

*“There’s a South Pole,” said Christopher Robin, “and I expect there’s an East Pole and a West Pole, though people don’t like talking about them.”*

Winnie-The-Pooh

## 8.1 Introduction

After discussing the static electric field and steady currents, we are now ready to take another significant step in the study of electromagnetics, the study of the static magnetic field. But what exactly is a magnetic field? This question will be answered gradually, but, for a simple description, we may say it is a new type of force field in the same sense that the electric field is a force field. Take, for example, a magnet. It attracts or repels other magnets and generates a “magnetic field” around itself. The permanent magnet generates a static (time-independent) magnetic field. A direct current can also generate a static magnetic field. How do we know that? As with many other aspects of electromagnetics, we know by experiment.

The various properties of the magnetic field will be discussed primarily from the point of view of currents. The importance of this for engineering design is twofold; first, it indicates that a static magnetic field can be generated to suit design purposes. Second, it provides a link between the electric field and the magnetic field. Thus, we can view the electric field as a source of current, which, in turn, is the source for the magnetic field and, therefore, at least a partial explanation for the term “electromagnetics.” The reason for the qualification is that the opposite does not happen in the case of static fields: a static magnetic field does not generate an electric field. We will see in **Chapters 10** and **11** that a time-dependent magnetic field does generate a time-dependent electric field, and at that point the link between the electric and magnetic field will be complete.

For many years, electric and magnetic fields were thought to be separate phenomena even though both were known since antiquity. It was not until 1819 when Hans Christian Oersted<sup>1</sup> found that the needle of a compass moved in the presence of a current-carrying wire that a link between the two fields was found. He concluded that the only way this can happen is if the current generates a magnetic field around the wire. He used a compass to “map” the behavior of the magnetic field around the wire. Following his initial discovery, Andre Marie Ampere<sup>2</sup> quickly established the correct relation between current and magnetic field in what is now known as Ampere’s law. With all that, as late as the beginning of the twentieth century, it was still common to use separate units for electric and magnetic quantities without any connection between the two systems.

<sup>1</sup> Hans Christian Oersted (1777–1851), Danish scientist and professor of physics. He tried for many years to establish the link between electricity and magnetism, a link that was suspected to exist by him and many other scientists of the same period. He finally managed to do so in his now famous experiment of 1819 in which he showed that a current in a wire affects a magnetic needle (compass needle). He disclosed his experiments, all made in the presence of distinguished witnesses, in 1820. Oersted was very careful to ensure that what he saw was, in fact, a magnetic phenomenon by repeating the experiments many times and with various “needles,” in addition to the magnetic needle (to show that the effect does not exist in conducting materials such as copper or insulating materials such as glass—only in magnetized materials). Intervening materials between the wire and needle were also tested. As was the custom of the day, his work was written and communicated in Latin in a pamphlet titled: “Experimenta circa efficaciam, conflictus electrici in acum magneticam.”

<sup>2</sup> See the footnote on page 341.

In this chapter, we discuss the relationships between the steady electric current and the static magnetic field. These will be in the form of basic postulates which, as in the case of the static electric field, are experimental in nature. Two important relations, the Biot–Savart and Ampere’s laws, will allow us to calculate the magnetic field due to electric currents.

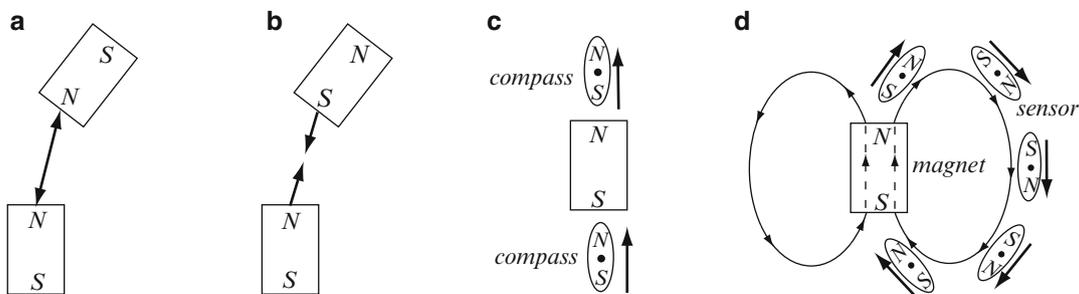
## 8.2 The Magnetic Field, Magnetic Field Intensity, and Magnetic

### Flux Density

Electrical\_Current.m

What, then, is the magnetic field? We might start to answer this question by playing with two magnets. The first effect to notice is that there is a force between the two magnets; the magnets either attract or repel each other as shown in **Figure 8.1**. Since attraction happens at a distance, each magnet must have a domain in which it attracts the other magnet. This is exactly what we called a field. That the field is a vector field we can establish using a compass as a measuring device. The direction of the force is established by the direction of the compass needle in space. Placing the compass at as many positions as we wish, a complete map of the vector field is established. This simple measurement establishes the following:

- (1) A field exists throughout space.
- (2) The field is stronger closer to the magnet.
- (3) The two ends of the magnet behave differently; one attracts the north pole of the compass and is labeled the south pole; the other attracts the south pole of the compass and is labeled the north pole (**Figure 8.1c**). This arbitrary identification is convenient because of its relation with the Earth’s magnetic field.
- (4) By placing the compass at different locations in space, we can map the magnetic field. One field line is shown in **Figure 8.1d**.



**Figure 8.1** The permanent magnet. (a) Two permanent magnets repel each other. (b) Two permanent magnets attract each other. (c) Identification of the poles of a magnet. (d) Mapping the magnetic field of a permanent magnet using a compass

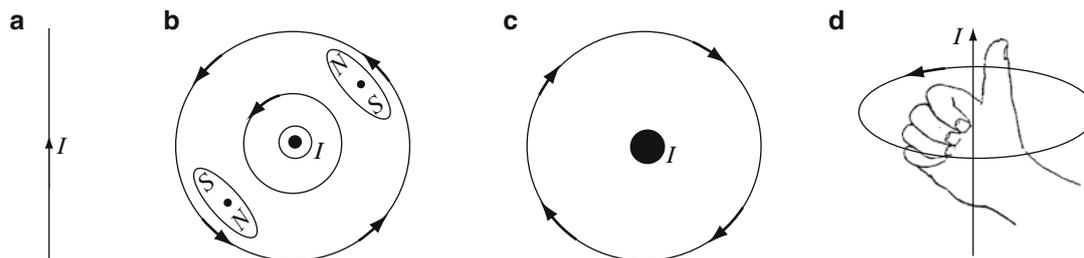
This simple experiment is all that is necessary to establish the existence of the magnetic field of a magnet. What we need to do now is to show that the same occurs due to steady electric currents and, more importantly, to find the exact relationship between the electric current and the magnetic field. For now, we will say that the magnetic field is a **field of force** acting on a magnet; the only qualification is that the magnet must be small (for example, the needle of a compass) to render the measurement valid.

The first of these, showing that a force exists due to the magnetic field produced by a current, is easily performed using the above idea but for a steady current. This is, in fact, what Oersted did in his historical experiment. Consider a straight wire, carrying a steady (direct) current  $I$  as shown in **Figure 8.2a**. Positioning the compass at various locations in space, we note that a force exists on the compass. Since this force cannot be an electrostatic force (no charges on the compass) and it certainly cannot be gravitational (since disconnecting the current will cancel the force), we must conclude that the force is similar to that between two magnets. Thus, the current in the wire has generated a magnetic field with properties identical to those of an equivalent magnet.

Experiment shows that the direction of the compass now is tangential to any circle centered at the current and depends on its direction. In **Figure 8.2b**, the direction is counterclockwise, whereas in **Figure 8.2c**, it is clockwise. The direction of the

compass (arrow pointing from S to N) is taken as the direction of the magnetic field. From this experiment, we establish two very general and important properties:

- (1) The direction of the magnetic field due to a current is defined by the right-hand rule shown in **Figure 8.2d**. The rule states that “if the thumb of the right-hand shows the direction of current, the fingers show the direction of the magnetic field.”



**Figure 8.2** Oersted’s experiment. (a) Current in a wire. (b) Direction of the compass needle for a current out of the page. (c) Direction of the compass for a current into the page. (d) The right-hand rule establishes the relation between the direction of current and the resulting magnetic field

- (2) The lines of magnetic field (i.e., the lines to which the compass is tangential) are always closed lines. We will return to this property of the magnetic field shortly, after we have a better definition of the magnetic field.

The above experiment is lacking in one respect: Nothing has been established as far as the strength of the force exerted on the compass. We can intuitively say that the further away from the current we are, the lower the force, but exact values of force, or its relationship with the current, cannot be established from this experiment.

Since a magnetic field is generated by a steady current but not by a static charge, we conclude that the magnetic field and, therefore, the force in the magnetic field are related to the motion of charges (current) or, more specifically, to the velocity of charges. We recall that in the case of a static electric charge, a force acted on the charge as  $\mathbf{F} = q\mathbf{E}$ . Similarly, there is a force acting on a moving charge due to the magnetic field. As with the electrostatic force, we can establish the force in a magnetic field by performing a series of experiments from which both the magnitude and direction of the force are found. These experiments lead to the following relation between the magnetic force  $\mathbf{F}_m$ , electric charge  $q$ , charge velocity  $\mathbf{v}$ , and a new quantity  $\mathbf{B}$ :

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \quad [\text{N}] \quad (8.1)$$

where  $\mathbf{B}$  is called the *magnetic flux density*. This relation is known as the Lorentz<sup>3</sup> force equation for a moving charge in the magnetic field, whereas  $\mathbf{F}_e = q\mathbf{E}$  is the Coulomb force for a charge  $q$  in the electric field. For a general field, which includes both electric and magnetic components, the force on a charge is the sum of the two forces and may be written as

$$\mathbf{F}_{total} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad [\text{N}] \quad (8.2)$$

This relation is known as the Lorentz–Coulomb force equation (sometimes only as the Lorentz force equation). Note that a stationary charge ( $\mathbf{v} = 0$ ) only experiences an electric force, whereas a moving charge experiences both an electric and a magnetic force.

The relation  $q\mathbf{v} \times \mathbf{B}$  may be viewed as the defining relation for the magnetic flux density  $\mathbf{B}$  since all other quantities are known or measurable. This is similar to using the relation  $\mathbf{F} = q\mathbf{E}$  to define the electric field intensity as  $\mathbf{E} = \mathbf{F}/q$  (force per unit charge). Although we will define the magnetic flux density in a different way shortly, the discussion here is useful in that it shows the physical meaning of the magnetic flux density.

<sup>3</sup> Hendrik Antoon Lorentz (1853–1928). Dutch physicist who is best known for his work on the effects of magnetism on radiation. For this he received the Nobel Prize in 1902. In his attempt to explain electricity, magnetism, and light, he arrived at the Lorentz transformation, which helped Albert Einstein in formulating his theory of relativity.

In terms of units, the magnetic flux density has units of (newton/coulomb)/ (meter/second) or newton/(ampere · meter). This unit is called the *tesla* [T] or weber/meter<sup>2</sup> [Wb/m<sup>2</sup>]. The weber is equal to newton.meter/ampere. The common SI unit is the tesla [T], whereas the [Wb/m<sup>2</sup>] is used sometimes to emphasize that  $\mathbf{B}$  is a flux density. Both units are appropriate and commonly used.

In addition to the magnetic flux density, we define the magnetic field intensity  $\mathbf{H}$  using the constitutive relation

$$\mathbf{B} = \mu \mathbf{H} \quad [\text{T}] \quad (8.3)$$

where  $\mu$  is called the *magnetic permeability* (or, in short, permeability) of materials. In free space, the relation is

$$\mathbf{B} = \mu_0 \mathbf{H} \quad [\text{T}] \quad (8.4)$$

where  $\mu_0$  is the *permeability of free space*. The magnitude of  $\mu_0$  is  $4\pi \times 10^{-7}$  and its unit is the henry/meter or [H/m]. This unit will become obvious as we discuss the magnetic field intensity  $\mathbf{H}$  and the magnetic flux density  $\mathbf{B}$ . The relations in Eqs. (8.3) and (8.4) are fundamental and we will discuss them at some length in the following chapter. At this point, it is sufficient to indicate that both can be shown to be correct by performing whatever experiments are necessary to do so. This approach will allow us to introduce the important laws of Biot and Savart and Ampere.

**Equation (8.1)** may be viewed as defining the magnetic flux density  $\mathbf{B}$ , but this definition is not convenient for calculation because it involves the vector product of  $\mathbf{v}$  and  $\mathbf{B}$  rather than  $\mathbf{B}$  directly and is defined on individual moving charges. We will, therefore, seek a more convenient mathematical relation, one that will allow calculation of the magnetic flux density directly from the current  $I$ . Also, the magnetic flux density is material dependent (through  $\mu$ ). For this reason, we calculate  $\mathbf{H}$  rather than  $\mathbf{B}$ , at least until we had a chance to discuss behavior of materials in the magnetic field, which we will do in the following chapter. For these reasons, most of the discussion in this chapter is in terms of  $\mathbf{H}$  rather than  $\mathbf{B}$ . This does not create much difficulty since  $\mathbf{B} = \mu \mathbf{H}$  can be used to calculate  $\mathbf{B}$ . We will return to using  $\mathbf{B}$  in latter parts of this chapter and in the next chapter because the force in the magnetic field depends on  $\mathbf{B}$  and force is fundamental in magnetic applications.

There are, in fact, two relations that accomplish the task of calculating the magnetic field intensity: the Biot–Savart and Ampere’s laws. We start by describing the Biot–Savart law since Ampere’s law may be derived from the Biot–Savart law.

### 8.3 The Biot–Savart Law

The law due to Biot and Savart<sup>4</sup> was deduced from a series of experiments on the effects of currents on each other and on permanent magnets (such as the compass needle Oersted used). The law gives a relation for the magnetic field intensity  $\mathbf{H}$  at a point in space due to a current  $I$ . It states that the element of magnetic field intensity  $d\mathbf{H}$  at a point  $P(x,y,z)$  due to an element of current  $I dl'$  located at a point  $P'(x',y',z')$  is proportional to the current element, the angle  $\psi$  between  $dl'$  and the position vector  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ , and inversely proportional to the distance  $R$  squared.

These quantities are shown in **Figure 8.3**. The magnitude of  $d\mathbf{H}$  is

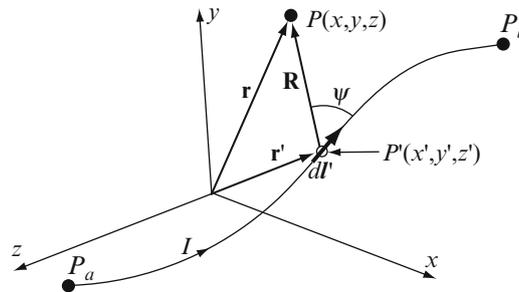
$$dH(x, y, z) = \frac{I(x', y', z') dl' (x', y', z') \sin \psi}{4\pi R^2} \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.5)$$

where the coordinates  $(x,y,z)$  indicate a general point at which the field is calculated (field point), while coordinates  $(x',y',z')$  indicate the location of the current element (source point). Thus,  $I$  and  $dl'$  only depend on the primed coordinates. The unit of  $\mathbf{H}$  is the ampere/meter [A/m], as can be seen from **Eq. (8.5)**.

<sup>4</sup>Jean-Baptiste Biot (1774–1862) was a professor of physics and astronomy. Among other topics, he worked on optics and the theory of light. Felix Savart (1791–1841) was a physician by training but later became professor of physics. This law is attributed to Biot and Savart because they were the first to present quantitative results for the force of a current on a magnet (in fact quantifying Oersted’s discovery). The experiments mentioned here are described in detail in James Clerk Maxwell’s *Treatise on Electricity and Magnetism* (Part IV, Chapter II), as Ampere’s experiments. The law was presented in 1820, about a year after the landmark experiment by Oersted established the link between current and the magnetic field.

To find the total magnetic field intensity at point  $P(x,y,z)$ , the above quantity is integrated along the current. For the segment in **Figure 8.3**, we get

$$dH(x, y, z) = \int_a^b \frac{I(x', y', z') dl'(x', y', z') \sin \psi}{4\pi R^2} \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.6)$$



**Figure 8.3** Derivation of the Biot–Savart law for a current element

Since the magnetic field intensity  $\mathbf{H}$  is a vector, it is more appropriate to calculate the vector  $\mathbf{H}$  rather than its magnitude. From the right-hand rule described above, we note that the magnetic field intensity  $\mathbf{H}$  is perpendicular to  $I$  (or  $d\mathbf{l}'$ ) as well as to the plane formed by  $\mathbf{r}$  and  $\mathbf{r}'$ . Thus, it must be described as a cross product between  $d\mathbf{l}'$  and  $\mathbf{r} - \mathbf{r}'$ . Using the cross product  $d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = \hat{\mathbf{n}} dlR \sin \psi$  in **Eq. (8.6)** gives

$$\mathbf{H}(x, y, z) = \frac{1}{4\pi} \int_a^b \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.7)$$

Also, the current  $I$  is constant in most cases we will treat and may be taken outside the integral. Alternatively, using the definition of the unit vector,  $\hat{\mathbf{R}} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ ,

$$\mathbf{H}(x, y, z) = \frac{1}{4\pi} \int_a^b \frac{I d\mathbf{l}' \times \hat{\mathbf{R}}}{|\mathbf{R}|^2} \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.8)$$

In either case, the direction of  $\mathbf{H}$  is perpendicular to the vectors  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  and  $d\mathbf{l}'$ , as required. If we use the definition of the magnetic flux density  $\mathbf{B} = \mu\mathbf{H}$ , the Biot–Savart law becomes

$$\mathbf{B}(x, y, z) = \frac{\mu}{4\pi} \int_a^b \frac{I d\mathbf{l}' \times \hat{\mathbf{R}}}{|\mathbf{R}|^2} \quad [\text{T}] \quad (8.9)$$

This is then the basic law we need to define and calculate the magnetic field intensity or magnetic flux density, given a current or current distribution. We must, however, note the following:

- (1) The current element is assumed to be infinitely thin. A conductor of this type is called a *filament*.
- (2) The shape of the contour (i.e., the shape of the filament or wire) is not important except for the evaluation of the line integral.
- (3) Strictly speaking, we must always have a closed contour along which we integrate; otherwise, there can be no current. However, it is permissible to calculate the contribution to the field due to a segment of the contour, assuming the current closes somehow.
- (4) The space in which the current flows and the field is calculated is assumed to be of the same material and homogeneous.

The application of the law for practical calculations is straightforward. One aspect that must be followed strictly is the fact that integration is always along the current (i.e., in primed coordinates such as  $(x',y',z')$  or  $(r',\phi',z')$ ), whereas the field is calculated at a fixed point such as  $(x,y,z)$  or  $(r,\phi,z)$ . The two systems of coordinates should not be confused.

The Biot–Savart law is rather general and can be used for any current configuration, including distributed currents and current densities. Before we discuss these, it is useful to review a few examples of calculation of fields due to filaments (thin conducting wires).

