14 Theory of Transmission Lines

O tell me, when along the line
From my full heart the message flows,
What currents are induced in thine?
One click from thee will end my woes.

—James C. Maxwell (1831–1879),
mathematician, physicist
Valentine from a telegraph clerk to a telegraph clerk,

14.1 Introduction

Hopefully, by now you have a good understanding of waves propagating in space and in materials, including reflection and transmission at interfaces. Although not mentioned often enough, there were a number of assumptions implicit in this type of propagation. The most important was the fact that only plane waves were treated. In most cases, we also assumed the waves only propagate forward from the source, although reflections from interfaces cause waves to also propagate backward toward the source and these were treated in **Chapter 13**. Whereas the existence of interfaces complicates treatment, it also allows for applications such as radar to be feasible. If we were to summarize the previous two chapters in a few words, we would say that all wave phenomena were treated in essentially infinite space; that is, plane waves were not restricted in space except for the occasional interface.

There are, however, many applications in which this type of propagation is either impractical, not feasible, or inefficient. For example, consider the following situation: A spacecraft is flying at a distance of two million kilometers from Earth toward a distant planet. How can we communicate with the spacecraft? It makes no sense to use plane waves for this purpose even if true plane waves could be generated. A more practical approach would be a narrow beam, perhaps not much larger than the spacecraft, tracking the vehicle. Doing so reduces the power requirements and minimizes interference with other systems.

In Earth-bound systems, there is a third approach: connect a pair of conductors between two points and transmit the information over the two conductors. We call a connection, of this sort a ***transmission line***. Although at first glance this approach seems like a very simple circuit, it is far from it. We have not yet defined the properties of the transmission line proposed here, but the following simple analysis of the line, an analysis that does not require knowledge of line properties, should point out the special properties of this type of line.

Consider the following example: A power transmission line connects a power station with a load at a very large distance. Neglect losses on the line. Circuit theory tells us that the distance between source and load is immaterial. However, there is an additional assumption implicit in treating this problem as a simple circuit, that of instantaneous propagation. In other words, we assume that any change in the load appears instantaneously at the generator. If we were to short the load, the generator will see a short circuit at the same instant.

We know that this is not true; all propagation of energy takes time. Even in DC circuits, we often take into account time constants because of the capacitive and inductive terms in the circuits which influence the transient characteristics of the circuits. What happens if we look at the wave propagating on the same line? For a power line, the frequency is 60 Hz. Therefore, assuming propagation at the speed of light (propagation on lines is much slower than this in most cases), the wavelength of the wave is 5,000 km. Imagine now that you could measure the voltage on the line at any location and at

N. Ida, *Engineering Electromagnetics*, DOI 10.1007/978-3-319-07806-9_14,

the generator and compare the two at any given time. The two voltages will be different. For example, using **Figure 14.1**, the voltage at a distance of 1,250 km (a quarter wavelength) is zero when the generator is at its peak. One-quarter of a cycle later (1/240 s), the generator voltage goes to zero and the voltage at point *A* is now maximum. There is clearly a delay of 1/240 s between the two locations—this is the time required for the voltage wave to propagate from the generator to point *A*.

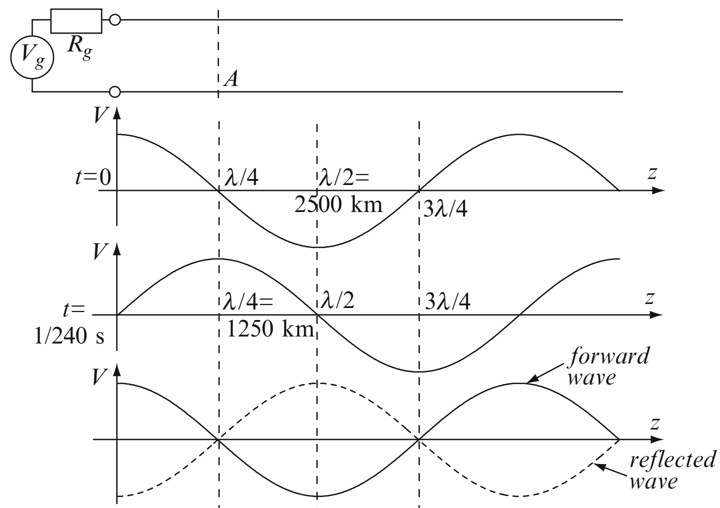


Figure 14.1 A very long power transmission line and conditions on the line at different times and locations

Now suppose we short the line at some point. In circuit theory, this is a disaster: the current will immediately rise to dangerous levels and destroy the circuit unless proper protection (such as a fuse) is available. However, for the above finite-speed line, suppose we short the circuit at exactly one-quarter wavelength. A short means that the energy propagated on the line cannot reach the load. Since the line is ideal, it cannot dissipate energy either. Thus, the energy must propagate back on the line toward the generator. The short is a disturbance, and by the time it reaches back to the generator, another quarter cycle has passed and now the generator is at negative maximum voltage. The voltage of the disturbance propagating back from the load is at positive maximum voltage. The total voltage at the generator is now zero. This does not look like such a disastrous event. Certainly, the generator will not be destroyed. Of course, if the short occurs at any other location, the result would be different. Also, we only looked at the short at a given instant in time. However, the point here is to show that the behavior of a transmission line is different than a circuit, because the assumptions we use are different.

Although this "analysis" leaves out more than it includes and certainly does not take into account all effects on the line; it indicates that propagation on lines is not the same as flow of power in simple circuits.

Consider another example with less "bang" to it but with similarly important consequences. Suppose you designed a circuit consisting of two sensors connected to an AND gate. The two sensors produce pulses as shown in **Figure 14.2a**. A normal circuit theory approach would give a "1" if both inputs are "1." The result in **Figure 14.2b** (c) is expected. Now, assume the inputs of the same gate are connected to the same sensors but one is a distance *a* and the second a distance *b* from the inputs (diagram). Because the speed of propagation of the pulse on lines is finite, the pulse on line *A* reaches the gate after a time *a*/*vp*. The second reaches it at a time *b*/*vp*. Thus, if the pulses are narrow, the two pulses reach the gate at different times and the output is "wrong" **[Figure 14.2b (d)]**. The longer line has a longer delay. The design is correct as far as circuits are concerned but may not operate properly because of delays on the lines. For these circuits to operate properly, both lines must be of the same length, such as strips of the same length on a printed circuit board, or the pulse on the shorter line must be delayed so that the two pulses reach the gate at the same time. This aspect of propagation is extremely important in high-speed computers. As the speed of computation increases, the limitation of propagation on physical lines, even within a board or a single chip, becomes more important and, in the end, imposes upper limits on computation speed.

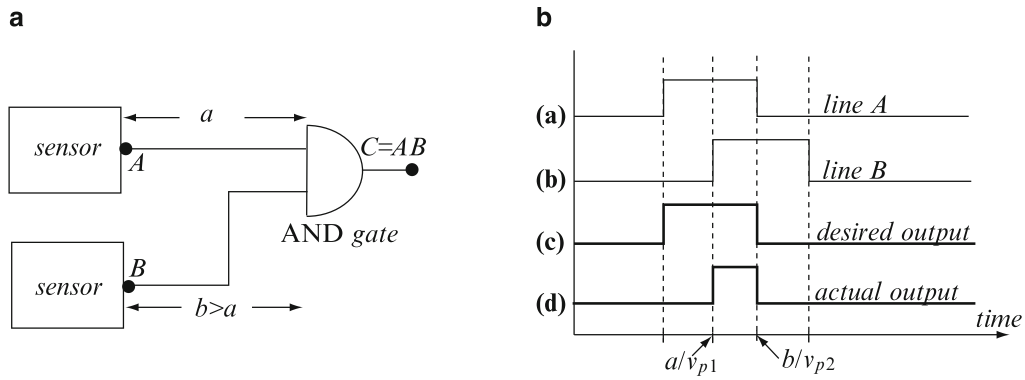


Figure 14.2 (**a**) A logic circuit with two input lines of different lengths. (**b**) Desired and actual output for a set of inputs

The two examples given suggest that transmission lines are not simple circuits, and for proper design, simple properties such as finite speed of propagation must be taken into account. The approach here is to view all lines as transmission lines, define their properties, and then see what the connection between transmission lines and circuits is. We will find that circuits are essentially transmission lines in which the distances are so short as to allow us to neglect the finite speed of propagation at the operating frequency. This, however, can only be correct at relatively low frequencies. In the case of the above power transmission line, a few kilometers is a very short distance because the wavelength is 5,000 km. On the other hand, at 1 GHz, the wavelength is only 0.3 m. Any circuit connection longer than a few centimeters will be a "long" connection and propagation effects cannot be neglected.

Thus, you may view the theory of transmission lines as a more general approach to treatment of transfer of energy on lines. Many of the methods used here will be familiar from circuit theory, and in most cases, the results will be in terms of voltages and currents on the line. The connection between voltages and currents and electric and magnetic fields also affords a different point of view of the field variables.

Example 14.1 Two memory boards are connected to the processor of a computer. Suppose we wish to add two values placed in the two memory banks. One memory bank is 0.1 m from the processor and the second is 0.15 m away. The processor can add the two values in 1 ns (a 1 GHz computer). Propagation on the transmission lines connecting the processor to memory banks is at $0.2c$ [m/s] (typical of copper lines). What is the minimum time needed for computation?

Solution: The total time of computation is the time needed for propagation (rounded to the nearest cycle since all computation is done in cycles) plus the time needed by the processor. In this case, computation can only start after the signal on the longer line reaches the processor. A delay of $0.05/0.2c$ must also be introduced on the shorter line so that both signals from memory reach the processor at the same time.

The delay due to propagation is

$$\Delta t = 0.15/0.2c = 0.15/\left(0.2 \times 3 \times 10^8\right) = 2.5 \times 10^{-9} \quad [\mathrm{s}]$$

Thus, the time required for computation is four cycles or 4 ns instead of 1 ns. In effect, the computer has been slowed down by a factor of 4. There are many ways of dealing with this problem, including shorter lines, cache memory on the processor chip, and propagation while the processor performs other tasks and other methods of scheduling.

14.2 The Transmission Line

What then is a transmission line? Well, it is no more than a physical connection between two locations through two conductors. We must indicate at the outset that any transmission of energy through conducting or nonconducting media may be considered a transmission line. Also, any guiding of energy by physical structures may be included in this general definition. However, we will restrict our discussion here to conducting lines with the following properties:

(1) The transmission line is made of two conductors in any configuration.
(2) The electric and magnetic field intensities in the line are perpendicular to each other and perpendicular to the direction of propagation of power. This type of propagation was defined in **Chapter 12** as transverse electromagnetic (TEM) propagation and has all the properties of plane waves.

Examples of lines that we may consider are parallel conducting wires such as the two-wire power cable used to power your toaster or the overhead power transmission line made of thick cables and suspended from towers. Similarly, a twisted pair of wires as used in some telephone lines is of this type. These three transmission lines are shown in **Figures 14.3a–14.3c**. Another common type of transmission line is the coaxial transmission line shown in **Figure 14.3d** (also discussed in **Chapter 9**). It is made of two coaxial conductors: an inner, thin, solid conductor and an outer hollow cylindrical conductor. The latter is usually stranded to allow flexibility and the two conductors are insulated with some dielectric material. Dimensions of coaxial cables and their properties vary, but a good example of an often used coaxial cable is the antenna cable on televisions, input cables for oscilloscopes, or input leads in audio equipment.

A third type of transmission line which we will concern ourselves with is the parallel strip line shown in **Figure 14.3e**. This line may be made of two strips, very close to each other, such as strips on printed circuit boards or of two parallel plates.

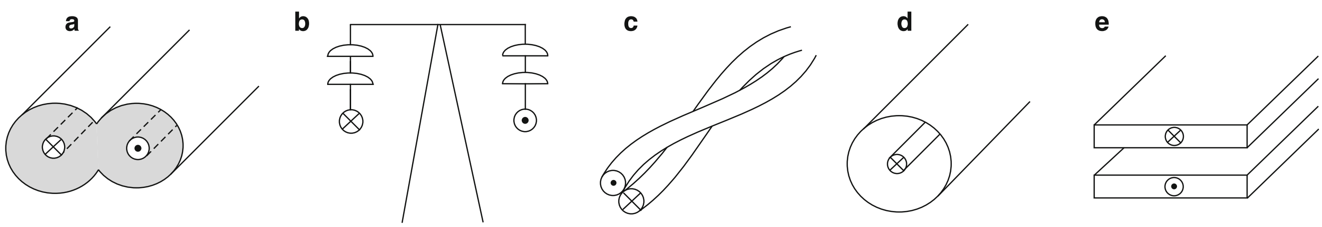


Figure 14.3 (**a**) Simple two-lead cable. (**b**) Overhead power line. (**c**) Twisted pair. (**d**) The coaxial transmission line. (**e**) Parallel plate transmission line (strip line)

Although each line has its own properties and parameters, our discussion will be general and will encompass all lines that satisfy the above requirements. In doing so, we first discuss infinite lines, followed by finite, load terminated lines. The lossless (ideal) line is discussed first since it is the simplest, followed by lossy or attenuating lines. In terms of sources connected to the line, we start with steady state sources but will also discuss transients and the effect of line parameters on propagation of these transients.

14.3 Transmission Line Parameters

A transmission line has three types of parameters:

(1) Dimensional parameters: These include length, dimensions of each conductor (thickness, width, diameter, etc.), spacing between lines, thickness of insulation, and the like. These parameters define the physical configuration of the line but also play a role in defining its electrical properties.
(2) Material parameters: The line is made of conductors and insulators. The electrical properties of these materials are their conductivities, permittivities, and permeabilities. These obviously affect the way a line performs its task.
(3) Electric parameters: These are the resistance, capacitance, inductance, and conductance per unit length of the line. Although we could calculate these parameters for the whole line (lumped parameters), we will have little use for lumped parameters. The reason for this was hinted at in the Introduction: The voltage and current vary along the line, making the use of lumped parameters useless. Instead, we will use distributed parameters. The four line parameters are as follows:

R: Series resistance of the line in ohms per unit length [Ω/m].
L: Series inductance of the line in henrys per unit length [H/m].
C: Shunt capacitance of the line in farads per unit length [F/m].
G: Shunt conductance of the line in siemens per unit length [S/m])

Before we discuss the properties of transmission lines, it is important to be able to define the various line parameters. These are evaluated from known electromagnetic relations and we, in fact, have performed these tasks in previous chapters. However, we will repeat the steps involved in these calculations here to review the principles involved. To do so, we consider as an example, the parallel plate transmission line in **Figure 14.3e**. The line is very long but we will evaluate the parameters for a length $l = 1$ m. The procedure given here is rather general and applies to many transmission lines although the details of evaluation of the expressions for different lines vary.

14.3.1 Calculation of Line Parameters

14.3.1.1 Resistance per Unit Length

Any transmission line, made of conducting materials, has a finite resistance because of the finite conductivity of the material. However, the resistance we usually use as series resistance in transmission lines is different than that obtained for a conductor at DC. The reason for this is shown in **Figure 14.4**. **Figure 14.4a** shows a conductor through which a direct current flows. The current density in the conductor is uniform and independent of material properties except conductivity. On the other hand, in **Figure 14.4b**, the current is sinusoidal at frequency f. Now, the current density depends on the skin depth. It decays exponentially from the surface inward, as we have seen in **Chapter 12**. For practical purposes, and especially at high frequencies, only a small depth of the material contains current (at a depth of 5 skin depths, the current density is less than 0.7 % the current density at the surface). At the frequencies normally used in transmission lines and in good conductors, this depth is often only a few micrometers. Therefore, we will call this current a surface current and the current density it produces, a surface current density. The resistance we need to worry about is, therefore, a surface resistance in the sense that only a small volume close to the surface contributes to this resistance. This resistance is the series resistance we need. To see how the series resistance R_s can be calculated for the parallel plate transmission line in **Figure 14.3e**, we use **Figure 14.4c**. The current in the lower conductor flows in the positive z direction. We will also assume initially that the thickness t tends to infinity, calculate the surface resistance per unit length of the lower plate, and then multiply the value of this resistance by 2 to obtain the total surface resistance per unit length of the line.

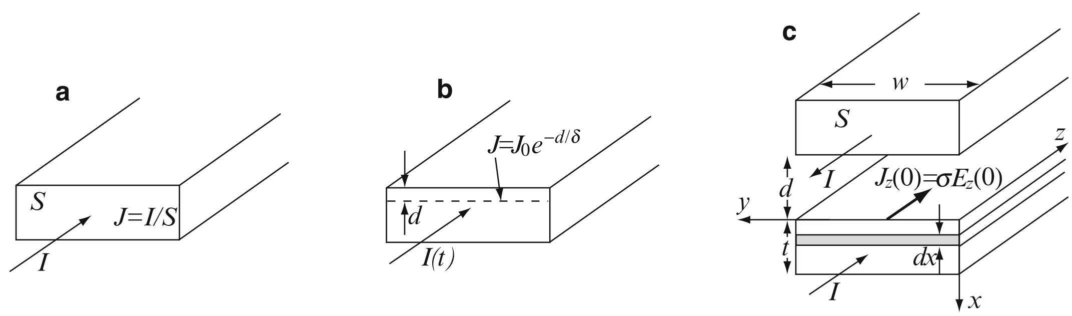


Figure 14.4 (**a**) Uniform DC current distribution in a conductor. (**b**) The AC distribution decays exponentially from the surface. (**c**) Calculation of the series resistance in a parallel plate transmission line based on the AC current distribution

For a current density to exist in the conductor, there must be an electric field intensity E_z inside the conductor (in the direction of flow of current) for any conductor except a perfect conductor. The electric field intensity inside the conductor decays exponentially with depth; that is, a wave propagating into the conductor produces an electric field:

$$E_z(x) = E_z(0)e^{-\alpha x}e^{-j\beta x} = E_z(0)e^{-x/\delta}e^{-jx/\delta} = E_z(0)e^{-(1+j)x/\delta} \quad [\text{V/m}] \qquad (14.1)$$

where $E_z(0)$ is the electric field intensity at the surface ($x = 0$) of the conductor. The attenuation and phase constants for the given conductor are α and β, respectively, and the skin depth is δ. For a good conductor, the constants α, β, and δ are [see **Eqs. (12.111)** and **(12.113)**]

$$\alpha = \sqrt{\pi f \mu_c \sigma_c} \quad \left[\frac{\text{Np}}{\text{m}}\right], \quad \beta = \sqrt{\pi f \mu_c \sigma_c} \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad \delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} \quad [\text{m}] \qquad (14.2)$$

In these relations, σ_c and μ_c indicate conductivity and permeability, respectively, of the conducting material to distinguish them from the material properties of the dielectric between the conductors.

The current density in the conductor is

$$J_z(x) = \sigma_c E_z(x) = \sigma_c E_z(0) e^{-(1+j)x/\delta} \quad \left[\text{A/m}^2\right] \tag{14.3}$$

To calculate the total current in the lower conductor, we note that the current density only varies with depth. Thus, an element of current $dI = J(x)wdx$ (**Figure 14.4c**) is defined. This is now integrated over the thickness of the conductor, which we took to be infinitely thick:

$$I = w\int_{x=0}^{x=\infty} J_z(x)dx = w\sigma_c E_z(0)\int_{x=0}^{x=\infty} e^{-(1+j)x/\delta}dx = \frac{w\sigma_c \delta E_z(0)}{1+j} \quad [\text{A}] \tag{14.4}$$

Note that the thickness t can be taken to be infinite since the current density after about 10 skin depths is so small as to contribute almost nothing to the total current. This approximation is permissible for any conductor which is thick compared to the skin depth.

Impedance of a length $l = 1$ m (the line is directed in the z direction) of the conductor is, by definition,

$$Z = \frac{V}{I} = \frac{1}{I}\int_{z=0}^{z=l} \mathbf{E} \cdot d\mathbf{l} = \int_{z=0}^{z=l} \left(\frac{E_z}{I}\right) dz \quad [\Omega] \tag{14.5}$$

where E_z is the electric field intensity in the direction of the current. The impedance per unit length is the ratio between the tangential component of the electric field intensity [V/m] at the surface of the conductor and the surface current [A], shown in parentheses in **Eq. (14.5)**:

$$Z_s = \frac{E_z(0)}{I} = \frac{E_z(0)(1+j)}{w\delta\sigma_c E_z(0)} = \frac{(1+j)}{w\delta\sigma_c} = \frac{1}{w}\left(\frac{1}{\delta\sigma_c} + \frac{j}{\delta\sigma_c}\right) \quad \left[\frac{\Omega}{\text{m}}\right] \tag{14.6}$$

where I from **Eq. (14.4)** was used. The expression in parentheses contains only quantities related to material properties, which are independent of dimensions and have units of [Ω]. The real part of this relation is the surface resistance of the conductor and is independent of dimensions—it is a property of the conductor:

$$\boxed{R_s = \frac{1}{\sigma_c \delta} \quad [\Omega]} \tag{14.7}$$

Multiplying the impedance in **Eq. (14.6)** by 2 to take into account the upper conductor, we obtain the resistance per unit length of the line:

$$\boxed{R = \frac{2}{w\sigma_c\delta} = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad \left[\frac{\Omega}{\text{m}}\right]} \tag{14.8}$$

The imaginary part of the surface impedance is due to the inductive nature of the conductor. We can write the surface inductance L_s and the inductance per unit length L as

$$L_s = \frac{1}{\sigma_c\delta\omega} \quad [\text{H}], \quad X = \frac{j}{w\sigma_c\delta} = j\omega L \quad \rightarrow \quad L = \frac{1}{w\sigma_c\delta\omega} \quad \left[\frac{\text{H}}{\text{m}}\right] \tag{14.9}$$

The inductance L obtained here is an internal series inductance per unit length of the lower conductor and *should not* be confused with the inductance per unit length of the transmission line. The latter is the external inductance which we will calculate shortly. The internal inductance is quite small, especially at very high frequencies and in good conductors. For this reason, we normally neglect this term in the analysis of transmission lines.

14.3.1.2 Inductance per Unit Length

The inductance per unit length of any transmission line can be calculated by calculating the magnetic flux density due to an assumed current in the line, calculating the total flux linkage with the line per unit length, and then dividing by the current to obtain the inductance. This method was described in detail in **Section 9.4**. To calculate the inductance per unit length, we use again the geometry in **Figure 14.4c**. First, we calculate the magnetic flux density from Ampere's law. Because $w \gg d$, the magnetic flux density between the plates can be assumed to be uniform and parallel to the plates **(Figure 14.5a)**. A contour is drawn around one of the conductors, as shown in **Figure 14.5a**. The magnetic field intensity outside the plates is zero since a contour enclosing both conductors encloses a zero net current. Thus, only the path section contained between the conductors ($a - d$) contributes to the flux density. From Ampere's law,

$$I = \oint \mathbf{H} \cdot d\mathbf{l} = Hw \quad \rightarrow \quad H = \frac{I}{w} \quad \rightarrow \quad B = \frac{\mu I}{w} \quad [\mathrm{T}] \tag{14.10}$$

where $\mathbf{B} = \mu\mathbf{H}$ was used and μ is the permeability of the dielectric material between the conducting plates.

Now, we need to calculate the total flux linkage. Since there is only one closed circuit (out of which we only calculate the flux for a 1 m section), the total flux contained between the two conductors is the flux linkage of the segment. This flux is calculated by integrating the flux density over the shaded area in **Figure 14.5b**, which shows the transmission line from a side view. Because the flux density is uniform, this gives

$$\Phi = \Lambda = BS = Bd1 = \frac{\mu I d}{w} \quad [\mathrm{Wb}] \tag{14.11}$$

Dividing by I gives the inductance per unit length of the transmission line:

$$\boxed{L = \frac{\Lambda}{I} = \frac{\mu d}{w} \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]} \tag{14.12}$$

The latter is only the external inductance per unit length of the transmission line. In general, we assume that the internal inductance given in **Eq. (14.9)** is small compared with the external inductance in **Eq. (14.12)** and, therefore, the internal inductance is usually neglected.

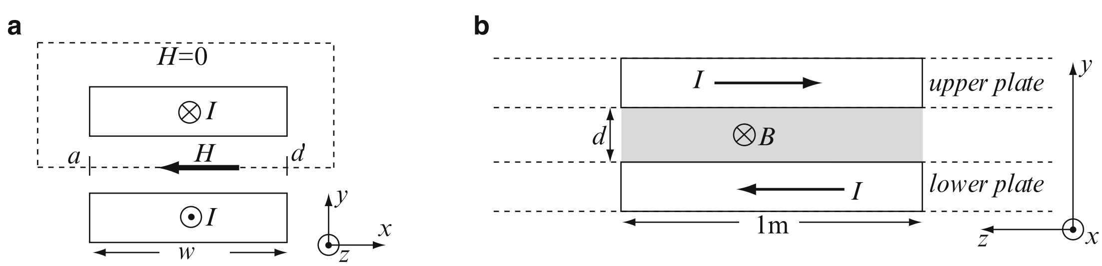


Figure 14.5 Calculation of inductance per unit length of the parallel plate transmission line. (**a**) Calculation of magnetic field intensity between the plates. (**b**) Calculation of flux per unit length

14.3.1.3 Capacitance per Unit Length

Capacitance of any system of two conductors is calculated by assuming a given charge or charge density on one conductor, equal and opposite charge on the second conductor, and then calculating the potential difference between the two conductors. From the calculated potential difference and charge, the capacitance is calculated as $C = Q/V$. This method of computation was discussed in detail in **Section 4.7.2**. We assume a total charge Q is uniformly distributed on the inner surface of the upper conductor and a total charge $-Q$ is uniformly distributed on the inner surface of the lower conductor. This forms a capacitor, with two plates, each of length 1 m and width w (**Figure 14.6**). Thus, the surface charge densities are Q/w on the upper conductor and $-Q/w$ on the lower conductor. Assuming no fringing ($w \gg d$), the use of Gauss's law gives

$$E = \frac{Q}{w\varepsilon} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{14.13}$$

where the Gaussian surface is shown in **Figure 14.6** and ε is the permittivity of the material between the conducting plates. The charge only exists on the inner surfaces of the conductors and we have also taken into account the fact that the electric field intensity is zero outside the plates. The potential difference between the plates is

$$|V| = \int_0^d E dl = \frac{Qd}{w\varepsilon} \quad [\text{V}] \tag{14.14}$$

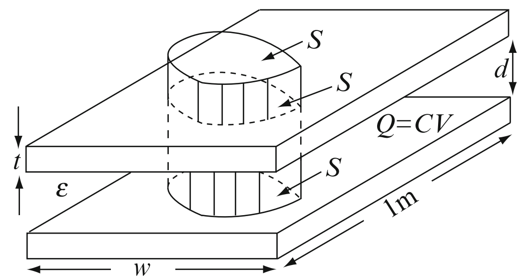


Figure 14.6 Calculation of the electric field intensity between the plates of the transmission line using Gauss's law with an assumed surface charge density on the upper plate equal to Q/w

The capacitance per unit length is therefore

$$\boxed{C = \frac{Q}{V} = \frac{w\varepsilon}{d} \quad \left[\frac{\text{C}}{\text{m}}\right]} \tag{14.15}$$

The same result may be obtained from the formula for parallel plate capacitors, but the method given here is more general.

