1 Vector Algebra

The vector analysis I use may be described either as a convenient and systematic abbreviation of Cartesian analysis... In this form it is not more difficult, but easier to work than Cartesians. Of course, you have to learn it. Initially, unfamiliarity may make it difficult...

—Oliver Heaviside (1850–1925), a self-taught mathematician and electrical engineer in his introduction to Vector Analysis, originally published in 1893 (Electromagnetic Theory, Chelsea Publishing Co., N.Y., 1971, Vol. 1, p. 135)

1.1 Introduction

Vector algebra[1] is the algebra of vectors: a set of mathematical rules that allows meaningful and useful operations in the study of electromagnetics. We will define vectors and the necessary operations shortly, but, for now, it is useful to remember the following axiom which will be followed throughout this book: Nothing will be defined, no quantity or operation will be used, unless it has some utility either in explaining the observed physical quantities or otherwise simplifies the discussion of a topic. This is important because, as we increase our understanding of the subject, topics may seem to be disconnected, particularly in this and the following chapter. The discussion of vector algebra and vector calculus will be developed separately from the ideas of the electromagnetic field but for the purpose of describing the electromagnetic field. It is also implicit in this statement that by doing so, we should be able to simplify the discussion of electromagnetics and, necessarily, better understand the physical properties of fields.

Vector algebra is a set of rules that apply to vector quantities. In this sense, it is similar to the algebra we are all familiar with (which we may call scalar algebra): it has rules, the rules are defined and then followed, and the rules are self-consistent.

Because at this point we know little about electromagnetics, the examples given here will be taken from other areas: mechanics, elementary physics, and, in particular, from everyday experience. Any reference to electric or magnetic quantities will be in terms of circuit theory or generally known quantities. The principle is not to introduce quantities and relations that we do not fully understand. It sometimes comes as a surprise to find that many of the quantities involved in electromagnetics are familiar, even though we may have never thought of them in this sense. All that the rules of vector algebra do is to formalize these rather loose bits of information and define their interactions. At that point, we will be able to use them in a meaningful way to describe the behavior of fields in exact terms using a concise notation.

It is worth mentioning that vector algebra (and vector calculus, which will be discussed in the following chapter) contains a very small number of quantities and operations. For this reason, the vector notation is extremely compact. There are only two quantities required: ***scalars*** and ***vectors***. Four basic operations are required for vectors: ***addition***, ***vector scaling***, ***scalar product***, and ***vector product***.

[1] Vector analysis, of which vector algebra is a subset, was developed simultaneously and independently by Josiah Willard Gibbs (1839–1903) and Oliver Heaviside (1850–1924) around 1881, for the expressed purpose of describing electromagnetics. The notation used throughout is more or less that of Heaviside. Vector analysis did not gain immediate acceptance. It was considered to be "useless" by Lord Kelvin, and many others thought of it as "awfully difficult," as Heaviside himself mentions in his introduction to vector algebra. Nevertheless, by the end of the nineteenth century, it was in general use.

N. Ida, *Engineering Electromagnetics*, DOI 10.1007/978-3-319-07806-9_1,

In addition, we will define distributions of vectors and scalars in space as ***vector*** and ***scalar fields*** and will introduce the commonly used ***coordinate systems***. The discussion in this chapter starts with the definition of scalars and vectors in Cartesian coordinates. The latter is assumed to be known and is used exclusively in the first few sections, until cylindrical and spherical coordinates are defined.

1.2 Scalars and Vectors

A quantity is a ***scalar*** if it has only a magnitude at any location in space for a given time. To describe the mass of a body, all we need is the magnitude of its mass or, for a distributed mass, the distribution in space. The same applies to the altitude of a mountain or the length of a road. These are all scalar quantities and, in particular, are static scalar quantities (independent of time). In terms of quantities useful in the study of electromagnetics, we also encounter other scalars such as work, energy, time, temperature, and electric potential (voltage). Scalar sources also play an important role: The electric charge or charge distribution (for example, charge distributed in a cloud) will be seen as sources of fields. The source of a 1.5 V cell is its potential and is a scalar source.

A ***vector***, on the other hand, is described by two quantities: a magnitude and a direction in space at any point and for any given time. Therefore, vectors may be space and time dependent. Common vectors include displacement, velocity, force, and acceleration. To see that the vector definition is important, consider a weather report giving wind speeds. The speed itself is only part of the information. If you are sailing, direction of the wind is also important. For a pilot, it is extremely important to know if the wind also has a downward component (shear wind), which may affect the flight plan. Sometimes, only the magnitude may be important: The electric generating capability of a wind-driven turbine is directly proportional to the normal (perpendicular to the turbine blades) component of the wind. Other times we may only be interested in direction. For example, the news report may say: "The rocket took off straight up." Here, the direction is the important information, and although both direction and magnitude are available, for one reason or another, the liftoff speed or acceleration is not important in this statement. The unit associated with a quantity is not part of the vector notation.

The use of vectors in electromagnetics is based on two properties of the vector. One is its ability to describe both magnitude and direction. The second is its very compact form, which allows the description of quantities with great economy in notation. This economy in notation eases handling of otherwise awkward expressions but also requires familiarity with the implications of the notation. In a way, it is like shorthand. A compact notation is used, but it also requires us to know how to read it so that the information conveyed is meaningful and unambiguous.

To allow instant recognition of a vector quantity, we denote vectors by a boldface letter such as **E**, **H**, **a**, and **b**. Scalar quantities are denoted by regular letters: E, H, a, and b. In handwriting, it is difficult to make the distinction between normal and boldface lettering. A common method is to use a bar or arrow over the letter to indicate a vector. Thus, $\overline{E}, \overline{H}, \overline{a}, \overline{b}$, are also vectors. If a quantity is used only as a vector, there is no need to distinguish it from the corresponding scalar quantity. Some vector operators (which will be discussed in the following chapter) are of this type. In these instances, neither boldface nor bar notation is needed since there is no room for confusion.

1.2.1 Magnitude and Direction of Vectors: The Unit Vector and Components of a Vector

The magnitude of a vector is that scalar which is numerically equal to the vector:

$$A = |\mathbf{A}| \tag{1.1}$$

The magnitude of a vector is its length and includes the units of the vector. Thus, for example, the magnitude of a velocity vector **v** is the speed v [m/s]. To define the direction of a vector **A**, we employ the idea of the unit vector. A unit vector $\hat{\mathbf{A}}$ is a vector of magnitude one (dimensionless) in the direction of **A**:

$$\boxed{\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}} = \frac{\mathbf{A}}{A} \tag{1.2}$$

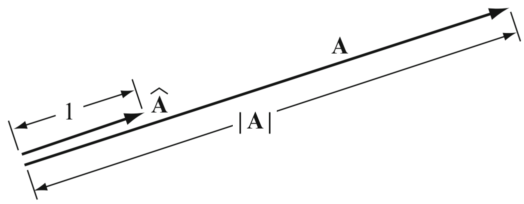


Figure 1.1 The relations between vector **A**, the unit vector $\hat{\mathbf{A}}$, and the magnitude of the vector $|\mathbf{A}|$

Thus, $\hat{\mathbf{A}}$ can be viewed as a new dimensionless vector of unit magnitude ($|\hat{\mathbf{A}}| = 1$) in the direction of, or parallel to, the vector **A**. **Figure 1.1** shows a vector, its magnitude, and its unit vector.

