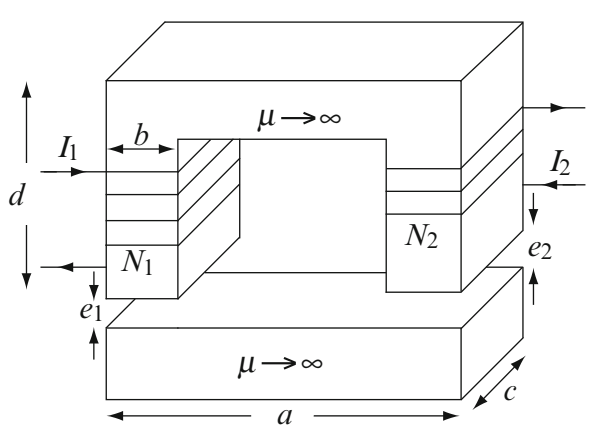


Figure 9.82

9.35 Mutual Inductance. An iron core is made as shown in **Figure 9.83**. A coil (L_1) with N_1 turns is wound on the right leg of the core and a coil (L_2) with N_2 turns is wound on the left leg of the core. Calculate the mutual inductance between the two coils.

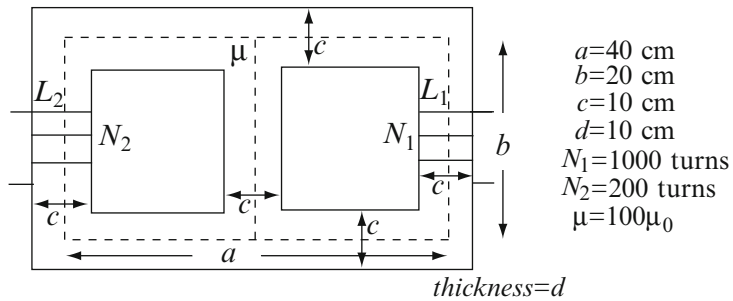


Figure 9.83

Forces

9.36 Force on a Current-Carrying Conductor. A thin bar 1 m long carries a current $I = 10$ A. What is the maximum force that the terrestrial magnetic field exerts on the bar? Assume the magnetic field of the planet is parallel to the surface of the earth and equal to $B = 50$ μT. Show the required orientation of the bar so that the force is maximum.

9.37 Force on a Loop. A current I_1 [A] flows in a thin conducting wire in the positive z direction. A rectangular loop carries a current I_2 [A] as shown in **Figure 9.84**. Dimensions are as shown and the loop and wire are on a plane. Assuming free space, find:

(a) The total net force on the loop (magnitude and direction).
(b) The total net force on the wire (magnitude and direction).

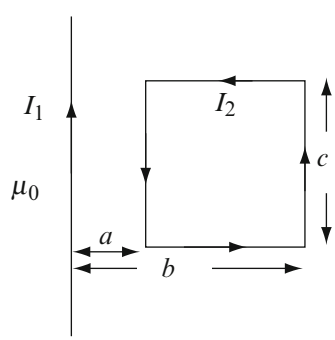


Figure 9.84

9.38 Forces on Thick, Current-Carrying Conductors. Two infinitely long conducting bars (shown in cross section in **Figure 9.85**) carry a current I [A] as shown.

(a) Write an expression for the force per unit length between the two conductors (do not evaluate the integrals).
(b) Is this an attraction or repulsion force? Explain.
(c) **Optional:** Integrate the expression in **(a)** for $a = 0.05$ m, $d = 0.2$ m, $b = 0.2$ m, $I = 1{,}000$ A, and $\mu = \mu_0$. **Note:** The integral is very tedious. You may want to write a computer program to accomplish the integration.