14.3.1.4 Conductance per Unit Length

Conductance is calculated in a manner similar to that for calculation of resistance in **Chapter 7**. We assume the material between the plates has a uniform conductivity σ and apply a known, arbitrary potential difference between the plates which generates a uniform electric field intensity $E = V/d$. This gives rise to a current density $\mathbf{J} = \sigma\mathbf{E}$. The total current is then calculated by integrating $\mathbf{J}$ over the area of the plates ($w \times 1$). From Ohm's law, we can now calculate the resistance and its reciprocal is the conductance G. To outline the method, consider **Figure 14.7**. The potential V produces an electric field and current density:

$$E = \frac{V}{d} \quad \rightarrow \quad J = \sigma E = \sigma\frac{V}{d} \quad \left[\frac{\text{A}}{\text{m}^2}\right] \tag{14.16}$$

This current density flows from the upper plate to the lower plate, and is uniform between the plates. Thus, the total current is this current density multiplied by the area of the plate:

$$I = JS = \sigma\frac{V}{d}w \quad [\text{A}] \tag{14.17}$$

Assuming the unknown resistance between the plates is R, we can write from Ohm's law

$$V = IR = R\sigma\frac{V}{d}w \quad \rightarrow \quad R = \frac{d}{w\sigma} \quad [\Omega] \tag{14.18}$$

The conductance per unit length is therefore

$$G = \frac{1}{R} = \frac{w\sigma}{d} \quad \left[\frac{\text{S}}{\text{m}}\right] \tag{14.19}$$

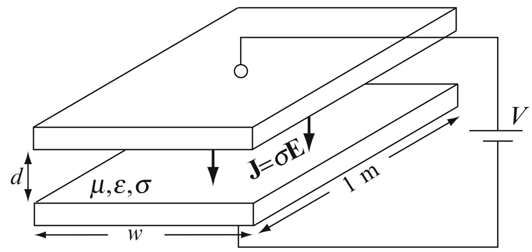


Figure 14.7 Calculation of conductance per unit length by calculating the resistance between upper and lower plates

The methods outlined in **Sections 14.3.1.1** through **14.3.1.4** are completely general and apply equally well to other types of transmission lines. **Table 14.1** shows the parameters of a number of transmission lines, including the parallel plate transmission line evaluated here.

Table 14.1 Transmission line parameters for some common transmission lines

Two-wire line (**Figure 14.3a**).	Coaxial line (**Figure 14.3d**).	Parallel plate line (**Figure 14.3e**).
a = radius of conductor, d = distance between centers of conductors.	a = radius of inner conductor, b = inner radius of outer conductor.	w = width of plates, d = distance between plates.
$R = \dfrac{1}{\pi a \delta \sigma_c}$	$R = \dfrac{1}{2\pi\delta\sigma_c}\left[\dfrac{1}{a} + \dfrac{1}{b}\right]$	$R = \dfrac{2}{w\delta\sigma_c} \quad \left[\dfrac{\Omega}{\text{m}}\right]$
$L = \dfrac{\mu}{\pi}\cosh^{-1}\dfrac{d}{2a}$	$L = \dfrac{\mu}{2\pi}\ln\dfrac{b}{a}$	$L = \dfrac{\mu d}{w} \quad \left[\dfrac{\text{H}}{\text{m}}\right]$
$G = \dfrac{\pi\sigma}{\cosh^{-1}(d/2a)}$	$G = \dfrac{2\pi\sigma}{\ln(b/a)}$	$G = \dfrac{\sigma w}{d} \quad \left[\dfrac{\text{S}}{\text{m}}\right]$
$C = \dfrac{\pi\varepsilon}{\cosh^{-1}(d/2a)}$	$C = \dfrac{2\pi\varepsilon}{\ln(b/a)}$	$C = \dfrac{w\varepsilon}{d} \quad \left[\dfrac{\text{F}}{\text{m}}\right]$

Note: If $(d/2a)^2 \gg 1$, $\cosh^{-1}(d/2a) \approx \ln(d/a)$. For widely separated, two-wire, thin lines, this approximation can be used to simplify the expressions. σ_c and μ_c are the conductivity and permeability of the conductor, respectively. σ, μ, and ε are the properties of the dielectric between the conductors.

In summary, the transmission line parameters are evaluated as any other lumped circuit parameters for a line of unit length.

Example 14.2 A two-wire transmission line is made of two bare, round wires and operates at 400 Hz. The conductors are made of copper and placed in free space. The two wires need to pass through a very thick wall made of alumina and into a hot oven to connect to a temperature sensor. Because copper does not withstand high temperatures very well, the section inside the wall is made of tungsten but with identical dimensions to the outside wire. The geometry is shown in **Figure 14.8**.

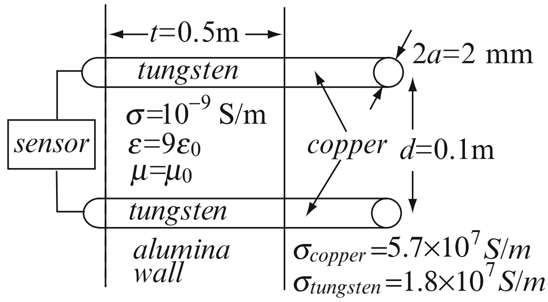


Figure 14.8 Two transmission line segments with different properties, connected in series

Calculate the line parameters for the segment inside the wall and the line outside the wall.

Solution: The line parameters of both sections are given in column 1 of **Table 14.1**, but they have different values.

(1) Copper line in air:

$$R = \frac{1}{\pi a \delta \sigma_c} = \frac{\sqrt{\pi f \mu_0 \sigma_c}}{\pi a \sigma_c} = \frac{1}{a}\sqrt{\frac{f \mu_0}{\pi \sigma_c}} = \frac{1}{0.001}\sqrt{\frac{400 \times 4 \times \pi \times 10^{-7}}{\pi \times 5.7 \times 10^7}} = 1.675\pi \times 10^{-3} \quad \left[\frac{\Omega}{\text{m}}\right]$$

Where $\delta = 1/\sqrt{\pi f \mu_0 \sigma_c}$, μ_0 is the permeability and σ_c is the conductivity of copper.

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a} \approx \frac{\mu_0}{\pi}\ln\frac{d}{a} = \frac{4 \times \pi \times 10^{-7}}{\pi}\ln\frac{0.1}{0.001} = 1.842 \quad \left[\frac{\mu\text{H}}{\text{m}}\right]$$

where the approximation $\cosh^{-1}(d/2a) \approx \ln(d/a)$ was used (since $(d/2a)^2 = 50^2 \gg 1$). The same approximation is used for the calculation of capacitance and conductance per unit length.

$$G = \frac{\pi\sigma}{\cosh^{-1}(d/2a)} \approx \frac{\pi\sigma}{\ln(d/a)} = 0 \quad \left[\frac{\text{S}}{\text{m}}\right]$$

The conductance in air is zero because conductivity of air is zero:

$$C = \frac{\pi\varepsilon}{\cosh^{-1}(d/2a)} \approx \frac{\pi\varepsilon_0}{\ln(d/a)} = \frac{\pi \times 8.854 \times 10^{-12}}{\ln 100} = 6.04 \quad \left[\frac{\text{pF}}{\text{m}}\right]$$

(2) Tungsten line in alumina

$$R = \frac{1}{a}\sqrt{\frac{f\mu_0}{\pi\sigma_t}} = \frac{1}{0.001}\sqrt{\frac{400 \times 4 \times \pi \times 10^{-7}}{\pi \times 1.8 \times 10^7}} = 2.98 \times 10^{-3} \quad \left[\frac{\Omega}{\text{m}}\right]$$

$$L \approx \frac{\mu_0}{\pi}\ln\frac{d}{a} = 1.842 \quad \left[\frac{\mu\text{H}}{\text{m}}\right]$$

$$G \approx \frac{\pi\sigma_a}{\ln(d/a)} = \frac{\pi \times 10^{-9}}{\ln 100} = 6.822 \times 10^{-10} \quad \left[\frac{\text{S}}{\text{m}}\right]$$

$$C \approx \frac{\pi 9\varepsilon_0}{\ln(d/a)} = \frac{\pi \times 9 \times 8.854 \times 10^{-12}}{\ln 100} = 54.36 \quad \left[\frac{\text{pF}}{\text{m}}\right]$$

Thus, while the inductance per unit length remains the same (because permeability of alumina and that of air are the same), all other parameters are different. We, therefore, expect these two line segments to have different properties, including different speeds of propagation. We shall see a little later that the speed of propagation in the tungsten segment is slower than in the copper segment because of the higher permittivity of alumina.

Exercise 14.1

(a) Calculate the line parameters for a coaxial line with inner radius $a = 1$ mm, outer radius $b = 4$ mm, and material parameters as in **Figure 14.9**. The line operates at 60 Hz.
(b) Does it matter how thick the outer conductor is? Assume frequency is 1 MHz and the material is copper.

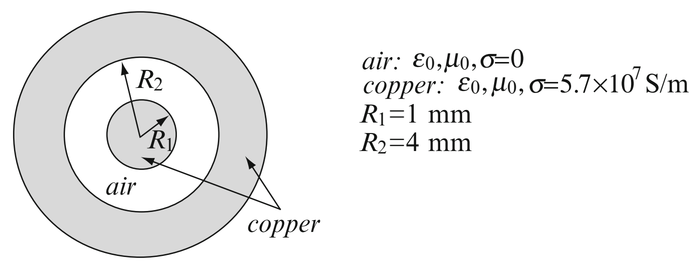


Figure 14.9 Cross section and properties of a coaxial transmission line

Answer **(a)** $R = 4.1 \times 10^{-4}$ Ω/m, $L = 0.277$ μH/m, $G = 0$, $C = 40.1$ pF/m. **(b)** Yes, Calculation of resistance per unit length assumes conductors are thick compared to skin depth.

14.4 The Transmission Line Equations

As discussed above, the lumped parameter approach to transmission lines is not feasible. Instead, we define the transmission line equations using a distributed parameter approach. The transmission line is viewed as being made of a large number of short segments, each of length Δl as shown in **Figure 14.10** which also shows the parameters of one segment. In this notation, $R\Delta l$ is the resistance of the line of length Δl, $L\Delta l$ is the inductance, $C\Delta l$ is the capacitance, and $G\Delta l$ is the conductance, where R, L, C, and G are given per unit length. The total series impedance of the line segment is therefore

$$Z = R\Delta l + j\omega L\Delta l \quad [\Omega] \tag{14.20}$$

and the parallel line admittance is

$$Y = G\Delta l + j\omega C\Delta l \quad [1/\Omega] \tag{14.21}$$

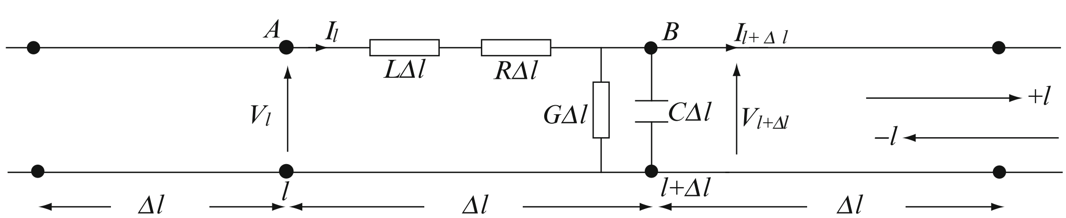


Figure 14.10 A transmission line viewed as a distributed parameter circuit built of segments of arbitrary but small length Δl One segment is shown in detail. Note the general direction l. Later, we will replace this with a specific coordinate

These parameters can now be used to build a transmission line of any length, as shown in **Figure 14.10**. The Δ notation was used to indicate that the segment of line used is arbitrary but must be small compared to wavelength. The circuit equations are written using Kirchhoff's laws for one of the segments to obtain the transmission line equations, assuming for the moment that both current and voltage are phasors. The voltage across the line segment of length Δl can be written in terms of the voltages at points A and B and the current in the segment. With the notation in **Figure 14.10**, we have,

$$V(l + \Delta l) - V(l) = -I(l)\left[R\Delta l + j\omega L\Delta l\right] \quad [\text{V}] \tag{14.22}$$

Dividing both sides by Δl

$$\frac{V(l+\Delta l)-V(l)}{\Delta l}=-I(l)[R+j\omega L] \tag{14.23}$$

The term on the left-hand side becomes the derivative of V with respect to l if we let Δl tend to zero. Thus, since Δl is arbitrarily small, we may write

$$\boxed{\frac{dV(l)}{dl}=-I(l)[R+j\omega L]} \tag{14.24}$$

This relation holds at any point on the line. Similarly, the current in the segment can be written in terms of the current at points A and B and the voltage at point B as

$$I(l+\Delta l)-I(l)=-V(l+\Delta l)[G\Delta l+j\omega C\Delta l] \quad [\mathrm{A}] \tag{14.25}$$

Following steps identical to **Eqs. (14.23)** and **(14.24)**, we get

$$\frac{dI(l)}{dl}=-V(l+\Delta l)[G+j\omega C] \tag{14.26}$$

To obtain an equation of the same form as for the voltage in **Eq. (14.24)**, we expand the term $V(l + \Delta l)$ in a Taylor series about l as $V(l + \Delta l) = V(l) + (dV(l)/dl)\Delta l/1! + (d^2V(l)/dl^2)(\Delta l)^2/2! + \ldots$ Neglecting all terms that contain Δl gives an approximation $V(l + \Delta l) \approx V(l)$. Substitution of this in **Eq. (14.26)** gives

$$\boxed{\frac{dI(l)}{dl}=-V(l)[G+j\omega C]} \tag{14.27}$$

The transmission line equations are the current and voltage relations in **Eqs. (14.24)** and **(14.27)**. These are two coupled first-order differential equations. Before attempting to solve for current and voltage, we can eliminate one of the variables and obtain separate equations for $V(l)$ and $I(l)$. To do so, we substitute $I(l)$ from **Eq. (14.24)** into **Eq. (14.27)** and $V(l)$ from **Eq. (14.27)** into **Eq. (14.24)**. From **Eq. (14.24)**,

$$I(l)=-\frac{dV(l)}{dl}\frac{1}{[R+j\omega L]} \quad [\mathrm{A}] \tag{14.28}$$

Substitution of this into **Eq. (14.27)** gives

$$\boxed{\frac{d^2V(l)}{dl^2}-V(l)[G+j\omega C][R+j\omega L]=0} \tag{14.29}$$

Similarly, substituting $V(l)$ from **Eq. (14.27)** into **Eq. (14.24)**, we get

$$\boxed{\frac{d^2I(l)}{dl^2}-I(l)[G+j\omega C][R+j\omega L]=0} \tag{14.30}$$

These two equations are wave equations of the same form as given in **Eq. (12.84)** for the electric field intensity **E** (see **Section 12.7.1**). In fact, we can rewrite **Eqs. (14.29)** and **(14.30)** as

$$\frac{d^2V}{dl^2}-\gamma^2V=0 \tag{14.31}$$

and

$$\frac{d^2I}{dl^2}-\gamma^2I=0 \tag{14.32}$$

where

$$\boxed{\gamma = \alpha + j\beta = \sqrt{[G + j\omega C][R + j\omega L]}} \tag{14.33}$$

The first of these is the wave equation for the voltage on the line and the second is the wave equation for current in the line. Therefore, γ is the propagation constant in analogy with the definition of the propagation constant in **Chapter 12 [Eq. (12.83)]**. This is fortunate because we can now use the solutions obtained in **Chapter 12** for plane waves. In fact, all we have to do is replace the electric field intensity in **Eq. (12.88)** by the voltage $V(l)$, the magnetic field intensity by $I(l)$, and the constant of propagation γ, by the term in **Eq. (14.33)**.

The propagation constant in **Eq. (14.33)** is complex. α is the attenuation constant along the line and β is the phase constant. The attenuation constant is given in nepers/m and the phase constant in radians/meter.

Based on the form of these equations and the similarity to the equations for plane waves **[Eq. (12.82)]**, we can now solve them by simply performing the above substitutions and using the solutions for plane waves. Thus, for the general transmission line described here, the solution for voltage and current can be written with the aid of **Eq. (12.88)** as

$$\boxed{V(l) = V^+ e^{-\gamma l} + V^- e^{\gamma l} \quad [\text{V}]} \tag{14.34}$$

$$\boxed{I(l) = I^+ e^{-\gamma l} + I^- e^{\gamma l} \quad [\text{A}]} \tag{14.35}$$

Direct substitution of these solutions into **Eqs. (14.31)** and **(14.32)** shows they are correct. The solution to these equations has two parts: one propagating in the positive l direction, the other in the negative l direction, along the line, exactly as for plane waves. V^+ and V^- are the amplitudes of the voltage waves propagating in the positive and negative l directions, respectively. For the current solution, I^+ and I^- are the respective amplitudes of the current waves. The amplitudes of the forward and backward propagating waves, V^+ and V^-, can be calculated from the terminal voltages on the transmission line as we shall see shortly.

It is interesting to note here that whereas plane waves were a convenient simplification for wave propagation, their use in transmission line is exact; that is, the waves in transmission line behave exactly as plane waves.

So far, we have defined one characteristic quantity of the line: the propagation constant in **Eq. (14.33)**. Now that we obtained the voltages and currents on the line, we can define the second characteristic quantity of any transmission line: the characteristic line impedance.

The ***characteristic line impedance*** Z_0 of a transmission line is defined as the ratio between the forward-propagating voltage amplitude and the forward-propagating current amplitude:

$$\boxed{Z_0 = \frac{V^+}{I^+} \quad [\Omega]} \tag{14.36}$$

To evaluate the characteristic impedance in terms of the line parameters (since these are known and independent of line current), we substitute the general solution from **Eqs. (14.34)** and **(14.35)** into the transmission line relation in **Eqs. (14.24)** and **(14.27)**. Starting with **Eq. (14.24)**, we get

$$\frac{d\left(V^+ e^{-\gamma l} + V^- e^{\gamma l}\right)}{dl} = -\left(I^+ e^{-\gamma l} + I^- e^{\gamma l}\right)[R + j\omega L] \tag{14.37}$$

or, after evaluating the derivatives,

$$-\gamma V^+ e^{-\gamma l} + \gamma V^- e^{\gamma l} = -\left(I^+ e^{-\gamma l} + I^- e^{\gamma l}\right)[R + j\omega L] \tag{14.38}$$

Similarly, using **Eq. (14.27)**, we get

$$-\gamma I^+ e^{-\gamma l} + \gamma I^- e^{\gamma l} = -\left(V^+ e^{-\gamma l} + V^- e^{\gamma l}\right)[G + j\omega C] \tag{14.39}$$

Now, suppose, first, that only a forward-propagating wave exists by setting $V^- = 0, I^- = 0$ in **Eqs. (14.38)** and **(14.39)**. We get

$$\boxed{-\gamma V^{+}e^{-\gamma l}=-I^{+}e^{-\gamma l}[R+j\omega L] \quad \text{and} \quad -\gamma I^{+}e^{-\gamma l}=-V^{+}e^{-\gamma l}[G+j\omega C]} \tag{14.40}$$

Thus, the characteristic impedance can be written as

$$Z_0=\frac{V^{+}}{I^{+}}=\frac{R+j\omega L}{\gamma}=\frac{\gamma}{G+j\omega C} \quad [\Omega] \tag{14.41}$$

The first form is obtained from the first expression in **Eq. (14.40)** and the second from the second expression. Also, by substituting for γ from **Eq. (14.33)**, we obtain

$$\boxed{Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad [\Omega]} \tag{14.42}$$

Now suppose that only a backward-propagating wave exists. By setting $V^{+}=0$, $I^{+}=0$ in **Eqs. (14.38)** and **(14.39)**, we get

$$\gamma V^{-}e^{\gamma l}=-I^{-}e^{\gamma l}[R+j\omega L] \quad \text{and} \quad \gamma I^{-}e^{\gamma l}=-V^{-}e^{\gamma l}[G+j\omega C] \tag{14.43}$$

Dividing each of these two equations by I^{-}, we can write

$$\frac{V^{-}}{I^{-}}=-\frac{R+j\omega L}{\gamma}=-\frac{\gamma}{G+j\omega C}=-Z_0 \tag{14.44}$$

We can summarize these results as follows:

$$Z_0=\frac{V^{+}}{I^{+}}=-\frac{V^{-}}{I^{-}}=\frac{R+j\omega L}{\gamma}=\frac{\gamma}{G+j\omega C}=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad [\Omega] \tag{14.45}$$

The characteristic impedance Z_0 is independent of location on the line and only depends on line parameters. Thus, the name characteristic impedance. The characteristic impedance is, in general, a complex value. However, whereas all other line parameters are given in per meter units, the characteristic impedance is a line property, independent of length. In other words, for any given line, if we were to measure the characteristic impedance, the above value would be obtained for any length of line and at any location on the line.

Using **Eq. (14.45)**, the line current given in **Eq. (14.35)** can be written as

$$I(l)=\frac{V^{+}}{Z_0}e^{-\gamma l}-\frac{V^{-}}{Z_0}e^{\gamma l} \quad [\text{A}] \tag{14.46}$$

Finally, we also mention that the wavelength and phase velocity for any propagating wave are given as

$$\boxed{\lambda=\frac{2\pi}{\beta} \quad [\text{m}]} \quad \boxed{v_p=\frac{\omega}{\beta} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{14.47}$$

The quantity βl has units of radians. It is called the ***electrical length*** of the line and may be considered an additional line parameter.

The discussion in this section assumed time-harmonic quantities. This was done on purpose, since phasor calculations are usually simpler to perform and the final result is also simpler. More important, this choice allowed us to use the results already obtained for transverse electromagnetic wave propagation. In turn, this choice shows that propagation along transmission lines is similar to transmission in free space and other materials, as long as the basic assumptions of transverse electromagnetic waves are satisfied. Both plane waves in materials and waves in transmission lines satisfy these conditions. Thus, we can expect that other parameters such as reflection and transmission of energy as well as the reflection and transmission coefficients should be similar. We will discuss these topics separately.

Instead of using the time-harmonic forms for voltage and current, we could start with the time-dependent voltage and current to obtain the time-dependent transmission line equations following essentially identical steps as above (see **Exercises 14.2** and **14.3**). However, we will not use the time-dependent transmission line equations in this and the following chapter. One reason for this is that many of the properties we require, including phase and attenuation constants, wavelength, wave number, and the like, can only be properly defined for time-harmonic fields. However, we will take up the issue of time-dependent behavior in **Chapter 16** when we discuss transients on transmission lines and the following section shows the transmission line in the time domain for the sake of completeness.

14.4.1 Time-Domain Transmission Line Equations

Although we will use the transmission line equations exclusively in the frequency domain, it is nevertheless useful to derive here the time-domain transmission line equations. One can envision the use of the time-domain equations in instances when a single frequency cannot describe the behavior of the line. However, it should also be remembered that the line parameters themselves are frequency dependent (see, for example, the expression for R in **Table 14.1**) and a complete, exact analysis in the time domain is rather difficult. In most cases, it is easier to transform the time-domain signal into the frequency domain and analyze the transmission line at the individual harmonics, recalculating the line parameters at each harmonic if necessary. However, if we assume the line parameters to be constants, analysis in the time domain is possible.

To obtain the time domain transmission line equations we start with **Figure 14.10** but with time-dependent voltages $V(l,t)$, $V(l+\Delta l,t)$ $I(l,t)$, and $I(l+\Delta l,t)$. With these and with the fact that the potential across an inductor is $LdI(l,t)/dt$ and the current in a capacitor is $CdV(l,t)/dt$, we write by applying Kirchoff's laws:

$$V(l+\Delta l,t) - V(l,t) = -I(l,t)R\Delta l - L\Delta l\frac{dI(l,t)}{dt} \quad [\mathrm{V}] \tag{14.48}$$

$$I(l+\Delta l,t) - I(l,t) = -V(l,t)G\Delta l - C\Delta l\frac{dV(l,t)}{dt} \quad [\mathrm{A}] \tag{14.49}$$

Dividing each equation by Δl and allowing Δl to tend to zero we obtain

$$\frac{dV(l,t)}{dl} = -I(l,t)R - L\frac{dI(l,t)}{dt} \tag{14.50}$$

$$\frac{dI(l,t)}{dl} = -V(l,t)G - C\frac{dV(l,t)}{dt} \tag{14.51}$$

We can now rewrite these equations so that each is a function of a single variable as follows: First we take the derivative with respect to l on both sides of **Eqs. (14.50)** and **(14.51)**

$$\frac{d^2V(l,t)}{dl^2} = -R\frac{dI(l,t)}{dl} - L\frac{d}{dl}\left(\frac{dI(l,t)}{dt}\right) = -R\frac{dI(l,t)}{dl} - L\frac{d}{dt}\left(\frac{dI(l,t)}{dl}\right) \tag{14.52}$$

$$\frac{d^2I(l,t)}{dl^2} = -G\frac{dV(l,t)}{dl} - C\frac{d}{dl}\left(\frac{dV(l,t)}{dt}\right) = -G\frac{dV(l,t)}{dl} - C\frac{d}{dt}\left(\frac{dV(l,t)}{dl}\right) \tag{14.53}$$

Substituting $dI(l,t)/dl$ from **Eq. (14.51)** into **Eq. (14.52)** and $dV(l,t)/dl$ from **Eq. (14.50)** into **Eq. (14.53)**, we get

$$\frac{d^2V(l,t)}{dl^2} - LC\frac{d^2V(l,t)}{dt^2} - (LG+RC)\frac{dV(l,t)}{dt} - RGV(l,t) = 0 \tag{14.54}$$

$$\frac{d^2I(l,t)}{dl^2} - LC\frac{d^2I(l,t)}{dt^2} - (LG+RC)\frac{dI(l,t)}{dt} - RGI(l,t) = 0 \tag{14.55}$$

Equations (14.50) and (14.51) are equivalent to **Eqs.** (14.24) and (14.27) whereas **Eqs.** (14.54) and (14.55) are equivalent to **Eqs.** (14.29) and (14.30). In fact, one can obtain **Eqs.** (14.24), (14.27), (14.29), and (14.30) from **Eqs.** (14.50), (14.51), (14.54), and (14.55) by simply replacing d/dt by $j\omega$ and d^2/dt^2 by $(j\omega)^2 = -\omega^2$.

Exercise 14.2 Show that transformation of **Eqs.** (14.54) and (14.55) into the frequency domain results in **Eqs.** (14.29) and (14.30).

Exercise 14.3 Obtain the time-dependent wave equations for a transmission line for which $R = 0, G = 0$ (this type of line will be called a lossless line in the following section).

14.5 Types of Transmission Lines

The transmission line equations in **Section** 14.4 were obtained for a completely general transmission line. As can be seen, the equations are rather involved. The propagation constant as well as the line impedance are complex and are not always easy to evaluate. Both a phase constant and an attenuation constant exist; therefore, we can expect the waves along the line to decay due to attenuation as well as change their phases. The fact that both a forward- and backward-propagating wave exists indicates that the line may be finite in length whereby the backward-propagating wave is due to a reflection from the load, a connection on the line, or any other discontinuity that may exist.

For practical applications, we distinguish between a number of special types of transmission lines in addition to the above general lossy line. These include the ***lossless transmission line*** and the ***infinitely long transmission line*** as well as the so-called ***distortionless transmission line***. The wave characteristics on these lines are simplified because of the assumptions associated with the three types of lines, but, more importantly, they represent useful, practical lines. These are described next.

14.5.1 The Lossless Transmission Line

A lossless transmission line is a line for which both the series resistance and the shunt conductance are zero ($R = 0, G = 0$). In practice, this implies that the line is made of perfect conducting materials and perfect dielectrics. Although no practical line satisfies these conditions exactly, many lines satisfy them approximately. The implications of these conditions are that the attenuation constant is zero, the propagation constant is purely imaginary, and the characteristic impedance of the line is real.

If we substitute $R = 0$ and $G = 0$ in the propagation constant in **Eq.** (14.33), we get

$$\gamma = j\beta = j\omega\sqrt{LC} \tag{14.56}$$

Similarly, the characteristic impedance of the line **[from Eq.** (14.45)**]** is real and equal to

$$\boxed{Z_0 = \sqrt{\frac{L}{C}} \quad [\Omega]} \tag{14.57}$$

A number of propagation parameters can now be easily evaluated. The phase and attenuation constants are found from the propagation constant:

$$\boxed{\beta = \omega\sqrt{LC} \quad [\text{rad/m}], \quad \alpha = 0} \tag{14.58}$$

The wavelength is defined as

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \quad [\mathrm{m}]} \tag{14.59}$$

and the speed of propagation of the wave along the line (phase velocity) is

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]} \tag{14.60}$$

Because the dielectric is lossless, the phase velocity may also be written as

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{14.61}$$

From this, the following relation is obtained:

$$\boxed{\mu\varepsilon = LC} \tag{14.62}$$

In particular, the phase constant and the phase velocity only depend on the inductance and capacitance per unit length. The voltage or current waves propagate along the line without attenuation at a speed dictated by the inductance and capacitance per unit length of the line.