**Example 8.1 Field Intensity Due to a Short, Straight Segment** The thin, finite-length wire in **Figure 8.4a** carries a current  $I = 1$  A:

- Calculate the magnetic field intensity at point  $A$  shown in **Figure 8.4a**.
- Calculate the magnetic field intensity at point  $B$  shown in **Figure 8.4a**.
- What is the magnetic field intensity at a distance  $h = 1$  m from the wire if the wire is infinitely long?

**Solution:** Because the wire produces a circular field, the problem is best solved in cylindrical coordinates  $(r,\phi,z)$ . An element of length  $d\mathbf{l}' = \hat{\mathbf{z}}dz'$  is identified at point  $(0,0,z')$  in **Figure 8.4b** or **8.4c**. The magnetic field intensity at point  $P(r,\phi,z)$  is calculated using the Biot–Savart law. The solution is independent of the  $\phi$  coordinate, simply from symmetry considerations:

- At point  $A$ , the coordinates are  $(r = 1, z = 0)$ . The magnetic field intensity at point  $A$  due to the current in element  $d\mathbf{l}' = \hat{\mathbf{z}}dz'$  is

$$d\mathbf{H}(r, z) = \frac{I\hat{\mathbf{z}}dz' \times \hat{\mathbf{R}}}{4\pi|\mathbf{R}|^2} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The vector  $\mathbf{R}$  (see **Figure 8.4b**) and the unit vector  $\hat{\mathbf{R}}$  are, respectively,

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \hat{\mathbf{r}}h - \hat{\mathbf{z}}z', \quad \hat{\mathbf{R}} = \frac{\hat{\mathbf{r}}h - \hat{\mathbf{z}}z'}{\sqrt{h^2 + z'^2}}$$

Thus,  $d\mathbf{H}$  is

$$d\mathbf{H}(r, z) = \frac{I\hat{\mathbf{z}}dz' \times (\hat{\mathbf{r}}h - \hat{\mathbf{z}}z')}{4\pi(h^2 + z'^2)(\sqrt{h^2 + z'^2})} = \hat{\boldsymbol{\phi}} \frac{Ihdz'}{4\pi(h^2 + z'^2)^{3/2}} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The total magnetic field intensity at  $A$  is found setting  $h = 1$  m and integrating from  $z' = -1$  m to  $z' = +1$  m:

$$\mathbf{H}_A = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{z'=-1}^{z'=1} \frac{dz'}{(1+z'^2)^{3/2}} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi} \int_{z'=0}^{z'=1} \frac{dz'}{(1+z'^2)^{3/2}} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi} \frac{z'}{\sqrt{z'^2+1}} \Big|_0^1 = \hat{\boldsymbol{\phi}} \frac{I}{2\sqrt{2}\pi} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The magnetic field intensity is in the  $\phi$  direction, as required.

- To calculate the magnetic field intensity at point  $B$ , it is convenient to shift the  $r$  axis so that point  $B$  is on the axis. This allows the use of the previous result, with a change in the limits of integration from  $(-1,+1)$  to  $(0,2)$ , as shown in **Figure 8.4c**:

$$\mathbf{H}_B = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{z'=0}^{z'=2} \frac{dz'}{(1+z'^2)^{3/2}} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \frac{z'}{\sqrt{z'^2+1}} \Big|_0^2 = \hat{\boldsymbol{\phi}} \frac{I}{2\sqrt{5}\pi} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

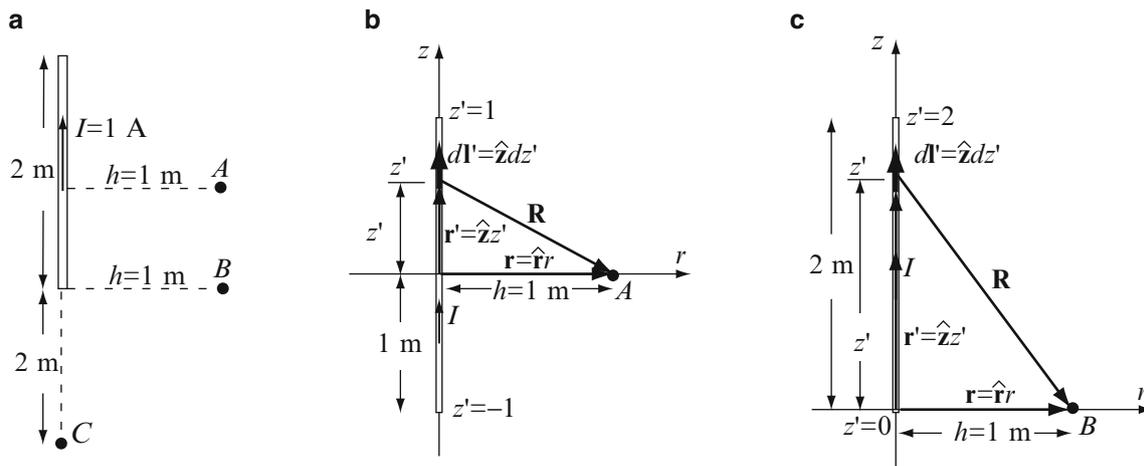
Note that the magnetic field intensity at point  $B$  is lower than at point  $A$ .

- (c) For an infinitely long wire, we use the result in (a) but integrate between  $-\infty$  and  $+\infty$ . The magnetic field intensity at a distance  $h$  from the wire is

$$\mathbf{H}_h = \hat{\phi} \frac{I}{4\pi} \int_{z'=-\infty}^{z'=\infty} \frac{h dz'}{(h^2 + z'^2)^{3/2}} = \hat{\phi} \frac{I}{2\pi} \frac{z'}{h\sqrt{h^2 + z'^2}} \Big|_0^{\infty} = \hat{\phi} \frac{I}{2\pi h} \left[ \frac{\text{A}}{\text{m}} \right]$$

For  $I = 1 \text{ A}$  and  $h = 1 \text{ m}$ ,

$$\mathbf{H} = \hat{\phi} \frac{1}{2\pi} \left[ \frac{\text{A}}{\text{m}} \right].$$



**Figure 8.4** Magnetic field intensity due to a short, straight segment carrying current  $I$ . (a) Geometry and dimensions. (b) Calculation of  $\mathbf{H}$  at point A. (c) Calculation of  $\mathbf{H}$  at point B

**Exercise 8.1** For the geometry in **Example 8.1a**:

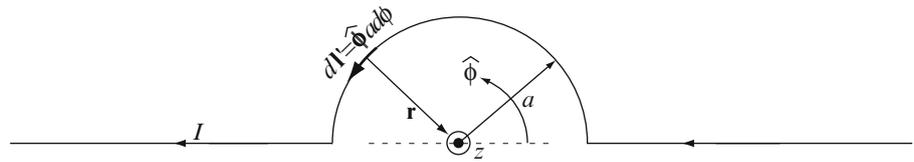
- (a) Calculate the magnetic field intensity at  $h = 0$  in the middle of the segment in **Figure 8.4a**.  
 (b) Calculate the magnetic field intensity at any point along the axis of the wire, but outside the wire. Use, for example, point C in **Figure 8.4a**.

**Answer**

- (a) 0. (b) 0.

**Example 8.2 Magnetic Field Intensity and Magnetic Flux Density Due to a Half-Loop** A current  $I$  [A] flows in the circuit shown in **Figure 8.5**. Calculate the magnetic flux density and the magnetic field intensity at the center of the semicircle assuming the circuit is in free space.

**Figure 8.5** Calculation of the magnetic flux density at the center of a semicircular current loop



**Solution:** The Biot–Savart law is used to integrate around the semicircular loop. The position vector  $\mathbf{r}$  is always in the negative  $r$  direction, and the magnetic flux density is in the direction perpendicular to the loop (out of the page for any point inside the loop). The contribution of the straight wire is zero since for any point on the straight wire,  $\sin\phi$  in Eq. (8.5) is zero (see Exercise 8.1).

The two vectors necessary are  $d\mathbf{l}'$  (in the direction of current) and  $\mathbf{r}$  (pointing to the center of the loop). Taking the current to flow in the positive  $\phi$  direction, the two vectors are

$$\mathbf{r} = -\hat{\mathbf{r}}a, \quad d\mathbf{l}' = \hat{\boldsymbol{\phi}}a d\phi' \quad \rightarrow \quad d\mathbf{l}' \times \mathbf{r} = \hat{\boldsymbol{\phi}}a d\phi' \times (-\hat{\mathbf{r}}a) = \hat{\mathbf{z}}a^2 d\phi'$$

Now the Biot–Savart law gives

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{a^2 d\phi'}{|\mathbf{r}|^3} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{d\phi'}{a} = \hat{\mathbf{z}} \frac{\mu_0 I}{4a} \quad [\text{T}]$$

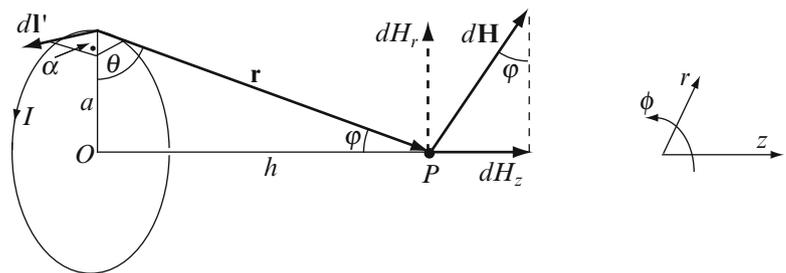
Thus,

$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu_0 I}{4a} \quad [\text{T}] \quad \text{and} \quad \mathbf{H} = \hat{\mathbf{z}} \frac{I}{4a} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

### Example 8.3 Magnetic Field Intensity of a Circular Loop

- Calculate the magnetic field intensity  $\mathbf{H}$  at point  $P$  in Figure 8.6 generated by the current  $I$  [A] in the loop. Point  $P$  is at a height  $h$  [m] along the axis of the loop.
- Calculate the magnetic field intensity at the center of the loop (point  $O$ ).

**Figure 8.6** Calculation of the magnetic field intensity at height  $h$  above a current-carrying loop



**Solution:** The solution is most easily carried out in cylindrical coordinates. From Figure 8.6, the element of length is  $d\mathbf{l}' = \hat{\boldsymbol{\phi}}a d\phi'$  and is directed in the positive  $\phi$  direction. This produces a magnetic field intensity perpendicular to  $\mathbf{r}$ . This magnetic field intensity has an axial component and a component perpendicular to the axis. The latter cancels because of symmetry of the current: An element  $d\mathbf{l}'$  diametrically opposed to the element shown produces an identical field but the component normal to the axis is opposite in direction. Since the only nonzero field intensity is in the  $z$  direction, the calculations may be carried out in scalar components.

(a) The magnetic field intensity due to current in an element of the loop of length  $dl'$  is

$$d\mathbf{H} = \frac{Id\mathbf{l}' \times \mathbf{r}}{4\pi r^3} \quad \rightarrow \quad dH = \frac{Idl'}{4\pi r^2} \sin\alpha = \frac{Idl'}{4\pi r^2} \left[ \frac{\text{A}}{\text{m}} \right]$$

where  $\alpha$  is the angle between  $d\mathbf{l}'$  and  $\mathbf{r}$  and  $dl' = ad\phi'$ . In this case,  $\alpha = 90^\circ$ . The component  $dH_z$  is

$$dH_z = \frac{Idl'}{4\pi r^2} \sin\phi$$

The angle  $\phi$  is constant for any position on the loop and equals

$$\sin\phi = \frac{a}{\sqrt{a^2 + h^2}}$$

Thus,

$$dH_z = \frac{Idl'}{4\pi(a^2 + h^2)} \frac{a}{\sqrt{a^2 + h^2}} = \frac{Iadl'}{4\pi(a^2 + h^2)^{3/2}} \left[ \frac{\text{A}}{\text{m}} \right]$$

Substituting for  $dl'$  and integrating over the loop, we get

$$H_z = \frac{Ia^2}{4\pi(a^2 + h^2)^{3/2}} \int_{\phi=0}^{\phi=2\pi} d\phi = \frac{Ia^2}{2(a^2 + h^2)^{3/2}} \left[ \frac{\text{A}}{\text{m}} \right]$$

or, in a more formal notation,

$$\mathbf{H}(0, 0, h) = \hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + h^2)^{3/2}} \left[ \frac{\text{A}}{\text{m}} \right].$$

(b) At the center of the loop,  $h = 0$ . Thus,

$$\mathbf{H}(0, 0, 0) = \hat{\mathbf{z}} \frac{I}{2a} \left[ \frac{\text{A}}{\text{m}} \right]$$

Comparison with **Example 8.2** shows this result to be twice the intensity of the half-loop.

**Example 8.4 Magnetic Field Intensity Due to a Rectangular Loop: Superposition of Fields** A rectangular loop carries a current  $I$  [A] as shown in **Figure 8.7a**. Calculate the magnetic field intensity at the center of the loop.

**Solution:** To calculate the magnetic field intensity, we use the Biot–Savart law for each of the segments forming the loop, following a slightly different method than in **Example 8.1**. **Figures 8.7b** and **8.7c** show how the field of each segment is calculated separately. Superposition of the four fields completes the solution.

Starting with the left, vertical segment, which we denote as segment (1) (see **Figure 8.7b**), and using scalar components, we get at the center of the loop (see also **Example 8.1** for the magnetic field intensity of a short, straight segment)

$$dH_1 = \frac{I|d\mathbf{l}' \times \hat{\mathbf{r}}|}{4\pi r^2} = \frac{Idl'}{4\pi r^2} \sin\theta = \frac{Idl'}{4\pi r^2} \cos\varphi \left[ \frac{\text{A}}{\text{m}} \right]$$

where  $|\hat{\mathbf{r}}| = 1$  was used. Instead of using the method in **Example 8.1**, the variables are changed to be a function of  $\varphi$  as follows. From **Figure 8.7b**,

$$\tan \varphi = \frac{l'}{b} \rightarrow l' = b \tan \varphi \rightarrow \frac{dl'}{d\varphi} = b \sec^2 \varphi \rightarrow dl' = b \sec^2 \varphi d\varphi$$

In addition,  $\cos \varphi = b/r$ , which gives  $r = b/\cos \varphi$ . Substituting  $dl'$  and  $r$  in the expression for  $dH$  and noting that the integration is symmetric about the centerline, we can write

$$H_1 = \frac{I}{4\pi b} \int_{-\varphi_1}^{\varphi_1} \cos \varphi d\varphi = \frac{I \sin \varphi}{4\pi b} \Big|_{-\varphi_1}^{\varphi_1} = \frac{I \sin \varphi_1}{2\pi b} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

where angles  $+\varphi_1$  and  $-\varphi_1$  are the limiting angles corresponding to points  $Q$  and  $R$  in **Figure 8.7b**. From **Figure 8.7b**, we can write  $\sin \varphi_1$  as

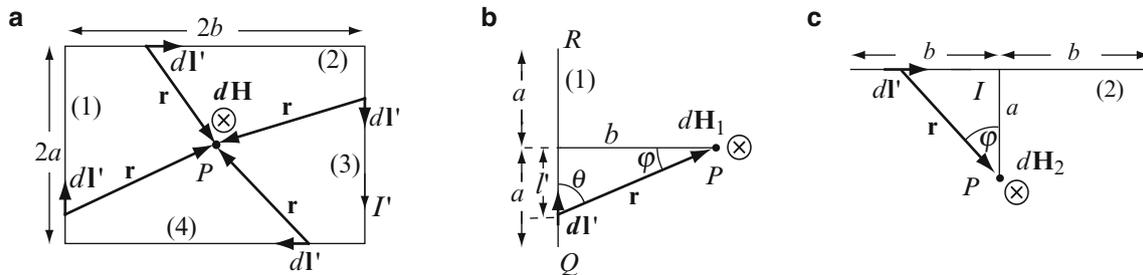
$$\sin \varphi_1 = \frac{a}{\sqrt{a^2 + b^2}}$$

Substituting this in the solution gives

$$H_1 = \frac{Ia}{2\pi b \sqrt{a^2 + b^2}} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The magnetic field intensity of segment (3) is identical in magnitude and direction. The magnetic field intensity of segment (2) is calculated by analogy since the only difference between segments (1) and (2) is that  $a$  and  $b$  are interchanged (see **Figure 8.7c**). Thus, the magnetic field intensity at the center of the loop due to segment (2) and therefore also due to segment (4) is

$$H_2 = H_4 = \frac{Ib}{2\pi a \sqrt{a^2 + b^2}} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$



**Figure 8.7** (a) Calculation of the magnetic field intensity at the center of a rectangular loop. (b) Calculation of the magnetic field intensity due to the left segment in (a). (c) Calculation of the magnetic field intensity due to the top segment

The total field is  $H_t = 2H_2 + 2H_1$ , or

$$H_t = \frac{I}{\pi \sqrt{a^2 + b^2}} \left( \frac{a}{b} + \frac{b}{a} \right) = \frac{I}{\pi ab} \sqrt{a^2 + b^2} \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (\text{directed into the page})$$

**Exercise 8.2** Consider again **Example 8.4** and calculate the magnetic field intensity at one of the corners of the loop in **Figure 8.7a**.

**Answer**  $H = \frac{I}{8\pi ab} \sqrt{a^2 + b^2} \left[ \frac{\text{A}}{\text{m}} \right]$  (directed into the page).

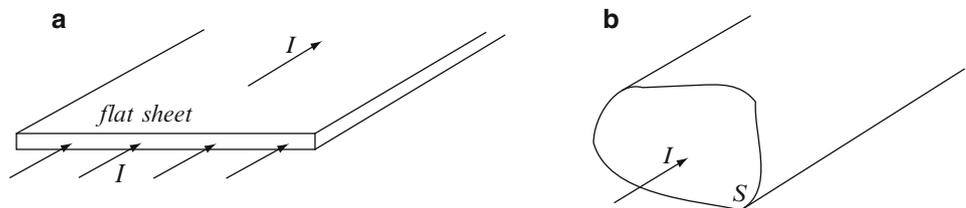
### 8.3.1 Applications of the Biot–Savart Law to Distributed Currents

The fact that the Biot–Savart law is written as a line integral does not mean that current distributions (say, current in a thin sheet of metal or current in a volume) cannot be treated as well. In fact, all that is required is to view the current distribution as an assembly of thin wires and calculate the field as a superposition of the fields due to each thin wire using **Eq. (8.8)** or **(8.9)**. In practice, each thin wire is a wire of differential size, and the contribution is found by integration rather than by summation.