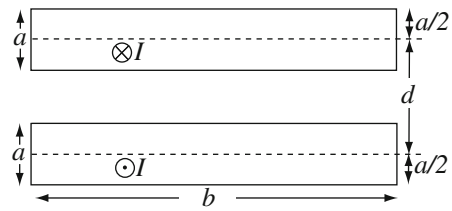


Figure 9.85

9.39 Force Between Loops. Two square loops are parallel to each other and carry currents as shown in **Figure 9.86**.

(a) Show, without calculations, that the two loops attract each other.
(b) Write the expression for the total force on one of the loops.
(c) **Optional:** Evaluate the force in **(b)** for $a = d = 1$ m, $b = 2$ m, and $I_1 = I_2 = 1$ A.

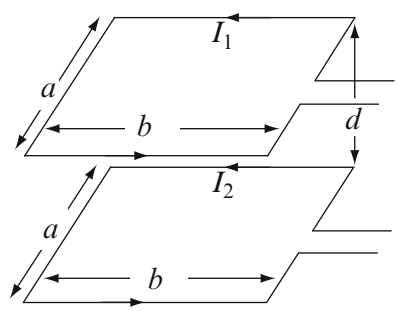


Figure 9.86

9.40 Forces in a Magnetic Circuit. A lifting electromagnet designed to lift steel bars is built as shown in **Figure 9.87**. To protect the poles, they are coated with a thin polymer of thickness t so that the minimum gap between the magnet and the bar being lifted equals t. Assuming the permeability of both the electromagnet core and the steel bar equal $200\mu_0$, calculate:

(a) The weight per unit current in the coil that the device can hold.
(b) What is the current needed to hold a 1 ton bar.

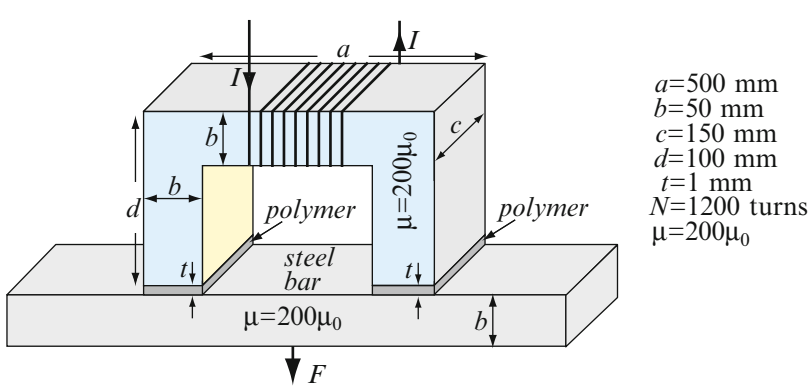


Figure 9.87

9.41 Application: Forces on Magnetic Poles of a Gap. In **Figure 9.88**, calculate the force that exists between the two faces of the gap. The magnetic path has a permeability equal to μ [H/m] and the core is d [m] thick. a, b, and g are the average path lengths of the corresponding magnetic path sections.

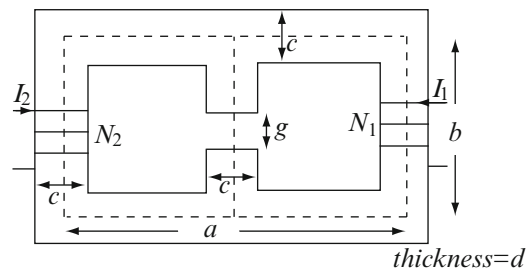


Figure 9.88

Torque

9.42 Application: Torque on a Current-Carrying Loop. Calculate the torque on the loop in **Figure 9.84**. Assume the loop and wire are on a plane in free space.

9.43 Torque on a Current-Carrying Bar. A thin bar of length $a = 1$ m carries a current I and is placed in a uniform magnetic field. The magnetic flux density is $\hat{\mathbf{B}} = \hat{\mathbf{z}} B_0$ where $B_0 = 0.4$ T and the current in the bar is 0.1 A. The bar is pivoted at its center and connected to the current as shown in **Figure 9.89a**.

(a) Calculate the torque on the bar. Show the direction of forces.
(b) Suppose the source is connected as in **Figure 9.89b**. What is now the torque on the bar?