Example 14.3 Application: Antenna Down-Cables A common transmission line is the antenna cable used for rooftop TV antennas. The cable is made of two wires separated by a thin dielectric in the form of a flat cable. The characteristic impedance of these cables is 300 Ω. If the conductors are made of copper, separated by air (free space), and are 1 mm thick, calculate:

(a) The required distance between the two wires to produce a 300 Ω impedance.
(b) Calculate the phase velocity and the phase constant when receiving VHF channel 3 (63 MHz) and UHF channel 69 (803 MHz).

Solution: The characteristic impedance is given in **Eq. (14.57)**. From this and the relations for L and C in **Table 14.1**, column 1, we calculate the required distance.

(a) The intrinsic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\frac{\mu}{\pi}\cosh^{-1}(d/2a)}{\pi\varepsilon/\cosh^{-1}(d/2a)}} = \frac{1}{\pi}\sqrt{\frac{\mu}{\varepsilon}}\left(\cosh^{-1}\frac{d}{2a}\right) = 300 \quad [\Omega]$$

For $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ and with $a = 0.0005$ m

$$\cosh^{-1}\frac{d}{2a} = \frac{300\pi}{\sqrt{\mu_0/\varepsilon_0}} = \frac{300\pi}{376.99} = 2.5 \quad \rightarrow \quad d = 2a\cosh 2.5 = 0.00613 \quad [\mathrm{m}]$$

The distance between the wires should be 6.13 mm.

(b) To calculate phase velocity and phase constant, we need the capacitance and inductance per unit length. However, since $LC = \mu_0\varepsilon_0$, the phase velocity must be that of free space, regardless of frequency ($v_p = c$). The phase constant depends on frequency. From **Eq. (14.58)**

$$\beta = \frac{\omega}{c} \quad \rightarrow \quad \beta_{63\,\mathrm{MHz}} = \frac{2 \times \pi \times 63 \times 10^6}{3 \times 10^8} = 1.32 \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right]$$

$$\beta_{803\,\mathrm{MHz}} = \frac{2 \times \pi \times 803 \times 10^6}{3 \times 10^8} = 16.82 \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right].$$

Example 14.4 Application: CAble TeleVision (CATV) Cables A cable TV coaxial cable is designed with a characteristic impedance of 75 Ω. The inner conductor is 0.5 mm thick and the internal diameter of the outer conductor is 8 mm.

(a) Calculate the dielectric constant required for the material between the conductors to produce this impedance. Assume permeability of free space.
(b) Calculate the phase velocity on the line.

Solution:

(a) The characteristic impedance is given in **Eq. (14.57)** and the capacitance and inductance per unit length in column 2 in **Table 14.1**.

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0(\ln(b/a))^2}{4\pi^2\varepsilon}} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}} = 75 \quad [\Omega]$$

Solving for ε, with all other values known

$$\varepsilon = \left(\frac{\ln(b/a)}{2\pi}\right)^2 \frac{\mu_0}{75^2} = \left(\frac{\ln\dfrac{0.004}{0.00025}}{2 \times \pi}\right)^2 \frac{4 \times \pi \times 10^{-7}}{75^2} = 43.5 \times 10^{-12} \quad \left[\frac{\mathrm{F}}{\mathrm{m}}\right]$$

or in terms of relative permittivity $\varepsilon_r = \varepsilon/\varepsilon_0 = 43.5 \times 10^{-12}/8.854 \times 10^{-12} = 4.92$. This relative permittivity may be attained with some plastics, although, in actual design, it is just as likely to choose the dielectric first and work with the other parameters around it to obtain the required impedance.

(b) To calculate the phase velocity, we use the permeability and permittivity of the material:

$$v_p = \frac{1}{\sqrt{\mu_0\varepsilon}} = \frac{1}{\sqrt{4 \times \pi \times 10^{-7} \times 43.5 \times 10^{-12}}} = 1.353 \times 10^8 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

Thus, the speed of propagation in the cable is 2.22 times (actually $\sqrt{4.92}$) times slower than in free space, or in the same cable but with air as the dielectric.

14.5.2 The Long Transmission Line

A long transmission line is a line that for practical purposes may be considered to be infinite. The infinite transmission line is characterized by transmission without backward-propagating waves since, as we have seen in **Chapter 13**, a backward-propagating wave can only exist if the incident wave is reflected from a discontinuity in the wave's path. The long line may be lossy or lossless. For a lossy line, the voltage and current waves are found from **Eqs. (14.34)** and **(14.46)** by removing the backward-propagating wave:

$$V(l) = V^{+}e^{-\gamma l} \quad [\text{V}] \quad \text{and} \quad I(l) = I^{+}e^{-\gamma l} = \frac{V^{+}}{Z_0}e^{-\gamma l} \quad [\text{A}] \tag{14.63}$$

The propagation constant γ is given in **Eq. (14.33)** and the characteristic impedance of the line is given in **Eq. (14.45)**. If the long line is lossless, the voltage and current waves are

$$V(l) = V^{+}e^{-j\beta l} \quad \text{and} \quad I(l) = \frac{V^{+}}{Z_0}e^{-j\beta l} \tag{14.64}$$

The phase constant is given in **Eq. (14.58)** and the characteristic impedance in **Eq. (14.57)**.

The infinite transmission line cannot be realized physically, but it will prove to be a convenient approximation for very long lines or for short lines before the forward wave has reached the load.

Example 14.5 Application: Propagation and Attenuation in TV Cables The line in **Example 14.4** is used to connect a cable TV distribution center to a TV 20 km away. Assume that the material between the conductors has an attenuation of 1 dB/km, which may be considered a low-loss line.

The permeability of the dielectric in the line is μ_0 [H/m] and its permittivity is $4.92\varepsilon_0$ [F/m]. The frequency is 80 MHz (approximately the middle frequency of VHF channel 5).

(a) Calculate the propagation constant of the wave.
(b) Write the voltage and current everywhere on the line. Assume the voltage at the generator is 1 V and the line is matched (no reflection of waves anywhere on the line).
(c) If a TV requires a signal of at least 100 mV to receive properly, what must be the signal amplitude at the generator?

Solution: We calculate the phase constant from the relations for low-loss dielectrics given in **Section 12.7.2**. The attenuation constant is calculated directly from the attenuation given.

(a) The attenuation is 1 dB/km. Since we require the attenuation constant in nepers/m and one Np/m equals 8.69 dB/m, the attenuation constant is

$$\alpha = \frac{1}{8.69 \times 1000} = 1.15 \times 10^{-4} \quad \left[\frac{\text{Np}}{\text{m}}\right]$$

The phase velocity of a low-loss dielectric is approximately the same as that in the lossless dielectric. Thus, the phase constant is approximately

$$\beta \approx \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{2 \times \pi \times 80 \times 10^{6} \times \sqrt{4.92}}{3 \times 10^{8}} = 3.716 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

These give the propagation constant as

$$\gamma = \alpha + j\beta = 1.15 \times 10^{-4} + j3.716.$$

(b) Because the line is matched, there is no reflected wave. Assuming zero phase at the generator and the generator is taken as the origin ($l = 0$), we write for the voltage

$$V(l) = 1e^{-\gamma l} = e^{-1.15\times 10^{-4} l}e^{-j3.71l} \quad [\text{V}]$$

The current is

$$I(l) = \frac{V(l)}{Z_0} = \frac{1}{75}e^{-1.15\times 10^{-4} l}e^{-j3.71l} \quad [\text{A}].$$

(c) The distance between the generator and the TV is 20,000 m. The required signal at the TV is 100 mV. Thus, we write

$$V(l = 20,000\ \ m) = V(l = 0)e^{-1.15\times10^{-4}\times20,000}e^{-j3.71\times20,000}$$
$$= V(l = 0)\times 0.1e^{-j74,200}\quad [\text{V}]$$

The magnitude of the voltage at the generator is, therefore, V(l = 0) = 1 V. This is an extremely low-loss line. Practical lines have much higher losses (see the following exercise). The very large change in phase (3.71 rad/m) means that in practice the change in phase over large distances as in this example is not very useful.

Exercise 14.4 Suppose the line in **Example 14.5** has an attenuation of 10 dB/km.

(a) What is the required voltage amplitude at the generator to produce a signal of 10 mV at the TV a distance of 10 km away.
(b) Would you characterize this line as a low-loss or a high-loss line?

Answer **(a)** 987.16 V. **(b)** High loss.

14.5.3 The Distortionless Transmission Line

The propagation constant and characteristic impedance for general lossy lines were obtained in **Eqs. (14.33)** and **(14.42)**, respectively. These are rather complicated expressions and are frequency dependent. Whenever transmission lines are used for propagation of a single frequency wave (monochromatic wave), the fact that the line impedance and propagation constant are frequency dependent is less important, but when a wave has a range of frequencies, such as in the communication of information, each frequency component will be attenuated differently, the phase of each component will propagate at different speeds, and each component will see a different line impedance. This inevitably leads to distortion of the wave (see **Sections 12.7.4.1** through **12.7.4.3**).

The question is: How can we design a general lossy line so that the attenuation constant, phase velocity, and characteristic impedance of the line are independent of frequency? If we can do that, we would obtain a distortionless transmission line. To do so, we note that if $R/L = G/C$, the propagation constant in **Eq. (14.33)** becomes

$$\gamma = j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}}\sqrt{1+\frac{R}{j\omega L}} = j\omega\sqrt{LC}\left[1+\frac{R}{j\omega L}\right] = j\omega\sqrt{LC} + R\sqrt{\frac{C}{L}} \tag{14.65}$$

From this, the attenuation and phase constants are

$$\boxed{\alpha = R\sqrt{\frac{C}{L}}\quad\left[\frac{\text{Np}}{\text{m}}\right],\quad \beta = \omega\sqrt{LC}\quad\left[\frac{\text{rad}}{\text{m}}\right]} \tag{14.66}$$

and, therefore, the phase velocity is

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}\quad\left[\frac{\text{m}}{\text{s}}\right]} \tag{14.67}$$

Thus, the first two conditions (i.e., that the attenuation constant and phase velocity are independent of frequency) are satisfied. What about the characteristic impedance? If we substitute the condition $R/L = G/C$ in **Eq. (14.42)**, we get

$$\boxed{Z_0 = \sqrt{\frac{R + j\omega L}{RC/L + j\omega C}} = \sqrt{\frac{L}{C}} \quad [\Omega]} \tag{14.68}$$

The characteristic impedance is also constant and the above requirements are satisfied. Thus, for a line to be distortionless, the line parameters must be designed so that

$$\boxed{\frac{R}{L} = \frac{G}{C}} \tag{14.69}$$

With this condition, the distortionless transmission line[1] has the same phase constant and characteristic impedance as the lossless line but a nonzero, constant attenuation.

Example 14.6 A line is made of two parallel conductors embedded in a low-loss dielectric, as shown in **Figure 14.11**. Material properties and dimensions are given in the figure. The design calls for a distortionless transmission line because the line is intended for use with modems at high speeds. Assume the frequency used is 100 MHz, and that the dielectric extends far from the conductors.

(a) Calculate the required distance d between the wires to produce a distortionless line at the given frequency.
(b) What are the characteristic impedance of the line and its attenuation constant?
(c) If, reduction of at most 40 dB is allowed before an amplifier is required, calculate the distance between each two amplifiers on the line.

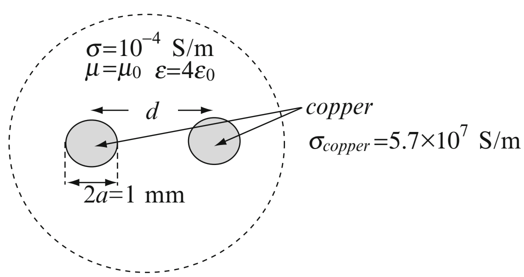


Figure 14.11 A two-wire transmission line. The distance d is designed so that the line is distortionless

Solution: From the distortionless line requirement,

$$\frac{R}{L} = \frac{G}{C}$$

(a) Substituting the parameters of the two-conductor line from **Table 14.1** (column 1),

$$\frac{1/\pi a \delta \sigma_c}{\dfrac{\mu}{\pi}\cosh^{-1}(d/2a)} = \frac{\dfrac{\pi\sigma}{\cosh^{-1}(d/2a)}}{\dfrac{\pi\varepsilon}{\cosh^{-1}(d/2a)}} \quad \rightarrow \quad \frac{1}{a\mu_0\delta\sigma_c\cosh^{-1}(d/2a)} = \frac{\sigma}{\varepsilon}$$

[1] The formula for distortionless lines is due to Oliver Heaviside. It was devised in 1897 as a solution to distortions on long (intercontinental) telephone lines. Following this, telephone lines were routinely "loaded" with additional series inductance at regular intervals to adjust their parameters so that distortionless lines are obtained. This practice is now rare. Lines are produced with parameters that guarantee they are distortionless.

where $\delta = 1/\sqrt{\pi f \mu_c \sigma_c}$ and the index c indicates conductor material properties:

$$\cosh^{-1}\frac{d}{2a} = \frac{\varepsilon}{a\sigma}\sqrt{\frac{\pi f}{\mu_0 \sigma_c}} = \frac{4 \times 8.854 \times 10^{-12}}{0.0005 \times 10^{-4}}\sqrt{\frac{\pi \times 10^8}{5.7 \times 10^7 \times 4 \times \pi \times 10^{-7}}} = 1.483$$

Thus,

$$\frac{d}{2a} = \cosh(1.483) = 2.317 \quad \rightarrow \quad d = 2 \times 2.317 \times 0.0005 = 2.317 \times 10^{-3} \quad [\text{m}]$$

The two wires must be separated by 2.317 mm to produce a distortionless line.

(b) To calculate the characteristic impedance and the attenuation constant, the line parameters are needed. These are obtained from **Table 14.1** and, with the above dimensions, are

$$R = 1.675 \quad [\Omega/\text{m}], \quad L = 0.592 \quad [\mu\text{H/m}],$$
$$G = 2.12 \times 10^{-4} \quad [\text{S/m}], \quad C = 75 \quad [\text{pF/m}]$$

It is worth verifying that these parameters indeed make a distortionless line:

$$\frac{R}{L} = \frac{1.675}{0.592 \times 10^{-6}} = 2.83 \times 10^6, \quad \frac{G}{C} = \frac{2.12 \times 10^{-4}}{75 \times 10^{-12}} = 2.83 \times 10^6$$

Therefore, the conditions for distortionless operation are satisfied. Now, the characteristic impedance and attenuation constant are

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.592 \times 10^{-6}}{75 \times 10^{-12}}} = 88.84 \quad [\Omega]$$
$$\alpha = R\sqrt{\frac{L}{C}} = 1.675\sqrt{\frac{75 \times 10^{-12}}{0.592 \times 10^{-6}}} = 0.0188 \quad \left[\frac{\text{Np}}{\text{m}}\right].$$

(c) The attenuation in the line is 0.0188 Np/m. 1 Np/m = 8.69 dB/m and, therefore, the attenuation is 0.1634 dB/m. For a total of 40 dB the distance is

$$d = \frac{40}{0.1634} = 244.84 \quad [\text{m}]$$

An amplifier is required every 245 m or so. This means the line is too lossy. Note, also, that this type of line is not normally used at high frequencies; coaxial lines are more common.

14.5.4 The Low-Resistance Transmission Line

It was mentioned before that a transmission line is made of two conductors in a given configuration. In a line of this type, it is often possible to assume that the conductivity of the conductor is so high as to have negligible resistance. In other words, the propagation on the transmission line is not affected by the conductor itself. The conductors are required only to guide the waves, but all propagation parameters are affected by the properties of the dielectric alone. Substituting $R = 0$ in **Eqs. (14.33)** and **(14.42)**, we get

$$\gamma = j\omega\sqrt{LC}\sqrt{1+\frac{G}{j\omega C}} \tag{14.70}$$

$$Z_0 = \sqrt{\frac{j\omega L}{G+j\omega C}} \tag{14.71}$$

Since the conductor's effect can be neglected, we can view this as a transverse electromagnetic wave propagating in a lossy dielectric material with properties ε_r, μ_r, and σ as if the conductors were not there.

For a general lossy dielectric, we obtained the propagation constant in **Eq. (12.83)** as

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{\left[1+\frac{\sigma}{j\omega\varepsilon}\right]} \tag{14.72}$$

The propagation constants in the transmission line and in the general dielectric are of exactly the same form. Direct comparison between **Eqs. (14.72)** and **(14.70)** gives the following two relations:

$$\boxed{LC = \mu\varepsilon \quad \text{and} \quad \frac{\sigma}{\varepsilon} = \frac{G}{C}} \tag{14.73}$$

where σ is the conductivity of the dielectric between the conductors. These two relations are important for two reasons:

(1) They hold for lossless and lossy transmission lines even if the series resistance is not zero. This can be easily verified for the three transmission lines listed in **Table 14.1**.

(2) The relations provide one of the simplest methods of evaluating the parameters of the line. If, for example, C is known, L and G can be evaluated directly. This is useful because in many cases, one of the line parameters is easier to evaluate than the other two. In such cases, these two relations provide a simple means of finding the line parameters.

Note also that if, in addition, $G = 0$, the line becomes lossless.

Example 14.7 Application: Superconducting Power Lines A transmission line designed for power transmission at 60 Hz is made with superconducting cables. The two conductors that make the transmission line are separated by 3 m. The size of the wires is not known, but their inductance per unit length is known to be 0.5 μH/m. The permittivity and permeability are those of free space and the conductivity of air is 10^{-7} S/m. Calculate:

(a) The attenuation constant on the line.
(b) The characteristic impedance of the line.

Solution: The attenuation constant, which is entirely due to losses in air, is calculated as the real part of **Eq. (14.70)**, after the capacitance per unit length is calculated from **Eq. (14.73)**. The characteristic impedance is given in **Eq. (14.71)**. Because this is a superconducting transmission line, the series resistance of the line is zero.

(a) The capacitance per unit length **[Eq. (14.73)]** is

$$C = \frac{\mu_0\varepsilon_0}{L} = \frac{1}{c^2L} = \frac{1}{9\times10^{16}\times0.5\times10^{-6}} = 22.22\times10^{-12} \quad \left[\frac{\text{F}}{\text{m}}\right]$$

The conductance is also calculated from **Eq. (14.73)**

$$G = \frac{\sigma C}{\varepsilon_0} = \frac{10^{-7}\times22.22\times10^{-12}}{8.854\times10^{-12}} = 2.51\times10^{-7} \quad \left[\frac{\text{S}}{\text{m}}\right]$$

Substituting these, we get the propagation constant:

$$\gamma = j\omega\sqrt{LC}\sqrt{1+\frac{G}{j\omega C}} = j\times 2\times\pi\times 60\times\sqrt{0.5\times 10^{-6}\times 22.22\times 10^{-12}}\times\sqrt{1+\frac{2.51\times 10^{-7}}{j\times 2\times\pi\times 60\times 22.22\times 10^{-12}}}$$

$$= j1.257\times 10^{-6}\sqrt{1-j29.96} = (4.783+j4.946)\times 10^{-6}$$

The attenuation constant is very small and equal to 4.783×10^{-6} Np/m or 4.16×10^{-5} dB/m. Since the attenuation is frequency dependent, this value would change at other frequencies.

(b) The characteristic impedance of the line is

$$Z_0 = \sqrt{\frac{j\omega L}{G+j\omega C}} = \sqrt{\frac{j\times 2\times\pi\times 60\times 0.5\times 10^{-6}}{2.51\times 10^{-7}+j\times 2\times\pi\times 60\times 22.22\times 10^{-12}}} = 19.6926+j19.0463\approx 19.7(1+j)\quad [\Omega].$$

Note: Although we treated this problem as a wave problem, and, in fact, waves do exist at power frequencies, they are much less important than effects such as the induction of eddy currents and flux leakage. In spite of the fact that at low frequencies, distributed parameters are not normally necessary, their use is correct.

14.6 The Field Approach to Transmission Lines

The discussion in the previous sections was in terms of general line parameters and therefore applies to any transmission line. Unlike previous chapters, the primary variables here were the voltage and current of the line. This choice is natural if we view the line as a distributed parameter circuit. It is, however, possible to arrive at exactly the same results from a field point of view. In this case, the primary variables are the electric and magnetic field intensities and the discussion is much the same as that for plane waves. One advantage of using field variables is that these are vectors and, therefore, the direction of propagation at any point is always available and indicates the direction in which power is transferred. To demonstrate this approach, we look now at the wave characteristics on a parallel plate transmission line.

Suppose the transmission line shown in **Figure 14.12** is given. The line is very long and $w \gg d$. The material between the plates is a general dielectric. At a given instant in time, the potential between the two plates and the currents in the plates are as shown. For the given condition, the electric field intensity points from the upper plate to the lower plate (x direction) and the magnetic field intensity is parallel to the plates, pointing in the y direction. Because of our assumption that $w \gg d$, we may assume that the electric field intensity is everywhere perpendicular to the plates (no fringing at the edges) and the magnetic field intensity is everywhere parallel to the plates.

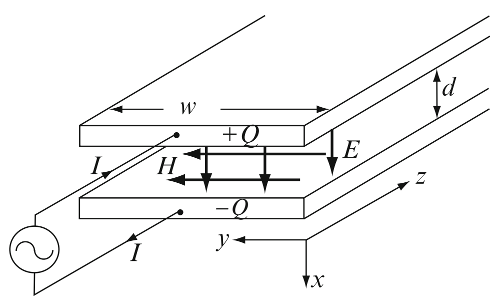


Figure 14.12 Relations between current and charge on the conductor and the electric and magnetic field intensities in the dielectric of a parallel plate transmission line

We know that the two fields are a solution to the source-free wave equation since there are no sources in this domain and propagation takes place; that is, the fields obey the general Maxwell equations. Also, because the transmission line is infinite in extent, in the z direction, there can only be a forward-propagating wave. Without knowing what the electric field intensity amplitude is, we can write, in general terms:

$$\mathbf{E} = \hat{\mathbf{x}} E_0 e^{-\gamma z} \quad [\text{V/m}] \tag{14.74}$$

where we have replaced the generic coordinate l with z. The magnetic field intensity is perpendicular to the electric field intensity and, using the intrinsic impedance of the dielectric, we can write

$$\mathbf{H} = \hat{\mathbf{y}} \frac{E_0}{\eta} e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{14.75}$$

where η is the intrinsic impedance of the dielectric between the plates. The wave is a transverse electromagnetic wave (**E** and **H** are perpendicular to each other and to the direction of propagation). The direction of propagation of the wave is in the positive z direction, as shown by the Poynting vector:

$$\mathcal{P}(z) = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{x}} E_0 e^{-\gamma z} \times \hat{\mathbf{y}} \frac{E_0}{\eta} e^{-\gamma z} = \hat{\mathbf{z}} \frac{E_0^2}{\eta} e^{-2\gamma z} \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{14.76}$$

The electric field intensity E_0 was arbitrarily chosen, but, in practice, its sources are the charge distribution on the conducting surfaces and the current density in the conducting plates. The voltage between the two plates can be written as

$$V = \int_{l_1} \mathbf{E} \cdot d\mathbf{l}_1 \quad [\text{V}] \tag{14.77}$$

and the current in one of the plates (upper) as

$$I = \int_{l_2} \mathbf{H} \cdot d\mathbf{l}_2 \quad [\text{A}] \tag{14.78}$$

where the contours l_1 and l_2 are shown in **Figure 14.13b**. These are the line voltage and current and may be substituted in **Eqs. (14.34)** and **(14.35)** to obtain the transmission line voltage and current in terms of the electric and magnetic field intensities. This approach will be used, indirectly, in **Chapter 17**, but we will not pursue it here.

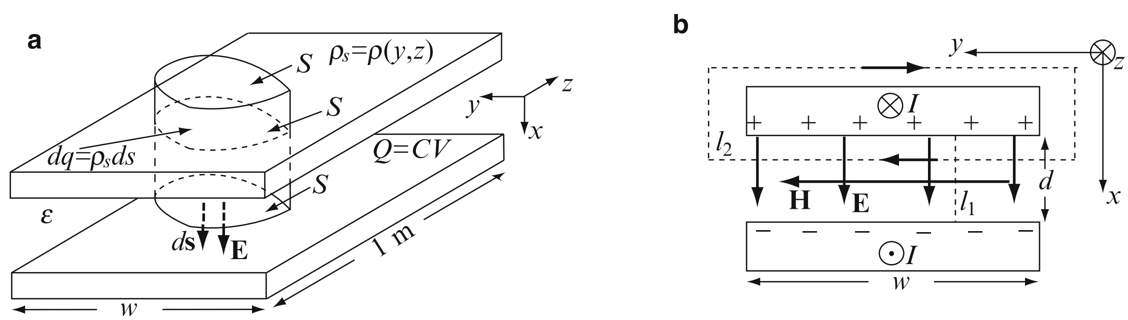


Figure 14.13 (**a**) Calculation of charge density using Gauss's law. (**b**) Calculation of current density using Ampere's law

We can now calculate the charge density and the current density in the conductors that will produce the required electric and magnetic fields from **Eqs. (14.77)** and **(14.78)**. This calculation is not absolutely necessary for the discussion here, but it emphasizes two important points:

(1) The sources of the fields produced by the transmission line are the charges and currents in the line.
(2) The charge and current distributions must be of a form that produces these fields; not all charge and current distributions will produce a propagating wave in the transmission line.

Suppose that a charge distribution exists on the upper and lower plates as shown in **Figure 14.13a**. To calculate the electric field intensity, we use Gauss's law. A small volume, with two surfaces parallel to the upper plate, is defined as shown in **Figure 14.13a**. The electric field intensity outside the plates is zero (as for parallel plate capacitors) and the electric field intensity between the plates is given by **Eq. (14.74)**. Taking a surface S as shown, we get from Gauss's law

$$\int_s \mathbf{E} \cdot d\mathbf{s} = \int_s (\hat{\mathbf{x}} E_0 e^{-\gamma z}) \cdot (\hat{\mathbf{x}}\, ds) = \frac{1}{\varepsilon}\int_s \rho ds \tag{14.79}$$

or

$$\rho(y, z) = \varepsilon E_0 e^{-\gamma z} \quad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right] \tag{14.80}$$

Thus, the charge density is uniform in the y direction (independent of y) but varies along the line. This variation is better seen if the charge density is written in the time domain as

$$\rho(y, z, t) = \mathrm{Re}\left\{\varepsilon E_0 e^{-(\alpha + j\beta)z} e^{j\omega t}\right\} = \varepsilon E_0 e^{-\alpha z} \cos(\omega t - \beta z) \quad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right] \tag{14.81}$$

In other words, the charge distribution must be cosinusoidal in the z direction. The attenuation constant produces a decaying charge density magnitude with distance. If propagation is without attenuation, then $\alpha = 0$ and there is no decay in amplitude of the electric field intensity. The charge density distribution on the lower plate is the same as on the upper plate but opposite in sign.

The current density in the line is calculated from Ampere's law. Using the upper plate again and assuming some current density in the plate, we can enclose this current density with an arbitrary contour as shown in **Figure 14.13b**. The magnetic field intensity outside the plates is zero and between the plates is given by **Eq. (14.75)**. In our case, **H** is in the positive y direction, as is $d\mathbf{l}$. Thus, the current density is in the positive z direction (**H** and **J** are always perpendicular to each other). Since the current is uniform in the y direction in this case, we can write $I(y,z) = wJ(y,z)$ and, performing the integration in **Eq. (14.78)** with the field in **Eq. (14.75)**, we get

$$\frac{E_0}{\eta} e^{-\gamma z} = J(y, z) \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{14.82}$$

This gives the magnitude of the current density in the upper plate. This current must be in the positive z direction to produce a magnetic field intensity in the positive y direction (based on our notation in **Figure 14.12**); therefore,

$$\mathbf{J}(y, z) = \hat{\mathbf{z}} \frac{E_0}{\eta} e^{-\gamma z} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{14.83}$$

The current density in the lower plate is the same in magnitude but in the negative z direction. The variation of current density along the line is also cosinusoidal, as for the charge density.

Note that the same results could be obtained from the boundary conditions for a perfect conductor as discussed in **Section 11.4.2**. If we do so, the required conditions at the surface of the conductors are given in **Table 11.5**.