There are two specific current configurations which are particularly important and will be treated here. One is a planar distribution of currents shown in **Figure 8.8a** and the second is a volume distribution shown in **Figure 8.8b**. These represent thin flat conductors and thick conductors, respectively. Consider first a very thin, flat conductor of length  $2L$  and width  $2d$  as shown in **Figure 8.9a**. A total current  $I$  flows in the conductor. Note that the conductor must be part of a larger closed circuit for the current to flow, but we wish to calculate the magnetic field intensity due to this section alone. If we “cut” out of the sheet, a wire of width  $dx'$  and the same thickness as the sheet, we obtain the element shown in **Figure 8.9b**. This is exactly the same configuration as in **Figure 8.3** for a straight, thin wire. The current in this differential wire is

$$dI = \left( \frac{I}{2d} \right) dx' \quad [\text{A}] \quad (8.10)$$

**Figure 8.8** (a) Surface current distribution. (b) Volume current distribution



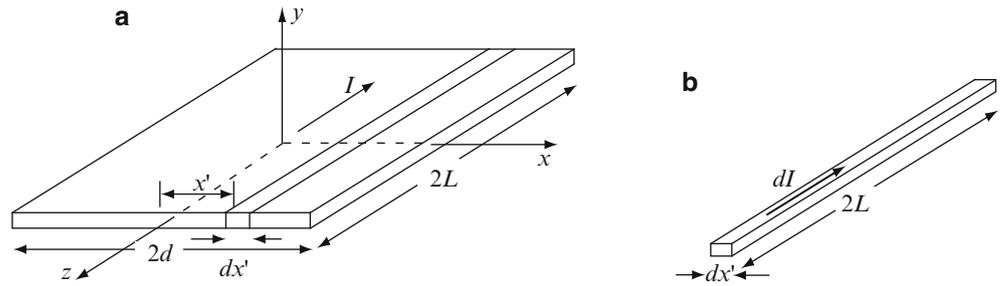
and the total contribution due to this differential wire is found using **Eq. (8.8)**:

$$d\mathbf{H}(x, y, z) = \left[ \int_{z'=-L}^{z'=+L} \frac{I dx'}{2d} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi |\mathbf{r} - \mathbf{r}'|^2} \right] \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.11)$$

where integration is on  $d\mathbf{l}$  and  $\mathbf{r}$  is the vector connecting  $d\mathbf{l}$  and  $P(x, y, z)$  (see **Figure 8.3**). To obtain the total field intensity, we integrate over the width of the current sheet in **Figure 8.9**. We get

$$\mathbf{H}(x, y, z) = \int_{x'=-d}^{x'=+d} \left[ \int_{z'=-L}^{z'=+L} \frac{I}{2d} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi |\mathbf{r} - \mathbf{r}'|^2} \right] dx' \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.12)$$

**Figure 8.9** (a) Flat sheet with a current distribution along its width. (b) An element of current used to apply the Biot–Savart law

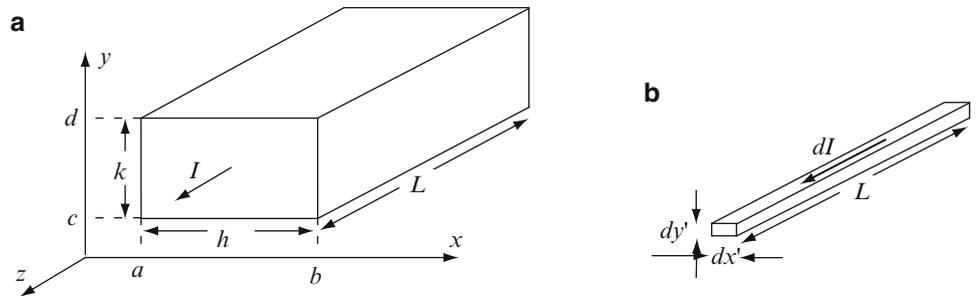


Now, we must define the various vectors,  $d\mathbf{l}'$ ,  $\mathbf{r}'$ ,  $\mathbf{R}$ , and perform the integration. An example of how this may be done is shown in **Example 8.5**. In **Eq. (8.12)**, the term  $I/2d$  has units of current per unit length [A/m]. This is a surface current density and is a convenient term when dealing with thin sheets of current since, in this case, the thickness of the current sheet is small and fixed.

If the current is distributed throughout a conductor of finite volume, we proceed in the same fashion: Define a differential wire and integrate the contributions of the individual filaments over the cross section of the conductor. To see how this is done, consider the conductor in **Figure 8.10a**. The current  $I$  is assumed to be uniformly distributed throughout the cross-sectional area of the conductor. Thus, referring to **Figure 8.10b**, the geometry is that of a thin wire of cross-sectional area  $dx'dy'$  and a current  $dI$ :

$$dI = \frac{I}{kh} dx'dy' \quad [\text{A}] \quad (8.13)$$

**Figure 8.10** (a) A thick conductor with a current distributed throughout its cross section. (b) An element of current in the form of a wire of differential cross-sectional area



where the quantity  $I/kh$  is the current density in the wire [A/m<sup>2</sup>]. Now, we substitute **Eq. (8.13)** into **Eq. (8.8)** to find the contribution of the differential wire:

$$d\mathbf{H}(x, y, z) = \int_{z'=0}^{z'=L} \frac{I}{kh} dx'dy' \frac{d\mathbf{l}' \times \hat{\mathbf{R}}}{4\pi|\mathbf{r} - \mathbf{r}'|^2} \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.14)$$

where, based on **Figure 8.10**, we substituted  $d\mathbf{l}' = \hat{\mathbf{z}} dz'$  (integrating in the direction of the current). To find the total magnetic field intensity due to the conductor, we integrate over the cross-sectional area of the conductor to account for the total current:

$$\mathbf{H}(x, y, z) = \int_{y'=c}^{y'=d} \left[ \int_{x'=a}^{x'=b} \left[ \int_{z'=0}^{z'=L} \frac{I}{kh} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{R}}}{4\pi|\mathbf{r} - \mathbf{r}'|^2} dz' \right] dx' \right] dy' \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.15)$$

where  $d\mathbf{l}$ ,  $\mathbf{r}$ ,  $\mathbf{r}'$ , and  $\mathbf{R}$  are defined in **Figure 8.3**. These expressions may look intimidating, but they are nothing more than the integration of the magnetic field intensity of a differential area wire over the cross-sectional area of a thick conductor [**Eq. (8.15)**] or width of a flat conductor [**Eq. (8.12)**].

**Example 8.5 Magnetic Field Intensity Due to a Long Thin Sheet: Application as a Ground Plane** A thin sheet of conducting material,  $b = 1$  m wide, and very long carries a current  $I = 100$  A. Calculate the magnetic field intensity at a distance  $h = 0.1$  m above the center of the conductor (see **Figure 8.11a**).

**Solution:** The magnetic field intensity may be evaluated by viewing the sheet as an assembly of thin conductors, each  $dx'$  [m] wide, infinitely long, and carrying a current  $(I/b)dx'$ . The magnetic field intensity for a thin segment was found in **Example 8.1** by direct application of the Biot–Savart law. The field of the infinitesimally thin conductor is integrated over the width of the sheet to find the magnetic field intensity due to the sheet.

The current  $dI$  in an infinitesimally thin section of the sheet (cut lengthwise as in **Figure 8.11b**) is

$$dI = \frac{I}{b} dx' = 100 dx' \quad [\text{A}]$$

The magnetic field intensity at a distance  $r$  from a long line carrying a current  $dI$  (see **Example 8.1c**) is

$$d\mathbf{H} = \hat{\phi} \frac{dI}{2\pi r} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

Now we place this infinitesimal segment at a general location  $x'$  on the sheet, as shown in **Figure 8.11c**. The distance  $r$  is the distance to the point at which we wish to calculate the magnetic field intensity. Because as  $x'$  changes, the angle  $\theta$  and, therefore, the direction of the field intensity change, it is easier to calculate the vertical and horizontal components separately and then integrate each. From symmetry alone, we know that the vertical component must cancel because symmetric segments have equal and opposite vertical components. The horizontal component is

$$dH_x = \frac{dI}{2\pi r} \cos(90 - \theta) = \frac{dI}{2\pi r} \sin\theta = \frac{h dI}{2\pi r^2} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

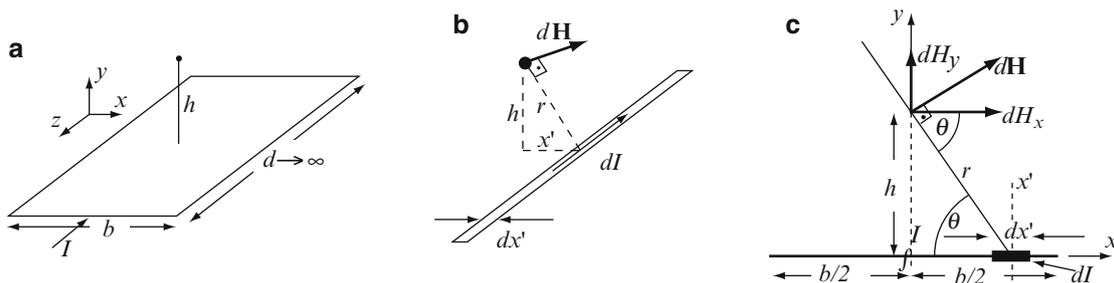
where  $\sin\theta = h/r$  was used. Substituting  $r = (x'^2 + h^2)^{1/2}$  and rearranging terms, we get

$$dH_x = \frac{h dI}{2\pi(x'^2 + h^2)} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

Substituting  $dI = 100 dx'$  and  $h = 0.1$  and integrating between  $x' = -b/2 = -0.5$  m and  $x' = b/2 = 0.5$  m gives

$$H_x = \frac{10}{2\pi} \int_{-0.5}^{0.5} \frac{dx'}{(x'^2 + 0.1^2)} = \frac{10}{2\pi} \left[ \frac{1}{0.1} \tan^{-1} \frac{x'}{0.1} \right] \Big|_{-0.5}^{0.5} = \frac{50 \times 2.7468}{\pi} = 43.72 \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The resulting magnetic field intensity is parallel to the sheet (in the  $x$  direction in **Figure 8.11c**).



**Figure 8.11** Magnetic field intensity due to a conducting ground plane. (a) Configuration. (b) Calculation of the magnetic field intensity of a differential strip. (c) Calculation of the field of the ground plane as an assembly of infinitesimal strips

## 8.4 Ampere's Law

Ampere's law,<sup>5</sup> also called the Ampere circuital law, states that

*“the circulation of  $\mathbf{H}$  around a closed path  $C$  is equal to the current enclosed by the path.”*

That is,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} \quad [\text{A}] \quad (8.16)$$

The circulation is defined by the line integral around a closed contour of the scalar product  $\mathbf{H} \cdot d\mathbf{l}$ . Thus, in fact, only the component of  $\mathbf{H}$  tangential to the contour of integration is included in the calculation.

Why do we need another law that, as stated, may be derived from the Biot–Savart law (even though we show no proof)? The reason is simple and is the same we used earlier to justify Gauss' law as opposed to the use of Coulomb's law; Ampere's law is much easier to apply to some problems. In particular, highly symmetric current configurations are very easy to evaluate using Ampere's law, whereas they may be more complex using the Biot–Savart law. It should be remembered, however, that Ampere's law is universally applicable and, in the next section, we will show that Ampere's law is, in fact, one of the postulates of the magnetic field. In a practical sense, Ampere's law is another tool which we can use whenever it makes sense to do so.

When applying Ampere's law, we must remember that under normal circumstances, the unknown quantity is  $\mathbf{H}$ . Since  $\mathbf{H}$  is inside the integral sign, we must find a closed contour, enclosing the current  $I$  such that the component of  $\mathbf{H}$  tangential to the contour is constant along the contour. Under these conditions,  $\mathbf{H} \cdot d\mathbf{l} = H_{\text{tan}} dl$  and  $H_{\text{tan}}$  can be taken outside the integral sign. Evaluation of the integral is now possible. Thus, the requirements here are that  $\mathbf{H}$  be tangential to a contour and constant along the contour. These conditions are satisfied for highly symmetric current configurations. These include the following:

- (1) Current in an infinite (or as an approximation, in a very long) filament.
- (2) Current or current density in an infinite (or very long) solid or hollow cylindrical conductor. Normally, the current is uniformly distributed in the conductor, but any radially symmetric current distribution is allowed.
- (3) Infinite sheet of current (or very large, flat current sheet).
- (4) Multiple conductors in a symmetric configuration.
- (5) Nonsymmetric current distributions that are superpositions of symmetric current distributions as described in (1) through (3).

The key requirement for the application of Ampere's law is symmetry of current or current density distribution. The law applies to any current configurations, but it is not generally possible to find a contour over which the tangential magnetic field intensity is constant and, therefore, evaluate the field. The following examples show various important aspects of application of Ampere's law.

**Example 8.6 Application: Field Intensity Due to a Single, Thin Wire. Magnetic Field of Overhead Transmission Lines** Calculate the magnetic field intensity due to a long filamentary conductor carrying a current  $I$  at a distance  $h$  from the wire. The conductor is very long (infinite). Compare this result with the result in **Example 8.1c**.

**Solution:** Since the magnetic field intensity of the wire is circular, a circle of radius  $h$  (in the  $r - \phi$  plane) may be used as the contour in Ampere's law. Taking the current to be in the  $z$  direction for convenience (see **Figure 8.2b** or **Figure 8.4b**), the magnetic field intensity is in the  $\phi$  direction. Taking  $d\mathbf{l} = \hat{\phi} h d\phi$  we get from Ampere's law

$$\oint \mathbf{H} \cdot d\mathbf{l} = \oint \hat{\phi} H \cdot \hat{\phi} h d\phi = Hh \int_0^{2\pi} d\phi = 2\pi Hh = I \quad \rightarrow \quad H = \frac{I}{2\pi h} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

This result is identical to the result obtained in **Example 8.1c** but is much easier to obtain.

<sup>5</sup> Ampere's law is named after Marie Andre Ampere (see footnote on page 341). Ampere derived this law from study of the solenoidal (circular) nature of the magnetic field of straight wires.

**Example 8.7 Application: Field Intensity Due to a Long, Thick Conductor—The Transmission Line** A wire of radius  $a = 10$  mm carries a current  $I = 400$  A. The wire may be assumed to be infinite in length.

- (a) Calculate the magnetic field intensity everywhere in space.  
 (b) Plot the magnetic field intensity as a function of distance from the center of the conductor.

**Solution:** The solution in **Example 8.6** is the magnetic field intensity at any distance outside a wire carrying a current  $I$ . We also recall that only the current enclosed by the contour contributes to the magnetic field intensity. Thus, outside the conductor, at a distance  $r > a$ , the magnetic field intensity is given in **Example 8.6**. Inside the wire ( $r \leq a$ ), we still use the same relation, but the current is only that enclosed by the contour.

- (a) For  $0 < r \leq a$ : The current density in the wire is the total current divided by the cross-sectional area of the wire (see **Figure 8.12a**):

$$J = \frac{I}{\pi a^2} \quad \left[ \frac{\text{A}}{\text{m}^2} \right]$$

The current enclosed by a contour of radius  $r$  equals the current density multiplied by the cross-sectional area enclosed by the contour:

$$I_r = JS_r = \frac{I\pi r^2}{\pi a^2} = \frac{I r^2}{a^2} \quad [\text{A}]$$

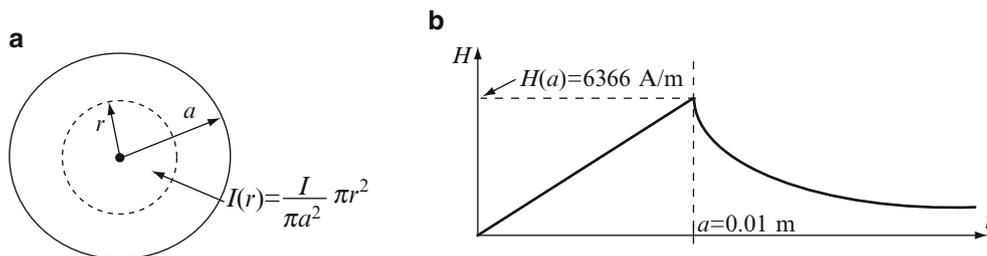
The magnetic field intensity at a distance  $r$  from the center of the wire ( $0 < r \leq a$ ) is

$$H = \frac{I_r}{2\pi r} = \frac{I r}{2\pi a^2} = \frac{400r}{2 \times \pi \times (0.01)^2} = 6.366 \times 10^5 r \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

For  $r > a$  the result of **Example 8.6** gives

$$H = \frac{I_r}{2\pi r} = \frac{I}{2\pi r} = \frac{400}{2 \times \pi \times r} = \frac{63.66}{r} \quad \left[ \frac{\text{A}}{\text{m}} \right].$$

- (b) The plot of  $H$  versus distance is shown in **Figure 8.12b**. Note that the magnetic field intensity is zero at the center of the wire and rises linearly up to the wire surface where the magnetic field intensity is 6366 A/m. Then, it decays as  $1/r$  until it is zero at infinity. The magnetic field intensity is circular, following the right-hand rule. If the current is in the  $z$  direction,  $\mathbf{H}$  is in the  $\phi$  direction. The plot in **Figure 8.12b** gives only the magnitude of  $\mathbf{H}$ .



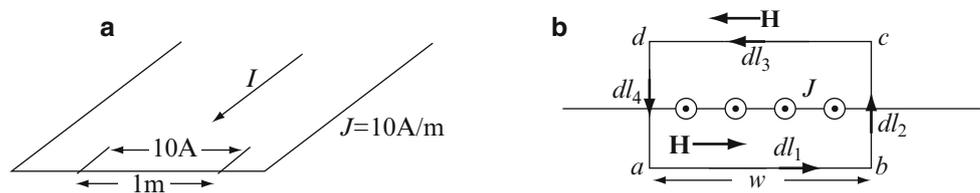
**Figure 8.12** Magnetic field intensity due to a thick conductor carrying a uniformly distributed current. (a) Method of calculation of  $\mathbf{H}$  inside the conductor. (b) Plot of  $H$  versus radial distance from the center of the conductor

**Example 8.8 Application: Field Intensity Due to an Infinite Sheet—Application as a Ground Plane for Lightning Protection** When it becomes necessary to protect devices from overvoltages such as produced by lightning, ground planes are often employed. Underneath power substations and in computer rooms, it is common to use a grid of conductors (as an approximation to a continuous, conducting plane). The example that follows shows how the magnetic field intensity due to a large ground plane is calculated.

A very large sheet of conducting material may be assumed to be infinite and carries a surface current density  $J_1 = 10 \text{ A/m}$  as shown in **Figure 8.13a**. Calculate the field intensity everywhere in space.

**Solution:** To calculate the field, we draw a contour on which the magnetic field intensity is either perpendicular or parallel to the contour. From symmetry considerations, the magnetic field intensity at any two points at equal distance from the plane must be the same. The proper contour is a rectangular loop of arbitrary size, as shown in **Figure 8.13b**. The dimensions  $bc$  and  $da$  are not important since the same amount of current is enclosed by the contour for any dimension  $bc$  or  $da$ .