Vectors may have components in various directions. For example, a vehicle moving at a velocity **v** on a road that runs *SE* to *NW* has two equal velocity components, one in the *N* direction and one in the *W* direction, as shown in **Figure 1.2a**. We can write the velocity of the vehicle in terms of two velocity components as

$$\mathbf{v} = \hat{\mathbf{N}}\, v_N + \hat{\mathbf{W}}\, v_W = \hat{\mathbf{N}}\, v\frac{\sqrt{2}}{2} + \hat{\mathbf{W}}\, v\frac{\sqrt{2}}{2} \tag{1.3}$$

The two terms on the right-hand side ($\hat{\mathbf{N}}\, v\sqrt{2}/2$ and $\hat{\mathbf{W}}\, v\sqrt{2}/2$) are called the ***vector components*** of the vector. The components of the vectors can also be viewed as scalars by taking only their magnitude. These are called ***scalar components***. This definition is used extensively when standard systems of coordinates are used and the directions in space are known. In this case, the scalar components are $v\sqrt{2}/2$ in the *N* and *W* directions. To avoid confusion as to which type of component is used, we will always indicate specifically the type of component unless it is obvious which type is meant.

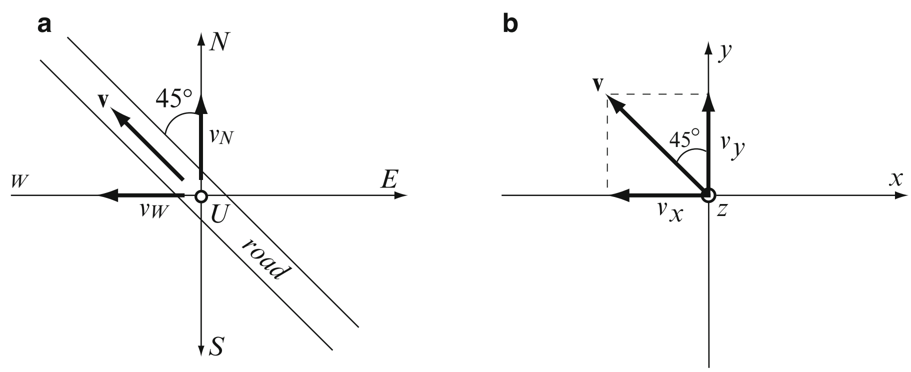


Figure 1.2 (**a**) A convenient coordinate system. (**b**) A more "standard" coordinate system

We chose here a particular system of coordinates to demonstrate that the system of coordinates is a matter of choice. The same can be accomplished by laying a standard system of coordinates, say the rectangular coordinate system over the road map shown in **Figure 1.2a**. This action transforms the road map into a standard coordinate system, and now, using **Figure 1.2b**, we can write

$$\mathbf{v} = -\hat{\mathbf{x}}\, v_x + \hat{\mathbf{y}}\, v_y = -\hat{\mathbf{x}}\, v\frac{\sqrt{2}}{2} + \hat{\mathbf{y}}\, v\frac{\sqrt{2}}{2} \tag{1.4}$$

The components of the vector are in the *x* and *y* directions. The magnitude of the vector is *v*, and this is written directly from the geometry in **Figure 1.2b** as

$$v = |\mathbf{v}| = \left| -\hat{\mathbf{x}}\, v_x + \hat{\mathbf{y}}\, v_y \right| = \sqrt{v_x^2 + v_y^2} \tag{1.5}$$

The unit vector is in the direction of **v** and is given as

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\hat{\mathbf{x}}\, v_x + \hat{\mathbf{y}}\, v_y}{\left|-\hat{\mathbf{x}}\, v_x + \hat{\mathbf{y}}\, v_y\right|} = \frac{-\hat{\mathbf{x}}\, v_x + \hat{\mathbf{y}}\, v_y}{\sqrt{v_x^2 + v_y^2}} = -\hat{\mathbf{x}}\,\frac{\sqrt{2}}{2} + \hat{\mathbf{y}}\,\frac{\sqrt{2}}{2} \tag{1.6}$$

It is important to note that although the unit vector $\hat{\mathbf{v}}$ has unit magnitude, its components in the x and y directions do not. Their magnitude is $\sqrt{2}/2$. This may seem to be a minor distinction, but, in fact, it is important to realize that the vector components of a unit vector are not necessarily of unit magnitude. Note, also, that the magnitude of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ is one since these are the unit vectors in the direction of the vector components of $\mathbf{v}$, namely, x and y.

In this case, both the vector and the unit vector were conveniently written in terms of a particular coordinate system. However, as a rule, any vector can be written in terms of components in other coordinate systems. An example is the one used to describe directions as N, S, W, and E. We shall discuss this separately, but from the above example, some systems are clearly more convenient than others. Also to be noted here is that a general vector in space written in the Cartesian system has three components, in the x, y, and z directions (see below). The third dimension in the above example of velocity gives the vertical component of velocity as the vehicle moves on a nonplanar surface.

In the right-handed Cartesian system (or right-handed rectangular system), we define three coordinates as shown in **Figure 1.3**. A point in the system is described as $P(x_0,y_0,z_0)$, and the general vector $\mathbf{A}$, connecting two general points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, is given as

$$\mathbf{A}(x,y,z) = \hat{\mathbf{x}}A_x(x,y,z) + \hat{\mathbf{y}}A_y(x,y,z) + \hat{\mathbf{z}}A_z(x,y,z) \tag{1.7}$$

where the scalar components A_x, A_y, and A_z are the projections of the vector on the x, y, and z coordinates, respectively. These are

$$A_x = x_2 - x_1, \quad A_y = y_2 - y_1, \quad A_z = z_2 - z_1 \tag{1.8}$$

The length of the vector (i.e., its magnitude) is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1.9}$$

and the unit vector in the direction of vector $\mathbf{A}$ is

$$\hat{\mathbf{A}} = \frac{\mathbf{A}(x,y,z)}{A(x,y,z)} = \frac{\hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{1.10}$$

or

$$\hat{\mathbf{A}} = \hat{\mathbf{x}}\frac{(x_2 - x_1)}{A} + \hat{\mathbf{y}}\frac{(y_2 - y_1)}{A} + \hat{\mathbf{z}}\frac{(z_2 - z_1)}{A} \tag{1.11}$$

We will make considerable use of the unit vector, primarily as an indicator of direction in space. Similarly, the use of components is often employed to simplify analysis.

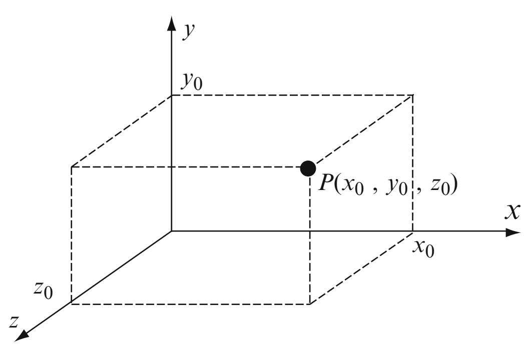


Figure 1.3 A point in the Cartesian system of coordinates

Example 1.1 A vector is given as $\mathbf{A} = -\hat{\mathbf{x}}\,5 - \hat{\mathbf{y}}\,(3x+2) + \hat{\mathbf{z}}$. Calculate:

(a) The scalar components of the vector in the x, y, and z directions.
(b) The length of the vector.
(c) The unit vector in the direction of **A**.

Solution: The solution makes use of **Eqs. (1.8)** through **(1.11)**. In this case, the vector (and all its properties) depends on the variable x alone, although it has components in the y and z directions:

(a) The scalar components of the vector are the coefficients of the three unit vectors:

$$A_x = -5, \quad A_y = -(3x+2), \quad A_z = 1$$

Note: The negative sign is part of the scalar component, not the unit vector.