Example 14.8 Parallel Plate Transmission Line Consider the parallel plate transmission line shown in **Figure 14.14**. The distance between the plates is very small compared to the width of the line ($w \gg d$); the plates are perfectly conducting and separated by free space. A voltage is applied to one end of the line: $V = V_0 \cos \omega t$, where $V_0 = 12$ V and $\omega = 3 \times 10^9$ rad/s. Calculate:

(a) The surface charge density on the plates. Calculate the minimum and maximum charge density.
(b) The surface current density in the plates.
(c) The time-averaged power propagated in the line if all power is contained within the cross-sectional area of the line (i.e., no fields exist outside the line).

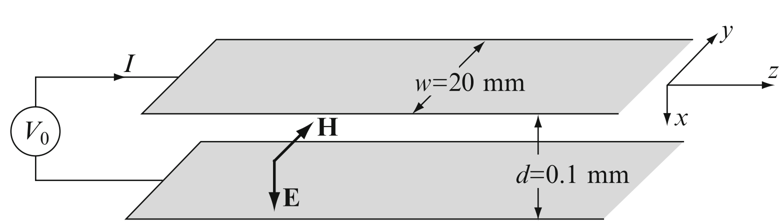


Figure 14.14 A parallel plate transmission line and the electric and magnetic field intensities between the plates

Solution: Since there is no fringing of the fields, the electric field intensity anywhere on the line equals the potential divided by the separation d as can be seen from **Figure 14.13b** and **Eq. (14.77)**, exactly like in a parallel plate capacitor. The line is lossless and long; therefore, the propagation constant is $\gamma = j\beta_0$.

(a) At the generator, the electric and magnetic field intensities are:

$$E_0 = \frac{V_0}{d}\cos\omega t \quad \left[\frac{\text{V}}{\text{m}}\right] \quad \text{and} \quad H_0 = \frac{E_0}{\eta_0} = \frac{V_0}{\eta_0 d}\cos\omega t \quad \left[\frac{\text{A}}{\text{m}}\right]$$

If we assume the electric field to be in the positive x direction as in **Figure 14.14**, the magnetic field must be in the positive y direction for the wave to propagate in the positive z direction. Taking this convention, the electric and magnetic field intensity vector phasors (at $z = 0$) are

$$\mathbf{E} = \hat{\mathbf{x}}\frac{V_0}{d} \quad \left[\frac{\text{V}}{\text{m}}\right], \quad \mathbf{H} = \hat{\mathbf{y}}\frac{V_0}{\eta_0 d} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

These fields propagate in the positive z direction. At a distance z from the generator, the fields are

$$\mathbf{E}(z) = \hat{\mathbf{x}}\frac{V_0}{d}e^{-j\beta_0 z} \quad \left[\frac{\text{V}}{\text{m}}\right], \quad \mathbf{H}(z) = \hat{\mathbf{y}}\frac{V_0}{\eta_0 d}e^{-j\beta_0 z} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

From the electric field intensity, the charge density on the line is

$$\rho(z) = \varepsilon_0 E_0 e^{-\gamma z} = \varepsilon_0\frac{V_0}{d}e^{-j\beta_0 z} = 8.854 \times 10^{-12} \times \frac{12}{0.0001}e^{-j3\times 10^9 z/3\times 10^8} = 1.062 \times 10^{-6}e^{-j10z} \quad \left[\frac{\text{C}}{\text{m}^2}\right].$$

(b) The surface current density is

$$\mathbf{J} = \hat{\mathbf{z}}\frac{V_0}{\eta_0 d}e^{-j\beta_0 z} = \hat{\mathbf{z}}\frac{12}{377 \times 0.0001}e^{-j10z} = \hat{\mathbf{z}}\,318.3e^{-j10z} \quad \left[\frac{\text{A}}{\text{m}}\right].$$

(c) The time-averaged power density may be calculated anywhere on the line. However, because the line is lossless, it is best to calculate this at the generator. The time-averaged power density is

$$\boldsymbol{\mathcal{P}}_{av}(z) = \hat{\mathbf{z}}\frac{E_0^2}{2\eta_0} = \hat{\mathbf{z}}\frac{V_0^2}{2\eta_0 d^2} = \hat{\mathbf{z}}\,1.9098 \times 10^7 \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

and the total power is the power density multiplied by the cross-sectional area of the line. The latter is $S = wd$. Thus,

$$P = \mathcal{P}_{av}S = 1.9098 \times 10^7 \times 0.02 \times 0.0001 = 38.196 \quad [\text{W}]$$

The same result can be obtained by multiplying the time-averaged current by time-averaged voltage. This gives:

$$P = \frac{V_0 I_0}{2} = \frac{V_0 J w}{2} = \frac{12 \times 318.3 \times 0.02}{2} = 38.196 \quad [\text{W}]$$

14.7 Finite Transmission Lines

By a finite transmission line is meant a line of finite length with a generator of some sort at one end and a load at the other. Both the generator and the load should be viewed in generic terms: The load may actually be a short circuit, an open circuit, or another transmission line. The generator may be an actual source, the output of another transmission line, or, perhaps, a receiving antenna. The configuration we discuss here is shown in **Figure 14.15**.

Until now, we discussed only infinite lines or made no specific reference to the length of the line. Now, we have to discuss the distance on the line with respect to the fixed points of the line: These are the locations of the load and the generator. Thus, we seek a reference point to which to relate all our calculations. We could choose either the generator or the load for this purpose, but it is common to use the load as a reference point. This choice is partly arbitrary, partly based on convenience, and mostly on convention. At any rate, the only important point here is to be consistent and not flip between points of reference.

We must be careful now: Inspecting **Figure 14.15**, the positive z direction is toward the generator. On the other hand, the positive direction of propagation of power must be from the generator toward the load since energy is naturally transferred from generator to load. We recall that the wave solutions on a general transmission line are

$$V(l) = V^+ e^{-\gamma l} + V^- e^{\gamma l} \quad [\text{V}] \tag{14.84}$$

$$I(l) = I^+ e^{-\gamma l} + I^- e^{\gamma l} \quad [\text{A}] \tag{14.85}$$

In these equations, l is positive toward the load ($l = 0$ at the generator). The first term is the forward-propagating wave (from generator to load) and the second is the backward-propagating wave (from load to generator). This convention was used for infinitely long transmission lines.

Our new convention for the finite transmission lines requires that the forward-propagating wave propagates in the negative z direction and the backward propagating wave propagates in the positive z direction. Thus, to create our reference system, we replace $+l$ by $-z$ and $-l$ by $+z$:

$$\boxed{V(z) = V^+ e^{\gamma z} + V^- e^{-\gamma z} \quad [\text{V}]} \quad \text{and} \quad \boxed{I(z) = I^+ e^{\gamma z} + I^- e^{-\gamma z} \quad [\text{A}]} \tag{14.86}$$

The first term in each relation is still the forward-propagating wave and the second is the backward-propagating wave. The relation between the z and l notation and the forward- and backward-propagating waves is shown in **Figure 14.15** for a transmission line of length d, connected to a generator and a load. The two sets of equations also indicate what is involved in choosing a particular point of reference.

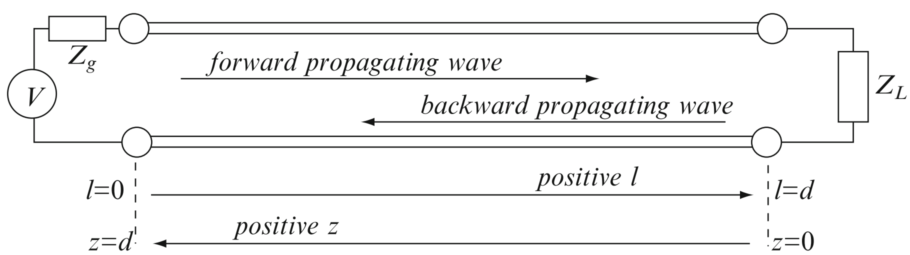


Figure 14.15 A finite transmission line with the reference shifted to the load

Example 14.9 The Amplitudes of the Forward and Backward Propagating Waves The amplitudes V^+ and V^- have been used so far assuming they are known. They can be calculated from the terminal voltages on the transmission line.

(a) Given the lossless line in **Figure 14.16**, calculate the amplitudes of the forward and backward propagating waves if the load voltage is given, equal to $V_L = 50$ V.

(b) Given the lossless line in **Figure 14.16**, calculate the amplitudes of the forward and backward propagating waves if the line input voltage and currents are given as $V_i = 50$ V and $I_i = 1$A.

Solution: We use the general relations in **Eq. (14.86)**.

(a) In **Figure 14.16**, the load voltage is given. Since our reference point is at the load, we substitute $z = 0$ in **Eq. (14.86)** and get

$$V_L = V^+ + V^- \quad \text{and} \quad I_L = \frac{V^+}{Z_0} - \frac{V^-}{Z_0}$$

For the data given

$$V_L = V^+ + V^- = 50 \quad \text{and} \quad I_L = \frac{V^+}{100} - \frac{V^-}{100} = \frac{50}{50} = 1 \quad \rightarrow \quad V^+ - V^- = 100$$

Solving for V^+ and V^- we get

$$V^+ = 50 \quad [\text{V}], \quad V^- = -25 \quad [\text{V}].$$

(b) In this case the terminal voltage is given at the line input. Since the input is at a distance d from the load, we write

$$V_i = V^+ e^{j\beta d} + V^- e^{-j\beta d} = 50 \quad [\text{V}] \quad \text{and} \quad I_i = \frac{V^+}{Z_0} e^{j\beta d} - \frac{V^-}{Z_0} e^{-j\beta d} = 1 \quad [\text{A}]$$

For the given data (with $\beta = 2\pi/\lambda$)

$$V^+ e^{j2\pi\times2.3} + V^- e^{-j2\pi\times2.3} = 50 \quad \text{and} \quad V^+ e^{j2\pi\times2.3} - V^- e^{-j2\pi\times2.3} = 100$$

Solving for V^+ and V^- we get

$$V^+ = 75e^{-j4.6\pi} \quad \text{and} \quad V^- = -25e^{-j4.6\pi} \quad [\text{V}]$$

or

$$V^+ = 75\angle108^\circ \quad \text{and} \quad V^- = 25\angle-72^\circ \quad [\text{V}].$$

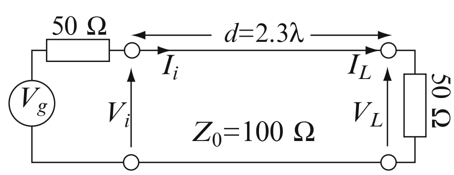


Figure 14.16

Exercise 14.5

(a) Calculate the line input voltage and current (V_i,I_i) in the line in **Figure 14.16** if the load voltage is given as $V_L = 50$ V.

(b) Calculate the load voltage and current (V_L,I_L) in **Figure 14.16** if the line input voltage and currents are given as $V_i = 50$ V and $I_i = 1$A.

Answer

(a) $V_i = -16.025 + j95.1 \quad [\text{V}], \quad I_i = -0.314 + j0.475 \quad [\text{A}]$
(b) $V_L = -4.161 - j25.61 \quad [\text{V}], \quad I_L = -0.0832 - j0.5122 \quad [\text{A}]$

14.7.1 The Load Reflection Coefficient

Waves.m

First, we recall the definition of the characteristic impedance Z_0. This was defined for an infinite transmission line as the ratio between the forward-propagating voltage wave and the forward-propagating current wave. Thus, for any line, the characteristic impedance is

$$Z_0 = \frac{V^+ e^{\gamma z}}{I^+ e^{\gamma z}} = \frac{V^+}{I^+} = -\frac{V^-}{I^-} \quad [\Omega] \tag{14.87}$$

as was shown in **Eq. (14.45)**. This impedance is characteristic of the line and has nothing to do with generator or load. Similarly, the propagation constant γ is independent of load or generator, as are the parameters R, L, G, and C.

Since the load is very important for our analysis and since it is one of the few variables an engineer has any control over, it is only natural that we should wish to analyze the transmission line behavior in terms of the load impedance and the line's variables. Thus, we first calculate the load impedance:

$$Z_L = \frac{V_L}{I_L} \quad [\Omega] \tag{14.88}$$

where V_L and I_L are the total load voltage and total load current. By total voltage and current is meant the sum of forward and backward voltages and currents, respectively.

The load is located at $z = 0$. In terms of the current and voltage of the line, this becomes

$$Z_L = \frac{V(0)}{I(0)} = \frac{V^+ + V^-}{I^+ + I^-} = \frac{V^+ + V^-}{V^+/Z_0 - V^-/Z_0} = Z_0 \frac{V^+ + V^-}{V^+ - V^-} \quad [\Omega] \tag{14.89}$$

Note that if only forward-propagating waves exist ($V^- = 0$), the load impedance must be equal to the characteristic impedance of the line. This condition defines matching between load and line. Matching in transmission lines only requires that the load and line impedances be equal, unlike circuits where matching also means maximum transfer of power (conjugate matching). Under matched conditions ($Z_L = Z_0$), there are no backward propagating waves.

On the other hand, if $Z_L \neq Z_0$, there will be both forward-propagating and backward-propagating waves. At the load ($z = 0$) we can calculate the backward propagating wave amplitude V^- from **Eq. (14.89)** as

$$V^- = V^+ \frac{Z_L - Z_0}{Z_L + Z_0} \quad [\mathrm{V}] \tag{14.90}$$

The backward-propagating wave is due to the reflection of the forward-propagating wave at the load. Thus, we define the ***load reflection coefficient*** as

$$\boxed{\Gamma_L = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}} \tag{14.91}$$

It is important to remember that this is the reflection coefficient at the load only. At other locations on the line, the reflection coefficient is, in general, different and we should never confuse the load reflection coefficient with any other reflection coefficient that may be convenient to define. The load reflection coefficient will always be denoted with a subscript L as in **Eq. (14.91)**. Note also that in general, the load reflection coefficient is a complex number since it is the ratio of the complex amplitudes V^- and V^+. Thus, we can also write the reflection coefficient as

$$\boxed{\Gamma_L = |\Gamma_L| e^{j\theta_\Gamma}} \tag{14.92}$$

where θ_Γ is the phase angle of the reflection coefficient. This form will become handy later in our study.

Example 14.10 Application: Mismatched Antenna and Line A transmission line used to connect a transmitter to its antenna has characteristic impedance $Z_0 = 50\ \Omega$. The antenna, with impedance $Z_L = 50 + j50$, is connected as a load to the line. Calculate the load reflection coefficient.

Solution: Using **Eq. (14.91)**

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j50 - 50}{50 + j50 + 50} = \frac{j50}{100 + j50} = \frac{1 + j2}{5}$$

or in terms of magnitude and phase

$$\Gamma_L = \frac{1}{5} + j\frac{2}{5} = \frac{1}{\sqrt{5}} e^{j0.352\pi}$$

This mismatch is not very healthy for the transmitter because of the backward propagating waves (and, therefore, power) returning to the generator and, therefore, should be avoided.

Example 14.11 A long power transmission line supplies 1,500 MW at 750 kV (rms) to a matched load (i.e., the load impedance equals the line impedance).

(a) Suppose the load is disconnected. What is the reflection coefficient at the load?
(b) Because of a fault on the line, the load changes from the matched condition to $Z_L = 200 + j100\ \Omega$. What is the reflection coefficient at the load now?

Solution: The load impedance is calculated from the load power and the reflection coefficient is then calculated from **Eq. (14.91)**.

(a) The load impedance under matched conditions is

$$P = \frac{V_L^2}{2Z_L} \quad \rightarrow \quad Z_L = \frac{V_L^2}{P} = \frac{(750,000)^2}{2 \times 1.5 \times 10^9} = 187.5 \quad [\Omega]$$

The characteristic line impedance is $Z_0 = 187.5\ \Omega$.
If the load is disconnected, the load impedance becomes infinite ($Z_L \gg Z_0$) and the load reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{Z_L}{Z_L} = +1$$

(b) The load reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{200 + j100 - 187.5}{200 + j100 + 187.5} = \frac{-12.5 + j100}{387.5 + j100} = 0.0827 + j0.2341 = 0.2518 e^{j0.38\pi}$$

The magnitude of the reflection coefficient is $|\Gamma_L| = 0.2518$ and the phase angle of the load reflection coefficient is $\theta_L = 0.38\pi$.

14.7.2 Line Impedance and the Generalized Reflection Coefficient

After calculating the characteristic impedance and the reflection coefficient at the load, we can now tackle the question of the impedance at any other point on the line. This is an important question because it will allow us to connect the line to, say, a generator, ensuring that the line is matched to the generator, or to connect one line to another. These are questions of practical engineering importance. The simple example in **Figure 14.17** shows the concepts involved. A loudspeaker is to be connected to a power amplifier through a transmission line. We know that for optimal operation, the output of the amplifier must be matched to the load. At the amplifier, the load consists of the speaker and the line and the amplifier must be matched to the line. We defer the question of matching until the next chapter, but for any attempt at matching, we must be able to calculate the input impedance of the line.

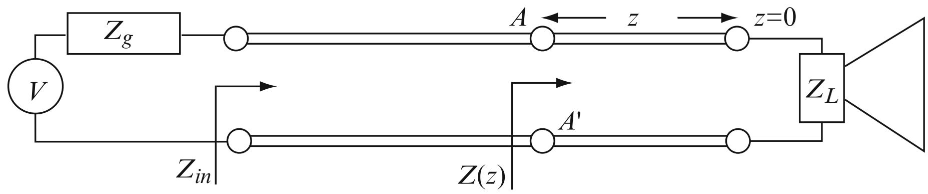


Figure 14.17 Distinction between load, input, and line impedances

This input impedance, which, in general, is different than the characteristic impedance of the line, must in some way depend on the load impedance. That this must be so should be obvious from our experience: suppose the above amplifier is matched to the line for the given load. If we now change the load, say by shorting the speaker, the system is not matched any more. In fact, by shorting the load, we may well have damaged the amplifier. It is, therefore, important to be able to calculate the line impedance for any load condition. Before continuing, we distinguish between two terms associated with impedance of the line. These are as follows:

Input line impedance is the impedance at the input or generator side of the line. In the above example, this impedance is the impedance of the line at the end, which is connected to the source (amplifier in this case). This impedance will always be denoted as Z_{in}.

Line impedance is the impedance at any point on the line. The distinction between the two terms is shown in **Figure 14.17**. The line impedance will be denoted as $Z(z)$. The distinction is not terribly important since if we were to cut the line at the points A–A', the line impedance would then become the input impedance. We will, however, distinguish between the two terms wherever appropriate.

To calculate the line impedance, we need to calculate the total voltage and total current at any point on the line and divide the voltage by current. Using **Figure 14.18a** as a guide, the voltage and current at point z on the line are

$$V(z) = V^+ e^{\gamma z} + V^- e^{-\gamma z} \quad [\mathrm{V}], \quad I(z) = \frac{V^+}{Z_0} e^{\gamma z} - \frac{V^-}{Z_0} e^{-\gamma z} \quad [\mathrm{A}] \tag{14.93}$$

where we made use of **Eq. (14.46)** to rewrite the current in terms of voltage and characteristic impedance. We can divide $V(z)$ by $I(z)$ to obtain $Z(z)$, but this would not be very helpful now because the result would be in terms of both the forward and backward waves. Instead, we use the load reflection coefficient in **Eq. (14.91)** to write

$$\boxed{V(z) = V^+ \left(e^{\gamma z} + \Gamma_L e^{-\gamma z}\right) \quad [\mathrm{V}]} \quad \text{and} \quad \boxed{I(z) = \frac{V^+}{Z_0} \left(e^{\gamma z} - \Gamma_L e^{-\gamma z}\right) \quad [\mathrm{A}]} \tag{14.94}$$

The line impedance at point z is

$$\boxed{Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{\left(e^{\gamma z} + \Gamma_L e^{-\gamma z}\right)}{\left(e^{\gamma z} - \Gamma_L e^{-\gamma z}\right)} \quad [\Omega]} \tag{14.95}$$

This expression is quite useful because it requires only knowledge of the reflection coefficient at the load, the characteristic impedance of the line, and the value of z (distance from the load). We will make considerable use of this expression here and in the following chapter.

Another way to look at the expression in **Eq. (14.95)** is to use the definition of the reflection coefficient in **Eq. (14.91)** and substitute it in **Eq. (14.95)**. Doing so and rearranging terms gives

$$Z(z) = Z_0 \frac{\left((Z_L + Z_0) e^{\gamma z} + (Z_L - Z_0) e^{-\gamma z}\right)}{\left((Z_L + Z_0) e^{\gamma z} - (Z_L - Z_0) e^{-\gamma z}\right)} = Z_0 \frac{Z_L \left(e^{\gamma z} + e^{-\gamma z}\right) + Z_0 \left(e^{\gamma z} - e^{-\gamma z}\right)}{Z_0 \left(e^{\gamma z} + e^{-\gamma z}\right) + Z_L \left(e^{\gamma z} - e^{-\gamma z}\right)} \quad [\Omega] \tag{14.96}$$

Now, we can use the identities $(e^{\gamma z} + e^{-\gamma z})/2 = \cosh\gamma z$ and $(e^{\gamma z} - e^{-\gamma z})/2 = \sinh\gamma z$ and write

$$\boxed{Z(z) = Z_0\frac{Z_L\cosh\gamma_z + Z_0\sinh\gamma_z}{Z_0\cosh\gamma_z + Z_L\sinh\gamma_z} = Z_0\frac{Z_L + Z_0\tanh\gamma z}{Z_0 + Z_L\tanh\gamma z}\quad [\Omega]} \tag{14.97}$$

where the relation $\tanh\gamma z = \sinh\gamma z/\cosh\gamma z$ was used. With these relations, we can now calculate the line impedance at any location, including at the input of the line.

Now, we can argue as follows: If the line impedance at a point on the line is equal to $Z(z)$, then cutting the line at this point and replacing the cut section by an equivalent load equal to $Z(z)$ should not change the conditions on the line to the left of the cut. This is shown in **Figure 14.18a**. The equivalent line in **Figure 14.18b** can be viewed as a new line with load impedance $Z(z)$. There is no reason we cannot calculate the reflection coefficient at this point on the line using **Eq. (14.91)** with $Z(z)$ instead of Z_L. Using **Eqs. (14.91)** and **(14.94)**, we get for the reflection coefficient at point z on the line

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V^+\Gamma_L e^{-\gamma z}}{V^+ e^{\gamma z}} = \frac{\Gamma_L e^{-\gamma z}}{e^{\gamma z}} = \Gamma_L e^{-2\gamma z} \tag{14.98}$$

or using the form in **Eq. (14.92)** and also the relation $\gamma = \alpha + j\beta$,

$$\boxed{\Gamma(z) = \Gamma_L e^{-2\gamma z} = \Gamma_L e^{-2\alpha z - j2\beta z} = |\Gamma_L| e^{-2\alpha z} e^{j\theta_\Gamma} e^{-j2\beta z}} \tag{14.99}$$

The reflection coefficient $\Gamma(z)$ is called the ***generalized reflection coefficient*** to distinguish it from the load reflection coefficient. The generalized reflection coefficient on a general, lossy line can be viewed as having an amplitude $|\Gamma_L|$ at the load, which decays exponentially (for a lossy line) as we move toward the generator, and a phase which varies linearly with z and is equal to

$$\boxed{\phi_{\Gamma(z)} = \theta_\Gamma - 2\beta z \quad [\text{rad}]} \tag{14.100}$$

Although these relations are rather general, we will, for the most part, use lossless transmission lines. This simply means that $\alpha = 0$ and $\gamma = j\beta$, but doing so will simplify analysis considerably.

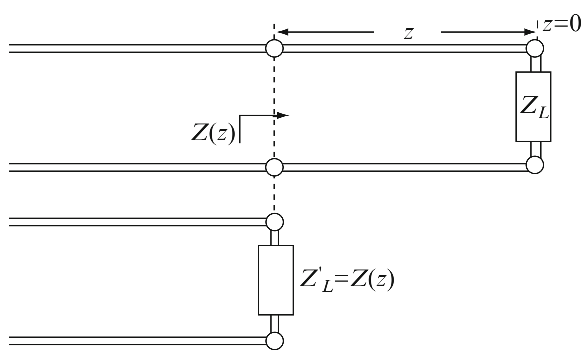


Figure 14.18 Method of calculation of the generalized reflection coefficient. (**a**) The impedance on the line at a general point, viewed as a new load to the line to the *left* of the point. (**b**) The new line and load

Example 14.12 A transmission line has propagation constant $\gamma = 0.01 + j0.05$, characteristic impedance $Z_0 = 50\ \Omega$, and a load $Z_L = 50 + j50\ \Omega$ is connected at one end. Calculate:

(a) The impedance on the line at the load.
(b) The impedance at a distance of 10 m from the load.
(c) Plot the line impedance as a function of distance from load.

Solution: The impedance on the line may be calculated from **Eq. (14.95)** or from **Eq. (14.97)**. The latter is usually more convenient.

(a) To find the impedance at the load, we set $z = 0$,

$$Z(z=0) = Z_0 \frac{Z_L + Z_0 \tanh(0)}{Z_0 + Z_L \tanh(0)} = Z_L = 50 + j50 \quad [\Omega]$$

This, of course, could have been guessed, but the calculation shows that the line impedance formula applies anywhere on the line.

(b) The impedance at a distance $z = 10$ m on the line may also be calculated using **Eq. (14.97)** by setting $z = 10$ m. However, we will use **Eq. (14.95)** to demonstrate its use. To do so, we first calculate the reflection coefficient at the load (see **Example 14.10**):

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 + j2}{5} = \frac{1}{\sqrt{5}} e^{j0.352\pi}$$

The line impedance at $z = 10$ m is

$$Z(z) = Z_0 \frac{\left(e^{\gamma z} + \dfrac{1}{\sqrt{5}} e^{j0.352\pi} e^{-\gamma z}\right)}{\left(e^{\gamma z} - \dfrac{1}{\sqrt{5}} e^{j0.352\pi} e^{-\gamma z}\right)} = 50 \frac{\left(\sqrt{5} e^{(0.1+j0.5)} + e^{(-0.1+j(0.352\pi-0.5))}\right)}{\left(\sqrt{5} e^{(0.1+j0.5)} - e^{(-0.1+j(0.352\pi-0.5))}\right)} = 50 \frac{2.91 + j1.7}{1.4249 + j0.6695}$$

$$= 106.68 + j9.53 \quad [\Omega]$$

where all angles were measured in radians.

(c) The line impedance (magnitude) as a function of distance from the load is shown in **Figure 14.19** for the first 200 m. Because of attenuation, the load's effect diminishes with distance.

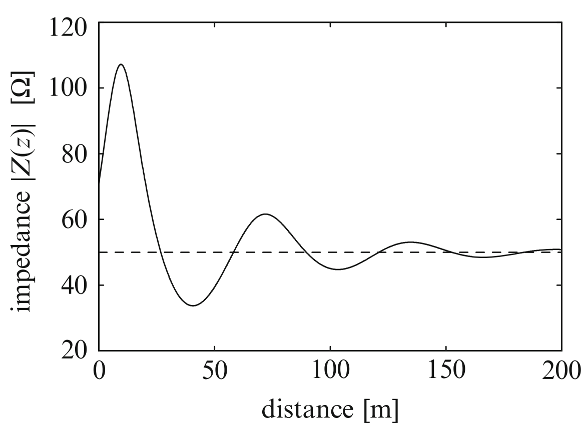


Figure 14.19 Impedance [Ω] along a lossy transmission line

14.7.3 The Lossless, Terminated Transmission Line

In all of the above relations, we assumed a general lossy transmission line in which the propagation constant is a general complex number. There was no reason to do otherwise since we could always replace γ by $\alpha + j\beta$ to obtain the expressions in terms of the attenuation and phase constants α and β for any condition. However, the expression in **Eq. (14.97)** requires the use of hyperbolic sine, cosine, and tangent functions. If the line is lossless, then $\alpha = 0$ and $\gamma = j\beta$. Under these conditions, the voltage and current on the line [setting $\gamma = j\beta$ in **Eq. (14.94)**] are

$$V(z) = V^+\left(e^{j\beta z} + \Gamma_L e^{-j\beta z}\right) \quad [\text{V}] \quad \text{and} \quad I(z) = \frac{V^+}{Z_0}\left(e^{j\beta z} - \Gamma_L e^{-j\beta z}\right) \quad [\text{A}] \tag{14.101}$$

Similarly, the line impedance of a lossless transmission line is found by setting $\gamma = j\beta$ in **Eq. (14.97)**:

$$\boxed{Z(z) = Z_0 \frac{Z_L + jZ_0 \tan\beta z}{Z_0 + jZ_L \tan\beta z} = Z_0 \frac{Z_L \cos\beta z + jZ_0 \sin\beta z}{Z_0 \cos\beta z + jZ_L \sin\beta z} \quad [\Omega]} \tag{14.102}$$

where $\tanh(j\beta z) = j\tan(\beta z)$ was used. In general, the line impedance is a complex value, as we should expect. The latter expression is also useful in that it indicates explicitly the periodic nature of the line impedance and that the period is directly related to the term βz, which in **Section 14.4** we called the ***electrical length*** of the transmission line. Not surprisingly, the electrical length of the line plays an important role in line behavior.