**Figure 8.13** Magnetic field intensity of a large (infinite) ground plane. (a) Geometry and dimensions. (b) The contour used for calculation using Ampere's law



We assume the magnetic field intensity on both sides of the sheet is in the direction of integration (right-hand rule) and evaluate the integral:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I = J_1 w$$

where  $J_1$  indicates that the current density is a surface current density. Performing the integration in segments gives

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_a^b H_1 dl_1 + \int_b^c H_2 dl_2 + \int_c^d H_3 dl_3 + \int_d^a H_4 dl_4$$

The second and fourth integrals are zero because  $\mathbf{H}$  and  $d\mathbf{l}$  are perpendicular to each other on these segments. The first and third integrals are evaluated observing that  $|H_1| = |H_3| = |H|$  and, from the symmetry consideration above, are constant along the segments  $ab$  and  $cd$ . Also, the distances  $ab$  and  $cd$  are equal to  $w$ . Thus,

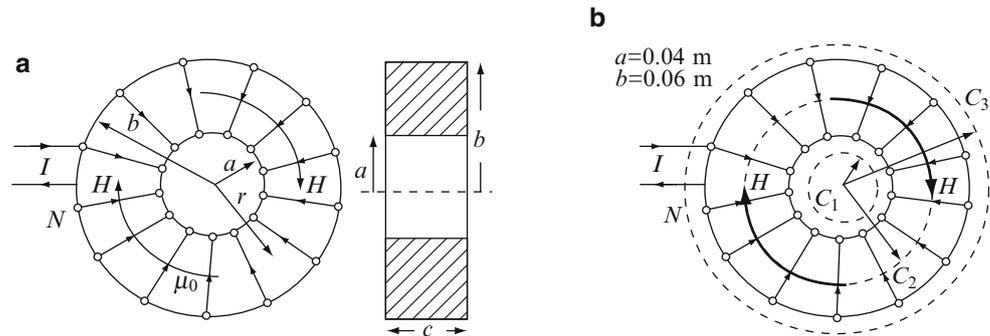
$$\oint \mathbf{H} \cdot d\mathbf{l} = H \int_a^b dl_1 + H \int_c^d dl_3 = 2Hw = J_1 w \quad \rightarrow \quad H = \frac{J_1}{2} = 5 \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The magnetic field intensity is constant in space, equals half the magnitude of the surface current density, and is in opposite directions on the two sides of the plate, following the right-hand rule. The magnetic field intensity is parallel to the surface of the sheet. The result above is exact for an infinite sheet but may be used as an approximation for large sheets or even for smaller sheets if the field very close to the sheet is required. Note also that the infinite sheet geometry may be viewed as being made of single, thin wires, very closely spaced, with  $N$  wires per unit width of the sheet, and each carrying a current  $I/N$  as shown in **Figure 8.13b**. If  $N$  tends to infinity, the continuous distribution in **Figure 8.13a** is obtained.

**Example 8.9 Application: Magnetic Field Intensity in a Toroidal Coil** A torus with the dimensions shown in **Figure 8.14a** is wound with  $N = 100$  turns of wire, uniformly wound around the torus. The coil thus formed carries a current  $I = 1 \text{ A}$ . Calculate the magnetic flux density everywhere in space. The torus has a rectangular cross section and the permeability of the core is  $\mu_0$  [H/m].

**Solution:** To use Ampere's law, we need to find a contour on which the magnetic field intensity is constant in magnitude and, preferably, in the direction of the contour. **Figure 8.14b** shows that any contour which is concentric with the torus' axis is symmetric about all conductors and, therefore, can be used in conjunction with Ampere's law. From the right-hand rule, applied to individual wires, the direction of the flux density is parallel to the contours, as shown in **Figure 8.14b**.

**Figure 8.14** A toroidal coil. (a) Dimensions and construction. (b) Contours used for calculation of the magnetic flux density



For  $r < a$ : Contour  $C_1$  does not enclose any current. Therefore, the magnetic field intensity and magnetic flux density are zero. For  $a < r < b$ : Contour  $C_2$  encloses a current equal to  $NI$ . From Ampere's law,

$$\oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = H \oint dl_2 = H2\pi r = NI \quad \rightarrow \quad H = \frac{NI}{2\pi r} = \frac{100}{2\pi r} = \frac{15.915}{r} \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The magnetic field intensity varies within the torus. It is higher toward the inner surface and lower toward the outer surface. Sometimes, it is convenient to approximate the magnetic field intensity in the torus as an average between the outer and inner fields and assume that this average field intensity exists everywhere within the torus. Calculating the magnetic field intensity at  $r = a$  and  $r = b$  and taking the average between the two values gives

$$H_{av} = \frac{NI}{4\pi} \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{100}{4\pi} \left( \frac{1}{0.04} + \frac{1}{0.06} \right) = 331.57 \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

This approximation is quite good if  $a$  and  $b$  are large compared to the radial thickness of the torus ( $b - a$ ). The magnetic flux density is found from the relation  $\mathbf{B} = \mu_0 \mathbf{H}$ :

$$B = \mu_0 H = \frac{\mu_0 NI}{2\pi r} = \frac{4 \times \pi \times 10^{-7} \times 100}{2 \times \pi \times r} = \frac{2 \times 10^{-5}}{r} \quad [\text{T}]$$

and the average magnetic flux density is

$$B_{av} = \frac{\mu_0 NI}{4\pi} \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{4 \times \pi \times 10^{-7} \times 100}{4 \times \pi} \left( \frac{1}{0.04} + \frac{1}{0.06} \right) = 4.167 \times 10^{-4} \quad [\text{T}]$$

For  $r > b$ : Contour  $C_3$  encloses  $N$  positive currents and  $N$  negative currents. Thus, the total net current enclosed is zero and the magnetic field intensity and magnetic flux density are zero.

#### Notes:

- (1) The magnetic field is contained entirely within the torus. This property is used in many applications, including high-quality transformers and other coils which require containment of the field. Because of the field containment, the magnetic field generated in a torus does not affect or interfere with other devices.
- (2) The effect of having 100 turns is to multiply the current by 100. In other words, from Ampere's law, the total current enclosed in the contour is  $100I$ . If this current is due to a single turn carrying 100 A, 10 turns carrying 10 A each, or 10,000 turns carrying 0.01 A each, the result is the same. For this reason, we will often use the term *ampere · turns* [ $\text{A} \cdot \text{t}$ ] to indicate the total current in a device.

- (3) This problem, although easily solved using Ampere's law, would be almost impossible to solve using the Biot–Savart law; integration along all current paths would be very tedious.

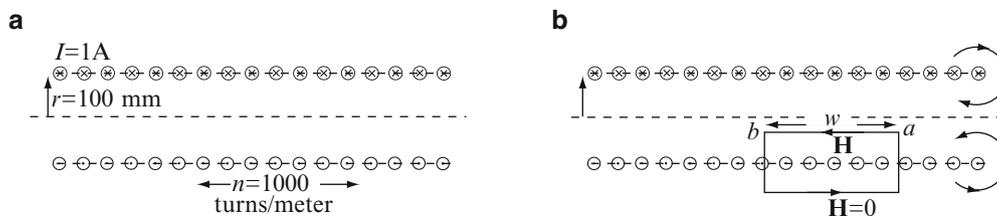
**Example 8.10 Application: Magnetic Field Inside a Long Solenoid** A solenoid is a coil, wound on a circular (sometimes rectangular) form. The turns of the coil are most often tightly wound and normally in a single layer, whereas the coil is very long. Short, multilayer solenoids are called coils.

A very long solenoid is wound with a turn density of 1 turn per mm length of the solenoid ( $n = 1,000$  turns/meter). The current in the turns of the solenoid is  $I = 1$  A. Calculate the magnetic field intensity and magnetic flux density everywhere in space. The current and dimensions are shown in **Figure 8.15a** (in axial cross section).

**Solution:** Although the coil is cylindrical, the cross section shown in **Figure 8.15a** is identical through any axial cut that includes the axis of the coil. The configuration looks as if we had two planar current sheets, with currents in opposite directions. From the right-hand rule, we note that the fields of the two opposing current layers are in opposite directions outside the solenoid but are in the same direction inside the solenoid. Thus, the field outside the solenoid must be zero.

A contour is chosen as shown in **Figure 8.15b**. The width of the contour is  $w$  and is arbitrary. The total current enclosed by the contour is  $nwI$ . Since the field outside the solenoid is zero and inside it must be along the axis (from the right-hand rule and symmetry considerations), Ampere's law gives

$$\oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = H \int_a^b dl_2 = Hw = nwI \rightarrow H = nI \quad \left[ \frac{\text{A}}{\text{m}} \right]$$



**Figure 8.15** A long solenoid. (a) Dimensions and properties (axial cross section shown). (b) Contour used for calculation

Thus, the magnetic field intensity inside the solenoid is

$$H = nI = 1000 \quad \left[ \frac{\text{A}}{\text{m}} \right]$$

The magnetic flux density is found by multiplying the magnetic field intensity by the permeability of free space:

$$B = \mu_0 H = \mu_0 nI = 4 \times \pi \times 10^{-7} \times 1000 = 1.257 \times 10^{-3} \quad [\text{T}]$$

Like the torus in **Example 8.9**, the field of a long solenoid is zero outside the solenoid. Inside the solenoid, the field is constant. Practical solenoids are finite in length and, for these, the result obtained here is only an approximation. Note also that an infinite solenoid may be viewed as a torus with infinite radius (see **Exercise 8.3**). Note, again, that the same result would be obtained if, for example, the turn density were 100 turns/m while the current were 10 A. What is important for the solution is the term  $nI$ , which is the number of ampere · turns per unit length of the solenoid. For this reason, a solenoid may also be made of a single, bent sheet, carrying a given current.

**Exercise 8.3** Show that the magnetic field intensity of a circular cross-sectional torus of average radius  $a$  [m] equals the field of an infinite solenoid as the average radius of the torus tends to infinity.

## 8.5 Magnetic Flux Density and Magnetic Flux

We started this chapter with the Lorentz force on a moving charge and showed that this force is proportional to the magnetic flux density  $\mathbf{B}$  [see Eq. (8.1)]. The relation between the magnetic field intensity  $\mathbf{H}$  and the magnetic flux density  $\mathbf{B}$  for general materials was also given as  $\mathbf{B} = \mu\mathbf{H}$  in Eq. (8.3). The permeability  $\mu$  is material related. We will discuss this in more detail in the following chapter. For now it is sufficient to say that every material has a given and measurable permeability  $\mu$  and we may assume that permeability is known, even though it may not be a constant value.

The units of permeability can now be defined in terms of  $\mathbf{B}$  and  $\mathbf{H}$ . The SI unit for  $\mathbf{B}$  is the tesla<sup>6</sup> [T], whereas that for  $\mathbf{H}$  is the ampere/meter [A/m]. Thus, permeability has units of tesla  $\cdot$  meter/ampere or weber  $\cdot$  meter/(meter<sup>2</sup>  $\cdot$  ampere). In the latter form, the weber/ampere is called a henry [H]. Therefore, the units of permeability are henry/meter [H/m]:

$$\mu \rightarrow \left[ \frac{\text{tesla} \cdot \text{meter}}{\text{ampere}} \right] = \left[ \frac{\text{weber} \cdot \text{meter}}{\text{meter}^2 \cdot \text{ampere}} \right] = \left[ \frac{\text{weber}}{\text{ampere} \cdot \text{meter}} \right] = \left[ \frac{\text{henry}}{\text{meter}} \right] = \left[ \frac{\text{H}}{\text{m}} \right]$$

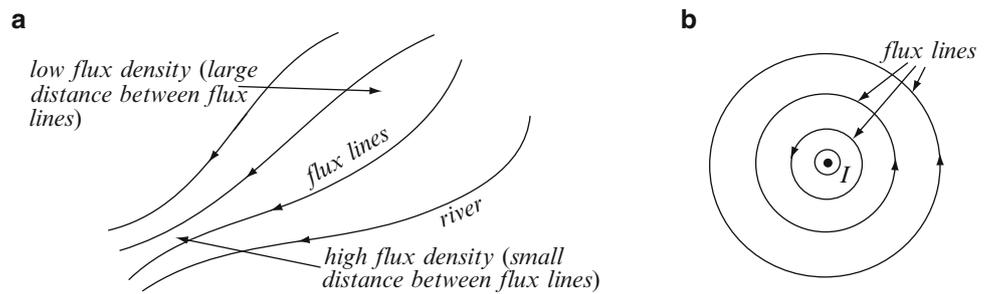
The magnetic flux density is an area density vector. In the case of the electric field, we also had a flux density ( $\mathbf{D}$ ), and in the case of steady currents, the current density ( $\mathbf{J}$ ) was also an area density vector. In both cases, we integrated the density function over the area to obtain the total flux through an area. For example, the surface integral of  $\mathbf{J}$  was the total current through the surface. Because  $\mathbf{B}$  is a flux per unit area, we can write

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{Wb}] \quad (8.17)$$

The quantity  $\Phi$  is called the *magnetic flux* and has units of tesla  $\cdot$  meter<sup>2</sup> [ $\text{T} \cdot \text{m}^2$ ] or weber [Wb]. Whereas the tesla is usually used for  $\mathbf{B}$ , the weber is more convenient for the flux. The concept of magnetic flux is shown in Figure 8.16. The analogy between the magnetic flux to any real flow is exactly that—an analogy. Nothing flows in the real sense. On the other hand, this analogy allows the use of analogous quantities. In flow of a fluid, we can define lines of flow or streamlines. These lines show the direction of flow. Similarly, a magnetic flux line is a line along which the magnetic flux density  $\mathbf{B}$  is directed; that is, the magnetic flux density is tangential to flux lines everywhere in the magnetic field. The direction of  $\mathbf{B}$  must be shown specifically by an arrow if required (see Figure 8.16b). Note also that flux lines do not imply that the flux density is constant. The magnitude of the flux density is indicated by the line density. At any point, the distance between adjacent flux lines indicates the magnitude of the flux density  $\mathbf{B}$ . The concept of flux or field lines is artificial. It serves to conceptualize the magnetic field just as it serves to conceptualize the flow in a river.

<sup>6</sup> After Nicola Tesla (1856–1943). Tesla is well known for his invention of AC electric machines and development of the multiphase system of AC power. This happened in the early 1880s, followed by a number of other important engineering designs, including the installation of the first, large-scale AC power generation and distribution system at Niagara Falls (approx. 12 MW), in 1895. His induction motor became a staple of AC power systems, but his patents (about 500 in all) include transformers, generators, and systems for transmission of power. Tesla also had designs of grandiose scale, including “worldwide aerial transmission of power,” high-frequency, high-voltage generators and many others. Tesla was always somewhat of a “magician” and liked to keep his audience guessing, but later in his life, he became more of an eccentric and a recluse. Although some of his designs were more dreams than engineering, he is considered by many as the greatest inventor in electrical engineering. His statement regarding his student pay that there are “too many days after the first of the month” is even more familiar than his inventions, even to those that never heard it.

**Figure 8.16** The concept of magnetic flux. (a) Flux in space. (b) Flux lines around a filamentary current



**Example 8.11 Application: Flux Through a Loop** A square loop is placed near a current-carrying wire as shown in **Figure 8.17a**. The loop and the wire are in the  $r$ - $z$  plane:

- (a) Calculate the flux through this loop.  
 (b) The loop is now turned around its vertical axis by  $90^\circ$  (**Figure 8.17b**) so that the loop is now perpendicular to the  $r$ - $z$  plane and symmetric about the wire. Calculate the flux through the loop.

**Solution:** The flux is calculated using **Eq. (8.17)**, whereas the flux density  $\mathbf{B}$  is calculated using Ampere's law in **Eq. (8.16)**. The magnetic field intensity of a wire was calculated in **Example 8.6**, but we will calculate it here anew.

- (a) To calculate the magnetic flux density, we define a contour of radius  $r$  and use cylindrical coordinates with the wire coinciding with the  $z$  axis. For a current in the positive  $z$  direction, the magnetic field intensity is in the positive  $\phi$  direction (right-hand rule, see **Figure 8.17d**). Thus, Ampere's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \hat{\phi} B \cdot \hat{\phi} r d\phi = Br \oint d\phi = 2\pi r B = \mu_0 I$$

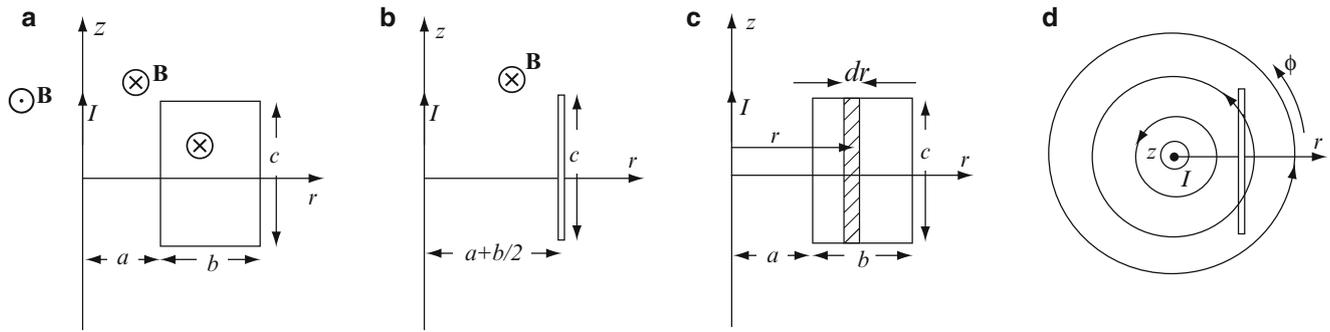
or

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{or} \quad \mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \quad [\text{T}]$$

To calculate the flux, we define an element of area  $ds = c dr$  as shown in **Figure 8.17c** and integrate this from  $r = a$  to  $r = a + b$ :

$$\Phi = \int_s \hat{\phi} \frac{\mu_0 I}{2\pi r} \cdot \hat{\phi} ds = \frac{\mu_0 I c}{2\pi} \int_a^{a+b} \frac{dr}{r} = \frac{\mu_0 I c}{2\pi} \ln r \Big|_{r=a}^{r=a+b} = \frac{\mu_0 I c}{2\pi} \ln \frac{a+b}{a} \quad [\text{Wb}].$$

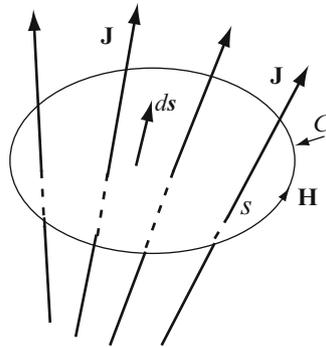
- (b) The loop is now perpendicular to the  $r$ - $z$  plane. Although direct integration as in (a) can be performed, the total flux through the loop must be zero. This can be seen from the fact that any flux that enters the loop equals the flux that exits it, as shown in **Figure 8.17d**. Note, however, that if the loop were not symmetric about the current, this argument would not be correct.



**Figure 8.17** (a) Loop placed near a current-carrying wire. Loop and wire are in a plane. (b) Loop perpendicular to the plane in (a). (c) Method of calculating the flux in the loop. (d) Total flux in (b) is zero

### 8.6 Postulates of the Static Magnetic Field

From the relations obtained so far, it is now possible to define a minimum set of required relations for the static magnetic field: the postulates. To do so, the curl and divergence of the field must be specified (Helmholtz’s theorem). However, instead of trying to define these directly, we will use the relations already obtained. One relation is Ampere’s law in Eq. (8.16). We can rewrite the enclosed current in Eq. (8.16) in terms of the current density  $\mathbf{J}$  using Figure 8.18.