(b) The length of the vector is given by **Eq. (1.9)**:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(-5)^2 + (-(3x+2))^2 + 1^2} = \sqrt{9x^2 + 12x + 30}$$

(c) The unit vector is calculated from **Eq. (1.11)**:

$$\hat{\mathbf{A}} = -\hat{\mathbf{x}}\frac{5}{\sqrt{9x^2+12x+30}} - \hat{\mathbf{y}}\frac{3x+2}{\sqrt{9x^2+12x+30}} + \hat{\mathbf{z}}\frac{1}{\sqrt{9x^2+12x+30}}$$

where the scalar components A_x, A_y, and A_z and the magnitude of **A** calculated in **(a)** and **(b)** were used.

Example 1.2 An aircraft takes off at a 60° angle and takeoff speed of 180 km/h in the *NE–SW* direction. Find:

(a) The velocity vector of the aircraft.
(b) Its direction in space.
(c) Its ground velocity (i.e., the velocity of the aircraft's shadow on the ground).

Solution: First, we choose a system of coordinates. In this case, *E–W*, *N–S*, and *D* (down)–*U* (up) is an appropriate choice. This choice describes the physics of the problem even though it is not the most efficient system we can use. (In the exercise that follows, the Cartesian system is used instead.) The components of velocity are calculated from the magnitude (180 km/h) of velocity and angle using projections on the ground and vertically, followed by the velocity vector and the unit vector:

(a) The aircraft velocity has two scalar components: the vertical component $v_u = 180\sin 60°$ and the ground component $v_g = 180\cos 60°$. These speeds are given in km/h. The SI units call for the second as the unit of time and the meter as the unit of distance. Thus, we convert these speeds to m/s. Since 180 km/h = 50 m/s, we get $v_g = 50\cos 60°$ and $v_u = 50\sin 60°$. The west and south components are calculated from v_g, as (see **Figure 1.4**)

$$v_w = 50\cos 60^\circ\cos 45^\circ, \quad v_s = 50\cos 60^\circ\sin 45^\circ \quad \left[\frac{\text{m}}{\text{s}}\right]$$

The third component is v_u. Thus, the velocity vector is

$$\mathbf{v} = \hat{\mathbf{W}}\,50\cos 60^\circ\cos 45^\circ + \hat{\mathbf{S}}\,50\cos 60^\circ\sin 45^\circ + \hat{\mathbf{U}}\,50\sin 60^\circ = \hat{\mathbf{W}}\,17.678 + \hat{\mathbf{S}}\,17.678 + \hat{\mathbf{U}}\,43.3 \quad [\text{m/s}]$$

(b) The direction in space is given by the unit vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\hat{\mathbf{W}}\,50\cos 60^\circ\cos 45^\circ + \hat{\mathbf{S}}\,50\cos 60^\circ\sin 45^\circ + \hat{\mathbf{U}}\,50\sin 60^\circ}{\sqrt{(50\cos 60^\circ\cos 45^\circ)^2 + (50\cos 60^\circ\sin 45^\circ)^2 + (50\sin 60^\circ)^2}} = \hat{\mathbf{W}}\frac{\sqrt{2}}{4} + \hat{\mathbf{S}}\frac{\sqrt{2}}{4} + \hat{\mathbf{U}}\frac{\sqrt{3}}{2} \quad \left[\frac{\text{m}}{\text{s}}\right]$$

(c) Ground velocity is the velocity along the ground plane. This is calculated by setting the vertical velocity of the aircraft found in **(a)** to zero:

$$\mathbf{v} = \hat{\mathbf{W}}\,50\cos60^\circ\cos45^\circ + \hat{\mathbf{S}}\,50\cos60^\circ\sin45^\circ = \hat{\mathbf{W}}\,17.678 + \hat{\mathbf{S}}\,17.678 \qquad \left[\frac{\text{m}}{\text{s}}\right]$$

Note: It is useful to convert the units to SI units at the outset. This way there is no confusion as to what units are used, and what the intermediate results are, at all stages of the solution.

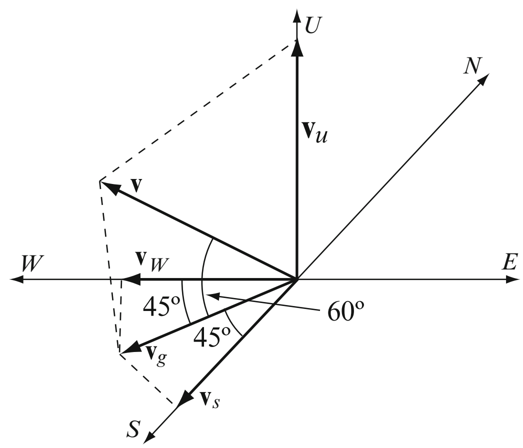


Figure 1.4 Velocity terms along the axes and on the ground

Exercise 1.1 Solve **Example 1.2** in the Cartesian system of coordinates with the positive x axis coinciding with E (east), positive y axis with N (north), and positive z axis with U (up).

Answer

(a) $\mathbf{v} = -\hat{\mathbf{x}}\,50\cos60^\circ\cos45^\circ - \hat{\mathbf{y}}\,50\cos60^\circ\sin45^\circ + \hat{\mathbf{z}}\,50\sin60^\circ \quad [\text{m/s}]$
(b) $\hat{\mathbf{v}} = -\hat{\mathbf{x}}\,\sqrt{2}/4 - \hat{\mathbf{y}}\,\sqrt{2}/4 + \hat{\mathbf{z}}\,\sqrt{3}/2 \qquad [\text{m/s}]$
(c) $\mathbf{v}_g = -\hat{\mathbf{x}}\,50\cos60^\circ\cos45^\circ - \hat{\mathbf{y}}\,50\cos60^\circ\sin45^\circ \quad [\text{m/s}]$

1.2.2 Vector Addition and Subtraction

Point_Charges.m

The first vector algebra operation that needs to be defined is vector addition. This is perhaps the most commonly performed vector operation.

The sum of two vectors results in a third vector

$$\boxed{\mathbf{A} + \mathbf{B} = \mathbf{C}} \tag{1.12}$$

To see how this operation is carried out, we use two general vectors $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$ in Cartesian coordinates and write

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right) + \left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right) = \left(\hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z\right) \tag{1.13}$$

Adding components in the same directions together gives

$$\mathbf{C} = \hat{\mathbf{x}}\left(A_x + B_x\right) + \hat{\mathbf{y}}\left(A_y + B_y\right) + \hat{\mathbf{z}}\left(A_z + B_z\right) \tag{1.14}$$

Figure 1.5 shows this process: In **Figure 1.5a**, vectors **A** and **B** are separated into their three components. **Figure 1.5b** shows that vector **C** is obtained by adding the components of **A** and **B**, which, in turn, are equivalent to translating the vector **B** (without changing its direction in space or its magnitude) so that its tail coincides with the head of vector **A**. Vector **C** is now the vector connecting the tail of vector **A** with the head of vector **B**. This sketch defines a general graphical method of calculating the sum of two vectors:

(1) Draw the first vector in the sum.
(2) Translate the second vector until the tail of the second vector coincides with the head of the first vector.
(3) Connect the tail of the first vector with the head of the second vector to obtain the sum.