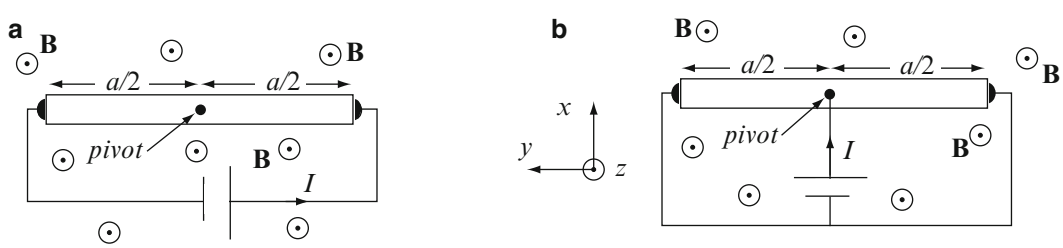


Figure 9.89

9.44 Application: Torque on Magnetic Dipoles. A loop of radius a [m] carries a current I [A]. A second, small loop of radius b [m] carries a current I_s [A] and is placed at point P_1, P_2, or P_3 as shown in **Figure 9.90**. In all cases, $h \gg b$, $a \gg b$, $h \gg a$, and the axes of the loops are parallel to each other. Calculate the torque:

(a) On the small loop when the small loop is at point P_1.
(b) On the small loop when the small loop is at point P_2.
(c) On the small loop when the small loop is at point P_3.
(d) On the large loop for the configuration in **(c)**.

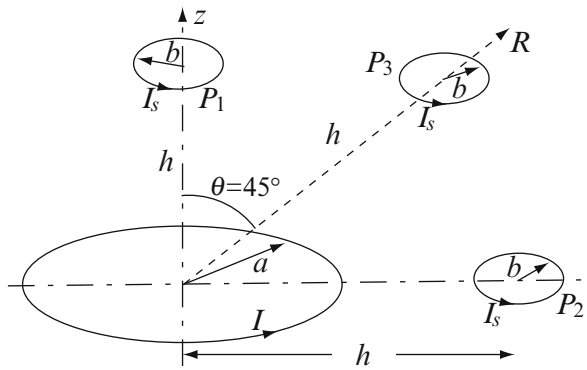


Figure 9.90

9.45 Torque on Small Loops. A small circular loop of radius a [m] is placed flat on a surface. A second, square loop, $a \times a$ [m^2], is placed on the same surface at a distance d [m] such that $d \gg a$ **(Figure 9.91a)**. The circular loop carries a current I_c [A] and the square loop a current I_s [A].

(a) Find the magnetic dipole moments of the circular and square loops in **Figure 9.91a**.

(b) Now, the square loop is rotated around its axis as in **Figure 9.91b** so that the side de is up without changing the directions of the currents. Calculate the torque on the round loop.

(c) Assuming the conditions in **(b)**, the number of turns in the round loop is increased to N. How does this affect the torque on the square loop?

(d) Assuming the conditions in **(c)**, the current in the round loop is changed to flow in the direction opposite that shown in **Figure 9.91b** and the number of turns in the square loop is also increased to N. Calculate the torque on the round loop.

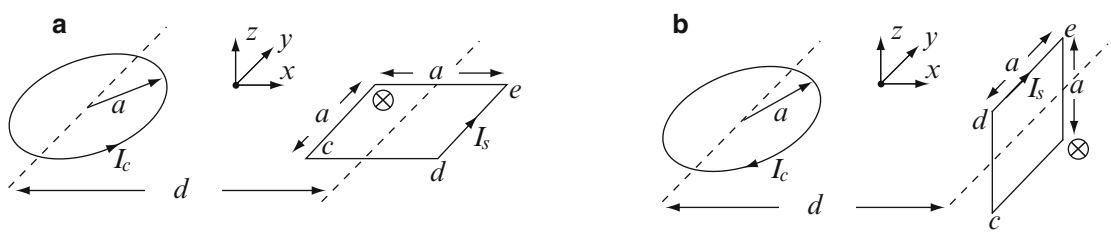


Figure 9.91