**Figure 8.18** Ampere’s law in terms of current density

The current enclosed by a closed contour  $C$  is the integral of the normal component of  $\mathbf{J}$  over the surface bounded by the contour  $C$ . Thus, using the scalar product to calculate the normal component, we can write

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{s} \tag{8.18}$$

Using Stokes’ theorem, we convert the closed contour integral to a surface integral as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_s \mathbf{J} \cdot d\mathbf{s} \tag{8.19}$$

Equating the integrands of the two surface integrals gives the required curl condition:

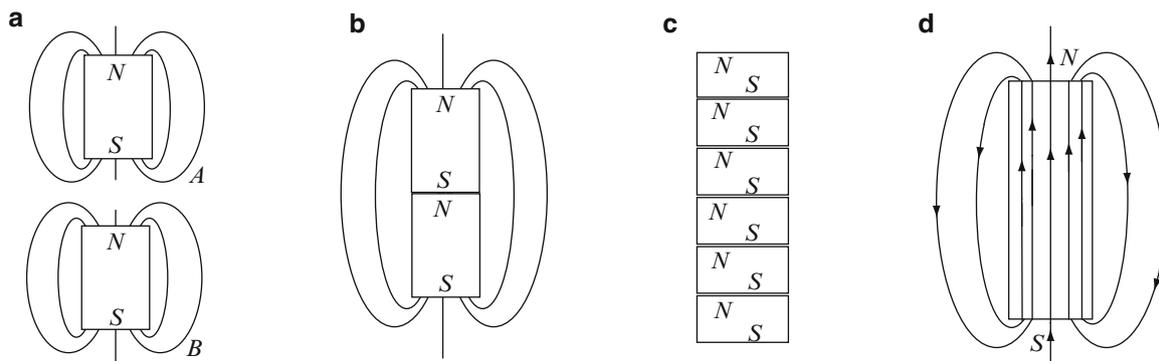
$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{8.20}$$

The second relation we must identify is the divergence of the magnetic field intensity or that of the magnetic flux density.

Before doing so, we will give two examples which will serve as an introduction to the general result. First, consider **Figure 8.16b**, in which a few flux lines of a line current are shown. If we draw any volume, anywhere in space, the total number of flux lines entering the volume must be equal to the total number of flux lines leaving the volume. If it were otherwise, some flux lines would either terminate in the volume or start in the volume. The conclusion is that the total net flux entering any volume must be zero; that is, there cannot be a source of flux (or a sink) inside the volume  $v$ . In other words, regardless of the volume, we choose, there cannot be a single magnetic pole (north or south) inside the volume although there can be pairs of poles. Thus, the conclusion is that the total flux through a closed surface (enclosing a volume  $v$ ) must be zero regardless of the shape or size of the surface:

$$\Phi = \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \quad (8.21)$$

The second example that is useful in this regard is that of a permanent magnet. Consider a number of small bar magnets, two of which are shown in **Figure 8.19a**, together with a representation of their magnetic fields using flux lines. North and south poles are shown on each magnet. For simplicity, we will assume that the magnets are identical in all respects. Now, we perform the following simple experiment: Place the south pole of magnet  $A$  on the north pole of magnet  $B$ . The result is shown in **Figure 8.19b**. It is clearly a single magnet and the  $N$  and  $S$  poles shown in the middle do not show a field of their own. In fact, these poles seem to have disappeared. If we now separate the two magnets again, each magnet returns to its original state. The two bar magnets are again as shown in **Figure 8.19a**; each has two poles and each produces a magnetic field identical to that of the composite magnet except, perhaps, to the strength of the field. We can repeat this process as many times as we wish with magnets as small as possible (**Figure 8.19c**). The same effect occurs: Each magnet has two poles regardless of size or how many smaller magnets make it. In the limit, the long magnet is made of differential length magnets, and since each of these also has two poles, we must conclude that magnetic poles can only exist in pairs. Because of this, the flux lines must close through the magnet as shown in **Figure 8.19d**. While the discussion above is not a proof, it is an experiment-supported conclusion and we will have to be satisfied with this.<sup>7</sup>



**Figure 8.19** Experiment that shows magnetic poles must exist in pairs. (a) Two permanent magnets and their fields. (b) Connecting the two magnets as shown results in a single magnet. (c) A permanent magnet as a stack of elementary magnets, each with two poles. (d) The magnetic field of a permanent magnet must close through the magnet

Applying the divergence theorem to **Eq. (8.21)** gives

$$\oint_s \mathbf{B} \cdot d\mathbf{s} = \int_v (\nabla \cdot \mathbf{B}) dv = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{B} = 0 \quad (8.22)$$

**Equations (8.20)** and **(8.22)** define the curl of  $\mathbf{H}$  and the divergence of  $\mathbf{B}$  and, through the constitutive relation  $\mathbf{B} = \mu\mathbf{H}$ , the curl of  $\mathbf{B}$  and the divergence of  $\mathbf{H}$ . We choose these two equations as the postulates of the magnetostatic field.

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0 \quad (8.23)$$

<sup>7</sup>The search for monopoles (single magnetic poles) is still continuing in basic physics research. No evidence of their existence has ever been found either on macroscopic or microscopic levels.

These relations were obtained from Eq. (8.16) or (8.19) and Eq. (8.21). The latter two equations are therefore the integral form of the postulates:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}}, \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad (8.24)$$

The first equation in each set is Ampere’s law, whereas the second equation simply represents the fact that no single magnetic pole may exist. From these equations, we note the following:

- (1) The magnetic field is nonconservative; the closed contour integral of the magnetic field intensity is nonzero.
- (2) The magnetic field is rotational; the curl of the magnetic field intensity is nonzero.
- (3) The magnetic field is solenoidal; the divergence of the magnetic flux density is zero.
- (4) The magnetic flux is conserved; the total net flux through any closed surface is zero.

## 8.7 Potential Functions

According to the previous section, the static magnetic field is completely defined by the curl and divergence given in Eq. (8.23) as required by the Helmholtz theorem. However, the theorem does not imply that the forms given above are the only possible forms and, more importantly, it does not give any clue as to which relations are easier to use. In fact, we have already seen that Ampere’s law is easier to use in some cases and the Biot–Savart law is more practical in others. We also recall from electrostatics that the use of the electric scalar potential (voltage) was one function that allowed simplification of the solutions of many otherwise complicated problems. The question now is: Are there any scalar or vector functions that can be used in conjunction with the magnetic field to effect the same results, that is, to simplify solution?

In fact, there are two functions that can be used. One is a vector function and can be used in general to describe the magnetic field. The second is a scalar function which may be used under certain conditions. These two functions are described next.

### 8.7.1 The Magnetic Vector Potential

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Since the divergence of the magnetic flux density is zero [Eq. (8.22)], we can invoke the vector identity  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$  (see Section 2.5) and write:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (8.25)$$

This is justified because when we substitute it back into the above vector identity we get  $\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \mathbf{B} = 0$ . This can always be done for a divergence-free (solenoidal) field  $\mathbf{B}$ . The vector  $\mathbf{A}$  is called the *magnetic vector potential*<sup>8</sup> and is defined through Eq. (8.25). It is important to note the following:

- (1) The magnetic vector potential is defined based on the divergence-free condition of  $\mathbf{B}$ .
- (2) The definition of  $\mathbf{A}$  is based entirely on the mathematical properties of the vector  $\mathbf{B}$ , not on its physical characteristics. In this sense,  $\mathbf{A}$  is viewed as an auxiliary function rather than a fundamental field quantity. Nevertheless, the magnetic

<sup>8</sup>The magnetic vector potential was considered by James Clerk Maxwell to be a fundamental quantity from which the magnetic flux density was derived. He called it the “Electrokinetic Momentum.” In fact, Maxwell used the scalar and vector potentials to define fields. The fields as we use them today were introduced as fundamental quantities by Oliver Heaviside and Heinrich Hertz. Oliver Heaviside, in particular, had some harsh words about potential functions. He considered the magnetic vector potential an “absurdity” and “Maxwell’s monster,” which should be “murdered.” Harsh words, but then Heaviside had harsh words for many people and topics. Although we use Heaviside’s form of electromagnetics in using field variables, we also make considerable use of potential functions. The guiding rule for us is simplicity and convenience.

vector potential is an important function with considerable utility. We will make considerable use of the magnetic vector potential here and in subsequent chapters.

- (3) Since the magnetic vector potential is a vector function, both its curl and divergence must be specified. The curl is defined in **Eq. (8.25)**, but we have said nothing about its divergence. At this stage, we will assume that the divergence of  $\mathbf{A}$  is zero  $\nabla \cdot \mathbf{A} = 0$  and delay the discussion of this quantity.
- (4) The magnetic vector potential does not have a simple physical meaning in the sense that it is not a measurable physical quantity like  $\mathbf{B}$  or  $\mathbf{H}$ . (We will try to give it some physical interpretation shortly.) It may seem a bit unsettling to define a physical quantity based on the mathematical properties of another function and then use this secondary function to evaluate physical properties of the magnetic field. In fact, there is nothing unusual about this process. You can view the definition of the magnetic vector potential as a transformation. As long as the inverse transformation is unique, there is nothing wrong in  $\mathbf{A}$  not having a readily defined physical meaning. We can use the magnetic vector potential in any way that is consistent with the properties of a vector field and the rules of vector algebra. If we then transform back to the magnetic flux density using **Eq. (8.25)**, all results thus obtained are correct.
- (5) Because the magnetic vector potential relates to the magnetic flux density through the curl, the magnetic vector potential  $\mathbf{A}$  is at right angles to the magnetic flux density  $\mathbf{B}$ .
- (6) The unit of  $\mathbf{A}$  is the Wb/m, as can be seen from **Eq. (8.25)**.

Now, it remains to be seen that the use of the magnetic vector potential does have some advantage in the calculation of the magnetic field. To show this, we start with the Biot–Savart law for the magnetic flux density [**Eq. (8.9)**]:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu I}{4\pi} \int_a^b \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad [\text{T}] \quad (8.26)$$

where  $a$  and  $b$  are two general points on the current contour. Note that we prefer to use  $\mathbf{r} - \mathbf{r}'$  instead of  $\mathbf{R}$  [as in **Eq. (8.9)**] to preserve the distinction between the source (primed) and field (unprimed) points. For evaluation, we will try to get the right-hand side into the form  $\nabla \times (\mathbf{F})$  such that we can then write  $\mathbf{A} = \mathbf{F}$ .

First, we note the following relation:

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad (8.27)$$

This relation, which is not immediately obvious, was derived in **Example 2.10**, but it may be shown to be correct by direct derivation. Substituting this in **Eq. (8.26)**,

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 I}{4\pi} \int_a^b d\mathbf{l}' \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \quad [\text{T}] \quad (8.28)$$

To transform this into the form required, we use the vector identity  $\nabla \times (\varphi \mathbf{f}) = \varphi(\nabla \times \mathbf{f}) + (\nabla \varphi) \times \mathbf{f}$  (where  $\mathbf{f}$  is any vector function and  $\varphi$  any scalar function). In our case,  $\mathbf{f} = d\mathbf{l}'$  and  $\varphi = 1/|\mathbf{r} - \mathbf{r}'|$ , and we get

$$\nabla \times \left( \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} (\nabla \times d\mathbf{l}') + \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times d\mathbf{l}' \quad (8.29)$$

The gradient is taken with respect to the general (unprimed) coordinates. Thus, the first term on the right-hand side must vanish since  $\nabla \times d\mathbf{l}' = 0$ . We therefore have

$$\nabla \times \left( \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right) = \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times d\mathbf{l}' \quad (8.30)$$

or since  $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$  for any two vector fields  $\mathbf{A}$  and  $\mathbf{B}$ , we get

$$d\mathbf{l}' \times \left( \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla \times \left( \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right) \quad (8.31)$$

Substituting this result for the integrand in **Eq. (8.28)** gives

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I}{4\pi} \int_a^b \nabla \times \left( \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right) \quad [\text{T}] \quad (8.32)$$

The integration is along the current and, therefore, on the primed coordinates; the curl inside the integral is on the unprimed coordinates. This means that the curl and the integral operators can be interchanged (the curl is independent of the primed coordinates). Also, since  $\mu_0 I / 4\pi$  is a constant, it can be moved inside the curl operator:

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[ \frac{\mu_0 I}{4\pi} \int_a^b \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right] \quad [\text{T}] \quad (8.33)$$

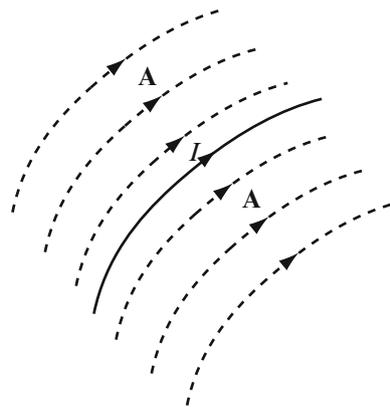
From this,

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_a^b \left( \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \right) \quad \left[ \frac{\text{Wb}}{\text{m}} \right]} \quad (8.34)$$

This is the Biot–Savart law in terms of the magnetic vector potential.

In comparison with the Biot–Savart law in **Eq. (8.26)**, this is simpler to evaluate since there is no need to evaluate the vector product in **Eq. (8.26)**. In addition, the magnetic vector potential  $\mathbf{A}$  is in the direction of  $d\mathbf{l}'$ , which, by definition, is taken in the direction of flow of current. Thus, the magnetic vector potential is always in the direction of the current (and perpendicular to  $\mathbf{B}$ ).

This property has prompted some to view it as a sort of current distributed in space. It is sometimes even called a “fuzzy current” since it is spread around the current, as shown in **Figure 8.20**. We will not take this interpretation too far, but it does serve to show how the magnetic vector potential and the current are related. After the magnetic vector potential is calculated, it may be used directly to evaluate other quantities or it may be used to calculate the magnetic flux density using **Eq. (8.33)**.



**Figure 8.20** An interpretation of the magnetic vector potential as a “fuzzy current” in space

The premise behind the magnetic vector potential is that it can serve in lieu of the magnetic flux density or the magnetic field intensity; that is, it is up to us which function to choose for calculation. The ultimate choice should be based on ease of use, but as long as we can evaluate the various expressions either function yields the correct results. This is important since

otherwise we must qualify the definition of  $\mathbf{A}$ . To see that solution in terms of the magnetic vector potential is equivalent to solution in terms of  $\mathbf{B}$  or  $\mathbf{H}$ , we now define Ampere's law and magnetic flux in terms of the magnetic vector potential. We start with Ampere's law in **Eq. (8.20)**. In free space,  $\mu_0$  is constant and we get

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \rightarrow \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (8.35)$$

Substituting the definition of the magnetic vector potential  $\mathbf{B} = \nabla \times \mathbf{A}$  gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} \quad (8.36)$$

The left-hand side can be evaluated directly by applying the curl twice. Instead, we can exploit the vector equality  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$  [see **Eq. (2.136)**]. Thus, by direct substitution into Ampere's law, we get

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \quad (8.37)$$

Now, we must decide on the divergence of  $\mathbf{A}$  which, until now, has not been defined. Taking  $\nabla \cdot \mathbf{A} = 0$  is one choice mentioned earlier. This condition is called *Coulomb's gauge*. As long as we do not violate the field equations (i.e., as long as we do not modify the properties of the magnetic field), the choice of the divergence of  $\mathbf{A}$  is arbitrary. Later, when discussing the time-dependent field, we will revisit this issue, but for static fields, the choice above is the best choice. Thus, we get

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \quad (8.38)$$

This is a vector Poisson equation. The term  $\nabla^2 \mathbf{A}$  is called the vector Laplacian and was discussed in **Chapter 2**. This equation can be solved for the vector field  $\mathbf{A}$  in a manner similar to that for the scalar Poisson equation (for the electric scalar potential) discussed in **Chapters 5** and **6**. If we write the vectors  $\mathbf{A}$  and  $\mathbf{J}$  explicitly (in Cartesian coordinates in this case), each has three vector components:

$$\nabla^2 (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) = -\mu_0 (\hat{\mathbf{x}}J_x + \hat{\mathbf{y}}J_y + \hat{\mathbf{z}}J_z) \quad (8.39)$$

Equating vector components on both sides of the equation, we obtain three separate, scalar equations, one for each scalar component of  $\mathbf{A}$ :

$$\boxed{\nabla^2 A_x = -\mu_0 J_x, \quad \nabla^2 A_y = -\mu_0 J_y, \quad \nabla^2 A_z = -\mu_0 J_z} \quad (8.40)$$

Each of these equations is a one-dimensional scalar Poisson equation and can be solved separately for the  $x$ ,  $y$ , and  $z$  components of  $\mathbf{A}$ . The question is, how do we obtain general solutions for these equations? One way to do so is to go back to the solution of the scalar Poisson equation for the electric field discussed in **Chapter 5**. Poisson's equation for the electric scalar potential was given in **Eq. (5.4)**:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0} \quad (8.41)$$

The general solution for this equation is [see **Eq. (4.33)** and **Section 5.4**]

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho_v(r') dv'}{|\mathbf{r} - \mathbf{r}'|} \quad [\text{V}] \quad (8.42)$$

Poisson's equations in **Eq. (8.40)** are similar in form to **Eq. (8.41)**. If we replace  $\rho_v/\epsilon_0$  by  $\mu_0 J$ , the equations are identical in form and, therefore, have the same form of solutions. Thus, we can write a solution for each component as

$$\begin{aligned}
 A_x(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{v'} \frac{J_x(r') dv'}{|\mathbf{r} - \mathbf{r}'|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right] \\
 A_y(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{v'} \frac{J_y(r') dv'}{|\mathbf{r} - \mathbf{r}'|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right] \\
 A_z(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{v'} \frac{J_z(r') dv'}{|\mathbf{r} - \mathbf{r}'|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right]
 \end{aligned} \tag{8.43}$$

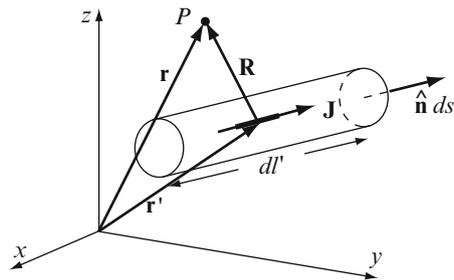
These can be combined in a vector solution by writing

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{J}(\mathbf{r}') dv'}{|\mathbf{r} - \mathbf{r}'|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right] \tag{8.44}$$

Now, consider a conductor of length  $d\mathbf{l}'$  and cross-sectional area  $ds'$  as shown in **Figure 8.21**. Writing  $dv' = d\mathbf{l}' \cdot \hat{\mathbf{n}} ds'$ , substituting this relation into **Eq. (8.44)**, and separating the volume integral into a surface and line integral, we get for any point outside the conductor

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_L \left[ \int_s \frac{\mathbf{J}(\mathbf{r}') \cdot \hat{\mathbf{n}} ds'}{|\mathbf{r} - \mathbf{r}'|} \right] d\mathbf{l}' = \frac{\mu_0 I}{4\pi} \int_L \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right] \tag{8.45}$$

where the integration of the current density over the cross-sectional area is independent of the distance of the volume element and the point at which the field is calculated.