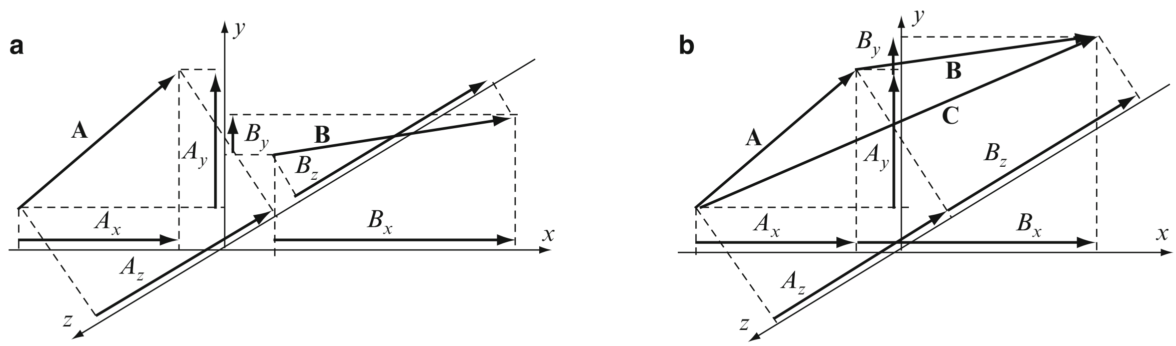


Figure 1.5 (**a**) Two vectors **A** and **B** and their x, y, and z components. (**b**) Addition of vectors **A** and **B** by adding their components

The process is shown in **Figure 1.6** in general terms. This method of calculating the sum of two vectors is sometimes called the ***head-to-tail*** method or rule. An alternative method is obtained by generating two sums **A** + **B** and **B** + **A** using the above method. The two sums are shown in **Figure 1.7a** as two separate vectors and as a single vector in **Figure 1.7b**. The result is a parallelogram with the two vectors, connected tail to tail forming two adjacent sides, and the remaining two sides are parallel lines to the vectors. This method is summarized as follows:

(1) Translate vector **B** so that its tail coincides with the tail of vector **A**.
(2) Construct the parallelogram formed by the two vectors and the two parallels to the vectors.
(3) Draw vector **C** with its tail at the tails of vectors **A** and **B** and head at the intersection of the two parallel lines (dashed lines in **Figure 1.8**).

This method is shown in **Figure 1.8** and is called the ***parallelogram rule***.

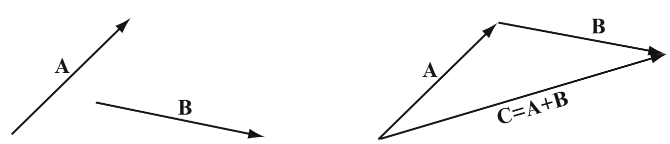


Figure 1.6 Addition of two vectors by translating vector **B** until its tail coincides with the head of vector **A**. The sum **A** + **B** is the vector connecting the tail of vector **A** with the head of vector **B**

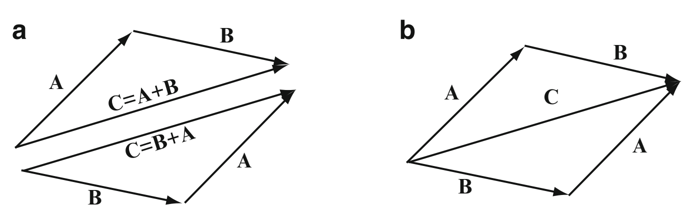


Figure 1.7 Calculating the sums **C** = **A** + **B** and **C** = **B** + **A**

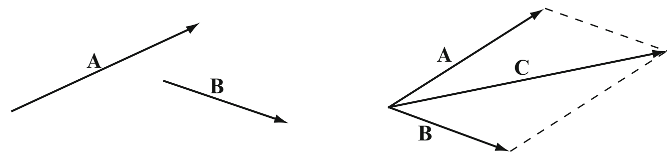


Figure 1.8 The parallelogram method. The *dashed lines* are used to show that opposite sides are equal and parallel

Vector subtraction is accomplished by noting the following:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \hat{\mathbf{A}}A + \left(-\hat{\mathbf{B}}\right)B \tag{1.15}$$

This indicates that vector subtraction is the same as the addition of a negative vector. In terms of the tail-to-head or parallelogram method, we must first reverse the direction of vector **B** and then perform summation of the two vectors. This is shown in **Figure** 1.9.

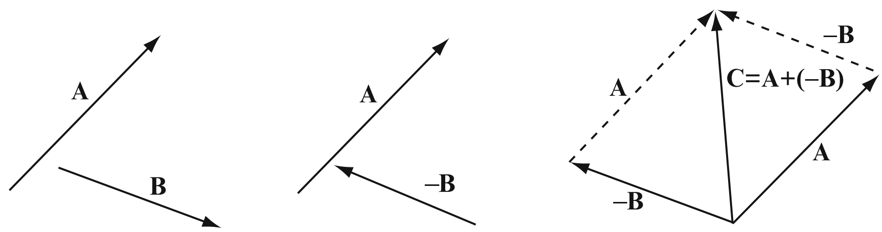


Figure 1.9 Subtraction of vector **B** from vector **A**

Summation or subtraction of more than two vectors should be viewed as a multiple-step process. For example:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{D} + \mathbf{C} \tag{1.16}$$

The sum $\mathbf{D} = \mathbf{A} + \mathbf{B}$ is calculated first using the above methods and then the sum $\mathbf{D} + \mathbf{C}$ is evaluated similarly. The same applies to subtraction.

Note: Any of the two graphical methods of calculating the sum of two vectors may be used, but, in computation, it is often more convenient to separate the vectors into their components and calculate the sum of the components. This is particularly true if we also need to calculate unit vectors. The graphical methods are more useful in understanding what the sum of the vector means and to visualize the direction in space.
Vector summation and subtraction are associative and commutative processes; that is:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(commutative)} \tag{1.17}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad \text{(associative)} \tag{1.18}$$

The vector addition is also distributive, but we will only show this in **Section** 1.2.3.

Example 1.3 Two vectors **A** and **B** (such as the velocity vectors of two aircraft) are $\mathbf{A} = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{x}}4 - \hat{\mathbf{z}}3$. Calculate:

(a) The sum of the two vectors.
(b) The difference **A** – **B** and **B** – **A** (these differences represent the relative velocities of **A** with respect to **B** and of **B** with respect to **A**).

Solution: **(a)** The vectors are placed on the system of coordinates shown in **Figure** 1.10a, and the components of **A** and **B** are found as shown. The components of vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ are now found directly from the figure. In **(b)**, we write the two expressions $\mathbf{D} = \mathbf{A} - \mathbf{B}$ and $\mathbf{E} = \mathbf{B} - \mathbf{A}$ and add together the components:

(a) Vector **A** has scalar components of 1, 2, and 3 in the x, y, and z directions, respectively. It may therefore be viewed as connecting the origin (as a reference point) to point $P_1(1,2,3)$, as shown in **Figure** 1.10a. Vector **B** is in the x–z plane and connects the origin to point $P_2(4,0,-3)$. The vectors may be translated anywhere in space as long as their lengths and directions are not changed. Translate vector **A** such that its tail touches the head of vector **B**. This is shown in **Figure 1.10b** in terms of the components (i.e., translation of vector **A** so that its tail coincides with the head of vector **B** is the same as translating its components so that their tails coincide with the heads of the corresponding components of vector **B**). The sum **C** = **A** + **B** is the vector connecting the tail of vector **B** with the head of vector **A**. The result is (writing the projections of vector **C** onto the *x*, *y*, and *z* axes):

$$\mathbf{C} = \hat{\mathbf{x}}5 + \hat{\mathbf{y}}2$$

(b) To calculate the differences, we add the vector components of the two vectors together, observing the sign of each vector component:

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = (\hat{\mathbf{x}}1 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) - (\hat{\mathbf{x}}4 - \hat{\mathbf{z}}3) = \hat{\mathbf{x}}(1-4) + \hat{\mathbf{y}}(2-0) + \hat{\mathbf{z}}(3-(-3))$$

or

$$\mathbf{A} - \mathbf{B} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}6$$

$$\mathbf{E} = \mathbf{B} - \mathbf{A} = (\hat{\mathbf{x}}4 - \hat{\mathbf{z}}3) - (\hat{\mathbf{x}}1 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3) = \hat{\mathbf{x}}(4-1) + \hat{\mathbf{y}}(0-2) + \hat{\mathbf{z}}((-3)-3)$$

or

$$\mathbf{B} - \mathbf{A} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}6$$

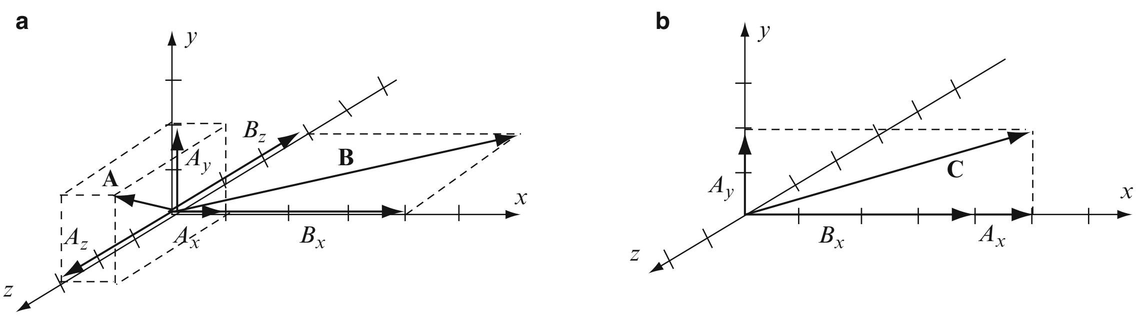


Figure 1.10 (**a**) Components of vectors **A** and **B**. (**b**) The sum **C** = **A** + **B** is obtained by summing the components of **A** and **B**

Exercise 1.2 Three vectors are given: $\mathbf{A} = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3, \mathbf{B} = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3$, and $\mathbf{C} = -\hat{\mathbf{x}}4$. Calculate:

(a) **A + B + C**.
(b) **A + B – 2C**.
(c) **A – B – C**.
(d) The unit vector in the direction of **A – 2B + C**.

Answer

(a) $\mathbf{A} + \mathbf{B} + \mathbf{C} = \hat{\mathbf{x}}1 + \hat{\mathbf{z}}6$.
(b) $\mathbf{A} + \mathbf{B} - 2\mathbf{C} = \hat{\mathbf{x}}13 + \hat{\mathbf{z}}6$.
(c) $\mathbf{A} - \mathbf{B} - \mathbf{C} = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}4$.
(d) $-\hat{\mathbf{x}}0.8538 + \hat{\mathbf{y}}0.4657 - \hat{\mathbf{z}}0.2328$.

1.2.3 Vector Scaling

A vector can be scaled by multiplying its magnitude by a scalar value. Scaling is defined as changing the magnitude of the vector:

$$\boxed{k\mathbf{A} = k\left(\hat{\mathbf{A}}A\right) = \hat{\mathbf{A}}\,(kA)} \tag{1.19}$$

The term "multiplication" for vectors is not used to avoid any confusion with vector products, which we define in the following section. Scaling of a vector is equivalent to "lengthening" or "shortening" the vector without modifying its direction if k is a positive constant, as shown in **Figure 1.11a**. Increasing the velocity of an aircraft (without change in direction) from 300 to 330 km/h scales the velocity vector by a factor of $k = 1.1$. If k is negative, the resulting scaled vector has a magnitude $|k|$ times its nonscaled magnitude but also a negative direction, as shown in **Figure 1.11b**.

Vector scaling is both associative and commutative but not distributive (simply because the product of two vectors has not been defined yet); that is,

$$k\mathbf{A} = \mathbf{A}k \quad \text{(commutative)} \tag{1.20}$$

$$k(p\mathbf{A}) = (kp)\mathbf{A} \quad \text{(associative)} \tag{1.21}$$

Also,

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B} \tag{1.22}$$

The latter shows that the vector sum is distributive.

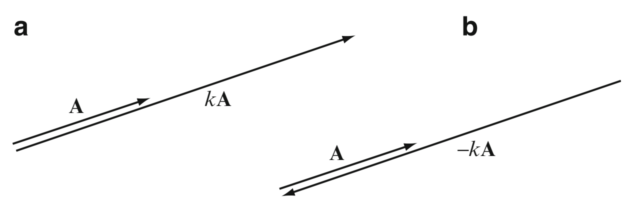


Figure 1.11 (**a**) Scaling of vector **A** by a positive scalar k. (**b**) Scaling of vector **A** by a negative scalar k

1.3 Products of Vectors

The multiplication of two vectors is called a ***product***. Here, we define two types of products based on the result obtained from the product. The first type is the scalar product. This is a product of two vectors which results in a scalar. The second is a vector product of two vectors, which results in a vector. Beyond the form of the product, these have important physical and geometrical meanings which make them some of the most useful and often encountered vector operations.

1.3.1 The Scalar Product

A ***scalar product*** of two vectors **A** and **B** is denoted as $\mathbf{A}\cdot\mathbf{B}$ and is defined as

"the product of the magnitudes of **A** and **B** and the cosine of the smaller angle between **A** and **B**"; that is,

$$\boxed{\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\phi_{AB}} \tag{1.23}$$

where the angle ϕ_{AB} is the smaller angle between **A** and **B**, as shown in **Figure 1.12**. The sign $\equiv$ indicates that **Eq. (1.23)** is the definition of the scalar product. The result is a scalar. The scalar product is often called a ***dot product*** because of the *dot* notation used. It has a number of properties that we will exploit later:

(1) For any angle $0 \leq \phi_{AB} < \pi/2$, the scalar product is positive. For angles above $\pi/2$ ($\pi/2 < \phi_{AB} \leq \pi$), the scalar product is negative.
(2) The scalar product is zero for any two perpendicular vectors ($\phi_{AB} = \pi/2$).
(3) For $\phi_{AB} = 0$ (parallel vectors), the scalar product equals AB, and for $\phi_{AB} = \pi$, the product is ($-AB$).
(4) The magnitude of the scalar product of two vectors is always smaller or equal to the product of their magnitudes ($|\mathbf{A} \cdot \mathbf{B}| \leq AB$).
(5) The product can be viewed as the product of the magnitude of vector **A** and the magnitude of the projection of vector **B** on **A** or vice versa ($\mathbf{A} \cdot \mathbf{B} = A(B\cos\phi_{AB}) = B(A\cos\phi_{AB})$).
(6) The scalar product is commutative and distributive:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative)} \tag{1.24}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive)} \tag{1.25}$$

The scalar product can be written explicitly using two vectors **A** and **B** in Cartesian coordinates as

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= \left(\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z\right) \cdot \left(\hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z\right) \\ &= \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} A_x B_x + \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} A_x B_y + \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} A_x B_z + \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} A_y B_x + \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} A_y B_y + \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} A_y B_z + \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} A_z B_x + \hat{\mathbf{z}} \cdot \hat{\mathbf{y}} A_z B_y + \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} A_z B_z\end{aligned} \tag{1.26}$$

From properties **(2)** and **(3)** and since unit vectors are of magnitude 1, we have

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{y}} = 0 \tag{1.27}$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1 \tag{1.28}$$

Therefore, **Eq. (1.26)** becomes

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \tag{1.29}$$

This form affords simple evaluation of the product from the components of the vectors rather than requiring calculation of the angle between the vectors. From this, we also note that

$$\mathbf{A} \cdot \mathbf{A} = AA\cos(0) = A^2 = A_x^2 + A_y^2 + A_z^2 \tag{1.30}$$

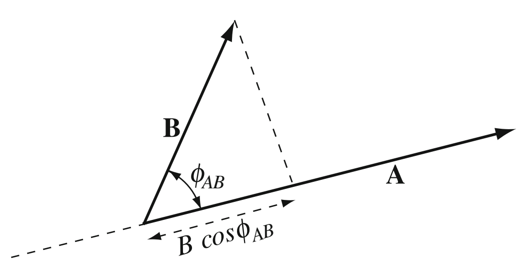


Figure 1.12 Definition of the scalar product between vectors **A** and **B**. The smaller angle between the vectors is used

Example 1.4 Calculate the projection of a general vector **A** onto another general vector **B** and the vector component of **A** in the direction of **B**.