The generalized reflection coefficient for lossless lines was obtained in **Eq. (14.99)**. Like the line impedance, the reflection coefficient is periodic along the line. This is best seen if the exponential function is written as $e^{-j2\beta z} = \cos 2\beta z - j\sin 2\beta z$. The generalized reflection coefficient now is

$$\boxed{\Gamma(z) = \Gamma_L e^{-j2\beta z} = |\Gamma_L| e^{j\theta_\Gamma} e^{-j2\beta z} = \Gamma_L\left(\cos(\theta_\Gamma - 2\beta z) - j\sin(\theta_\Gamma - 2\beta z)\right)} \tag{14.103}$$

Thus, the generalized reflection coefficient for lossless transmission lines can be viewed as having constant amplitude equal to that of $|\Gamma_L|$ but varying in phase along the line as

$$\boxed{\phi_{\Gamma(z)} = \theta_\Gamma - 2\beta z} \tag{14.104}$$

Because of this phase angle, the generalized reflection coefficient has maxima and minima along the line. However, it is more convenient to talk of maxima and minima in voltage or current, or both. Consider **Eq. (14.101)**. Rearranging the terms, we get the voltage on the line as

$$V(z) = V^+\left(e^{j\beta z} + \Gamma_L e^{-j\beta z}\right) = V^+ e^{j\beta z}\left(1 + \Gamma_L e^{-j2\beta z}\right) = V^+ e^{j\beta z}(1 + \Gamma(z)) \quad [\text{V}] \tag{14.105}$$

Similarly, the current on the line is

$$I(z) = \frac{V^+}{Z_0}\left(e^{j\beta z} - \Gamma_L e^{-j\beta z}\right) = \frac{V^+}{Z_0} e^{j\beta z}\left(1 - \Gamma_L e^{-j2\beta z}\right) = \frac{V^+}{Z_0} e^{j\beta z}(1 - \Gamma(z)) \quad [\text{A}] \tag{14.106}$$

Now, we can discuss the maximum and minimum magnitudes of the voltage. First, we note that the term $e^{j\beta z}$ varies between -1 and +1. Thus, its magnitude is 1. Similarly, the generalized reflection coefficient $\Gamma(z)$ varies between $-\Gamma(z)$ and $+\Gamma(z)$ because the term $e^{-j2\beta z}$ varies between -1 and +1. Thus, we can write the maximum and minimum magnitudes of voltage as

$$V_{max} = \left|V^+\right|\left(1 + \left|\Gamma(z)\right|\right) \quad [\text{V}] \tag{14.107}$$

$$V_{min} = \left|V^+\right|\left(1 - \left|\Gamma(z)\right|\right) \quad [\text{V}] \tag{14.108}$$

The same can be done for the current. Following identical steps but starting with **Eq. (14.106)**, we get

$$I_{max} = \frac{V_{max}}{|Z_0|} = \left|\frac{V^+}{Z_0}\right|\left(1 + \left|\Gamma(z)\right|\right) \quad [\text{A}] \tag{14.109}$$

$$I_{min} = \frac{V_{min}}{|Z_0|} = \left|\frac{V^+}{Z_0}\right|\left(1 - \left|\Gamma(z)\right|\right) \quad [\text{A}] \tag{14.110}$$

The ratio between the maximum and minimum voltage (or current) is called the ***standing wave ratio*** (SWR) and is defined as

$$\boxed{\mathrm{SWR}=\frac{V_{max}}{V_{min}}=\frac{I_{max}}{I_{min}}=\frac{1+|\Gamma(z)|}{1-|\Gamma(z)|}\quad[\text{dimensionless}]} \tag{14.111}$$

The standing wave ratio varies between 1 and ∞. If the reflection coefficient is zero (no reflected waves), the standing wave ratio is 1. If the magnitude of the reflection coefficient is 1, the standing wave ratio is ∞. Thus, a matched load produces no reflected waves and the line should have a standing wave ratio of 1.

Sometimes, the standing wave ratio is known or may be measured. In such cases, the magnitude of the generalized reflection coefficient can be calculated from the standing wave ratio as

$$\boxed{|\Gamma(z)|=\frac{\mathrm{SWR}-1}{\mathrm{SWR}+1}} \tag{14.112}$$

This expression can be substituted in **Eqs. (14.107)** and **(14.108)** to obtain the minimum and maximum voltage on the line in terms of the standing wave ratio:

$$V_{max}=|V^+|\quad(1+|\Gamma(z)|)=|V^+|\left(1+\frac{\mathrm{SWR}-1}{\mathrm{SWR}+1}\right)=|V^+|\quad\left(\frac{2\mathrm{SWR}}{\mathrm{SWR}+1}\right)\quad[\mathrm{V}] \tag{14.113}$$

$$V_{min}=|V^+|\quad(1-|\Gamma(z)|)=|V^+|\left(1-\frac{\mathrm{SWR}-1}{\mathrm{SWR}+1}\right)=|V^+|\quad\left(\frac{2}{\mathrm{SWR}+1}\right)\quad[\mathrm{V}] \tag{14.114}$$

From the last three equations, it is apparent that the effect of the standing wave ratio is as follows:

(1) The larger the standing wave ratio, the larger the maximum voltage and the lower the minimum voltage on the line.
(2) If SWR $=1$, the reflection coefficient is zero. In this case, $V_{max}=V_{min}=|V^+|$. The magnitude of the voltage on the line does not vary. The phase of course varies. This corresponds to a matched load.
(3) If SWR $=\infty$, the magnitude of the reflection coefficient equals 1 ($\Gamma(z)=-1$ or $\Gamma(z)=+1$). In this case, $V_{max}=2|V^+|$ and $V_{min}=0$. We will see shortly that this corresponds to either a short circuit ($\Gamma(z)=-1$) or an open circuit ($\Gamma(z)=+1$). This condition in plane waves was called a complete standing wave.

Now that we have all the tools to calculate the reflection coefficient anywhere on the line as well as the standing wave ratio, we can return to the equations for current and voltage and see how these behave along the line. The basis of calculation are **Eqs. (14.105)** and **(14.106)**. Voltage and current anywhere on the line (including at the load) are

$$\boxed{V(z)=V^+e^{j\beta z}\left(1+\Gamma_Le^{-j2\beta z}\right)=V^+e^{j\beta z}\left(1+|\Gamma_L|e^{j\theta_\Gamma}e^{-j2\beta z}\right)\quad[\mathrm{V}]} \tag{14.115}$$

$$\boxed{I(z)=\frac{V^+}{Z_0}e^{j\beta z}\left(1-\Gamma_Le^{-j2\beta z}\right)=\frac{V^+}{Z_0}e^{j\beta z}\left(1-|\Gamma_L|e^{j\theta_\Gamma}e^{-j2\beta z}\right)\quad[\mathrm{A}]} \tag{14.116}$$

We can also calculate the voltage and current at the load. These are obtained by setting $z=0$:

$$\boxed{V_L=V^+\left(1+|\Gamma_L|e^{j\theta_\Gamma}\right)\quad[\mathrm{V}]}\quad\text{and}\quad\boxed{I_L=\frac{V^+}{Z_0}\left(1-|\Gamma_L|e^{j\theta_\Gamma}\right)\quad[\mathrm{A}]} \tag{14.117}$$

To completely characterize the voltage and current waves, we must find the locations of the minima and maxima on the line. Suppose we plot the voltage and current starting at the load and going toward the generator. For any given load, the load reflection coefficient is known and we can calculate the voltage and current at the load **[Eq. (14.117)]** and the maximum and minimum voltage and current [from **Eqs. (14.107)** through **(14.110)**]. We could, in fact, use **Eqs. (14.115)** and **(14.116)** to plot the voltage and current directly. The only other bit of information needed is the location of minima and maxima in the voltage and current waves. These are found as follows.

From inspection of **Eqs. (14.115)** and **(14.116)**, the minimum in voltage must occur at locations on the line at which the phase $\theta_\Gamma - 2\beta z$ equals $-\pi, -3\pi, -5\pi$, etc. The general condition to be satisfied (taking z to be positive to the left and away from the load) is

$$\theta_\Gamma - 2\beta z = -(2n+1)\pi, \quad n = 0,\ 1,\ 2,\ \ldots \tag{14.118}$$

This condition can be verified by direct substitution in **Eq. (14.108)** or **Eq. (14.115)**. On the other hand, the current is maximum at this point because of the negative sign in front of Γ_L in **Eq. (14.110)** or **Eq. (14.116)**. The location of the first minimum in voltage (maximum in current) occurs at

$$\theta_\Gamma - 2\beta z_{min} = -\pi \quad \rightarrow \quad z_{min} = \frac{\theta_\Gamma + \pi}{2\beta} \quad [\mathrm{m}] \tag{14.119}$$

The next minimum occurs at

$$\theta_\Gamma - 2\beta z = -3\pi \quad \rightarrow \quad z = \frac{\theta_\Gamma + 3\pi}{2\beta} = \frac{\theta_\Gamma + \pi}{2\beta} + \frac{\pi}{\beta} \quad [\mathrm{m}] \tag{14.120}$$

From the definition of wavelength, we can also write these relations in terms of the wavelength by using the relation

$$\lambda = \frac{2\pi}{\beta} \quad \rightarrow \quad \frac{\pi}{\beta} = \frac{\lambda}{2} \quad \rightarrow \quad \frac{1}{2\beta} = \frac{\lambda}{4\pi} \tag{14.121}$$

Thus, the conditions for minima are

For the first minimum:

$$\boxed{z_{min} = \frac{\lambda}{4\pi}(\theta_\Gamma + \pi) \quad [\lambda]} \tag{14.122}$$

The unit [λ] shows that the distance is indicated in wavelengths. For any minimum:

$$\boxed{z_{min} = \frac{\lambda}{4\pi}(\theta_\Gamma + (2n+1)\pi) \quad [\lambda], \quad n = 0, 1, 2, \ldots} \tag{14.123}$$

This has the advantage of being described in terms of increments of $\lambda/2$.

The maxima occur at a distance of $\lambda/4$ on each side of a minimum. We know this must be so since the conditions on the line repeat at increments of $\lambda/2$. Between every two minima there is a maximum. Thus, we can calculate the location of the first voltage maximum by adding $\lambda/4$ in **Eq. (14.123)**. Voltage maxima (current minima) occur at

$$\boxed{z_{max} = \frac{\lambda}{4\pi}(\theta_\Gamma + (2n+1)\pi) + \frac{\lambda}{4} \quad \rightarrow \quad z = \frac{\lambda}{4\pi}(\theta_\Gamma + 2n\pi) \quad [\lambda], \quad n = 0, 1, 2, \cdots} \tag{14.124}$$

The complete description of voltage and current on the line is now shown in **Figure 14.20**. Note in particular that the minima are sharper than the maxima. In other words, the voltage or current do not vary sinusoidally. Whenever measurements of standing wave ratio are required, the minima are usually easier to identify. Note also that **Figure 14.20** assumes, arbitrarily, that $V_L > 0$ at the load. This does not have to be so: V_L can be negative or zero.

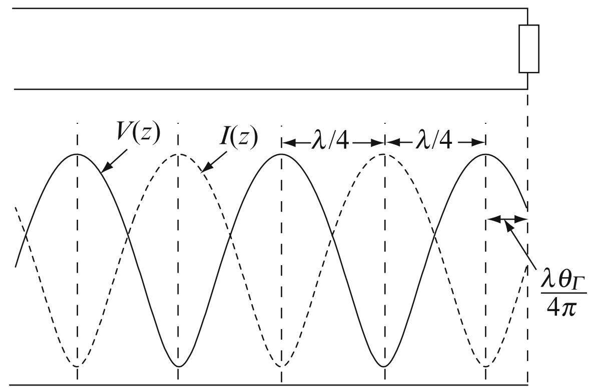


Figure 14.20 Locations of voltage maxima and minima on a transmission line and the relation between voltage and current minima and maxima on the line

The maxima in line impedance occur at locations of voltage maxima (current minima) and minima in line impedance occur at location of voltage minima (current maxima).

From the foregoing discussion, it is clear that voltage and current are highly dependent on the load reflection coefficient and they vary from point to point. From **Eq. (14.102)**, we can also tell that the line impedance varies from point to point. The above relations are general and apply to any load. The only restriction in the above discussion is that the line be lossless.

A number of particular solutions may be obtained for particular loads. These loads are useful because they lead to simple, practical solutions. These are as follows:

(1) Matched load: $Z_L = Z_0$. The load reflection coefficient is zero ($\Gamma_L = 0$).
(2) Short circuited load: $Z_L = 0$. The load reflection coefficient is $\Gamma_L = -1$.
(3) Open circuit load: $Z_L = \infty$. The load reflection coefficient is $\Gamma_L = +1$.
(4) Resistive load: $Z_L = R_L + j0$. The load reflection coefficient is real: $-1 < \Gamma_L < 1$.

These particular types of terminated transmission lines are discussed in the following sections.

Example 14.13 A transmission line with characteristic impedance of 100 Ω and a load of $50 - j50$ Ω is connected to a matched generator. The line is very long and the voltage measured at the load is 50 V. Calculate:

(a) The maximum and minimum voltage on the line (magnitude only).
(b) Location of maxima and minima of voltage on the line starting from the load.

Solution: The voltage on the line is best calculated from the standing wave ratio, which, in turn, is calculated from the load reflection coefficient. The location of a voltage maximum is that location at which the impedance is maximum. We first calculate the reflection coefficient and then the standing wave ratio. From these, the minimum and maximum amplitudes are calculated using **Eqs. (14.113)** and **(14.114)**. The location of minima and maxima is calculated from **Eqs. (14.123)** and **(14.124)**.

(a) The reflection coefficient at the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-50 - j50}{150 - j50} = \frac{-1 - j1}{3 - j1} = \frac{-1 - j2}{5} = \frac{1}{\sqrt{5}} e^{j0.6476\pi}$$

The standing wave ratio is calculated from the magnitude of the load reflection coefficient:

$$\text{SWR} = \frac{1 + |\Gamma(z)|}{1 - |\Gamma(z)|} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 1/\sqrt{5}}{1 - 1/\sqrt{5}} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} = 2.618$$

To calculate the minimum and maximum voltage on the line, we first need to calculate the forward wave amplitude. The total amplitude at the load is known. From **Eq. (14.117)**, we get at the load

$$V_L = V^+(1+\Gamma_L) \quad \rightarrow \quad V^+ = \frac{V_L}{(1+\Gamma_L)} = \frac{50}{(1+(-1-j2)/5)} = \frac{125}{2-j1} = 25(2+j1) \quad [\mathrm{V}]$$

The magnitude of the forward-propagating voltage is

$$|V^+| = |25(2+j1)| = \left|25\sqrt{5}\right| = 55.9 \quad [\mathrm{V}]$$

Thus, the maximum and minimum voltages are

$$V_{max} = |V^+|\left(\frac{2\mathrm{SWR}}{\mathrm{SWR}+1}\right) = 55.9\left(\frac{2\times 2.618}{3.618}\right) = 80.9 \quad [\mathrm{V}]$$

$$V_{min} = |V^+|\left(\frac{2}{\mathrm{SWR}+1}\right) = 55.9\left(\frac{2}{3.618}\right) = 30.9 \quad [\mathrm{V}].$$

(b) The first voltage minimum is calculated from **Eq. (14.122)**:

$$z_{min} = \frac{\lambda}{4\pi}(\theta_\Gamma + \pi) = \frac{\lambda}{4\pi}(0.6476\pi + \pi) = 0.412\lambda$$

All other minima are to the left, in increments of $\lambda/2$. Thus, the minima occur at $z = 0.412\lambda, 0.912\lambda, 1.412\lambda, 1.912\lambda, 2.412\lambda$, etc.

The maxima are one-quarter wavelength to the left and right of the above minima. These occur at $z = 0.162\lambda, 0.662\lambda, 1.162\lambda, 1.662\lambda, 2.162\lambda$, etc.

14.7.4 The Lossless, Matched Transmission Line

A matched transmission line is a line on which the load is equal to the characteristic impedance of the line:

$$\boxed{Z_L = Z_0} \tag{14.125}$$

Substitution of this condition in **Eq. (14.91)** results in a zero reflection coefficient at the load: $\Gamma_L = 0$. Thus, the line impedance anywhere on the line is

$$\boxed{Z(z) = Z_0\frac{Z_0 + jZ_0\tan\beta z}{Z_0 + jZ_0\tan\beta z} = Z_0 \quad [\Omega]} \tag{14.126}$$

Therefore, the impedance on the line for a matched load is constant and equal to Z_0.

The other relations on the line are also obtained by substituting $Z_L = Z_0$ and $\Gamma_L = 0$. Thus, the standing wave ratio on the line is SWR $= 1$ anywhere on the line. The voltage and current on the line are

$$\boxed{V(z) = V^+e^{j\beta z} \quad [\mathrm{V}]} \quad \text{and} \quad \boxed{I(z) = \frac{V^+}{Z_0}e^{j\beta z} \quad [\mathrm{A}]} \tag{14.127}$$

That is, the line voltage and current have only forward-propagating terms, as we expect with a zero reflection coefficient.

In summary, a matched load produces no reflected waves and, therefore, no standing waves on the line. All power on the line is transferred to the load.

14.7.5 The Lossless, Shorted Transmission Line

A shorted transmission line is characterized by $Z_L = 0$. From **Eq. (14.91)**, the reflection coefficient is $\Gamma_L = -1$. For the same reason, SWR $= \infty$. The line impedance is now

$$\boxed{Z(z) = Z_0 \frac{jZ_0 \tan\beta z}{Z_0} = jZ_0 \tan\beta z \quad [\Omega]} \tag{14.128}$$

The line impedance of a shorted transmission line is purely imaginary and varies between $-\infty$ and ∞. It has the following properties:

(1) $\Gamma_L = -1$, SWR $= \infty$.
(2) It is zero at the load and at any value $\beta z = n\pi$, $n = 1, 2, \ldots$. In terms of wavelength, the line impedance is zero at $z = n\lambda/2$, $n = 0, 1, 2, \ldots$ and is infinite at $z = n\lambda/2 + \lambda/4$, $n = 0, 1, 2, \ldots$.
(3) The line impedance is purely imaginary and alternates between positive and negative values, as shown in **Figure 14.21**. The impedance is positive (inductive) for $n\lambda/2 < z < n\lambda/2 + \lambda/4$ and negative (capacitive) between $n\lambda/2 + \lambda/4 < z < n\lambda/2 + \lambda/2$, $n = 0, 1, 2, \ldots$ The line impedance changes from $+\infty$ to $-\infty$ at $z = n\lambda/2 + \lambda/4$.
(4) A shorted transmission line behaves as an inductor or a capacitor, depending on the location on the line. A capacitance or an inductance may be designed by simply cutting a line of appropriate length as indicated in **(3)**. In this sense, shorted transmission lines are viewed as circuit elements.
(5) The conditions on a shorted transmission line repeat at intervals of $\lambda/2$; that is, if we add or remove a section of length $\lambda/2$ (or any integer multiple of $\lambda/2$), the line impedance does not change.

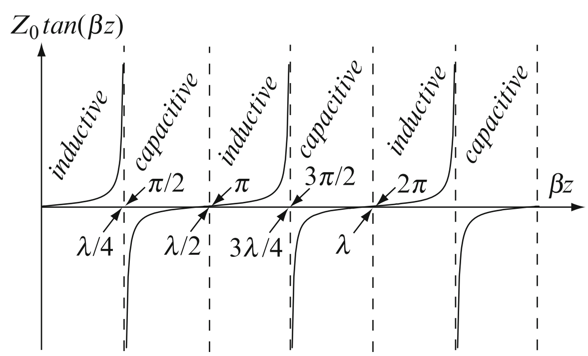


Figure 14.21 Line impedance on a shorted transmission line

The line voltage and line current are [setting $\Gamma_L = -1$ in **Eqs. (14.115)** and **(14.116)**]

$$\boxed{V(z) = V^+ e^{j\beta z}\left(1 - e^{-j2\beta z}\right) \quad [\text{V}]} \quad \text{and} \quad \boxed{I(z) = \frac{V^+}{Z_0} e^{j\beta z}\left(1 + e^{-j2\beta z}\right) \quad [\text{A}]} \tag{14.129}$$

In particular, at the load ($z = 0$), we get

$$V_L = 0 \quad [\text{V}] \quad \text{and} \quad I_L = \frac{2V^+}{Z_0} \quad [\text{A}] \tag{14.130}$$

Thus, whereas the voltage at the load must be zero, the current must be twice the forward-propagating current. This, of course, is a consequence of the fact that there is no transfer of power into the load and the reflected current is equal in magnitude and phase to the forward current.

14.7.6 The Lossless, Open Transmission Line

An open transmission line may be assumed to have an infinite impedance as load. Since $Z_L \to \infty$, the reflection coefficient at the load is $\Gamma_L = +1$. For the same reason, SWR $= \infty$. Substitution of Z_L into the line impedance in **Eq. (14.102)** gives (since $Z_L \gg Z_0$)

$$\boxed{Z(z) = Z_0 \frac{Z_L + jZ_0 \tan\beta z}{Z_0 + jZ_L \tan\beta z} = Z_0 \frac{Z_L}{jZ_L \tan\beta z} = -jZ_0 \cot\beta z \quad [\Omega]} \tag{14.131}$$

This result is very similar to the result for the shorted transmission line. The properties of this line are summarized as follows:

(1) $\Gamma_L = +1$, SWR $= \infty$.
(2) The line impedance is infinite at the load and at any value $\beta z = n\pi$, $n = 1, 2,\ldots$ In terms of wavelength, the line impedance is infinite at $z = n\lambda/2$, $n = 0, 1, 2,\ldots$ The line impedance is zero at $z = n\lambda/2 + \lambda/4$, $n = 0, 1, 2,\ldots$
(3) The line impedance is purely imaginary and alternates between positive and negative values, as shown in **Figure 14.22**. The impedance is negative (capacitive) for $n\lambda/2 < z < n\lambda/2 + \lambda/4$ and positive (inductive) between $n\lambda/2 + \lambda/4 < z < n\lambda/2 + \lambda/2$, $n = 0, 1, 2,\ldots$ The line impedance changes from $+\infty$ to $-\infty$ at $z = n\lambda/2$.
(4) An open transmission line behaves as an inductor or a capacitor, depending on the location on the line. A capacitance or an inductance may be designed by simply cutting a line of appropriate length as indicated in **(3)**. Open transmission lines may also be viewed as circuit elements.
(5) The conditions on an open transmission line repeat at intervals of $\lambda/2$; that is, if we add or remove a section of length $\lambda/2$ (or any integer multiple of $\lambda/2$), the line impedance is not affected.
(6) The conditions on an open transmission line are identical to those of a shorted transmission line if their length differs by an odd multiple of $\lambda/4$. This can be seen by direct comparison of **Figures 14.22** and **14.21**.

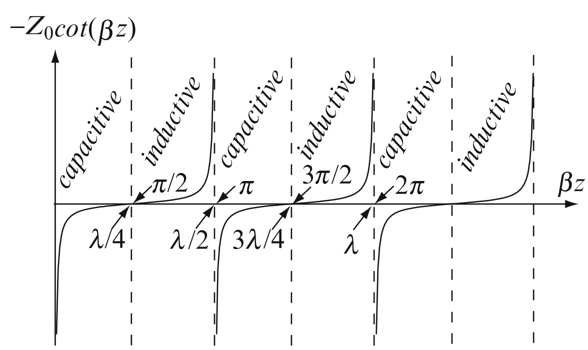


Figure 14.22 Line impedance on an open transmission line

The line voltage and line current on the open transmission line are

$$\boxed{V(z) = V^+ e^{j\beta z}\left(1 + e^{-j2\beta z}\right) \quad [\text{V}]} \quad \text{and} \quad \boxed{I(z) = \frac{V^+}{Z_0} e^{j\beta z}\left(1 - e^{-j2\beta z}\right) \quad [\text{A}]} \tag{14.132}$$

In particular, at the load ($z = 0$), we get

$$V_L = 2V^+ \quad [\text{V}] \quad \text{and} \quad I_L = 0 \tag{14.133}$$

Thus, maximum voltage occurs at the load, whereas maximum current occurs at $\lambda/4$ from the load. Again, there is no transfer of power into the load and the reflected voltage is equal to the forward-propagating voltage (and in the same phase).

Suppose now that we perform an experiment. First, we short a transmission line and obtain the line impedance at a point z. Then, we open the line and obtain the impedance at the same point. The shorted and open line impedances are those given in **Eqs. (14.128)** and **(14.131)**. If we take the product of these two impedances, we get

$$(jZ_0\tan\beta z)(-jZ_0\cot\beta z) = Z_0^2 \tag{14.134}$$

Perhaps a bit unexpected result but it gives us yet another way of calculating or measuring the characteristic impedance of a transmission line. The characteristic impedance of any lossless line is given as

$$\boxed{Z_0 = \sqrt{Z_{short}Z_{open}} \quad [\Omega]} \tag{14.135}$$

where Z_{short} is the line impedance with shorted load and Z_{open} is the line impedance with open load.

Exercise 14.6 The characteristic impedance of a transmission line is not known. To determine it, it is suggested to measure the short and open impedances at a point on the line; that is, the load is shorted and the impedance at a point on the line is measured. Then, the load is disconnected (open line) and the impedance at the same point on the line is again measured. These measurements give an open line impedance of $-j50$ and a shorted line impedance of $j75$. Calculate the characteristic impedance of the line.

Answer 61.24 Ω.

Exercise 14.7 Show that the relation in **Eq. (14.135)** holds for real or complex characteristic impedances.