**Figure 8.21** A conductor of length  $d\mathbf{l}'$  and cross-sectional area  $ds'$

This result is the same as the magnetic vector potential in **Eq. (8.34)**, which, of course, it should be. It indicates that the solution using Ampere's law or the Biot–Savart law is the same regardless of our starting point.

Finally, we look into the calculation of flux using **Eq. (8.17)**. Substituting  $\mathbf{B} = \nabla \times \mathbf{A}$ , we get

$$\Phi = \int_s \mathbf{B} \cdot ds = \int (\nabla \times \mathbf{A}) \cdot ds \quad [\text{Wb}] \tag{8.46}$$

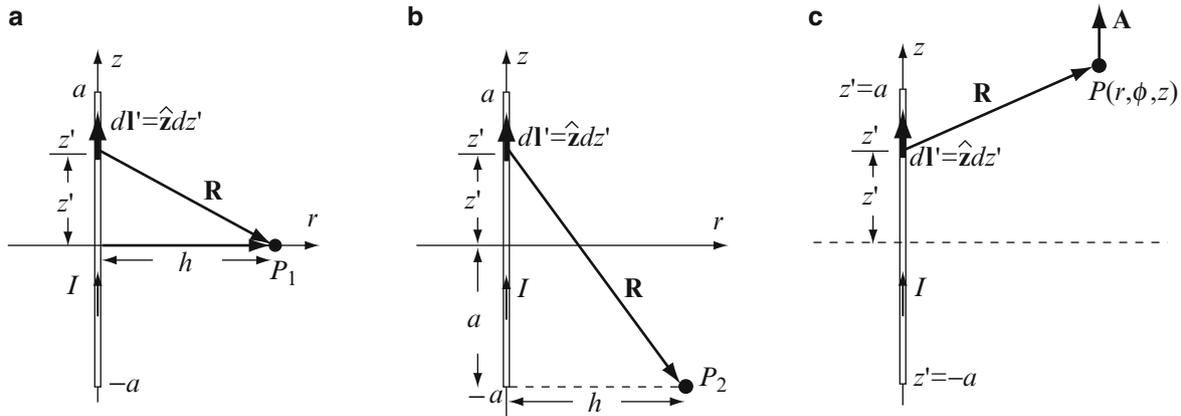
Using Stokes' theorem, we can convert this to a closed contour integral. The final result is

$$\boxed{\Phi = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad [\text{Wb}]} \tag{8.47}$$

where  $C$  is the contour bounding the surface  $s$ . This relation shows that the total flux through a surface  $s$  is equal to the line integral of the magnetic vector potential along the contour of the surface. In other words, we do not need to calculate the magnetic flux density  $\mathbf{B}$  if we wish to evaluate the total flux. This relation also gives a physical meaning to the magnetic vector potential as a measure of flux.

**Example 8.12 Magnetic Vector Potential Due to a Short, Straight Segment** A thin, finite-length wire as shown in **Figure 8.22a** carries a current  $I = 1$  A. Calculate:

- The magnetic vector potential at point  $P_1$  shown in **Figure 8.22a**.
- The magnetic vector potential at point  $P_2$  shown in **Figure 8.22b**.
- The magnetic vector potential at a general point  $P(r, \phi, z)$  in **Figure 8.22c**.



**Figure 8.22** Magnetic vector potential of a short segment. (a) Calculation of  $\mathbf{A}$  at  $P_1$ . (b) Calculation of  $\mathbf{A}$  at  $P_2$ . (c) Calculation of  $\mathbf{A}$  at a general point  $P(r, \phi, z)$

**Solution:** We solved for the magnetic flux density and magnetic field intensity for the same segment in **Example 8.1**. The approach here is similar: An element of length  $d\mathbf{l}' = \hat{\mathbf{z}} dz'$  is identified at point  $P(0, 0, z')$ . The magnetic vector potential at point  $P(r, \phi, z)$  is calculated using the Biot–Savart law in terms of the magnetic vector potential given in **Eq. (8.34)**. Since the magnetic vector potential is always in the direction of current, only a  $z$  component exists.

- At point  $P_1$ , the coordinates are  $(r = h, z = 0)$ . The magnetic vector potential at point  $P_1$  due to the current in element  $d\mathbf{l}' = \hat{\mathbf{z}} dz'$  is

$$d\mathbf{A}(r, z) = \hat{\mathbf{z}} \frac{\mu_0 I dz'}{4\pi |\mathbf{R}|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right]$$

The vector  $\mathbf{R}$  and its magnitude are

$$\mathbf{R} = \hat{\mathbf{r}} h - \hat{\mathbf{z}} z' \quad \rightarrow \quad R = \sqrt{h^2 + z'^2}$$

The magnetic vector potential is found by integrating over the length of the segment from  $z' = -a$  to  $z' = +a$ :

$$\mathbf{A}(h, 0) = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_{z'=-a}^{z'=a} \frac{dz'}{\sqrt{h^2 + z'^2}} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \left( z' + \sqrt{h^2 + z'^2} \right) \Big|_{z'=-a}^{z'=a} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \frac{a + \sqrt{h^2 + a^2}}{\sqrt{h^2 + a^2} - a} \quad \left[ \frac{\text{Wb}}{\text{m}} \right].$$

- (b) To calculate the magnetic vector potential at point  $P_2$ , we use **Figure 8.22b**.  $d\mathbf{l}'$  remains the same as in (a), but the vector  $\mathbf{R}$  and its magnitude become

$$\mathbf{R} = \hat{\mathbf{r}}h - \hat{\mathbf{z}}(z' + a) \quad \rightarrow \quad R = \sqrt{h^2 + (z' + a)^2}$$

Substituting this in the expression for  $d\mathbf{A}$  and integrating from  $z' = -a$  to  $z' = a$ ,

$$\begin{aligned} \mathbf{A}(h, -a) &= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_{z'=-a}^{z'=a} \frac{dz'}{\sqrt{z'^2 + 2az' + (h^2 + a^2)}} \\ &= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \left( 2\sqrt{z'^2 + 2az' + (h^2 + a^2)} + 2z' + 2a \right) \Big|_{z'=-a}^{z'=a} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{4a^2 + h^2} + 2a}{h} \quad \left[ \frac{\text{Wb}}{\text{m}} \right]. \end{aligned}$$

- (c) As in the previous two calculations, we place the element of length  $d\mathbf{l}'$  at a point  $z'$  along the element of current and calculate the length of the vector  $\mathbf{R}$  (see **Figure 8.22c**):

$$\mathbf{R} = \hat{\mathbf{r}}r - \hat{\mathbf{z}}(z - z') \quad \rightarrow \quad R = \sqrt{r^2 + (z - z')^2}$$

The magnetic vector potential at  $P(r, \phi, z)$  is

$$\begin{aligned} \mathbf{A}(r, \phi, z) &= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_{z'=-a}^{z'=a} \frac{dz'}{\sqrt{z'^2 - 2zz' + (r^2 + z^2)}} \\ &= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \left( 2\sqrt{z'^2 - 2zz' + (r^2 + z^2)} + 2z' - 2z \right) \Big|_{z'=-a}^{z'=a} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{a^2 - 2za + (r^2 + z^2)} + a - z}{\sqrt{a^2 + 2za + (r^2 + z^2)} - a - z} \quad \left[ \frac{\text{Wb}}{\text{m}} \right]. \end{aligned}$$

**Exercise 8.4** Using **Example 8.12**, calculate the magnetic flux density at point  $P_1$  using the definition of the magnetic vector potential in **Eq. (8.25)** and show that the result in **Example 8.1a** is obtained for  $h = a = 1$  m if we write  $\mathbf{H} = \mathbf{B}/\mu_0$ .

**Example 8.13 Flux Due to a Square Loop** A square loop carries a current  $I = 1$  A. The length of each side of the loop is  $2a = 1$  m. A smaller loop with dimensions  $2b = a = 0.5$  m is placed inside the larger loop as shown in **Figure 8.23a**. Calculate the total flux produced by the outer loop that passes through the inner loop.

**Solution:** In general, to calculate the magnetic flux, we need to calculate the magnetic flux density and then integrate over the area through which the flux flows. An alternative method is to calculate the magnetic vector potential along the boundaries of the surface and integrate this potential to obtain the flux using **Eq. (8.47)**. We show the latter method here. The magnetic vector potential due to the outer loop at the boundaries of the inner loop is first calculated and then integrated over the boundaries of the inner loop.

The magnetic vector potential due to a  $z$ -directed current-carrying segment of length  $2a$ , at a distance  $r$  from the segment, was calculated in **Example 8.12c**.

$$\mathbf{A}(r, \phi, z) = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{a^2 - 2za + (r^2 + z^2)} + a - z}{\sqrt{a^2 + 2za + (r^2 + z^2)} - a - z} = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \left[ \sinh^{-1} \left( \frac{a - z}{r} \right) + \sinh^{-1} \left( \frac{z + a}{r} \right) \right] \quad \left[ \frac{\text{Wb}}{\text{m}} \right]$$

where the relation  $\ln(x + (x^2 + 1)^{1/2}) = \sinh^{-1} x$  was used to write the second form of  $\mathbf{A}$ . Also, because  $\mathbf{A}$  does not depend on  $\phi$ , this dependence will be dropped for the remainder of calculations in this example.

We will use this expression as follows: Because the line integral in **Eq. (8.47)** requires only the component of the magnetic vector potential in the direction of integration, and the boundaries of the two loops are parallel to each other, we calculate the magnetic vector potential due to segments  $l'_1$  and  $l'_3$  at segments  $l_1$  and  $l_3$  since segments  $l'_2$  and  $l'_4$  generate components of the magnetic vector potential perpendicular to  $l_1$  and  $l_3$ . Similarly, the magnetic vector potential on segments  $l_2$  and  $l_4$  is due to  $l'_2$  and  $l'_4$ . Using **Figure 8.23b**, the magnetic vector potential at segment  $l_1$  due to segment  $l'_1$  is

$$\mathbf{A}(r_1, z) = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \left( \sinh^{-1} \frac{2(a - z)}{a} + \sinh^{-1} \frac{2(z + a)}{a} \right) \quad \left[ \frac{\text{Wb}}{\text{m}} \right]$$

The magnetic vector potential at segment  $l_1$  due to segment  $l'_3$  is

$$\mathbf{A}(r'_2, z) = -\hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \left( \sinh^{-1} \frac{2(a - z)}{3a} + \sinh^{-1} \frac{2(z + a)}{3a} \right) \quad \left[ \frac{\text{Wb}}{\text{m}} \right]$$

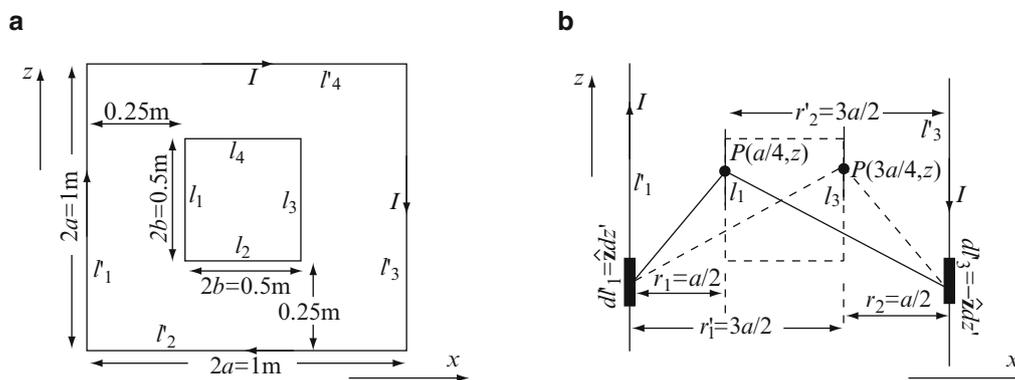
Note that this magnetic vector potential is in the opposite direction to that due to segment  $l_1$  because the currents are in opposite directions. The total magnetic vector potential at any location on segment  $l_1$  is

$$\mathbf{A}(r_1, z) = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \left( \sinh^{-1} \frac{2(a - z)}{a} + \sinh^{-1} \frac{2(z + a)}{a} - \sinh^{-1} \frac{2(a - z)}{3a} - \sinh^{-1} \frac{2(z + a)}{3a} \right) \quad \left[ \frac{\text{Wb}}{\text{m}} \right]$$

Similarly, interchanging between  $r_1$  and  $r_2$  in **Figure 8.23b**, the magnetic vector potential on segment  $l_3$  is identical in magnitude but in the opposite direction. Instead of calculating the magnetic vector potential on the upper and lower segments, we note from symmetry considerations that the magnitudes of the magnetic vector potentials on these segments must be the same as for segments  $l_1$  and  $l_3$  even though the direction of  $\mathbf{A}$  is horizontal.

The total flux is calculated by integrating the magnetic vector potential along the boundaries of the inner loop. Going along the current, all four segments contribute equal and positive values to the integral:

$$\Phi = 4 \int_{z=a/2}^{z=3a/2} \mathbf{A}_{l_1}(r) \cdot d\mathbf{l}_1 = \frac{\mu_0 I}{\pi} \int_{z=a/2}^{z=3a/2} \left( \sinh^{-1} \frac{2(a - z)}{a} + \sinh^{-1} \frac{2(z + a)}{a} - \sinh^{-1} \frac{2(a - z)}{3a} - \sinh^{-1} \frac{2(z + a)}{3a} \right) dz \quad [\text{Wb}]$$



**Figure 8.23** Calculation of flux through the inner loop, generated by the outer loop. (a) Geometry and dimensions. (b) Calculation of the magnetic vector potential due to segments  $l'_1, l'_3$  on  $l_1$  and due to segments  $l'_1$  and  $l'_3$  on  $l_3$

Performing the integration of the four terms above gives

$$\begin{aligned}
 \Phi &= \frac{\mu_0 I}{\pi} \left[ -(a-z) \sinh^{-1} \frac{2(a-z)}{a} + \sqrt{(a-z)^2 + \frac{a^2}{4}} + (a+z) \sinh^{-1} \frac{2(a+z)}{a} - \sqrt{(a+z)^2 + \frac{a^2}{4}} + (a-z) \sinh^{-1} \frac{2(a-z)}{3a} \right. \\
 &\quad \left. - \sqrt{(a-z)^2 + \frac{9a^2}{4}} - (a+z) \sinh^{-1} \frac{2(a+z)}{3a} + \sqrt{(a+z)^2 + \frac{9a^2}{4}} \right]_{z=a/2}^{z=3a/2} \\
 &= \frac{\mu_0 I}{\pi} \left( \frac{a}{2} \sinh^{-1}(-1) + \frac{a}{\sqrt{2}} + \frac{5a}{2} \sinh^{-1} 5 - \frac{\sqrt{26}a}{2} - \frac{a}{2} \sinh^{-1} \left( -\frac{1}{3} \right) - \frac{\sqrt{10}a}{2} - \frac{5a}{2} \sinh^{-1} \left( \frac{5}{3} \right) + \frac{\sqrt{34}a^2}{2} \right) \\
 &\quad - \frac{\mu_0 I}{\pi} \left( -\frac{a}{2} \sinh^{-1} 1 + \frac{a}{\sqrt{2}} + \frac{3a}{2} \sinh^{-1} 3 - \frac{\sqrt{10}a}{2} + \frac{a}{2} \sinh^{-1} \frac{1}{3} - \frac{\sqrt{10}a}{2} - \frac{3a}{2} \sinh^{-1} 1 + \frac{\sqrt{18}a}{2} \right) \\
 &= \frac{\mu_0 I}{\pi} \left( \frac{\sqrt{10}a}{2} - \frac{\sqrt{26}a}{2} + \frac{\sqrt{34}a}{2} - \frac{\sqrt{18}a}{2} + \frac{5a}{2} \left[ \sinh^{-1} 5 - \sinh^{-1} \frac{5}{3} \right] + \frac{3a}{2} \left[ \sinh^{-1} 1 - \sinh^{-1} 3 \right] \right) \quad [\text{Wb}]
 \end{aligned}$$

For the values given ( $a = 0.5 \text{ m}$ ,  $I = 1 \text{ A}$ ,  $\mu_0 = 4\pi \times 10^{-7}$ ), the total flux is

$$\Phi = 2 \times 10^{-7} \left( \frac{\sqrt{10}}{2} - \frac{\sqrt{26}}{2} + \frac{\sqrt{34}}{2} - \frac{\sqrt{18}}{2} + \frac{5}{2} \left[ \sinh^{-1} 5 - \sinh^{-1} \frac{5}{3} \right] + \frac{3}{2} \left[ \sinh^{-1} 1 - \sinh^{-1} 3 \right] \right) = 1.984 \times 10^{-7} \quad [\text{Wb}]$$

**Note:** The flux through the inner loop may also be calculated by integrating the flux density produced by the outer loop over the area of the inner loop using **Eq. (8.17)**. However, the method shown here is much simpler for the given configuration. In other cases, **Eq. (8.17)** may be easier to apply.

## 8.7.2 The Magnetic Scalar Potential

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The condition for a scalar potential to be defined is that the vector field must be curl-free; that is, the field must be a conservative field. Any vector field  $\mathbf{F}$  that satisfies the curl-free condition  $\nabla \times \mathbf{F} = 0$  may be described as the gradient of a scalar function  $\varphi$  by

$$\mathbf{F} = -\nabla\varphi. \quad (8.48)$$

The basis of this statement is that now we can substitute this back into the curl and obtain  $\nabla \times \mathbf{F} = \nabla \times (-\nabla\phi) = 0$ , based on one of the vector identities defined in **Chapter 2 [Eq. (2.112)]**. Whenever this is possible, it has the very distinct advantage of allowing calculation in terms of a scalar function rather than in terms of a vector function. Note that it would have been appropriate to define  $\mathbf{F} = \nabla\phi$  instead of  $\mathbf{F} = -\nabla\phi$ . The negative sign is introduced by convention, as was done for the electric scalar potential in **Section 4.4.3**.

Inspection of **Eq. (8.20)** shows that the magnetic field intensity is not curl-free in general and, therefore, we cannot describe it in terms of a scalar function. There are, however, a number of important applications in magnetics in which a magnetic field exists, but there are no current densities involved. The most obvious are those involving permanent magnets. In this case,  $\mathbf{J} = 0$  and we can write

$$\nabla \times \mathbf{H} = 0 \quad \rightarrow \quad \mathbf{H} = -\nabla\psi \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.49)$$

where  $\psi$  is called the **magnetic scalar potential**. This potential has properties similar to those of the electric potential since it was defined in exactly the same way. Thus, the closed contour integral of  $\mathbf{H} \cdot d\mathbf{l}$  is now zero (the magnetic field intensity is conservative because  $\mathbf{J} = 0$ ). Also, we may define the magnetic scalar potential difference as

$$\psi_{ba} = \psi_b - \psi_a = -\int_a^b \mathbf{H} \cdot d\mathbf{l} \quad [\text{A}] \quad (8.50)$$

where  $\psi_a$  may be viewed as a reference magnetic scalar potential. The unit of the magnetic scalar potential is the ampere [A].