Solution: The projection of vector **A** onto **B** is $A\cos\phi_{AB}$. This is

$$A\cos\phi_{AB} = \frac{\mathbf{A}\cdot\mathbf{B}}{B} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{B_x^2 + B_y^2 + B_z^2}}.$$

To calculate the vector component of **A** in the direction of **B**, we note that the magnitude of this component is the projection calculated above, whereas the direction of the component is that of the unit vector in the direction of **B**. The latter is

$$\hat{\mathbf{B}} = \frac{\mathbf{B}}{|B|} = \frac{\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z}{\sqrt{B_x^2 + B_y^2 + B_z^2}}.$$

The vector component of vector **A** in the direction of vector **B** is therefore

$$\begin{aligned}\mathbf{A}_B = \hat{\mathbf{B}}A\cos\phi_{AB} &= \frac{\hat{\mathbf{x}}B_x(\mathbf{A}\cdot\mathbf{B}) + \hat{\mathbf{y}}B_y(\mathbf{A}\cdot\mathbf{B}) + \hat{\mathbf{z}}B_z(\mathbf{A}\cdot\mathbf{B})}{B_x^2 + B_y^2 + B_z^2}\\ &= \hat{\mathbf{x}}\frac{B_x\left(A_xB_x + A_yB_y + A_zB_z\right)}{B_x^2 + B_y^2 + B_z^2} + \hat{\mathbf{y}}\frac{B_y\left(A_xB_x + A_yB_y + A_zB_z\right)}{B_x^2 + B_y^2 + B_z^2} + \hat{\mathbf{z}}\frac{B_z\left(A_xB_x + A_yB_y + A_zB_z\right)}{B_x^2 + B_y^2 + B_z^2}.\end{aligned}$$

Example 1.5 Two vectors are given as $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$ and $\mathbf{B} = -\hat{\mathbf{x}} + \hat{\mathbf{y}}5 + \hat{\mathbf{z}}$. Find the angle between the two vectors.

Solution: Using the scalar product, the cosine of the angle between the vectors is evaluated from **Eq. (1.23)** as

$$\cos\phi_{AB} = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} \quad\rightarrow\quad \phi_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right).$$

The magnitudes of **A** and **B** are

$$A = |\mathbf{A}| = \sqrt{1+25+1} = \sqrt{27} = 3\sqrt{3}, \quad B = |\mathbf{B}| = \sqrt{1+25+1} = \sqrt{27} = 3\sqrt{3}$$

The scalar product of **A** and **B** is

$$\mathbf{A}\cdot\mathbf{B} = (\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}})\cdot(-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 + \hat{\mathbf{z}}) = -1 + 25 - 1 = 23.$$

Thus

$$\cos\phi_{AB} \equiv \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{23}{3\sqrt{3}\times 3\sqrt{3}} = 0.85185 \quad\rightarrow\quad \phi_{AB} = \cos^{-1}(0.85185) = 31^\circ 35'.$$

Example 1.6 Application: The Cosine Formula The two vectors of the previous example are given and drawn schematically in **Figure 1.13**:

(a) Show that the distance between points P_1 and P_2 is given by

$$d = \sqrt{A^2 + B^2 - 2AB\cos\phi_{AB}}.$$

(b) Calculate this length for the two vectors.

Solution: This example is recognizable as the application of the cosine formula and, in fact, may be viewed as its derivation. Assuming a third vector pointing from P_1 to P_2 as shown in **Figure 1.13**, we calculate this vector as $\mathbf{C} = \mathbf{B} - \mathbf{A}$. The scalar product $\mathbf{C}\cdot\mathbf{C}$ gives the distance C^2. This is the distance between P_1 and P_2 squared. Taking the square root gives the required result:

(a)

$$C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) = \mathbf{B}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{A} - 2\mathbf{B}\cdot\mathbf{A}$$

Since $\mathbf{B}\cdot\mathbf{B} = B^2$, $\mathbf{A}\cdot\mathbf{A} = A^2$, and $\mathbf{B}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{B} = BA\cos\phi_{BA} = AB\cos\phi_{AB}$, we get

$$C^2 = A^2 + B^2 - 2AB\cos\phi_{AB} \quad \rightarrow \quad C = \sqrt{A^2 + B^2 - 2AB\cos\phi_{AB}}$$

(b) For the two vectors in **Example 1.5**

$$\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}\,5 - \hat{\mathbf{z}}, \qquad \mathbf{B} = -\hat{\mathbf{x}} + \hat{\mathbf{y}}\,5 + \hat{\mathbf{z}}.$$

we calculated

$$A = 3\sqrt{3}, \quad B = 3\sqrt{3}, \quad \cos\phi_{AB} = 0.85185$$

The distance between P_2 and P_1 is therefore

$$d = \sqrt{A^2 + B^2 - 2AB\cos\phi_{AB}} = \sqrt{27 + 27 - 2\times 27\times 0.85185} = 2.828$$

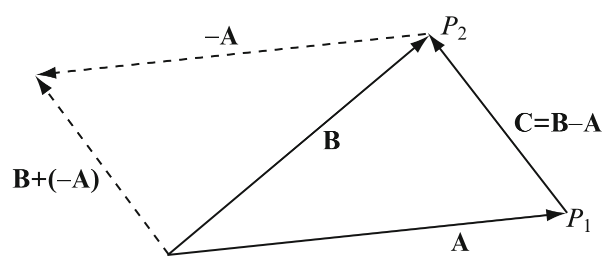


Figure 1.13 Diagram used to prove the cosine formula

Exercise 1.3 An airplane flies with a velocity $\mathbf{v} = \hat{\mathbf{x}}\,100 + \hat{\mathbf{y}}\,500 + \hat{\mathbf{z}}\,200$. Calculate the aircraft's velocity in the direction of the vector $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$.

Answer $\mathbf{v}_A = \hat{\mathbf{x}}\,800/3 + \hat{\mathbf{y}}\,800/3 + \hat{\mathbf{z}}\,800/3$

1.3.2 The Vector Product

The vector product[2] of two vectors **A** and **B**, denoted as $\mathbf{A}\times\mathbf{B}$, is defined as

"*the vector whose magnitude is the absolute value of the product of the magnitudes of the two vectors and the sine of the smaller angle between the two vectors while the direction of the vector is perpendicular to the plane in which the two vectors lie*";

that is,

$$\boxed{\mathbf{A}\times\mathbf{B} \equiv \hat{\mathbf{n}}\,|AB\sin\phi_{AB}|} \tag{1.31}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane formed by vectors **A** and **B** and ϕ_{AB} is, again, the smaller angle between the vectors. The normal unit vector gives the direction of the product, which is obviously a vector. For this reason, it is called a ***vector product*** or a ***cross product*** because of the cross symbol used in the notation.