14.7.7 The Lossless, Resistively Loaded Transmission Line

The discussion in **Sections 14.7.1** and **14.7.2** was in terms of a general load, but it applies equally well for a resistive load: $Z_L = R_L + j0$. The reflection coefficient at the load is real: $-1 < \Gamma_L < 1$:

$$\boxed{\Gamma_L = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0}} \tag{14.136}$$

and since Z_0 is also real for lossless lines [see **Eq. (14.57)**], the reflection coefficient is real. It can be either positive or negative depending on the relative magnitudes of R_L and Z_0. The line impedance is now given as

$$\boxed{Z(z) = Z_0\frac{R_L + jZ_0\tan\beta z}{Z_0 + jR_L\tan\beta z} = Z_0\frac{R_L\cos\beta z + jZ_0\sin\beta z}{Z_0\cos\beta z + jR_L\sin\beta z} \quad [\Omega]} \tag{14.137}$$

This impedance is maximum at locations of maximum voltage and minimum at locations of minimum voltage, as described in **Section 14.7.3**. The main difference between a resistive load and a general load is that for a general load, the phase angle of the load reflection coefficient can have any value. On the other hand, for a resistive load, the phase angle can be either zero or $-\pi$. This can be seen from **Eq. (14.136)**. There are two possible situations:

(1) $R_L > Z_0$. In this case, Γ_L is real, positive and we can write

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \quad \rightarrow \quad \Gamma_L = |\Gamma_L|e^{j0} \tag{14.138}$$

Now, if we substitute $\theta_\Gamma = 0$ in **Eqs. (14.115)** and **(14.116)**, we obtain the general voltage and current waves on the line:

$$\boxed{V(z) = V^+e^{j\beta z}\left(1 + \Gamma_L e^{-j2\beta z}\right) \quad [\text{V}]} \tag{14.139}$$

$$\boxed{I(z) = \frac{V^+}{Z_0} e^{j\beta z}\left(1 - \Gamma_L e^{-j2\beta z}\right) \quad [\mathrm{A}]} \tag{14.140}$$

The voltage and current at the load are

$$V_L = V^+(1 + \Gamma_L) \quad [\mathrm{V}] \quad \text{and} \quad I_L = \frac{V^+}{Z_0}(1 - \Gamma_L) \quad [\mathrm{A}] \tag{14.141}$$

The locations of voltage minima are now **[see Eqs. (14.122) and (14.123)]**

$$\boxed{z_{min} = \frac{\lambda}{4\pi}(2n + 1)\pi \quad [\lambda], \quad n = 0, 1, 2, \ldots} \tag{14.142}$$

Thus, the first minimum in voltage occurs at $n = 0$:

$$z_{min} = \frac{\pi}{2\beta} = \frac{\lambda}{4} \quad [\lambda] \tag{14.143}$$

Similarly, the locations of voltage maxima are **[Eq. (14.124)]**

$$z_{max} = \frac{\lambda}{4\pi} 2n\pi \quad [\lambda], \quad n = 0, 1, 2, \ldots \tag{14.144}$$

The first voltage maximum is at the load ($z = 0$). The following voltage maxima (current minima) are at increments of $\lambda/2$ from the load. The voltage and current minima and maxima are shown in **Figure 14.23a**.

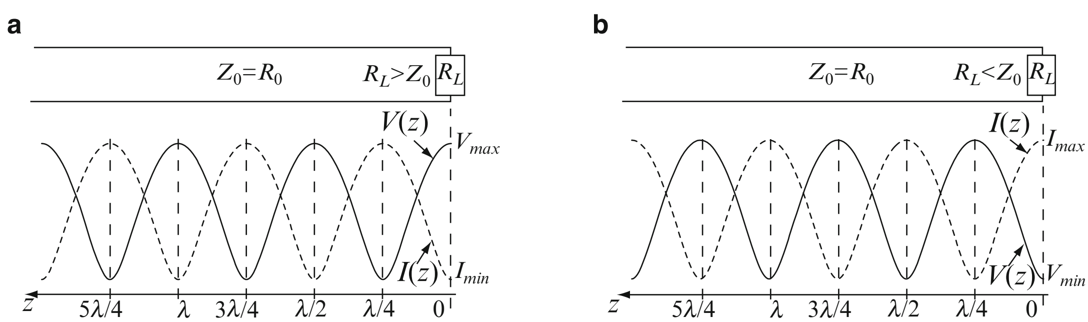


Figure 14.23 (**a**) Voltage and current maxima and minima for $R_L > Z_0$. (**b**) Voltage and current maxima and minima for $R_L < Z_0$

(2) $R_L < Z_0$. In this case, Γ_L is real and negative and we can write

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \quad \rightarrow \quad \Gamma_L = -|\Gamma_L| = |\Gamma_L| e^{-j\pi} \tag{14.145}$$

Now, if we substitute $\theta_\Gamma = -\pi$ in **Eqs. (14.115)** and **(14.116)**, we obtain the general voltage and current waves on the line:

$$\boxed{V(z) = V^+ e^{j\beta z}\left(1 + |\Gamma_L| e^{-j\pi} e^{-j2\beta z}\right) \quad [\mathrm{V}]} \tag{14.146}$$

$$\boxed{I(z) = \frac{V^+}{Z_0} e^{j\beta z}\left(1 - |\Gamma_L| e^{-j\pi} e^{-j2\beta z}\right) \quad [\mathrm{A}]} \tag{14.147}$$

The voltage and current at the load are

$$V_L = V^+(1 - |\Gamma_L|) \quad [\text{V}] \quad \text{and} \quad I_L = \frac{V^+}{Z_0}(1 + |\Gamma_L|) \quad [\text{A}] \tag{14.148}$$

The locations of minima in voltage are

$$\boxed{z_{min} = \frac{\lambda}{4\pi}(2n\pi) \quad [\lambda], \quad n = 0, 1, 2, \ldots} \tag{14.149}$$

Thus, the first minimum in voltage occurs at $z = 0$. Subsequent minima occur at intervals of $\lambda/2$ from the load. The first voltage maximum occurs at $z = \lambda/4$ and the general relation for voltage maxima is

$$\boxed{z_{max} = \frac{\lambda}{4\pi}(2n - 1)\pi \quad [\lambda], \quad n = 0, 1, 2, \ldots} \tag{14.150}$$

The complete description of voltage and current on the line for $R_L < Z_0$ is shown in **Figure 14.23b**. Note also that the maximum and minimum line impedance are given as

$$\boxed{Z_{max} = Z_0\frac{(1 + |\Gamma_L|)}{(1 - |\Gamma_L|)}} = Z_0\text{SWR} \quad [\Omega] \tag{14.151}$$

$$\boxed{Z_{min} = Z_0\frac{(1 - |\Gamma_L|)}{(1 + |\Gamma_L|)}} = \frac{Z_0}{\text{SWR}} \quad [\Omega] \tag{14.152}$$

The properties of line impedance on a resistively loaded line are

(1) $-1 < \Gamma_L < +1$, $1 < \text{SWR} < \infty$.
(2) The line impedance is maximum at locations of voltage maxima and minimum at locations of voltage minima. These locations are given in **Eqs. (14.142)** and **(14.144)** for $R_L > Z_0$ and in **Eqs. (14.149)** and **(14.150)** for $R_L < Z_0$.
(3) The line impedance can be complex as can be seen from **Eq. (14.137)**, but it is always real at locations of voltage maxima and voltage minima for any lossless line. The impedance at voltage maxima is $Z_{max} = Z_0\text{SWR}$, whereas at voltage minima (current maxima), it is $Z_{min} = Z_0/\text{SWR}$.
(4) For $R_L > Z_0$, the first voltage maximum occurs at the load ($z = 0$) and the first voltage minimum at a distance $\lambda/4$ from the load. All conditions on the line repeat at intervals of $\lambda/2$.
(5) For $R_L < Z_0$, the first voltage minimum occurs at the load and the first voltage maximum at a distance $\lambda/4$ from the load. All conditions on the line repeat at intervals of $\lambda/2$.

In effect, the main difference between a general load and a resistive load is the location of the minima and maxima. If the load is such that the magnitude of the reflection coefficient at the load is the same for resistive and arbitrary loads, the voltage and current on the line will be the same in both cases but displaced by the value of z_{min} in **Eq. (14.142)** or **(14.122)**. In other words, if we take an arbitrary load and calculate all circuit parameters, we obtain the standing wave pattern for the line. The line can now be shortened by the magnitude of z_{min} or lengthened by $\lambda/2 - z_{min}$ to obtain an identical circuit but with a resistive loading which has the same reflection coefficient magnitude. We will use this property of transmission lines in the following chapter.

14.8 Power Relations on a General Transmission Line

The power relation on a line can be written directly from the current and voltage on the line. The power at a distance z_0 from the load can be calculated by assuming that the load is at $z = 0$ and the input is at $z = z_0$ as shown in **Figure 14.24**. For this condition, the line voltage and current for a general lossy line are given in **Eq. (14.94)**. Setting $z = z_0$ gives the voltage and current as

$$V(z_0) = V^+(e^{\gamma z_0} + \Gamma_L e^{-\gamma z_0}) \quad [\text{V}] \quad \text{and} \quad I(z_0) = \frac{V^+}{Z_0}(e^{\gamma z_0} - \Gamma_L e^{-\gamma z_0}) \quad [\text{A}] \tag{14.153}$$

where Z_0 is the line characteristic impedance given in **Eq. (14.42)** and is, in general, a complex number. Now, the power entering this section of the transmission line is calculated from the current and voltage on the line at this point:

$$P_i = \frac{1}{2}\text{Re}\left\{V_{z_0} I_{z_0}^*\right\} = \frac{1}{2}\text{Re}\left\{[V^+(e^{\gamma z_0} + \Gamma_L e^{-\gamma z_0})]\left[\frac{V^+}{Z_0}(e^{\gamma z_0} - \Gamma_L e^{-\gamma z_0})\right]^*\right\}$$
$$= \frac{|V^+|^2}{2}\text{Re}\left\{\left(e^{(\alpha+j\beta)z_0} + |\Gamma_L|e^{j\theta_{z_0}}e^{-(\alpha+j\beta)z_0}\right)\frac{\left(e^{(\alpha+j\beta)z_0} - |\Gamma_L|e^{-j\theta_{z_0}}e^{-(\alpha+j\beta)z_0}\right)}{Z_0^*}\right\}$$
$$= \frac{|V^+|^2}{2|Z_0|}\text{Re}\left\{\left(e^{2\alpha z_0} + |\Gamma_L|e^{j\left(\theta_{z_0}-\beta z_0\right)} - |\Gamma_L|e^{-j\left(\theta_{z_0}-\beta z_0\right)} - |\Gamma_L|^2 e^{-2\alpha z_0}\right)e^{-j\theta_{Z_0}}\right\}$$
$$= \frac{|V^+|^2}{2|Z_0|}\text{Re}\left\{\left(e^{2\alpha z_0} + j2|\Gamma_L|\sin(\theta_{z_0} - \beta z_0) - |\Gamma_L|^2 e^{-2\alpha z_0}\right)e^{-j\theta_{z_0}}\right\}$$
$$= \frac{|V^+|^2}{2|Z_0|}\left(e^{2\alpha z_0} - |\Gamma_L|^2 e^{-2\alpha z_0}\right)\cos(\theta_{Z_0}) \quad [\text{W}] \tag{14.154}$$

where θ_{Z_0} is the phase angle of the characteristic impedance and θ_{Z_0} is the phase angle at $Z = Z_0$. To summarize

$$\boxed{P_{z_0} = \frac{|V^+|^2}{2|Z_0|}\left(e^{2\alpha z_0} - |\Gamma_L|^2 e^{-2\alpha z_0}\right)\cos(\theta_{Z_0}) \quad [\text{W}]} \tag{14.155}$$

This relation has two components: The first is forward-propagating toward the load (in the negative z direction according to our convention which defines the load as reference) and the second in the positive z direction (from load to generator). Both are real powers and the total power is the sum of the two.

The power entering any section of the line may now be evaluated by setting the correct value for z_0. The power at the load may be found by setting $z_0 = 0$:

$$P_{load} = \frac{|V^+|^2}{2|Z_0|}\left(1 - |\Gamma_L|^2\right)\cos(\theta_{Z_0}) \quad [\text{W}] \tag{14.156}$$

The general power relation in **Eq. (14.155)** may be simplified under certain conditions. If there is only a forward-propagating wave, the forward-propagating voltage, current, and power are

$$V^+(z_0) = V^+ e^{\gamma z_0} \quad [\text{V}], \quad I^+(z_0) = \frac{V^+}{Z_0}e^{\gamma z_0} \quad [\text{A}], \quad P^+(z_0) = \frac{|V^+|}{2|Z_0|}e^{2\alpha z_0}\cos(\theta_{Z_0}) \quad [\text{W}] \tag{14.157}$$

For the backward propagating wave alone

$$V^-(z_0) = V^+\Gamma_L e^{\gamma z_0} \quad [\text{V}], \quad I^-(z_0) = -\frac{V^+}{Z_0}\Gamma_L e^{\gamma z_0} \quad [\text{A}],$$
$$P^-(z_0) = \frac{|V^+|^2|\Gamma_L|^2}{2|Z_0|}e^{-2\alpha z_0}\cos(\theta_{Z_0}) \quad [\text{W}] \tag{14.158}$$

For lossless lines, the attenuation constant is zero and the characteristic impedance is real. The power at any point on the line is therefore

$$\boxed{P_{z_0} = \frac{|V^+|^2}{2Z_0}\left(1 - |\Gamma_L|^2\right) \quad [\text{W}]} \tag{14.159}$$

It is worth mentioning again that this power is positive propagating from generator to load. Note that if the reflection coefficient is zero, all power on the line is transferred to the load, although this does not imply maximum power transfer from generator to load.

The instantaneous power on the line is calculated similarly by multiplying the instantaneous voltage by instantaneous current.

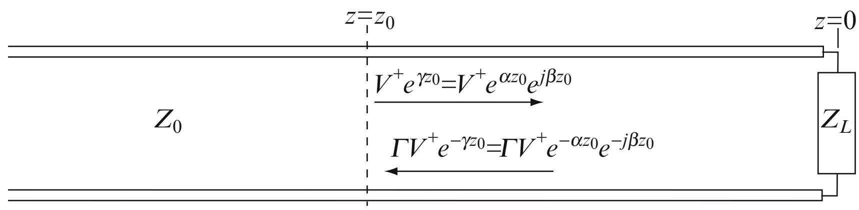


Figure 14.24 Notation used to calculate power relations on the transmission line

Example 14.14 Consider, again, the transmission line in **Example 14.13** but with a resistive load of 50 Ω. The characteristic impedance of the line is 100 Ω. The line is very long and the voltage measured at the load is 50 V. Calculate:

(a) Maximum and minimum voltage on the line (magnitude only).
(b) Location of voltage maxima and minima on the line (starting from the load).
(c) The minimum and maximum impedance on the line. Where do these occur?
(d) Power transmitted to the load.

Solution: First, we calculate the reflection coefficient and standing waves ratio. From the reflection coefficient, we calculate the minimum and maximum voltage using **Eqs. (14.113)** and **(14.114)**. From the standing wave ratio, we calculate the minimum and maximum impedance using **Eqs. (14.151)** and **(14.152)**. The power transmitted to the load is given in **Eq. (14.159)**.

(a) The reflection coefficient at the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - 100}{50 + 100} = -\frac{1}{3} = \frac{1}{3}e^{-j\pi} \quad \rightarrow \quad |\Gamma_L| = \frac{1}{3}, \quad \theta_L = -\pi$$

The standing wave ratio is

$$\text{SWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{3+1}{3-1} = 2$$

To calculate the minimum and maximum voltage on the line, we first need to calculate the forward wave amplitude. The total amplitude at the load is known. From **Eq. (14.117)**, we get at the load

$$V_L = V^+(1 + \Gamma_L) \quad \rightarrow \quad V^+ = \frac{V_L}{(1 + \Gamma_L)} = \frac{50}{(1 - 1/3)} = \frac{50}{2/3} = 75 \quad [\text{V}]$$

Thus, the minimum and maximum voltages are

$$V_{max} = |V^+| \left(\frac{2\text{SWR}}{\text{SWR} + 1} \right) = 75 \times \frac{4}{3} = 100 \quad [\text{V}],$$

$$V_{max} = |V^+| \left(\frac{2}{\text{SWR} + 1} \right) = 75 \times \frac{2}{3} = 50 \quad [\text{V}]$$

Note that the minimum voltage occurs at the load, as required for $R_L < Z_0$ ($V_{min} = V_L = 50$ V).

(b) The first voltage minimum is at the load ($R_L < Z_0$). The minima therefore occur at $z = 0, 0.5\lambda, 1.0\lambda, 1.5\lambda, 2.0\lambda$, etc. The voltage maxima are one-quarter wavelength to the left of the minima. These occur at: $z = 0.25\lambda, 0.75\lambda, 1.25\lambda, 1.75\lambda, 2.25\lambda$, etc.

(c) The minimum and maximum impedances on the line are

$$Z_{min} = \frac{Z_0}{\text{SWR}} = 50, \quad Z_{max} = Z_0\text{SWR} = 200 \quad [\Omega]$$

Minimum impedance occurs at the points of minimum voltage and maximum impedance at the points of maximum voltage.

(d) Power transmitted into the load [from **Eq. (14.159)**] is

$$P_L = \frac{|V^+|^2}{2Z_0}\left[1 - |\Gamma_L|^2\right] = \frac{75^2}{200}\left[1 - \left(\frac{1}{3}\right)^2\right] = 25 \quad [\text{W}]$$

This is about 89 % of the power of the incident wave and about 11 % of the power is reflected.

Exercise 14.8 Repeat **Example 14.14** but with the load and characteristic impedance interchanged. That is, $Z_L = 100\ \Omega$ and $Z_0 = 50\ \Omega$. The voltage measured at the load is 50 V.

Answer

(a) $\Gamma_L = 1/3$, $\theta_L = 0$, SWR $= 2$, $V^+ = 37.5$ V, $V_{max} = 50$ V (at load), $V_{min} = 25$ V.

(b) Voltage maxima (current minima) at $z = 0, 0.5\lambda, 1.0\lambda, 1.5\lambda, 2.0\lambda$, etc. Voltage minima (current maxima) at $z = 0.25\lambda, 0.75\lambda, 1.25\lambda, 1.75\lambda, 2.25\lambda$, etc.

(c) $Z_{min} = 25\ \Omega$, $Z_{max} = 100\ \Omega$.

(d) $P_L = 12.5$ W.

14.9 Resonant Transmission Line Circuits

We have already described the shorted and open transmission lines. These lines have an impedance which is purely imaginary and can be either positive or negative. Thus, a segment of shorted transmission line of appropriate length will behave as an inductor or as a capacitor. Similarly, an impedance of almost any value may be obtained by appropriate choice of lines and loads. It is, therefore, possible to use these line segments to build particular circuits with given properties. We will take advantage of this aspect of transmission lines in the following chapter, where we make use of shorted, open, and loaded transmission lines to match line to line or load to line.

To demonstrate the use of transmission lines as circuit elements, we discuss here the design of resonant transmission lines. We recall that for a circuit to resonate, it must have the following properties:

(1) A capacitance and inductance must be present.

(2) The capacitance and inductance may be connected in series to form a series resonator (**Figure 14.25a**) or in parallel to form a parallel resonator (**Figure 14.25b**).

(3) Lossy elements in the form of resistances or conductances may exist in either case **(Figures 14.25c** and **14.25d)**.
(4) At resonance, the impedance of the resonant circuit is real.
(5) As the frequency changes from below to above resonance, the nature of the circuit changes from capacitive to inductive or vice versa.

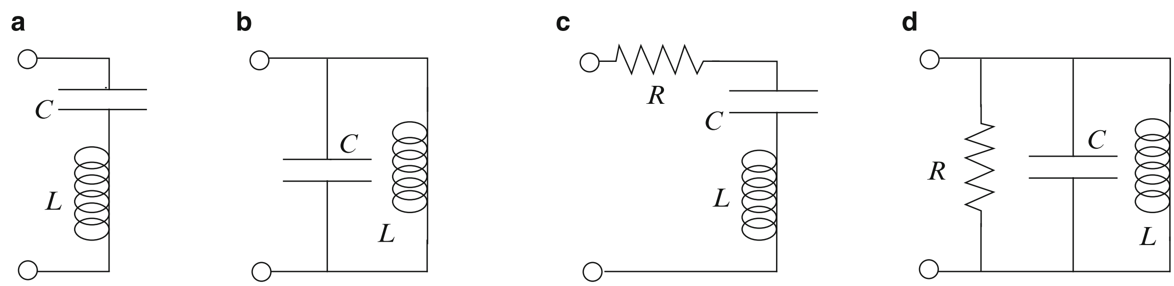


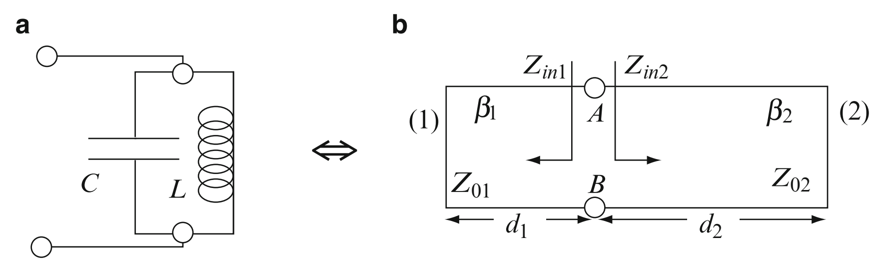
Figure 14.25 Resonating circuits. (**a**) Series resonator. (**b**) Parallel resonator. (**c**) Lossy series resonator. (**d**) Lossy parallel resonator

A parallel resonant circuit is shown in **Figure 14.26** together with its transmission line implementation. The lengths d_1 and d_2 must be determined to satisfy the required conditions for resonance. We will use lossless transmission lines for the calculations that follow. Suppose that the left branch in **Figure 14.26b** is made so that it is equivalent to a capacitance C. The input impedance of this segment must be

$$Z_{in1} = jZ_{01}\tan\beta_1 d_1 = \frac{1}{j\omega C} \quad [\Omega] \tag{14.160}$$

where Z_{01} is the characteristic impedance of the shorted line forming this segment and β_1 is the phase constant of the segment.

Figure 14.26 (**a**) A parallel resonant circuit. (**b**) The transmission line implementation of the parallel resonant circuit



Line (2), which is also shorted, must behave as an equivalent inductor of inductance L. Its input impedance is

$$Z_{in2} = jZ_{02}\tan\beta_2 d_2 = j\omega L \quad [\Omega] \tag{14.161}$$

At resonance, the impedance of the circuit is ∞ or, alternatively, the admittance of the parallel circuit is zero. Using the latter, we can write

$$\frac{1}{Z_{in1}} + \frac{1}{Z_{in2}} = \frac{1}{jZ_{01}\tan\beta_1 d_1} + \frac{1}{jZ_{02}\tan\beta_2 d_2} = 0 \tag{14.162}$$

Rearranging terms, we get the required condition for resonance:

$$\boxed{Z_{01}\tan\beta_1 d_1 + Z_{02}\tan\beta_2 d_2 = 0} \tag{14.163}$$

or in terms of the frequency itself ($\beta = \omega/v_p = 2\pi f/v_p$), we can write

$$Z_{01}\tan\frac{2\pi f d_1}{v_{p1}} + Z_{02}\tan\frac{2\pi f d_2}{v_{p2}} = 0 \tag{14.164}$$

This is a transcendental equation and we cannot solve it explicitly. However, since β_1, β_2, Z_{01}, and Z_{02} are known from the line parameters, all that remains to be defined are d_1 and d_2. This can be done in two ways: If the frequency is given, then a relation between d_1 and d_2 is obtained. We fix one value and find the second such that it satisfies the relation. Alternatively, we can fix both d_1 and d_2 and find the frequencies at which the resulting circuit resonates. Any method of solving the transcendental equation in **Eqs. (14.163)** or **(14.164)** is acceptable for solution.

Note that we should expect multiple solutions from the periodic nature of the tangent functions. The resonant circuit resonates at an infinite number of discrete frequencies.

Series resonant circuits can also be built, although not with all types of transmission lines. A simple series resonant circuit is shown in **Figure 14.27** together with its equivalent implementation in terms of shorted, lossless transmission lines. The line shown is a two-wire line, but other transmission lines may be used. Note, however, that there is no proper mechanism to connect two coaxial lines in series. This circuit is limited to open lines such as the parallel plate transmission line or the two-conductor open line.

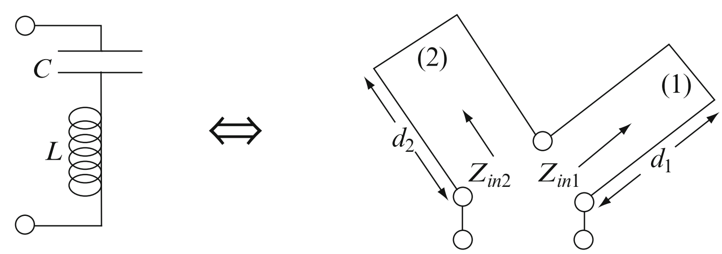


Figure 14.27 A series resonant circuit and its transmission line implementation

Following the same process as for the parallel resonant circuit, one segment, say segment (1), must be of length d_1 to make it capacitive and therefore will have the impedance in **Eq. (14.160)**. The second segment is made to be inductive and will have the impedance in **Eq. (14.161)**. Now, the total impedance of the line is zero and we write

$$Z_{in1} + Z_{in2} = jZ_{01}\tan\beta_1 d_1 + jZ_{02}\tan\beta_2 d_2 = 0 \tag{14.165}$$

Thus, the equation that must be satisfied is, again,

$$\boxed{Z_{01}\tan\beta_1 d_1 + Z_{02}\tan\beta_2 d_2 = 0} \tag{14.166}$$

Also, instead of using shorted transmission lines, open transmission lines may be used as well since the impedances of shorted and open transmission lines are the same if one line is shortened or lengthened by one-quarter wavelength. Thus, once a resonator is designed with either type of line, it is a simple matter to find a resonator made of the second type or a combination of the two.

In general, shorted transmission lines are preferred for a variety of reasons, including noise, but sometimes, especially when the line is used to measure external conditions, an open line is more practical. For example, resonant coaxial transmission lines are often used to measure water content in snow to evaluate runoff levels and water reserves. An open resonant circuit is most useful since it can then simply be pushed into the snow pack for measurement purposes. Similarly, a resonator that measures pollutants in air must be open in one way or another. If coaxial lines are used, the resonator must be made of open segments. If, on the other hand, parallel plates are used, shorted lines may be used because the structure itself is open.

Example 14.15 It is required to design a transmission line resonator made of a parallel plate transmission line. The dimensions and material properties are shown in **Figure 14.28a**. The length of the line at left is 0.4 m. The length of the line at right is 0.2 m.

(a) Calculate the first two resonant frequencies of the resonator.
(b) Suppose the short on the right-hand side is removed (**Figure 14.28b**). What are now the two lowest resonant frequencies?

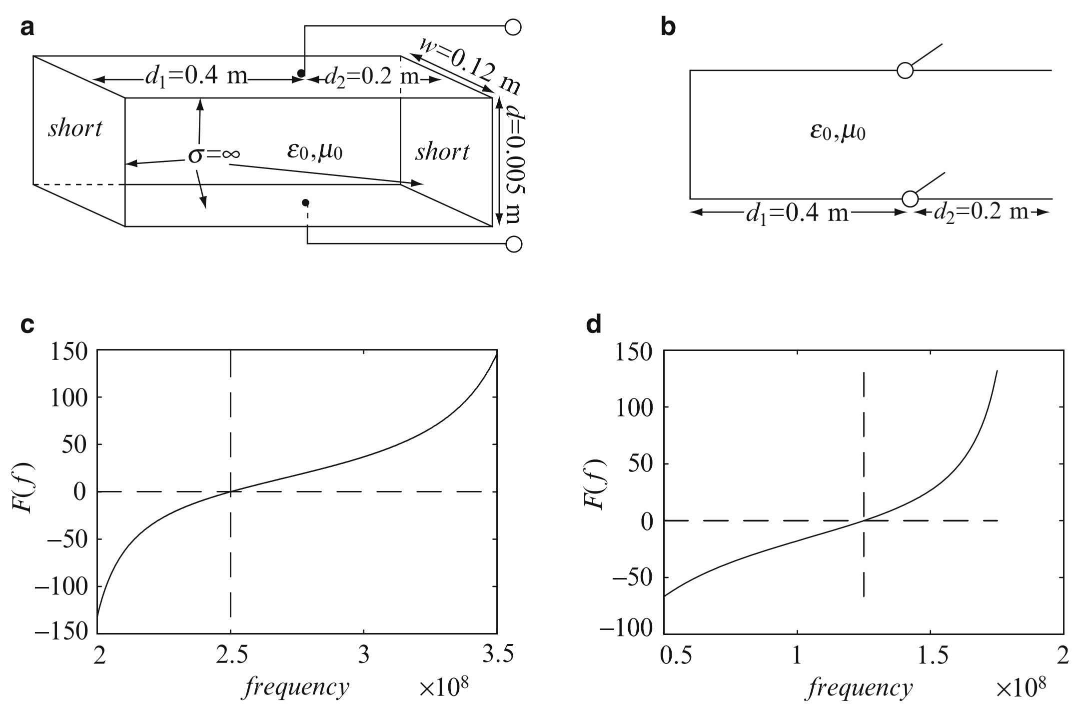


Figure 14.28 (**a**) A transmission line resonator with both ends shorted. (**b**) Transmission line resonator with right side open. (**c**) A resonant frequency for the circuit in (**a**). (**d**) A resonant frequency for the circuit in (**b**)

Solution: We will use **Eq. (14.164)**. The intrinsic line impedance and phase velocity are calculated first. Because both line segments are identical in properties, the characteristic impedance is not needed, but we will use it nevertheless for generality. With these, we calculate the expression in **Eq. (14.164)** for a range of frequencies (say from 1 MHz to 3,000 MHz, in increments of 1 MHz) and plot the result. The intersections with the real axis gives the resonant frequencies, provided the transition is smooth.