The magnetic scalar potential satisfies the Laplace equation exactly like the electric scalar potential:

$$\boxed{\nabla^2\psi = 0} \quad (8.51)$$

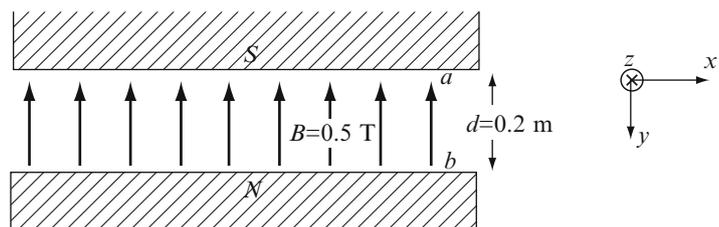
Thus, whenever a magnetic problem can be described such that there are no current sources in the region, the properties of the electric scalar potential can be invoked for the magnetic scalar potential.

The following example shows the use of the magnetic scalar potential in a simple geometry.

**Example 8.14 The Magnetic Scalar Potential Due to a Magnet** A very large magnet is made as shown in **Figure 8.24**, with the two poles separated a distance,  $d = 0.2$  m. Because of the size of the magnet, we may assume the field to be perpendicular to the surface of the poles. Suppose the magnetic flux density between the poles is known to be 0.5 T:

- Calculate the magnetic scalar potential everywhere between the poles.
- Calculate the magnetic scalar potential difference between the two poles.
- Suppose we now move the poles apart to twice the distance. What is the magnetic flux density if the magnetic scalar potential remains the same?

**Figure 8.24** Gap between the poles of a large magnet and the magnetic flux density between them



**Solution:** This problem is solved by letting the upper plate be at a reference magnetic scalar potential of zero and then integrating against the magnetic flux density from the upper plate toward the lower plate using Eq. (8.50). In (c), we assume that since the potential difference remains the same, the magnetic flux density must diminish in the same way the electric field diminishes when the plates of a capacitor are moved apart while keeping the potential difference constant:

- (a) Using Eq. (8.50) and assuming the magnetic field intensity is in the negative  $y$  direction as shown in Figure 8.24, we integrate from the upper pole toward the lower pole between  $y = 0$  and a general point  $y$  between the poles:

$$\psi_{ya} = \psi_y - \psi_a = -\int_0^y (-\hat{y}H) \cdot (\hat{y} dy) = \int_0^y \frac{B}{\mu_0} dy = \frac{By}{\mu_0} \quad [\text{A}]$$

The magnetic scalar potential varies linearly with the distance from the upper pole.

- (b) The magnetic scalar potential difference between the two poles is

$$\psi_{ba} = \psi_b - \psi_a = -\int_{y=a}^b (-\hat{y}H) \cdot (\hat{y} dy) = \int_{y=0}^d \frac{B}{\mu_0} dy = \frac{Bd}{\mu_0} = \frac{0.5 \times 0.2}{4\pi \times 10^{-7}} = 7.96 \times 10^4 \quad [\text{A}]$$

**Note:** The positive direction for potential is against the field. Therefore, the magnetic scalar potential at  $b$  is higher than at  $a$ .

- (c) The scalar potential difference remains the same and, because the poles are large, the magnetic flux density remains uniform between the poles. Using the result in (b), we get

$$\psi_{ab} = \frac{Bd}{\mu_0} = 7.96 \times 10^4 \quad \rightarrow \quad B = \frac{\psi_{ab}\mu_0}{2d} = \frac{7.96 \times 10^4 \times 4\pi \times 10^{-7}}{0.4} = 0.25 \quad [\text{T}].$$

## 8.8 Applications

**Application: Magnetic Prospecting—Geomagnetism** It is well known that a magnetic compass will not function properly in certain environments. For example, near volcanoes, where basalt rocks are present, there is a distortion in the terrestrial magnetic field due to concentration of iron in basalt. Similar occurrences can be observed in the presence of large deposits of iron ore, especially in the presence of magnetite. These observations are the basis of a specialized form of prospecting called magnetic prospecting or magnetic surveying. Since most rocks contain small amounts of magnetic materials (mostly magnetite,  $\text{Fe}_3\text{O}_4$ , and hematite  $\text{Fe}_2\text{O}_3$ ), measurement of variations in the geomagnetic field can detect deposits or lack of deposits in the crust. Measurements can be simple, such as direct measurement and recording of the surface field variations, or may require specialized techniques, such as the use of special magnetometers (a magnetometer is a sensitive instrument for measurement of magnetic fields). In some cases, the existence of ore may be detected directly. In other applications, specific materials are found in rock layers which exhibit detectable magnetic properties. For example, gold is often found in igneous rocks, which may be traced with a magnetometer. Diamonds are often found in kimberlite veins which also have specific magnetic anomalies. Similarly, the lack of any anomaly may indicate very deep base sedimentary rocks which are essential for accumulation of oil. Magnetic surveying of this type, which may be called passive magnetic surveying, is often done from aircraft, especially when large areas need to be surveyed. The method can also be used in archaeological and geological research. There are also active methods of prospecting which are often used.

**Application: The Twisted Pair** You may have noticed that telephone wires in telephone cables are twisted in pairs in the form of a braid. Similarly, if you were to cut an electric cable, you may find that the wires inside are tightly twisted together. This serves two purposes: First, in the case of telephone wires, it allows separation of pairs in cables, which may contain hundreds of pairs of wires, into individual lines. More important, the closer together the two wires are, the lower the total external magnetic field intensity they produce. A similar observation was made by Ampere in his series of experiments to

prove that the magnetic field is due to current in conductors. This method is very useful both in wiring of equipment and in distribution of power. One of the simplest methods of minimizing the magnetic field due to conductors is to keep the pairs close together. This is particularly important where one pair of wires may affect another pair (the effect is due to induction, a time-dependent effect which we will introduce in **Chapter 10**).

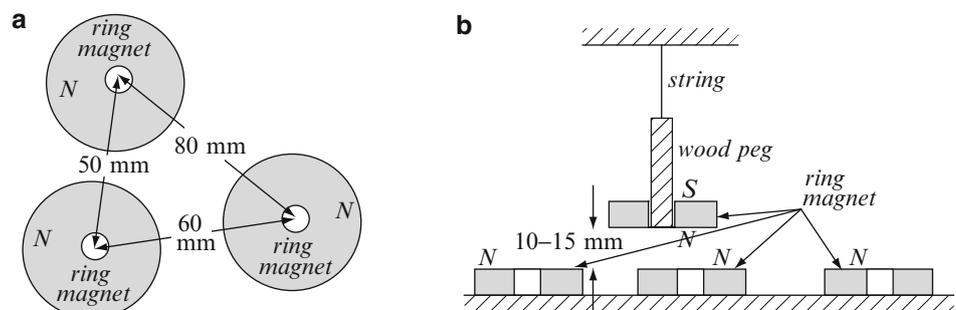
**Application: High-Field Coils** It is sometimes necessary to produce very high magnetic fields. One application is in magnetic resonance imaging (MRI) devices. In these applications, a magnetic flux density of between 1 and 6 T is required. To produce fields of this magnitude, a solenoid, about 1 m long and at least 0.5 m in diameter is made and a current, sufficiently high to produce the field, is passed through the coil. The currents are extremely high and the losses in the coils cannot be dissipated under normal conditions. Thus, it becomes necessary to cryogenically cool the solenoids through use of liquid nitrogen ( $77^\circ\text{K}$ ) or even at superconducting temperatures through use of liquid helium ( $4.2^\circ\text{K}$ ). Usually the higher temperatures are preferred because most metals used for coils become extremely brittle at superconducting temperatures. Only a few alloys can operate at these very low temperatures (niobium–titanium is the most commonly used).

**Application: High-Voltage Overhead Transmission Lines** A pair of overhead transmission lines generates a magnetic field intensity in space. A growing concern in society is the effect these magnetic (and electric) fields have on living organisms. In particular, some studies have implicated high magnetic fields due to overhead transmission lines with higher than average cancer rates for those exposed. Most of these studies suggest, without being conclusive, that low-frequency fields, such as those produced by AC distribution lines, are at fault. As power requirements grow, there is an increasing need to use higher voltages on transmission lines. Power distributions above 1 MV (million volts) already exist (for example, the trans-Siberian line in Russia uses 1.2 MV). What are the magnetic field intensities we can expect at ground levels? Typically, the distance between lines is a few meters to prevent the lines from touching during storms, and the lines may be as high as 20–25 m. This produces magnetic field intensities at ground level that may exceed 5 A/m.

## 8.9 Experiments

**Experiment 1 (Demonstrates: Magnetic forces: Chaotic Motion in a Nonuniform Magnetic Field).** A simple demonstration of properties of the magnetic field can be accomplished by a few ring magnets (of the type usually used to hold down notes). Place three magnets on a nonmagnetic table in an irregular pattern as shown in **Figure 8.25a**, making sure all are with their *N* poles up or all are with their *S* poles up. Insert a plastic pen or a piece of wood into the center hole of a fourth magnet so that it fits tightly and suspend this magnet on a string above the center of the three-magnet pattern. The string and pen should be about 40 to 50 cm long, and the fourth magnet should be suspended about 10 to 15 mm above the three fixed magnets. Make sure the magnets on the table are fixed (sticky tape is enough to hold them in place). Place the poles so that the magnets on the table repel the suspended magnet. The pen gives the magnet rigidity so it does not tumble into an attraction mode. Now, give the suspended magnet a push and observe its motion. The motion is quite irregular in a kind of chaotic pattern. Try different patterns and a different number of magnets in the pattern. Try to improve on this curious pendulum by using a rigid suspension (such as a long, round peg instead of the string and pen). The forces involved are quite complicated because of the irregular pattern (nonuniform field).

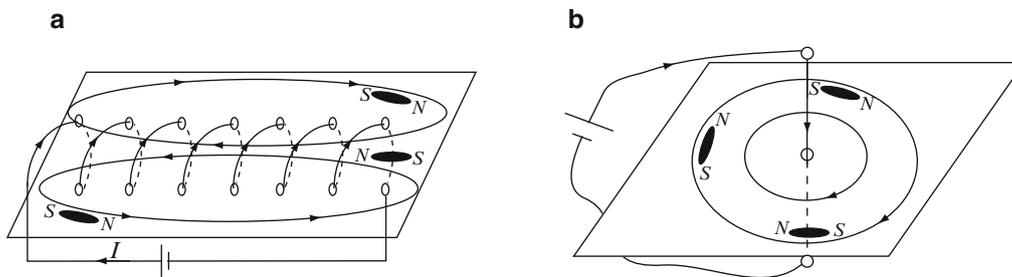
**Figure 8.25** A chaotic pendulum. (a) The pattern of fixed magnets. (b) The suspended magnet above the magnets in (a)



**Experiment 2 (The Nature of Permanent Magnets. Demonstrates: The Nonexistence of Monopoles).** The fact that a magnet always has two poles may be demonstrated with a number of small, ring (ceramic) magnets. Each individual magnet has two poles. By stacking any number of these rings, a single magnet with two poles is made. Removing the magnets restores the poles to their original condition.

**Experiment 3 (Demonstrates: The Magnetic Field of a Solenoid).** It is rather easy to see the general pattern of the field of a solenoid. Build a solenoid about 30 mm in diameter by passing a thick copper wire through holes in a flat piece of wood or cardboard as shown in **Figure 8.26a**. Connect the solenoid to a DC source and increase the current until you get at least 1 A. A DC, regulated, current-protected power supply works best, but a large battery may also be used. Using a magnetic compass, trace the magnetic field lines inside and outside the solenoid. You can also use a metal “slinky” by extending it so that the coils do not touch each other and connecting it to a source, as above. A piece of cardboard inserted inside the coil will allow placement of the compass to measure the field.

**Experiment 4 (Magnetic Field of a Straight Wire. Demonstrates: Oersted’s Experiment, Sensing of Low Magnetic Fields).** This is similar to the experiment performed by Oersted. Pass a current of at least 1 A through the wire. Use a small magnetic compass and mark on the cardboard the direction of the field at a few locations on the board. Connecting these locations, you should obtain the characteristic circular pattern of the field. If the current is large enough and the compass sensitive enough, you should have no difficulty in obtaining the pattern. The compass is one of the simplest magnetic field measuring devices and yet it is also one of the more sensitive. It also has the advantage of direct indication of the magnetic field as a vector and shows its relation to force.



**Figure 8.26** (a) Demonstration of the magnetic field of a solenoid. (b) Demonstration of the magnetic field of a straight, current-carrying wire

## 8.10 Summary

The relation between currents and the magnetic field is now explored, primarily through the Biot–Savart and Ampere’s laws and serves as an introduction to magnetostatics. The starting point is the magnetic field intensity  $\mathbf{H}$  [A/m] due to a filament segment carrying current  $I$  [A] and is calculated using the **Biot–Savart law** (see **Figure 8.3**)

$$\mathbf{H}(x, y, z) = \frac{1}{4\pi} \int_a^b \frac{Id\mathbf{l} \times \hat{\mathbf{R}}}{|\mathbf{R}|^2} \quad \left[ \frac{\text{A}}{\text{m}} \right] \quad (8.8)$$

The Biot–Savart law calculates the magnetic field intensity  $\mathbf{H}$  due to filamentary currents but can be used in thick conductors by stipulating a filament with differential cross section and integration over all such filaments. The magnetic flux density  $\mathbf{B}$  [T] is related to  $\mathbf{H}$  as  $\mathbf{B} = \mu\mathbf{H}$  where  $\mu$  is the permeability of the medium.

**Ampere’s law** is the circulation of the magnetic field intensity around a closed contour:

$$\oint_c \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} \quad [\text{A}] \quad (8.16)$$

For this to be useful in calculation of  $\mathbf{H}$  (or  $\mathbf{B}$ ), we require that  $\mathbf{H}$  be constant and either parallel or perpendicular to  $d\mathbf{l}$  along the path of integration. That is, we must find a contour, enclosing the current  $I$ , on which the scalar product in the integrand can be evaluated a priori so that  $H$  can be taken outside the integral. This relation is particularly useful for calculation of fields of very long conductors, solenoids, and toroidal coils.

**Magnetic Flux** Currents produce magnetic fields and magnetic fields produce flux,  $\Phi$

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{Wb}] \quad (8.17)$$

**Postulates** The relations above lead to the postulates of the magnetostatic field, specifying its curl and divergence:

$$\overbrace{\nabla \times \mathbf{H} = \mathbf{J} \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0}^{\text{differential form}} \quad \text{or} \quad \overbrace{\oint_c \mathbf{H} \cdot d\mathbf{l} = I_{enc.} \quad \text{and} \quad \oint_s \mathbf{B} \cdot d\mathbf{s} = 0}^{\text{integral form}} \quad (8.23) \text{ and } (8.24)$$

**Magnetic Vector Potential** From  $\nabla \cdot \mathbf{B} = 0$  we can define a magnetic vector potential as follows:

Since  $\nabla \cdot \mathbf{B} = 0$ , we have (see **Section 2.5** and the Helmholtz theorem):

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (8.25)$$

Substitution of this into **Eqs. (8.8)** and **(8.17)** leads to the Biot–Savart law and the flux in terms of the magnetic vector potential. These are often easier to calculate:

$$\mathbf{A} = \frac{\mu I}{4\pi} \int_a^b \frac{d\mathbf{l}'}{|\mathbf{R}|} \quad \left[ \frac{\text{Wb}}{\text{m}} \right] \quad \text{and} \quad \Phi = \oint_c \mathbf{A} \cdot d\mathbf{l} \quad [\text{Wb}] \quad (8.34) \text{ and } (8.47)$$

A magnetic scalar potential  $\psi$  is defined if  $\mathbf{J} = 0$  in **Eq. (8.23)** based on the Helmholtz theorem:

$$\text{If } \nabla \times \mathbf{H} = 0 \quad \rightarrow \quad \mathbf{H} = -\nabla\psi \quad (8.49)$$

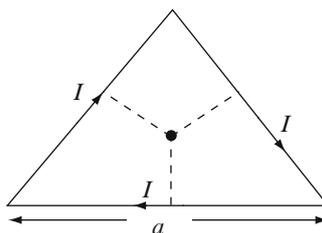
$\psi$  is used in the same fashion as the electric scalar potential  $V$ , but its units are the ampere.

**Reminder** Permeability of free space is  $\mu_0 = 4\pi \times 10^{-7}$  [H/m]

## Problems

### The Biot–Savart Law

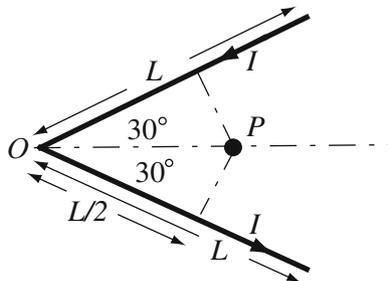
**8.1 Magnetic Flux Density Due to Filamentary Currents.** A current  $I$  [A] flows in a conductor shaped as an equilateral triangle (**Figure 8.27**). Calculate the magnetic flux density at the center of gravity of the triangle (where the three normals to the sides meet).



**Figure 8.27**

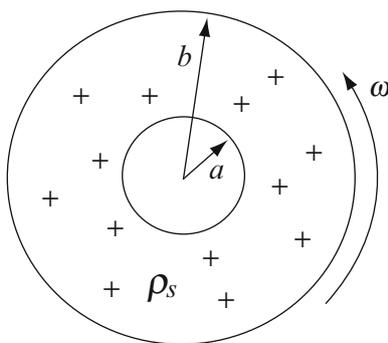
**8.2 Flux Density of Current Segments.** A conductor of length  $2L$  [m] carries current  $I$  [A] and is bent as in **Figure 8.28**:

- (a) Calculate the magnetic flux density at point  $P$ .
- (b) Calculate the magnetic flux density at point  $O$ .



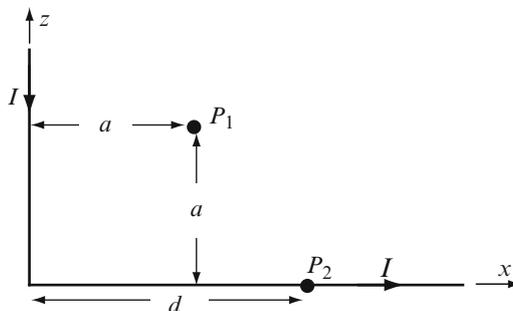
**Figure 8.28**

**8.3 Application: Magnetic Field of Moving Charges.** A thin insulating disk of outer radius  $b$  has a uniform charge density  $\rho_s$  [C/m<sup>2</sup>] distributed over the surface between  $r = a$  [m] and  $r = b$  [m]. The charges are bound and cannot move from their location. If the disk is rotating at an angular velocity  $\omega$ , calculate the magnetic field intensity at the center of the disk (**Figure 8.29**). **Note.** The first experiment to show the equivalency between moving charges and magnetic field was produced by Henry A. Rowland (1849–1901) in 1875. He showed that a charge placed on a disk (rotating at 3660 rpm) produced a magnetic field exactly like that of a closed-loop current.