[2] The vector product was defined by Sir William Rowan Hamilton (1805–1865) as part of his theory of quaternions around 1845. James Clerk Maxwell made use of this theory when he wrote his *Treatise on Electricity and Magnetism* in 1873, although he was critical of quaternions. Modern electromagnetics uses the Heaviside–Gibbs vector system rather than the Hamilton system.

The unit vector may be in either direction perpendicular to the plane, and to define it uniquely, we employ the right-hand rule, as shown in **Figure 1.14**. According to this rule, if the right-hand palm is placed on the first vector in the product and rotated toward the second vector through an angle ϕ_{AB}, the extended thumb shows the correct direction of the cross product. This rule immediately indicates that moving the palm from vector **B** to vector **A** gives a direction opposite to that moving from vector **A** to **B**. Thus, we conclude that the vector product is not commutative:

$$\boxed{\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \text{(noncommutative)}} \tag{1.32}$$

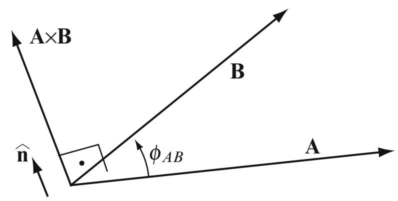


Figure 1.14 The vector product between vectors **A** and **B**

In addition to the noncommutative property of the vector product, the following properties are noted:

(1) The vector product is always perpendicular to the plane of the two vectors; that is, it is perpendicular to both vectors.
(2) For two vectors which are perpendicular to each other ($\phi_{AB} = \pi/2$), the magnitude of the vector product is equal to the product of the magnitudes of the two vectors ($\sin\phi_{AB} = 1$) and is always positive.
(3) The vector product of two parallel vectors is always zero ($\sin\phi_{AB} = 0$).
(4) The vector product of a vector with itself is always zero ($\sin\phi_{AA} = 0$).
(5) The vector product is not associative (this will be discussed in the following section because it requires the definition of a triple product).
(6) The vector product is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.33}$$

(7) The magnitude of the vector product represents the area bounded by the parallelogram formed by the two vectors and two lines parallel to the vectors, as shown in **Figure 1.15**.

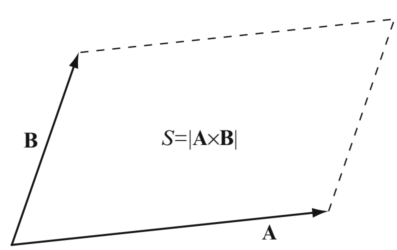


Figure 1.15 Interpretation of the magnitude of the vector product as a surface

Evaluation of the vector product is performed similarly to that for the scalar product: We write the product explicitly and expand the expression based on **Eq. (1.33)**. Using two general vectors **A** and **B** in Cartesian coordinates, we get

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right) \times \left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right) \\ &= (\hat{\mathbf{x}} \times \hat{\mathbf{x}})A_xB_x + (\hat{\mathbf{x}} \times \hat{\mathbf{y}})A_xB_y + (\hat{\mathbf{x}} \times \hat{\mathbf{z}})A_xB_z \\ &\quad + (\hat{\mathbf{y}} \times \hat{\mathbf{x}})A_yB_x + (\hat{\mathbf{y}} \times \hat{\mathbf{y}})A_yB_y + (\hat{\mathbf{y}} \times \hat{\mathbf{z}})A_yB_z \\ &\quad + (\hat{\mathbf{z}} \times \hat{\mathbf{x}})A_zB_x + (\hat{\mathbf{z}} \times \hat{\mathbf{y}})A_zB_y + (\hat{\mathbf{z}} \times \hat{\mathbf{z}})A_zB_z\end{aligned} \tag{1.34}$$

Because the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are perpendicular to each other, and using the right-hand rule in **Figure 1.14**, we can write

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \tag{1.35}$$

from property 4 above. Similarly, using property 2 and the right-hand rule, we can write

$$\begin{array}{lll} \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, & \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, & \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}, & \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}, & \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}} \end{array} \tag{1.36}$$

Substitution of these products and rearranging terms gives

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}\left(A_yB_z - A_zB_y\right) + \hat{\mathbf{y}}\left(A_zB_x - A_xB_z\right) + \hat{\mathbf{z}}\left(A_xB_y - A_yB_x\right) \tag{1.37}$$

This is a rather straightforward operation, although lengthy. To avoid having to go through this process every time we use the vector product, we note that the expression in **Eq. (1.37)** has the form of the determinant of a 3 × 3 matrix:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{x}}\left(A_yB_z - A_zB_y\right) + \hat{\mathbf{y}}\left(A_zB_x - A_xB_z\right) + \hat{\mathbf{z}}\left(A_xB_y - A_yB_x\right) \tag{1.38}$$

In the system of coordinates used here (right-hand Cartesian coordinates), the vector product is cyclic; that is, the products in **Eq. (1.36)** are cyclical, as shown in **Figure 1.16**. This is a simple way to generate the signs of the components of the cross product: a cross product performed in the sequence shown by the arrows in **Figure 1.16a** is positive; if it is in the opposite sequence (**Figure 1.16b**), it is negative.

The vector product is used for a number of important operations. They include finding the direction of the vector product, calculation of areas, evaluation of normal unit vectors, and representation of fields.

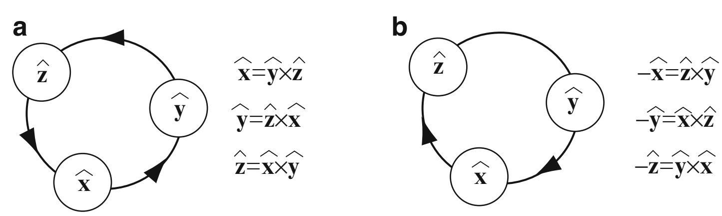


Figure 1.16 The cyclical relations between the various vector products of the unit vectors in Cartesian coordinates. (**a**) Positive sequence. (**b**) Negative sequence

Example 1.7 Application: Vector Normal to a Plane

(a) Find a vector normal to a plane that contains points $P_1(0,1,0)$, $P_2(1,0,1)$, and $P_3(0,0,1)$.
(b) Find the normal unit vector.

Solution: This is a common use for the vector product. Because the vector product of two vectors is normal to both vectors, we must first find two vectors that lie in the plane. Their vector product gives the normal vector. Calculation of the normal unit vector can be done either using the definition of the unit vector in **Eq. (1.2)** or through the use of the scalar and vector products.

Two vectors in the plane can be defined using any two pairs of points. Using P_1 and P_2, we define a vector (from P_1 to P_2) as

$$\mathbf{A} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}\left(y_2 - y_1\right) + \hat{\mathbf{z}}\left(z_2 - z_1\right) = \hat{\mathbf{x}}(1-0) + \hat{\mathbf{y}}\left(0-1\right) + \hat{\mathbf{z}}\left(1-0\right) = \hat{\mathbf{x}}1 - \hat{\mathbf{y}}1 + \hat{\mathbf{z}}1$$

Similarly for a second vector, we choose the vector between P_1 and P_3. This gives

$$\mathbf{B} = \hat{\mathbf{x}}(x_3 - x_1) + \hat{\mathbf{y}}\left(y_3 - y_1\right) + \hat{\mathbf{z}}\left(z_3 - z_1\right) = \hat{\mathbf{x}}(0-0) + \hat{\mathbf{y}}\left(0-1\right) + \hat{\mathbf{z}}\left(1-0\right) = -\hat{\mathbf{y}}1 + \hat{\mathbf{z}}1$$