(a) The line is lossless; therefore, the speed of propagation and intrinsic impedance on the line (from **Table 14.1**) is

$$v_p = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times 10^8 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0 d/w}{w\varepsilon_0/d}} = \frac{d}{w}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{0.005}{0.12}377 = 15.7 \quad [\Omega]$$

Both lines are identical in their properties; therefore, the condition for resonance is

$$F(f) = Z_{01}\tan\frac{2\pi f d_1}{v_p} + Z_{02}\tan\frac{2\pi f d_2}{v_p} = 15.7\tan\frac{2\times\pi\times f\times 0.4}{3\times 10^8} + 15.7\tan\frac{2\times\pi\times f\times 0.2}{3\times 10^8} = 0$$

To find the resonant frequencies, we plot this expression as a function of frequency and identify the zero crossings. Those crossings which are smooth and change the nature of the circuit (from capacitive to inductive or vice versa) are resonant frequencies. Others, such as when the value of the function tends to $\pm\infty$, are not resonant frequencies. **Figure 14.28c** shows a plot between 200 and 350 MHz with resonance at 250 MHz. A similar plot (not shown) results in a second resonant frequency at 500 MHz.

(b) In this case, one end is open. This means that a short circuit exists (in terms of conditions on the line), a distance $\lambda/4$ from the open circuit on either side. Thus, if we take the distance d_1 to be the distance from the shorted end and $d_2 = 0.2 \pm \lambda/4$ the distance from the artificial new short end (i.e., a short end that exists at either side of the open end), we get the condition for resonance:

$$F(f) = \tan\frac{2\pi f d_1}{v_p} + \tan\frac{2\pi f d_2}{v_p} = \tan\frac{2\times\pi\times f\times 0.4}{3\times 10^8} + \tan\frac{2\times\pi\times f\times\left(0.2 - 3\times 10^8/4f\right)}{3\times 10^8} = 0$$

where $\lambda/4 = c/4f = 3\times 10^8/4f$ was used and the shorter section on the right side was chosen.

Again plotting this relation, we get the two resonant frequencies as 125 MHz and 375 MHz. **Figure 14.28d** shows the first of these resonant frequencies. Note that one resonant frequency is lower than for the shorted line, the second higher. These correspond to "lengthening" of the line since in both cases, one-quarter wavelength is larger than (or equal to) the length of the open line ($\lambda/4 = 0.6$ m at 125 MHz, and $\lambda/4 = 0.2$ m at 375 MHz). Higher resonant frequencies may correspond to either "lengthening" or "shortening" of the line since open lines may behave as being shorted on either side of the open.

Exercise 14.9 In **Example 14.15**, calculate the next two resonant frequencies for **(a)** and **(b)**.

Answer **(a)** 750 MHz, 1,000 MHz. **(b)** 625 MHz, 875 MHz.

Exercise 14.10 Consider again **Figure 14.28a**. However, now a dielectric with relative permittivity 4 fills the space between the plates. The ends of the resonator are shorted.

(a) What is the condition for resonance?
(b) Calculate the first three resonant frequencies.

Answer **(a)** $F(f) = \tan(1.6755\times 10^{-8}f) + \tan(0.8377\times 10^{-8}f) = 0$. **(b)** 125 MHz, 250 MHz, and 375 MHz.

14.10 Applications

Application: Frequency Domain Reflectometry on Transmission Lines The following procedure may be used to detect conditions on a transmission line such as short circuits, discontinuities, and the like. Suppose a line is shorted at some point and it is desirable to find this location. A matched generator is connected to the line and the line voltage or current is measured at some location between the generator and load, usually close to the generator, as shown in **Figure 14.29**. Now, the frequency of the generator is increased or decreased until a maximum or minimum in voltage (or current) is detected by the measuring instrument. This then represents a maximum or a minimum in the standing wave pattern produced by the forward- and backward-propagating waves along the line. Usually, the minima are used because they are sharper than the peaks. Because the number of minima between the given location and the load is not known, we may assume there are n minima, the distance between each two being one-half wavelength at the given frequency. Now, the frequency is increased until the next minimum appears at the measuring instrument. There are now $n + 1$ minima and, again, the distance between

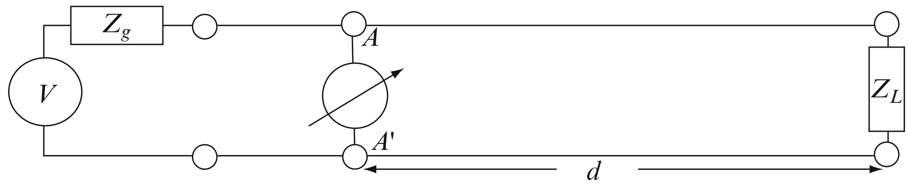


Figure 14.29 Frequency domain reflectometry. The signal is measured at A–A'

every two minima is $\lambda/2$ at the new, higher frequency. By measuring the two frequencies, the following may be written for the distance to the fault on the line:

$$d = \frac{\lambda_1}{2} n = \frac{\lambda_2}{2}(n+1) \tag{14.167}$$

where d is the distance between the measuring instrument and the short. Because $\lambda f = v_p$, we can write

$$\frac{v_{p_1}}{2f_1} n = \frac{v_{p_2}}{2f_2}(n+1) \tag{14.168}$$

Assuming the phase velocity does not change with frequency $\left(v_{p_1} = v_{p_2}\right)$, the number of minima, n, is

$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{f_1}{f_2 - f_1} \tag{14.169}$$

Substituting for n in **Eq. (14.167)**, the length of the line is

$$d = \frac{\lambda_2 \lambda_1}{2(\lambda_1 - \lambda_2)} = \frac{v_p}{2(f_2 - f_1)} \tag{14.170}$$

If the phase velocity on the line is known (i.e., if the line properties are known), the distance to the discontinuity may be calculated. However, there is a small difficulty here: A shorted or open line will give identical indications, but they will be a distance $\lambda/4$ away. For this reason, the location of the discontinuity may only be found to within $\pm\lambda/4$. Often, this is quite sufficient. In cable TV applications, this means about 1.5 m at the lowest frequency (50 MHz). The higher the frequency used for measurement, the closer the discontinuity may be located, thus the use of high frequencies for location of discontinuities.

Application: Transmission Line Methods of Oil Recovery An interesting proposed application of transmission lines is recovery of oil in oil shale deposits. The idea is rather simple: Vertical wells are drilled at given distances which then lean horizontally into the shale deposits. Cables are introduced and connected to sources in the range of a few MHz. A transverse electromagnetic (TEM) wave propagates in the line, with the shale serving as the lossy dielectric between the lines. Because the loss is rather high and the thermal insulating nature of shale and soil, considerable heat is produced which melts the shale (actually, reduces its viscosity). The shale is then pumped out by pressurized steam or water. A schematic view of this method is shown in **Figure 14.30**. Multiple transmission lines can heat large areas of the deposits at almost any depth.

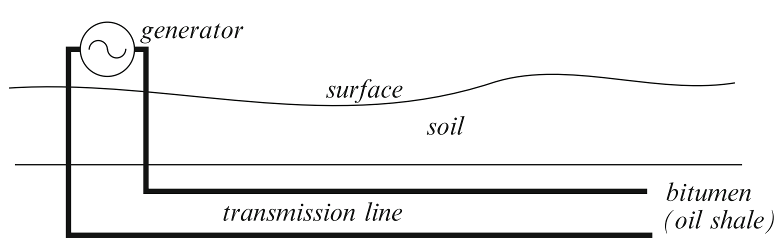


Figure 14.30 Transmission line method of oil recovery from oil shale

Application: Monitoring of Dielectrics and Lossy Dielectrics Resonant transmission lines are often used to monitor or measure the permittivity of materials during production. A simple device designed to monitor the thickness of a known dielectric (such as plastic film or paper) is shown in **Figure 14.31**. A parallel plate transmission line resonates at a frequency f for nominal thickness. The frequency is adjusted and measured. Any deviation in the material thickness changes the effective permittivity in the resonator and, therefore, changes the resonant frequency of the device. An increase in thickness reduces the frequency, whereas a decrease in thickness increases the frequency. A device of this kind can be easily incorporated in automatic production, with feedback to the appropriate control devices.

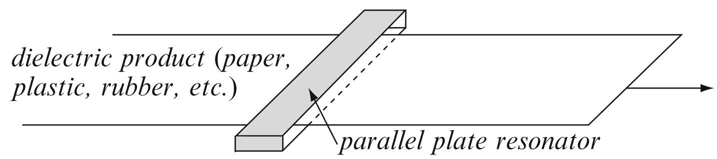


Figure 14.31 A transmission line resonator used to monitor material thickness during production

Similar devices can be used to monitor curing or drying of materials. The method is based on the fact that wet materials have higher permittivities than dry or cured products. As the water or solvent dries, the relative permittivity is reduced and the resonant frequency of the device increases. Since resonance is usually quite sharp very sensitive measurements can be made. This method may be used in monitoring of polymers, paper production, drying of grain, or even baked products.

The same method is used to measure water content in snow packs for prediction of runoff and water shortages. A transmission line resonator, usually in the form of a coaxial, open-ended resonator, is calibrated at a given frequency. Then, the device is pushed into the snow until the snow fills the resonator. Now, the frequency is lower because snow has a higher permittivity than air. The lower the frequency, the higher the water content since wetter snow has higher permittivity than dryer snow.

14.11 Experiment

Experiment 1 (Simulated transmission line. Demonstrates: Properties of Transmission Lines) The properties of transmission lines may be easily simulated by connecting resistances, capacitances, and inductances. The "trick" is to create a cell to represent a unit length of the transmission line. Unit length here may mean 1 m, 1 km, or any convenient unit. **Figure** 14.32 shows a simulated transmission line with a 3 Ω series resistance, 3300 pF capacitance, 0.01 S conductance (100 Ω parallel resistance), and 86 μH inductance. Each cell may represent a 1 km section of a power transmission line or a 1 m section of a higher-frequency line. Calculate the characteristic impedance on the line. Connect the line to a signal generator and connect an oscilloscope at its end. See how the output changes with frequency. Most properties and effects of transmission lines may be demonstrated using this or a similar line.

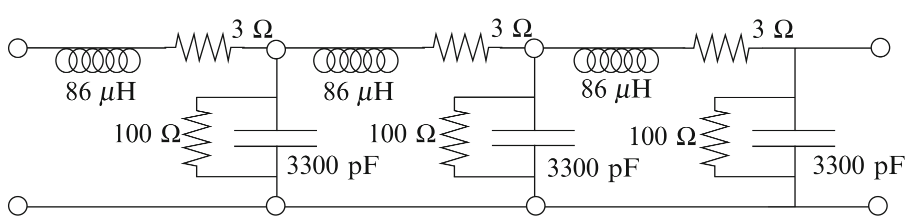


Figure 14.32 A simulated transmission line

14.12 Summary

The current chapter treats transmission lines in the frequency domain. The behavior of voltages and currents on the line behaves as plane waves propagating along the line. Many of the relations in **Chapter** 12 find new use here.

Reminders: Impedance: $Z = V/I$ [Ω], admittance: $Y = 1/Z = I/V$ [(1/Ω) = S].

Phase velocity $v_p = \omega/\beta = \lambda f$ [m/s], $\lambda = 2\pi/\beta$ [m]. In vacuum $v_p = 3 \times 10^8$ [m/s]

Transmission Line Parameters Series parameter, L, R given as resistance/m, and inductance/m.

Shunt parameters G, C given as capacitance/m and conductance/m. Each parameter is calculated for a 1 m length of the transmission line based on principles in **Chapters 4**, **7**, and **9**. **Table 14.1** summarizes the parameters for three common lines.

Transmission line equations are obtained by application of Kirchoff's laws (see **Figure 14.10**) for a section of length dl at an arbitrary location, l on the line. In the frequency domain

$$\frac{dV(l)}{dl} = -I(l)[R + j\omega L] \quad (14.24) \qquad \frac{dI(l)}{dl} = -V(l)[G + j\omega C] \quad (14.27)$$

Wave equations for voltage and current on the line, with reference to $l = 0$ at the generator:

$$\frac{d^2V(l)}{dl^2} - \gamma^2 V(l) = 0 \quad (14.31) \qquad \frac{d^2I(l)}{dl^2} - \gamma^2 I(l) = 0 \quad (14.32)$$

where

$$\gamma = \alpha + j\beta = \sqrt{[G + j\omega C][R + j\omega L]} \quad (14.33)$$

$\gamma =$ ***propagation constant***, $\alpha =$ ***attenuation constant***, $\beta =$ ***phase constant***.

Solutions for voltage and current on the line

$$V(l) = V^+ e^{-\gamma l} + V^- e^{\gamma l} \quad [\text{V}] \quad (14.34) \qquad I(l) = I^+ e^{-\gamma l} + I^- e^{\gamma l} \quad [\text{A}] \quad (14.35)$$

The first term in each solution is the forward (generator to load) propagating wave, and the second is the backward (load to generator) propagating wave.

General properties

$$Z_0 = \frac{V^+}{I^+} = -\frac{V^-}{I^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad [\Omega] \quad (14.45) \qquad \lambda = \frac{2\pi}{\beta} \quad [\text{m}], \quad v_p = \frac{\omega}{\beta} \quad \left[\frac{\text{m}}{\text{s}}\right] \quad (14.47)$$

Properties of particular types of lines

	Attenuation constant α [Np/m]	Phase constant β [rad/m]	Z_0 [Ω]	v_p [m/s]	Other relations	Equations
Lossless line	$\alpha = 0$	$\beta = \omega\sqrt{LC}$	$Z_0 = \sqrt{L/C}$	$v_p = 1/\sqrt{LC}$	$R = 0, G = 0$ $\mu\varepsilon = LC$	(14.56)–(14.62)
Long line	Can be any type of line. The defining property is the lack of reflection from the load					
Distortionless line	$\alpha = R\sqrt{C/L}$	$\beta = \omega\sqrt{LC}$	$Z_0 = \sqrt{L/C}$	$v_p = 1/\sqrt{LC}$	$R/L = G/C$	(14.65)–(14.69)
Low resistance line	$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\sqrt{1 + \frac{G}{j\omega C}}$		$Z_0 = \sqrt{\frac{j\omega L}{G + j\omega C}}$	$v_p = 1/\sqrt{LC}$	$R = 0$ $\mu\varepsilon = LC$ $\sigma/\varepsilon = G/C$	(14.70)–(14.73)

Field approach to transmission line analysis follows the ideas of plane waves in **Chapter 12** and is therefore not summarized here. For a discussion and examples see **Section 14.6**.

Finite transmission lines. By referencing all calculations to the load rather than generator, that is, the load is placed at $z = 0$ and the generator is in the positive z-direction, we get (see **Figure 14.15**)

$$V(z) = V^+ e^{\gamma z} + V^- e^{-\gamma z} \quad [\text{V}] \quad \text{and} \quad I(z) = I^+ e^{\gamma z} + I^- e^{-zl} \quad [\text{A}] \quad (14.86)$$

Load reflection coefficient

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{14.91}$$

Line impedance on a general lossy line at an arbitrary distance z from the load:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{e^{\gamma z} + \Gamma_L e^{-\gamma z}}{e^{\gamma z} - \Gamma_L e^{-\gamma z}} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z)}{Z_0 + Z_L \tanh(\gamma z)} \quad [\Omega] \qquad (14.95, 14.97)$$

Generalized reflection coefficient on a general lossy line at an arbitrary distance z from the load:

$$\Gamma(z) = \Gamma_L e^{-2\gamma z} = |\Gamma_L| e^{-2\alpha z} e^{j\theta_\Gamma} e^{-j2\beta z} \tag{14.99}$$

Phase on line (starting from load)

$$\theta_{\Gamma(z)} = \theta_\Gamma - 2\beta z \quad [\text{rad}] \tag{14.100}$$

where θ_Γ is the phase angle of the reflection coefficient.

Lossless line:

$$Z(z) = Z_0 \frac{Z_L + jZ_0 \tan\beta z}{Z_0 + jZ_L \tan\beta z} = Z_0 \frac{Z_L \cos\beta z + jZ_0 \sin\beta z}{Z_0 \cos\beta z + jZ_L \sin\beta z} \quad [\Omega] \tag{14.102}$$

$$\Gamma(z) = \Gamma_L e^{-j2\beta z} = |\Gamma_L| e^{-j(2\beta z - \theta_\Gamma)} \qquad (14.103) \qquad\qquad \theta_{\Gamma(z)} = \theta_\Gamma - 2\beta z \quad [\text{rad}] \qquad (14.104)$$

Standing Wave Ratio—SWR

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + |\Gamma(z)|}{1 - |\Gamma(z)|} \tag{14.111}$$

Maximum and minimum voltage (magnitude) on the line

$$V_{max} = |V^+|(1 + |\Gamma(z)|) = |V^+| \left(\frac{2\text{SWR}}{\text{SWR} + 1} \right) \quad [\text{V}] \tag{14.113}$$

$$V_{min} = |V^+|(1 - |\Gamma(z)|) = |V^+| \left(\frac{2}{\text{SWR} + 1} \right) \quad [\text{V}] \tag{14.114}$$

Voltage and current anywhere on the line

$$V(z) = V^+ e^{j\beta z} \left(1 + \Gamma_L e^{-j2\beta z}\right) \quad [\text{V}] \qquad (14.115) \qquad\qquad I(z) = \frac{V^+}{Z_0} e^{j\beta z} \left(1 - \Gamma_L e^{-j2\beta z}\right) \quad [\text{A}] \qquad (14.116)$$

At the load ($z = 0$):

$$V_L = V^+ \left(1 + |\Gamma_L| e^{j\theta_\Gamma}\right) \quad [\text{V}] \quad \text{and} \quad I_L = \frac{V^+}{Z_0} \left(1 - |\Gamma_L| e^{j\theta_\Gamma}\right) \quad [\text{A}] \tag{14.117}$$

Locations of voltage maxima (current minima)

$$z_{max} = \frac{\lambda}{4\pi} (\theta_\Gamma + 2n\pi), \quad n = 0, 1, 2, \ldots \quad [\text{m}] \tag{14.124}$$

Location of impedance or voltage minima (current maxima): $z_{max} \pm \lambda/4$:
Lossless matched line [Eqs. (14.125) through (14.127)]

$$Z_L = Z_0, \quad Z(z) = Z_0 \quad [\Omega], \quad V(z) = V^+ e^{j\beta z} \quad [\text{V}], \quad I(z) = \frac{V^+}{Z_0} e^{j\beta z} \quad [\text{A}]$$

Lossless Shorted and Open Transmission Lines The main relations for shorted and open lines are summarized in the table below **[Eqs. (14.128)** through **(14.133)]**.

	Z_L [Ω]	$Z(z)$ [Ω]	V_L [V]	I_L [A]	$V(z)$ [V]	$I(z)$ [A]
Shorted line	0	$jZ_0 \tan\beta z$	0	$\frac{2V^+}{Z_0}$	$V^+ e^{j\beta z}(1 - e^{-j2\beta z})$	$\frac{V^+}{Z_0} e^{j\beta z}\left(1 + e^{-j2\beta z}\right)$
Open line	∞	$-jZ_0 \cot\beta z$	$2V^+$	0	$V^+ e^{j\beta z}(1 + e^{-j2\beta z})$	$\frac{V^+}{Z_0} e^{j\beta z}\left(1 - e^{-j2\beta z}\right)$

The most significant properties of shorted and open lines are summarized in the table below.

	Γ_L	SWR	$Z(z) = 0$ *at*	$Z(z)=\infty$ *at*	$Z(z) > 0$ (inductive) for	$Z(z) < 0$ (capacitive) for
Shorted line	−1	∞	$z = n\lambda/2$, $n = 0, 1, 2, \ldots$	$n\lambda/2 + \lambda/4$,	$n\lambda/2 < z < n\lambda/2 + \lambda/4$	$n\lambda/2 + \lambda/4 < z < n\lambda/2 + \lambda/2$
Open line	1	∞	$n\lambda/2 + \lambda/4$, $n = 0, 1, 2, \ldots$	$z = n\lambda/2$	$n\lambda/2 + \lambda/4 < z < n\lambda/2 + \lambda/2$	$n\lambda/2 < z < n\lambda/2 + \lambda/4$

Notes:

(1) An open line behaves as if it were a shorted line lengthened or shortened by $\lambda/4$.
(2) A shorted line behaves as if it were an open line lengthened or shortened by $\lambda/4$.
(3) All properties and relations on any line repeat at intervals of $\lambda/2$.
Also, on any lossless transmission line

$$Z_0 = \sqrt{Z_{short} Z_{open}} \quad [\Omega] \tag{14.135}$$

Resistively Loaded Lossless Transmission Line The reflection coefficient is

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{14.136}$$

The following table summarizes the main relations on resistively loaded line

Load resistance	Γ_L	Voltage on the line [V]	Current in the line [A]	Minimum impedance occurs at: [wavelengths]
$R_L > Z_0$	$\Gamma_L > 0$ $\theta_\Gamma = 0$	$V^+ e^{j\beta z}(1 + \Gamma_L e^{-j2\beta z})$ **Eq. (14.139)**	$\frac{V^+}{Z_0} e^{j\beta z}\left(1 - \Gamma_L e^{-j2\beta z}\right)$ **Eq. (14.140)**	$z_{min} = \frac{\lambda}{4\pi}(2n+1)\pi$ $n = 0, 1, 2, \ldots$ **Eq. (14.142)**
$R_L < Z_0$	$\Gamma_L < 0$ $\theta_\Gamma = -\pi$	$V^+ e^{j\beta z}(1 + \lvert\Gamma_L\rvert e^{-j\pi} e^{-j2\beta z})$ **Eq. (14.146)**	$\frac{V^+}{Z_0} e^{j\beta z}\left(1 - \lvert\Gamma_L\rvert e^{-j\pi} e^{-j2\beta z}\right)$ **Eq. (14.147)**	$z_{min} = \frac{\lambda}{4\pi} 2n\pi$ $n = 0, 1, 2, \ldots$ **Eq. (14.149)**

Additional Properties

(1) $-1 < \Gamma_L < +1$, $1 < \text{SWR} < \infty$.
(2) The line impedance can be complex [see **Eq. (14.137)**],
(3) Impedance is real at locations of voltage maxima and voltage minima for any lossless line.

(4) The impedance at voltage maxima is $Z_{max} = Z_0\mathrm{SWR}$, whereas at voltage minima (current maxima), it is $Z_{min} = Z_0/\mathrm{SWR}$.
(5) For $R_L > Z_0$, the first voltage maximum occurs at the load ($z = 0$), the first voltage minimum at $\lambda/4$ from the load.
(6) For $R_L < Z_0$, the first voltage minimum occurs at the load, the first voltage maximum at $\lambda/4$ from the load.

Power on Transmission Lines Power at any point z_0 on a general, lossy line is

$$P_{z_0} = \frac{|V^+|^2}{2|Z_0|}\left(e^{2\alpha z_0} - |\Gamma_L|^2 e^{-2\alpha z_0}\right)\cos(\theta_{Z_0}) \quad [\mathrm{W}] \tag{14.155}$$

where θ_{Z_0} is the phase angle of the characteristic impedance.
Power at any point z_0 on a lossless line is

$$P_{z_0} = \frac{|V^+|^2}{2Z_0}\left(1 - |\Gamma_L|^2\right) \quad [\mathrm{W}] \tag{14.159}$$

Resonant Transmission Lines (See **Figure 14.25**) The resonant frequency is found by finding the frequencies at which the following function is satisfied:

$$Z_{01}\tan\frac{2\pi f d_1}{v_{p1}} + Z_{02}\tan\frac{2\pi f d_2}{v_{p2}} = 0 \tag{14.164}$$

where d_1, d_2 are the distances to the shorts from the connection point, Z_{01}, Z_{02} are the characteristic impedances of the two lines, and v_{p1}, v_{p2} the phase velocities on the two lines.
Reminder: 1 Np/m = 8.69 dB/m.

Problems

Transmission Line Parameters

14.1 Application: Coaxial Transmission Line. Two coaxial transmission lines, each made of an inner and outer shell, are given. The radius of the inner shell is $a = 10$ mm and the radius of the outer shell is $b = 20$ mm. The lines are shown in cross section in **Figure 14.33**. Permittivity and permeabilities are given. Calculate the line parameters for each line assuming perfect conductors.

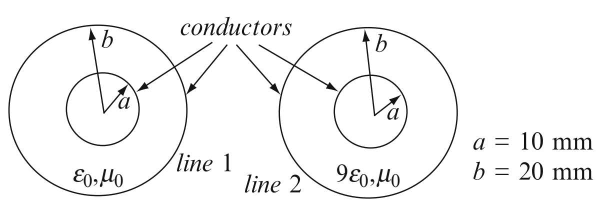


Figure 14.33

14.2 Application: Overhead Transmission Line with Ground Return. Two students live 1 km apart. They decide to connect between them a line so that they may use a private telephone without having to pay for the service. To do so, they need a transmission line and this requires two conductors. Since they do not have enough money, they decide to use a single wire and use the ground as the return conductor, assuming the ground is a good conductor. The wire is 1 mm thick and is strung at a height of 5 m above ground.

(a) Is this a sound approach?
(b) If so, calculate the line parameters for the transmission line made of the single conductor and ground, assuming a perfectly conducting ground.

(c) Suppose the installation was a bit flimsy and during a storm, their wire fell to the ground. The wire is insulated and the insulation is 0.5 mm thick. What are the line parameters now? Use permittivity and permeability of free space for the insulating material.

Long, Lossless Lines

14.3 Application: Parameters of Coaxial Line. The RG-11/U coaxial line has the following properties: $Z_0 = 75\ \Omega$ and $v_p = 2c/3$ m/s. Assuming the line to be lossless, calculate its inductance and capacitance per unit length.

14.4 Application: Coaxial Line and Delay on the Line. A coaxial transmission line is made of two circular conductors as shown in **Figure 14.34**. The inner conductor has diameter b and the outer conductor diameter a. The conductors are perfect conductors and are separated by a lossless ferrite (a material with finite permeability, finite permittivity, and zero conductivity). Given: $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_1 = 9\varepsilon_0$ [F/m], $\mu_1 = 100\mu_0$ [H/m], $a = 8$ mm, $b = 1$ mm.

(a) Calculate the characteristic impedance and the phase velocity on the line.

(b) Two coaxial transmission lines connect two memory banks to a processor. Line A is the same as calculated in part **(a)**. Line B has the same dimensions, but the ferrite is replaced with free space. What must be the minimum execution cycle (maximum frequency) of the processor if the two memory signals must reach the processor within one execution cycle. Assume the length of each cable is $d = 100$ mm. The execution cycle is defined by the slower of the signals.

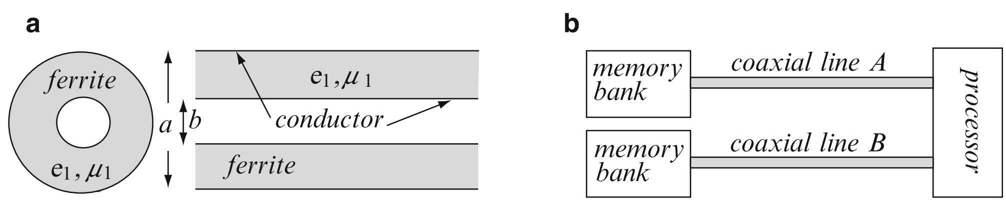


Figure 14.34

14.5 Application: Microstrip Line. A conducting strip with dimensions as shown in **Figure 14.35** is located above a conducting surface. The strip is very long and the conducting surface is infinite. Also, $w \gg d$. The strip and surface serve as a transmission line. Calculate assuming perfect conductors:

(a) Speed of propagation of waves on this line.