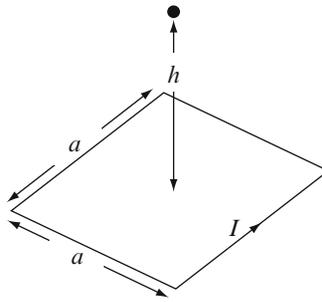
**Figure 8.29**

**8.4 Magnetic Flux Density of Semi-infinite Segments.** An infinitely long wire carrying a current  $I$  [A] is bent as shown in **Figure 8.30**. Find the magnetic flux density at points  $P_1$  and  $P_2$ . ( $P_2$  is at the center of the horizontal wire, at a distance  $d$  [m] from the bend.)



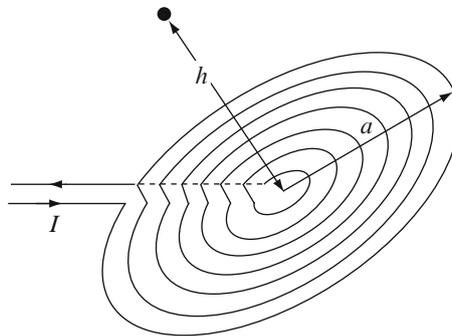
**Figure 8.30**

**8.5 Magnetic Flux Density of a Loop.** A square loop carries a current  $I$  [A]. Calculate the magnetic flux density (magnitude and direction) at a point  $h$  [m] high above the center of the loop (see **Figure 8.31**).



**Figure 8.31**

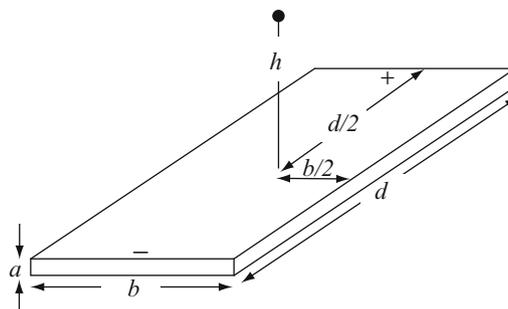
**8.6 Magnetic Flux Density of a Spiral Coil.** A wire is bent to form a flat spiral coil and carries a current  $I$  [A]. Calculate the flux density at a point  $h$  [m] high above the center of the spiral (**Figure 8.32**). The coil radius is  $a$  [m]. The spiral has  $N$  uniformly distributed turns. Assume each turn is a perfect circle.



**Figure 8.32**

**8.7 Magnetic Fields of Planar Structures.** A semiconductor material is made as shown in **Figure 8.33**. The long ends of the piece are connected to a voltage source. This causes electrons to move toward the positive connection and holes toward the negative connection. Holes and electrons move at the same velocity  $v$  [m/s] and have the same charge magnitude  $q$  [C] each. The hole and electron densities are equal to  $\rho$  holes/m<sup>3</sup> and  $\rho$  electrons/m<sup>3</sup>, respectively:

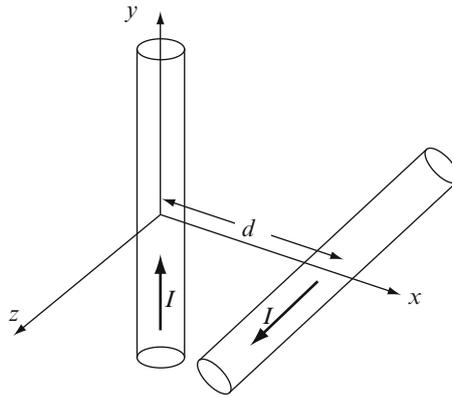
- Set up the integrals needed to calculate the magnetic flux density at a height  $h$  [m] from the center of the piece (see **Figure 8.33**). Do not evaluate the integrals.
- Optional:** Evaluate the integrals in (a) for the following data:  $a = 0.02$  m,  $b = 0.05$  m,  $d = 0.5$  m,  $h = 0.1$  m,  $\rho = 10^{20}$  (holes or electrons/m<sup>3</sup>),  $v = 1$  m/s,  $q = 1.6 \times 10^{-19}$  C. **Note.** The integration is very tedious. You may wish to perform the integration numerically.



**Figure 8.33**

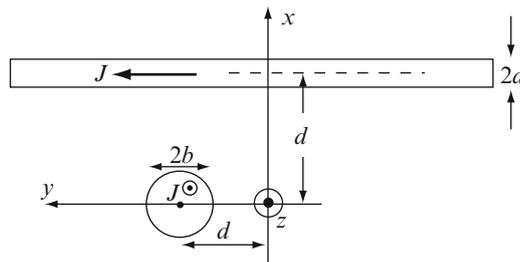
## Ampere's Law

**8.8 Magnetic Flux Density of Thick Conductors.** Two, infinitely long cylindrical conductors of radius  $a$  [m] carry a current  $I$  [A] each. One conductor coincides with the  $y$  axis, whereas the second is parallel to the  $z$  axis, in the  $x$ - $z$  plane, at a distance  $d$  from the origin as shown in **Figure 8.34**. The currents are in the positive  $y$  direction in the first conductor and in the positive  $z$  direction in the second conductor. Calculate the magnetic flux density (magnitude and direction) at all points on the  $z$  axis.



**Figure 8.34**

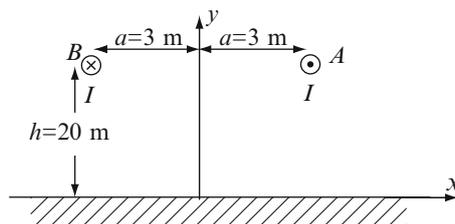
**8.9 Magnetic Flux Density of Thick Conductors.** Two infinitely long cylinders, one of radius  $a$  [m] and one of radius  $b$  [m], are separated as shown in **Figure 8.35**. One cylinder is directed parallel to the  $z$  axis, in the  $x$ - $z$  plane, the other parallel to the  $y$  axis, in the  $x$ - $y$  plane. A current density  $J$  [A/m<sup>2</sup>] flows in each, directed in the positive  $y$  and  $z$  directions as shown. Calculate the magnetic flux density at a general point  $P(x,y,z)$  outside the cylinders.



**Figure 8.35**

**8.10 Application: Magnetic Flux Density Due to Power Lines.** A high-voltage transmission line operates at 750 kV and 2000 A maximum. The towers used to support the lines are  $h = 20$  m high and the lines are separated a distance  $2a = 6$  m (**Figure 8.36**). Calculate:

- The magnetic flux density anywhere at ground level.
- The magnetic flux density (magnitude and direction) at ground level, midway between the two wires.

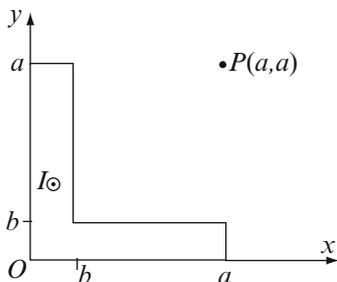


**Figure 8.36**

## Ampere's Law, Superposition

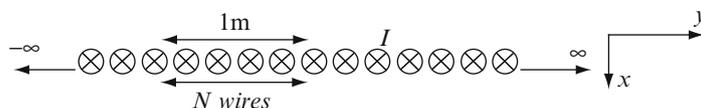
**8.11 Magnetic Flux Density Due to Volume Distribution of Currents.** A very long,  $L$ -shaped conductor with dimensions shown in cross section in **Figure 8.37** carries a total current  $I$  [A]. The current is in the positive  $z$  direction (out of the paper) and is uniformly distributed in the cross-section:

- (a) Set up the integrals needed to calculate the magnetic flux density (magnitude and direction) at point  $P(a,a)$ .  
 (b) **Optional:** Evaluate the integrals in (a) for the following data:  $a = 0.05$  m,  $b = 0.025$  m,  $I = 100$  A. **Note.** The integration is very tedious. You may wish to perform the integration numerically.



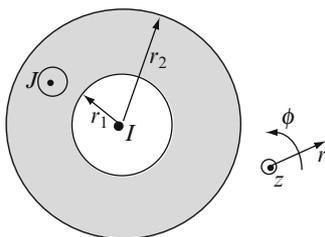
**Figure 8.37**

**8.12 Magnetic Field of a Current Sheet.** A thin layer of wires forms an infinite sheet of current. There are  $N$  wires per meter and each wire carries a current  $I$  [A] as shown in **Figure 8.38**. Calculate the magnitude and direction of the magnetic flux density everywhere in space.



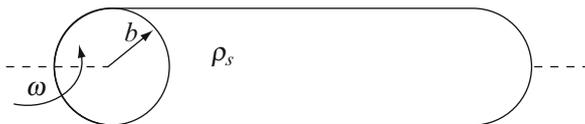
**Figure 8.38**

**8.13 Application: Magnetic Field of Thin and Thick Conductors.** An infinitely long thin wire is placed at the center of a hollow, infinitely long cylindrical conductor as shown in **Figure 8.39**. The conductor carries a current density  $J$  [A/m<sup>2</sup>] and the wire carries a current  $I$  [A]. The direction of  $J$  is out of the page. Find the magnetic flux density for the regions  $0 < r < r_1$ ,  $r_1 < r < r_2$ ,  $r > r_2$  for the following conditions: (a)  $I = 0$ . (b)  $I$  in the direction of  $J$ . (c)  $I$  in the direction opposing  $J$ .



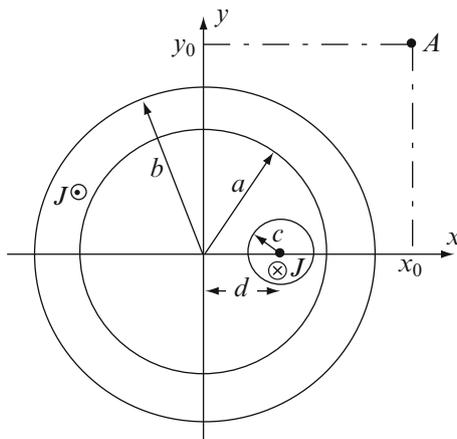
**Figure 8.39**

**8.14 Magnetic Flux Density Due to Rotating Surface Charge Density.** A long conducting cylinder of radius  $b$  [m] contains a surface charge density equal to  $\rho_s$  [C/m<sup>2</sup>] on its outer surface. The cylinder spins at an angular velocity  $\omega$  [rad/s] around its axis (**Figure 8.40**). Calculate the magnetic field intensity everywhere in space.



**Figure 8.40**

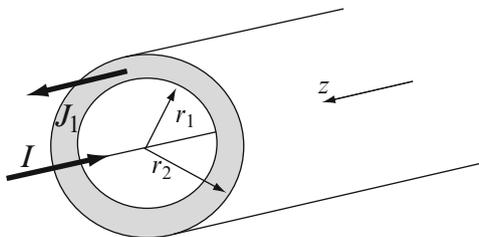
**8.15 Superposition of Magnetic Flux Densities Due to Thick Conductors.** A hollow, infinitely long cylindrical conductor has an outer radius  $b$  [m] and an inner radius  $a$  [m]. An offset cylinder of radius  $c$  [m] is located inside the large cylinder as shown in **Figure 8.41**. The two cylinders are parallel and their centers are offset by a distance  $d$  [m]. Assuming that the current density in each cylinder is  $J$  [A/m<sup>2</sup>] and is uniform, calculate the magnetic flux density at point  $A$  (outside the larger cylinder). The current in the larger cylinder is out of the page, and in the smaller cylinder, it is into the page.



**Figure 8.41**

**8.16 Application: Magnetic Field in a Coaxial Cable.** Given a very long (infinite) wire with current  $I$  [A] and a very long (infinite) cylindrical tube of thickness  $r_2 - r_1$  with uniform current density  $J_1$  [A/m<sup>2</sup>] as shown in **Figure 8.42**, find:

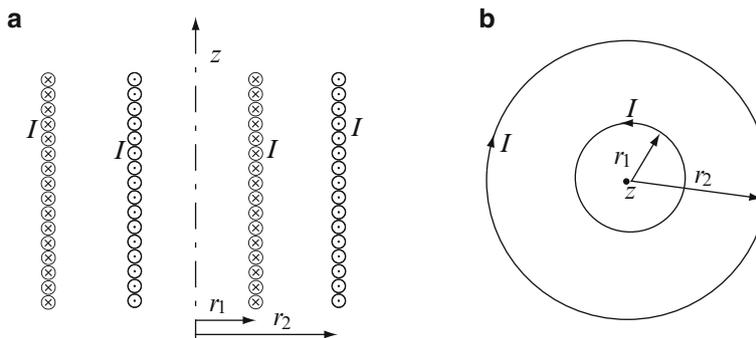
- (a)  $\mathbf{B}$  for  $0 < r < r_1$ , (b)  $\mathbf{B}$  for  $r_2 < r < \infty$ .



**Figure 8.42**

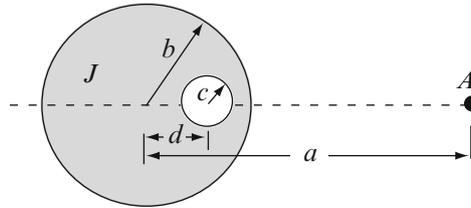
**8.17 Magnetic Field in Coaxial Solenoids.** A long solenoid (a single layer of very thin wires) of radius  $r_1$  [m] is placed inside a second long solenoid of radius  $r_2$  [m]. The currents in the solenoids are equal but in opposite directions, as shown in **Figure 8.43a** in axial cross section, and each solenoid has  $n$  turns per meter length:

- (a) Calculate the magnetic flux density for  $0 < r < r_1$ ,  $r_1 < r < r_2$  and  $r > r_2$ .
- (b) Repeat (a), if the current in the inner solenoid is reversed.



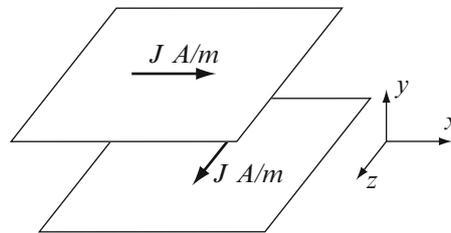
**Figure 8.43**

**8.18 Superposition of Solutions in Solid Conductors.** A very long (infinite) tubular conductor has radius  $b$  [m] and an offset hole of radius  $c$  as shown in **Figure 8.44**. The center of the hole is offset a distance  $d$  [m] from the center of the conductor. If the current density in the conductor flows out of the paper and equals  $J$  [ $\text{A}/\text{m}^2$ ], calculate the magnetic field intensity at  $A$ .



**Figure 8.44**

**8.19 Fields Due to Current Sheets.** Two infinite, thin sheets of current are arranged as shown in **Figure 8.45**. In the upper sheet, the current flows to the right, and in the lower, it flows out of the page. Calculate the magnetic field intensity everywhere. The current density is given in  $\text{A}/\text{m}$  meaning that the thickness of the sheets is negligible.



**Figure 8.45**

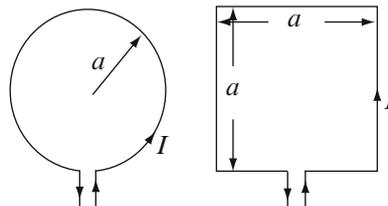
### Biot–Savart Law, Magnetic Vector Potential

**8.20 Magnetic Vector Potential of Current Segment.** A very long (but not infinite) straight wire carries a current  $I$  [A] and is located in free space:

- Calculate the magnetic vector potential at a distance  $a$  [m] from the center of the wire. Assume for simplicity that the wire extends from  $-L$  [m] to  $+L$  [m] and that  $L \gg a$ .
- From the result in (a), what must be the magnetic vector potential for an infinitely long current-carrying wire?

**8.21 Magnetic Vector Potential of Loops.** A circular loop and a square loop are given as shown in **Figure 8.46**:

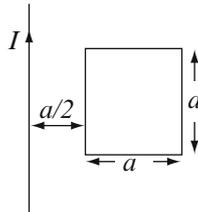
- Find the ratio between the magnetic flux densities at the center of the two loops.
- What is the magnetic vector potential at the center of each loop? Hint. Use Cartesian rather than cylindrical coordinates for the integration for both loops.



**Figure 8.46**

**8.22 Application: Magnetic Flux.** A straight wire carries a current  $I = 1$  A. A square loop is placed flat in the plane of the wire as shown in **Figure 8.47**. Calculate:

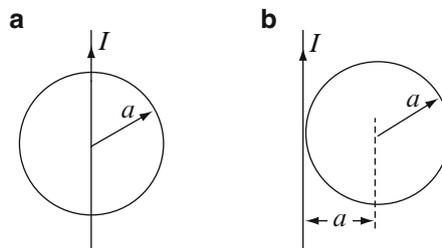
- (a) The flux, using the magnetic flux density.  
 (b) The flux, using the magnetic vector potential. (**Hint.** Use the result of **Problem 8.20**. Compare the results in (a) and (b).)



**Figure 8.47**

**8.23 Magnetic Flux.** A straight, infinitely long wire carries a current  $I = 10$  A and passes above a loop of radius  $a = 0.1$  m as shown in **Figure 8.48a**:

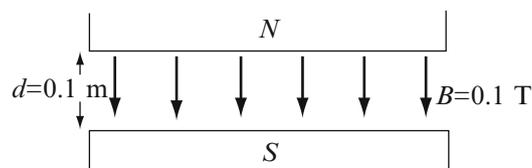
- (a) If the wire passes as in **Figure 8.48a**, show from symmetry considerations that the total flux through the loop is zero, regardless of how close the loop and the wire are.  
 (b) If the loop is moved sideways as in **Figure 8.48b**, calculate the total flux through the loop. Assume the loop and wire are in the same plane.



**Figure 8.48**

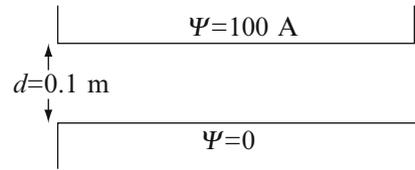
### Magnetic Scalar Potential

**8.24 Magnetic Scalar Potential Between the Poles of a Permanent Magnet.** The magnetic flux density between the poles of a large magnet is 0.1 T and is uniform everywhere between the poles in **Figure 8.49**. Calculate the magnetic scalar potential difference between the poles.



**Figure 8.49**

**8.25 Magnetic Scalar Potential Between the Poles of a Permanent Magnet.** The magnetic scalar potential difference between the poles of a magnet is 100 A (**Figure 8.50**). Calculate the magnetic field intensity if the space between the poles is air and the magnetic field intensity is uniform.



**Figure 8.50**