(b) Characteristic impedance of the transmission line for $w = 10$ mm, and $d = 0.1$ mm.

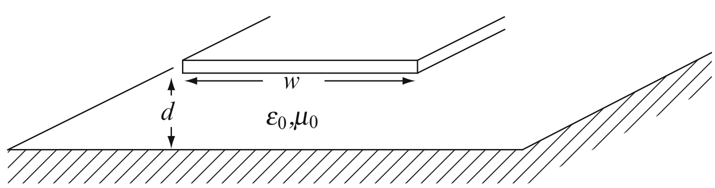


Figure 14.35

14.6 Application: Delay Line and Mismatched Transmission Lines. A lossless coaxial transmission line with dimensions as shown in **Figure 14.36a** is given. In a circuit implementation that includes the line, it becomes necessary to delay the signal on the line by an additional 1 μs. Two options exist:

(a) A length of line with properties identical to the given line is connected at the end of the line (between the line shown and the circuit). Calculate the required length of line to affect the required delay.

(b) A simulated line is made in the form of a ladder network of lumped parameter components as shown in **Figure 14.36b**. Design a network that will have the same characteristic impedance as the line and will provide the required delay. You are free to use as many sections as necessary to provide the delay.

(c) As an engineer, which method would you choose?

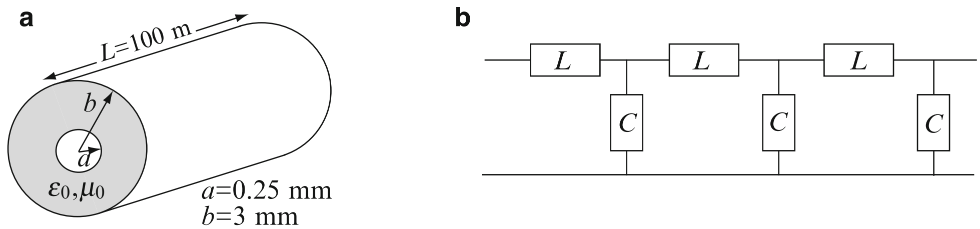


Figure 14.36

The Distortionless Transmission Line

14.7 Application: The Distortionless Line. A coaxial line is made as shown in **Figure 14.37**. Material properties and dimensions are given in the figure. The design calls for a distortionless transmission line at 250 MHz.

- **(a)** Assuming the outer radius must remain constant ($b = 20$ mm), what must be the radius a of the inner conductor for the line to be distortionless? Assume material properties μ_0, ε_0 for copper.
- **(b)** What is the characteristic impedance of the line and what is its attenuation constant for the design in **(a)**?
- **(c)** If reduction of at most 10 dB in amplitude is allowed before an amplifier is required, calculate the distance between each two amplifiers on the line.

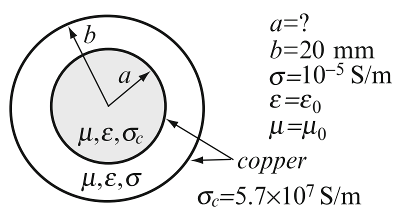


Figure 14.37

14.8 Application: Distortionless Parallel Plate Line. A parallel plate line is made as shown in **Figure 14.38**. Material properties and dimensions are given in the figure. The design calls for a distortionless transmission line operating at 1,000 MHz.

- **(a)** Calculate the required distance d between the two conductors to produce a distortionless line at the given frequency.
- **(b)** What are the characteristic impedance and attenuation constant of the line?
- **(c)** If a reduction of at most 75 dB in amplitude is allowed before an amplifier is required, calculate the distance between each two amplifiers on the line.

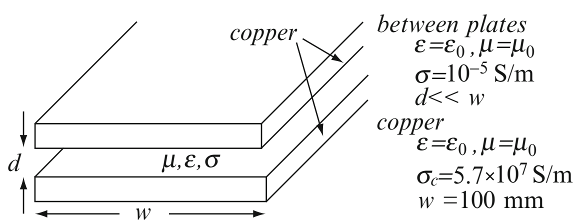


Figure 14.38

The Low-Resistance Transmission Line

14.9 Application: Low-Resistance Power Line. A power line is made of two round, parallel conductors, 20 mm in diameter, separated by 3 m, and suspended above ground, in air. Neglect any ground effects. Properties of air are: ε_0, μ_0, $\sigma = 10^{-5}$ S/m. The line is made of copper ($\mu = \mu_0$ [H/m], $\varepsilon = \varepsilon_0$ [F/m], $\sigma_c = 5.7 \times 10^7$ S/m) and operates at 60 Hz. Calculate:

(a) The propagation constant on the line. Show that this may be assumed to be a low-resistance line.
(b) Under the assumption of a low-resistance line, calculate the characteristic impedance of the line and its attenuation and phase constants.

14.10 Application: Properties of Telephone Lines. A two-wire, open-air telephone line is made of round wires, 1 mm in radius, and are supported 200 mm apart on insulating cups.

The line is made of copper, with conductivity of 5.7×10^7 S/m, and the surrounding space is air (with properties of free space). The line is 3 km long, may be assumed to be low loss, and is used to transmit data at 10 kHz.

(a) What is the load impedance if it must equal the characteristic impedance of the line (this is called matched impedance)?
(b) If the load requires 0.01 W to operate, how much power must be supplied to the line input?
(c) Suppose you are free to adjust the distance between the lines. Can the line be redesigned to be distortionless? If so, how?
(d) Is the assumption that the line is a low-loss line justified? Explain.

14.11 Properties of Long Lines. A long line has parameters R, L, C, $G = 0$, and operates at 100 kHz. The characteristic impedance is $Z_0 = 300 - j10\ \Omega$ and the phase velocity on the line is $v_p = 2c/3$ m/s. Find:

(a) The values of R, L, and C assuming the line is a low-loss line.
(b) Calculate the propagation constant on the line.
(c) Is the assumption in **(a)** that the line is low loss justified?

The Field Approach to Transmission Lines

14.12 Application: Power Relations on Coaxial Lines. Coaxial lines are usually used for transmission of signals at high frequencies. In some cases however, the lines are also required to carry considerable power at lower frequencies. One example is the cable TV coaxial line. These carry both the high-frequency video signals and 60 Hz power to operate devices on the line (such as amplifiers). Typical requirements are for the lines to carry up to 10 A at about 60 V. Consider the following example:

A coaxial cable with the dimensions shown in **Figure 14.39** has a characteristic impedance of 75 Ω. The line is connected to an AC source at 60 V (peak), 60 Hz. Neglect internal impedance in the source. The line may be considered long and the load is matched to the line. Calculate:

(a) The electric and magnetic field intensities in the dielectric of the line.
(b) Show that application of the Poynting theorem anywhere on the line gives the same time-averaged power as that obtained from the current voltage relations.

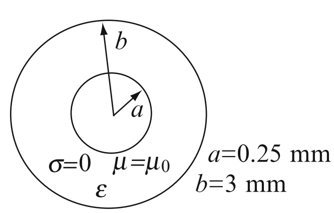


Figure 14.39

14.13 Application: Exposure to Electromagnetic Fields. One recent health concern is with exposure to both low- and high-frequency electromagnetic fields. These are believed by some to cause cancer. An AM station transmitting at 1 MHz decided to minimize exposure of personnel due to power on the transmission line leading from the generator to the antenna. The transmission line is a two-wire exposed (no insulation) line, 10 mm in diameter and separated by 0.2 m. The transmission line is parallel to the ground and leads to an antenna located at some distance from the transmitter, on top of a hill. The station decided that to minimize the risk due to the fields generated by the line, it will not allow the peak magnetic flux density at a distance 1 m from the center of the transmission line to exceed 10^{-6} T.

(a) What is the peak power the station can transmit without violating the maximum magnetic flux density requirement? Assume a lossless line and matched conditions at load and generator.
(b) Suppose the station needs to transmit more power than allowed in **(a)**. To do so, the engineers decide to move the two wires closer together so that now they are 0.1 m apart. How much power can the station transmit now without violating the maximum field requirement?
(c) What are the peak electric field intensities in **(a)** and **(b)** at the location of the peak magnetic flux density?

Finite Transmission Lines

14.14 Voltage and Current on Transmission Lines. The voltage and current at the center of the lossless line in **Figure 14.40** are given as $V_c=18\angle 72°$ V, $I_c=0.2\angle -36°$. If the wavelength is 1 m, calculate:

(a) The forward- and backward-propagating voltages.
(b) The forward- and backward-propagating currents.
(c) The load impedance.

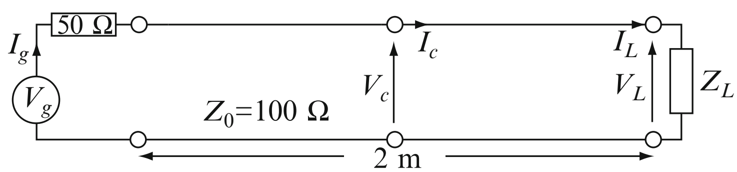


Figure 14.40

14.15 Voltage and Current on Transmission Lines. The voltage and current at the load of the lossless line in **Figure 14.40** are given as $V_L = 50$ V, $I_L = 0.2$ A. If the wavelength is 1 m, calculate:

(a) The voltage and current of the generator.
(b) The power supplied by the generator.

14.16 Terminated, Matched Line. A load impedance $Z_L = 50\ \Omega$ is connected to a generator of impedance $Z_G = 50\ \Omega$ through a transmission line with characteristic impedance $Z_0 = 200\ \Omega$.

(a) What must be the length d of the line (in wavelengths) for matched power transfer from generator to load?
(b) If the characteristic impedance of the line is changed to $Z_0 = (200 + j91)\ \Omega$, what is the length of the line (in wavelengths) for matched power transfer between generator and load?

14.17 Terminated, Mismatched Line. The line in **Figure 14.41** is given. The generator can be connected anywhere on the line. It is required that the generator be matched to the line so that there are no reflections into the generator. Find the closest location d (in wavelengths) to the load at which you can move the generator so that the generator is best matched to the line (i.e., the reflection coefficient at the generator is minimum). Assume the phase constant on the line is known as β_0 [rad/m].

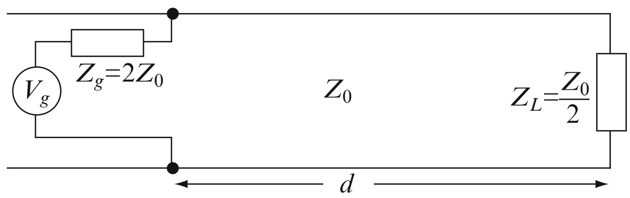


Figure 14.41

Line Impedance, Reflection Coefficient, Etc

14.18 Generalized Reflection Coefficient on a Line. Calculate the generalized reflection coefficient and the line impedance at a distance a wavelengths from the discontinuity shown in **Figure 14.42**. Assume the generator is matched to the line but the load is not. Dimensions are in wavelengths.

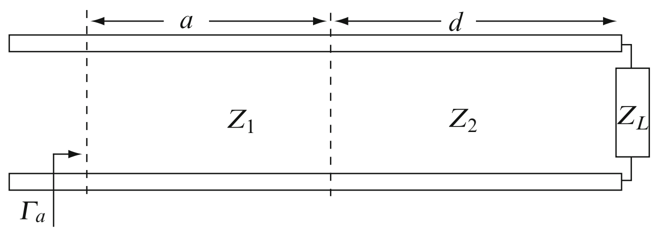


Figure 14.42

14.19 Application: Input Impedance of Microstrip Lines. A transmission line is made of two strips separated a distance d [m]. The width of the strips is w [m] and their thickness is b [m]. The strips are made of a conductor with conductivity σ [S/m]. The material between the strips is air (free space). A load Z_L [Ω] is connected at one end of the line (see **Figure 14.43**). Calculate the input impedance of the line at a distance r [m] from the load. Given: $w \gg d$, $\sigma_c = 1.0 \times 10^7$ S/m, $w = 0.1$ m, $d = 0.005$ m, $b \gg \delta$, $\mu = \mu_0$ [H/m], $\varepsilon = \varepsilon_0$ [F/m], $Z_L = 75$ Ω, $f = 1$ GHz.

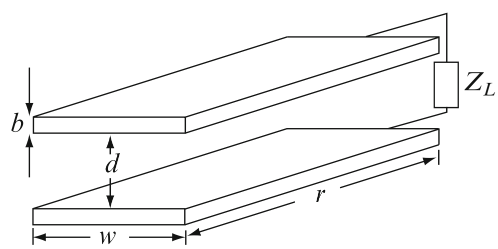


Figure 14.43

14.20 Line Impedance on Line with Multiple Loads. The transmission line in **Figure 14.44** represents a distribution line and a load that has been shorted at its end. Calculate the line impedance at a distance of 0.5λ to the left of the load.

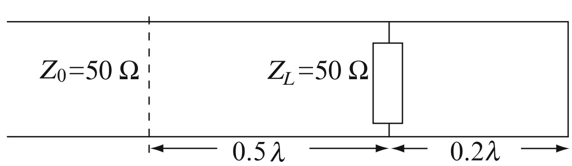


Figure 14.44

14.21 Periodically Loaded Transmission Line. A transmission line with characteristic impedance Z_0 [Ω] is loaded periodically at intervals d [m] with a load Z_L [Ω] as shown in **Figure 14.45**. There are N loads.

(a) Write an iterative procedure to calculate the line input impedance.
(b) Calculate the input impedance for $N = 3$, $Z_L = 2Z_0$ [Ω] and $\beta d = \pi/2$ [rad].

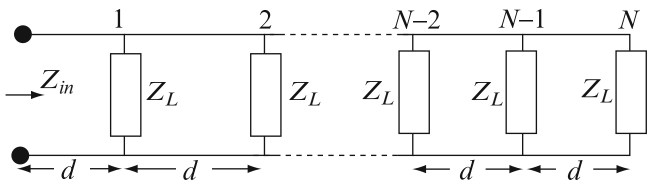


Figure 14.45

14.22 Line Impedance on Line with Multiple Loads. The transmission line in **Figure 14.46** is a distribution line normally operating at matched conditions. A second load at the end of the line has opened inadvertently. Calculate the line impedance at a distance 0.25λ from the load as shown.

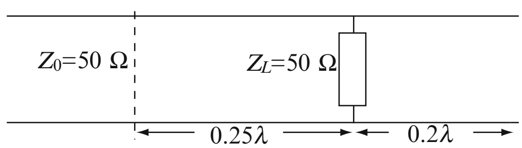


Figure 14.46

14.23 Connection of Transmission Lines in Parallel. A transmission line with characteristic impedance of 50 Ω is connected as shown in **Figure 14.47**. The measured impedance between points A–A' is 100 Ω. Calculate the two impedances Z_{L1} and Z_{L2} if the input impedance of the two lines when disconnected is equal.

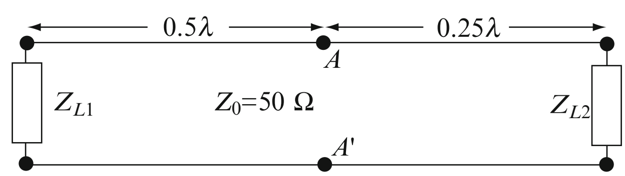


Figure 14.47

14.24 Transmission Lines in Series. Two lossless transmission lines are connected in series as in **Figure 14.48**. Write the expression for the input impedance Z_{in} of the structure.

The characteristic impedance of line 1 is Z_{01} [Ω] and of line 2 is Z_{02} [Ω]. The phase constant of both lines is β [rad/m] and the lengths l_1, l_2, and l_3 [m] are given. Line 1 and line 2 are loaded with load Z_L [Ω].

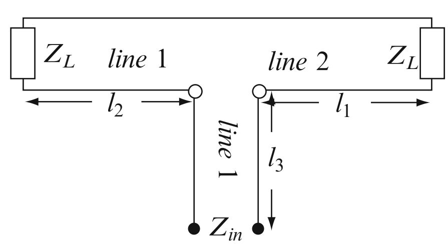


Figure 14.48

Shorted and Open Transmission Lines

14.25 Open and Shorted Transmission Lines. The two configurations in **Figure** 14.49 are given. The lines are lossless with air as insulator between the lines and operate at 300 MHz. The voltage of the generator is 12 V.

- **(a)** Calculate the current supplied by the generator in **Figure** 14.49a.
- **(b)** Calculate the current supplied by the generator in **Figure** 14.49b.
- **(c)** How much power does the generator supply in **(a)** and in **(b)**? Explain.

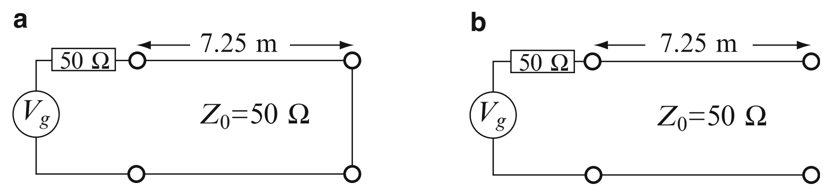


Figure 14.49

14.26 Application: Design of Network Elements. In the design of a transmission line network, it is required to design a reactance of $j100\ \Omega$ as an element in the network. Show how this can be accomplished with:

- **(a)** A lossless, shorted transmission line with characteristic impedance of 50 Ω at a wavelength of 1 m.
- **(b)** A lossless open transmission line with characteristic impedance of 75 Ω at a wavelength of 1 m.
- **(c)** If a capacitive reactance of $-j100\ \Omega$ is needed instead, what are the answers to **(a)** and **(b)**?

14.27 Application: Detection of Faults in Buried Lines. A buried cable has been cut at an unknown point. A signal generator is connected to the cable input and the frequency is swept. The first maximum at the generator occurs at a frequency f_1 [Hz] and the first minimum at a frequency $f_2 > f_1$. What is the distance from the generator to the short if the phase velocity is known and equals v_{p0} [m/s] at both frequencies. The generator is matched to the line.

14.28 Application: Detection of Faults in Buried Lines. A buried cable has water leaking into it, at a location along its length. The water leak manifests itself as an unknown impedance on the cable at the location of the leak. If the same results as in **Problem** 14.27 are obtained at the generator:

- **(a)** Is it possible with the data given in **Problem** 14.27 to find the exact location of the leak?
- **(b)** If yes, what is that location? If not, what is the minimum section that must be dug to ensure that the leak is found?

14.29 Application: Experimental Evaluation of Line Parameters. A transmission line is 10 km long and operates under matched conditions. An engineer requires the properties of the line; that is, its characteristic impedance, its attenuation constant, and its phase constant. It is not possible to subject the line to testing equipment, but the voltage and currents can be measured at the load and at the generator. These are given as $V = 10$ V, $I = 0.1$ A at the generator and $V = 3 - j2$ V at the load. Find:

- **(a)** The characteristic impedance, attenuation, and phase constants on the line.
- **(b)** The time-averaged power loss on the line.

14.30 Application: Measurements of Line Conditions. A lossless transmission line is connected to a matched load. The characteristic impedance of the line is 50 Ω. An additional load is connected to the line. To determine the additional load, the maximum voltage on the line is measured as 48 V and the minimum voltage on the line is measured as 30 V. From these measurements, calculate:

- **(a)** The additional load connected across the matched load.
- **(b)** What are the maximum and minimum line impedances and where do these occur?

14.31 Line Conditions on Shorted and Open Lines. The standing wave ratio on a lossless transmission line is measured and found to be infinite.

- **(a)** Can you tell from this measurement if the load is a short or an open circuit? Explain.
- **(b)** Suppose you can measure the line impedance directly and you find that at one point, the line impedance is zero. Moving toward the load a very short distance, you find that the line impedance is purely reactive and increases. Can you now tell if the line is shorted or open? If so, how?

14.32 **Design with Open and Shorted Transmission Lines.** The filter shown in **Figure 14.50** needs to be implemented with transmission line segments so that it may be inserted in a line circuit. Show how this may be done and calculate the line segment lengths in your implementation. The lines are lossless two-wire lines with air as insulator and a characteristic impedance of 75 Ω. Assume that the distance between the two wires in the line segments is very small compared to the wavelength. The frequency of operation is 600 MHz. Note that there is more than one solution possible.

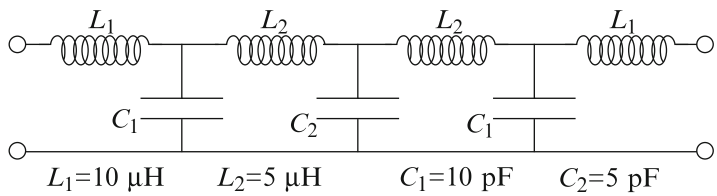


Figure 14.50

Resistive Loads on Transmission Lines

14.33 **Application: Standing Wave Ratio Measurements**. On a lossless transmission line, SWR = 5 and $Z_0 = 50\ \Omega$. To identify the conditions on the line, three measurements are made by sliding a voltmeter along the line. (1) The first maximum from the load is found at a distance of 0.25 m from the load. (2) The next maximum is found at 0.75 m from the load. (3) $V_L = 100$ V. Calculate:

(a) The load impedance.
(b) The amplitudes of the forward- and backward-propagating voltage waves.
(c) The amplitudes of the forward- and backward-propagating current waves.

14.34 **Application: Standing Wave Ratio and Minima and Maxima on the Line**. For a line with characteristic impedance $Z_0 = 50\ \Omega$ and a load $R = 220\ \Omega$, calculate:

(a) The standing wave ratio on the line.
(b) If the voltage at the load is 100 V, find the maximum and minimum voltage on the line.
(c) Where do the voltage maxima and minima occur?

14.35 **Application: Standing Wave Ratio and Minima and Maxima on the Line**. A line has a characteristic impedance $Z_0 = 50\ \Omega$ and a load $R = 25\ \Omega$. Calculate:

(a) The standing wave ratio on the line.
(b) If the voltage at the load is 100 V, find the maximum and minimum voltage on the line.
(c) Where do the voltage maxima and minima occur?

Capacitive and Inductive Loads on Transmission Lines

14.36 **Application: Line Impedance on a Capacitively Loaded Line**. A long lossless transmission line with characteristic impedance 50 Ω is terminated with a capacitance. At the frequency at which the line operates, the impedance of the load is $-j50\ \Omega$ and the phase constant on the line is 20π rad/m. Calculate and plot the line impedance as a function of distance from load.

14.37 **Application: Line Impedance on an Inductively Loaded Line**. A long lossless transmission line with characteristic impedance 50 Ω is terminated with an inductance. At the frequency at which the line operates, the reactance of the load is 50 Ω and the phase constant is 20π [rad/m]. Calculate and plot the line impedance as a function of distance from load. Compare this result with that in **Problem 14.36**.

14.38 Line Impedance on Line with General Load. A long lossless transmission line with characteristic impedance 50 Ω is terminated with a load as in **Figure 14.51**. The line operates in free space (material between the two lines is air) at 1 MHz. Calculate:

(a) The generalized reflection coefficient on the line.
(b) The standing wave ratio on the line.
(c) The location of the first minimum voltage on the line.
(d) The standing wave ratio as a function of frequency. Plot.

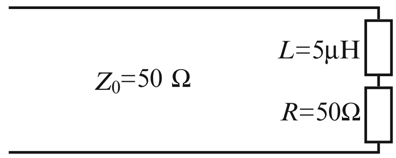


Figure 14.51

14.39 Line Impedance on Line with General Load. A long lossless transmission line with characteristic impedance 50 Ω is terminated with a load as in **Figure 14.52**. The line operates in free space (material between the two lines is air) at 1 MHz. Calculate:

(a) The generalized reflection coefficient on the line.
(b) The standing wave ratio on the line.
(c) The location of the first minimum voltage on the line.
(d) The standing wave ratio as a function of frequency.

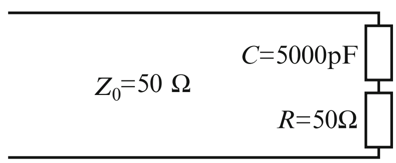


Figure 14.52

Power Relations on Lines

14.40 Power Relations on Transmission Lines. A transmission line of length l, characteristic impedance Z_0 [Ω], and propagation constant $\gamma = \alpha + j\beta$ operates at frequency f, with a load termination $Z_L \neq Z_0$ is given. Assume Z_L and Z_0 are complex and find:

(a) The input impedance.
(b) The reflection coefficient Γ_L at the load.
(c) The time averaged power P_L in the load given a forward-propagating voltage V^+.

14.41 Voltage and Power on a Coaxial Line. A coaxial line is made as follows **(Figure 14.53):** The inner wire diameter is 0.5 mm and the outer one is 6 mm. The material between the conductors is a plastic with relative permittivity of 4 and conductivity of 10^{-7} S/m. The line is matched at generator and load and is used as a cable TV line at 100 MHz. The peak input voltage to the line is 10 V and the line is 200 m long. Assuming ideal conductors, calculate:

(a) The voltage at the load.
(b) The time averaged power loss on the line.

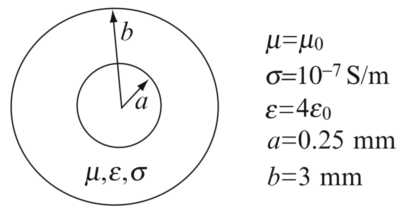


Figure 14.53

Resonant Transmission Lines

14.42 Application: Transmission Line Resonator. A transmission line resonator is made by connecting a 10 pF capacitor across the input of a 75 Ω lossless line and shorting the other end (**Figure 14.54**). What must be the length of the line in wavelengths so that the lowest resonant frequency is 800 MHz?

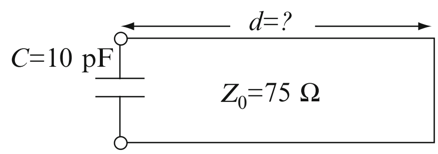


Figure 14.54

14.43 Application: Transmission Line Resonator. A lossless transmission line is cut into a length $b = 0.5$ m and shorted at both ends (**Figure 14.55**). Properties of the line are $Z_0 = 75\ \Omega$ and $v_p = c/3$ [m/s].

(a) If a connection is made at a point $x = b/2$, what is the impedance of the line at this point?
(b) A connection is now made on the line such that $x = 0.1$ m. Find the first four resonant frequencies for this line.

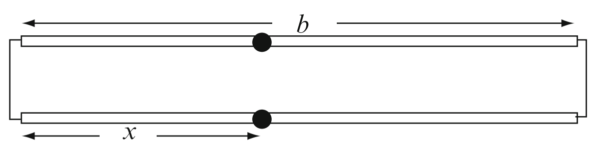


Figure 14.55

14.44 Application: Series Transmission Line Resonator. Two open transmission lines are connected as shown in **Figure 14.56**.

(a) Find the conditions for the system to resonate.
(b) What is the lowest resonant frequency for $Z_{01} = 50\ \Omega$, $Z_{02} = 75\ \Omega$, $l_1 = 0.2$ m, $l_2 = 0.4$ m, and $v_{p1} = v_{p2} = c/3$ [m/s]. The distance t is negligible.
(c) Find the next two resonant frequencies for the conditions in **(b)**.

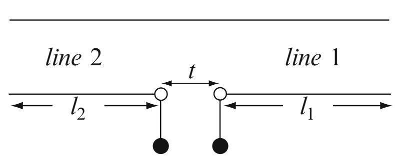


Figure 14.56

14.45 Application: Transmission Line Sensors. A transmission line resonator is made as shown in **Figure 14.57**. The device is used to sense moisture content of dough on a production line as the dough passes through the device. Dough has a permittivity given by $\varepsilon = \varepsilon_0(1 + 14k)$ [F/m], where k is the percentage of water in the dough. Calculate:

(a) The resonant frequency of the device when empty.
(b) The resonant frequency for 15 %, 10 %, and 5 % moisture in the dough.
(c) Comment on the applicability of this design for moisture measurements.

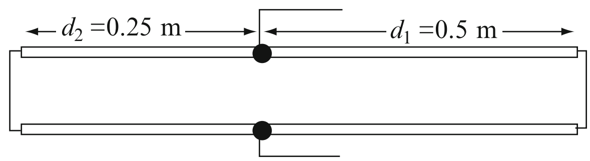


Figure 14.57