The cross product, **C**, is a vector normal to both **A** and **B** and, therefore, to the plane:

$$\mathbf{A}\times\mathbf{B} = (\hat{\mathbf{x}}\,1 - \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,1)\times\left(-\hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,1\right)$$
$$= \hat{\mathbf{x}}\,1\times(-\hat{\mathbf{y}}\,1) + \hat{\mathbf{x}}\,1\times\hat{\mathbf{z}}\,1 + \left(-\hat{\mathbf{y}}\,1\right)\times\left(-\hat{\mathbf{y}}\,1\right) + (-\hat{\mathbf{y}}\,1)\times\hat{\mathbf{z}}\,1 + \hat{\mathbf{z}}\,1\times\left(-\hat{\mathbf{y}}\,1\right) + \hat{\mathbf{z}}\,1\times\hat{\mathbf{z}}\,1$$

Using the identities in **Eqs. (1.35)** and **(1.36)**, we get

$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = -\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1$$

The unit vector can be found from **Eq. (1.2)** or from the definition of the vector product in **Eq. (1.31)**. We use the latter as an example of an alternative method:

$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|AB\sin\phi_{AB}|}$$

The angle ϕ_{AB} can be most easily calculated from the scalar product in **Eq. (1.23)** as

$$\phi_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

To do so, we need to evaluate the scalar product and the magnitude of the vectors. These are

$$\mathbf{A}\cdot\mathbf{B} = (\hat{\mathbf{x}}\,1 - \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,1)\cdot(-\hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,1) = 2, \quad A = \sqrt{3} \quad B = \sqrt{2}$$

Thus,

$$\phi_{AB} = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35^\circ 16'$$

The unit normal vector is now

$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|AB\sin\phi_{AB}|} = \frac{-\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1}{\left|\sqrt{6}\sin\left(35^\circ 16'\right)\right|} = \frac{-\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1}{1.4142} = -\hat{\mathbf{y}}\,0.7071 - \hat{\mathbf{z}}\,0.7071$$

The same result is obtained using **Eq. (1.2)**:

$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \frac{-\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1}{|-\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1|} = \frac{-\hat{\mathbf{y}}\,1 - \hat{\mathbf{z}}\,1}{\sqrt{2}} = -\hat{\mathbf{y}}\,0.7071 - \hat{\mathbf{z}}\,0.7071$$

Exercise 1.4 Vectors $\mathbf{A} = \hat{\mathbf{x}}\,1 - \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3$ and $\mathbf{B} = \hat{\mathbf{x}}\,3 + \hat{\mathbf{y}}\,5 + \hat{\mathbf{z}}\,1$ are in a plane, not necessarily perpendicular to each other. Vector $\mathbf{C} = \hat{\mathbf{x}}\,17 + \hat{\mathbf{y}}\,8 - \hat{\mathbf{z}}\,11$ is perpendicular to the same plane. Show that the vector product between **C** and **A** (or between **C** and **B**) must also be in the plane of **A** and **B**.

Example 1.8 Application: Area of a Triangle Find the area of the triangle with vertices at three general points $P_1(x_1, y_1, z_1)$, $P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$ (**Figure 1.17a**).

Solution: In this case, the vector nature of the vector product is irrelevant, but the magnitude of the vector product in **Eq. (1.31)** is equal to the area of the parallelogram formed by the two vectors. This can be seen from the fact that the magnitude of **A** × **B** is $A(B\sin\phi_{AB})$. This is the area of rectangle $abb'c'$ in **Figure 1.17b**. Since triangles acc' and bdb' are identical, this is also the area of parallelogram *abdc*. Since triangles *abc* and *cbd* are identical, the area of *abc* is equal to half the area of *abdc*. Calculation of the area of triangle *abc* is done by calculating the magnitude of the cross product of two of the vectors forming the sides of the triangle and dividing by 2:

$$S_{abc} = \frac{|\mathbf{A}\times\mathbf{B}|}{2}$$

From **Figure 1.17a**, vectors **A** and **B** are

$$\mathbf{A} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1),$$
$$\mathbf{B} = \hat{\mathbf{x}}(x_3 - x_1) + \hat{\mathbf{y}}(y_3 - y_1) + \hat{\mathbf{z}}(z_3 - z_1)$$

The vector product is obtained using **Eq. (1.38)**, and from this, the area of the triangle is

$$S_{abc} = \frac{|\mathbf{A}\times\mathbf{B}|}{2} = \frac{1}{2}\left\|\begin{matrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix}\right\|$$

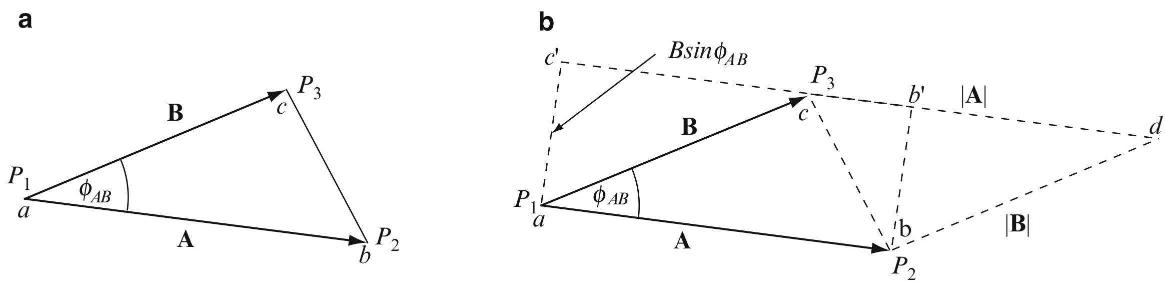


Figure 1.17 Area of a triangle. (**a**) A triangle with two of its sides shown as vectors. (**b**) The area of the triangle is half the area of the parallelogram *abdc*

Exercise 1.5 Find the area of the triangle formed by points (1,3,0), (1,2,1), and (3,5,2).

Answer $\sqrt{24}/2 = 2.4495/\text{m}^2$.

Example 1.9 Find a unit vector normal to both of the vectors

$$\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}2, \quad \mathbf{B} = \hat{\mathbf{x}}1 - \hat{\mathbf{y}}5.$$

Solution: The vector products **A** × **B** or **B** × **A** result in vectors normal to both **A** and **B**, respectively:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (\hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}2) \times (\hat{\mathbf{x}}1 - \hat{\mathbf{y}}5) \\ &= (\hat{\mathbf{x}}3)\times(\hat{\mathbf{x}}1) + (\hat{\mathbf{x}}3)\times(-\hat{\mathbf{y}}5) + (\hat{\mathbf{y}}1)\times(\hat{\mathbf{x}}1) \\ &\quad + (\hat{\mathbf{y}}1)\times(-\hat{\mathbf{y}}5) + (-\hat{\mathbf{z}}2)\times(\hat{\mathbf{x}}1) + (-\hat{\mathbf{z}}2)\times(-\hat{\mathbf{y}}5) \\ &= -\hat{\mathbf{x}}10 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}16\end{aligned}$$

The unit vector is

$$\hat{\mathbf{n}} = \frac{-\hat{\mathbf{x}}10 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}16}{|-\hat{\mathbf{x}}10 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}16|} = \frac{-\hat{\mathbf{x}}10 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}16}{\sqrt{360}} = \frac{-\hat{\mathbf{x}}5 - \hat{\mathbf{y}} - \hat{\mathbf{z}}8}{3\sqrt{10}}$$

Using the product **B** × **A** results in a unit vector $-\hat{\mathbf{n}}$ as can be shown by application of the right-hand rule.

Example 1.10 The general equation of a plane in Cartesian coordinates is $ax + by + cx + d = 0$. The equation of the plane may be found as $f(x,y,z) = \mathbf{n} \cdot \mathbf{C} = 0$ where **n** is the normal vector to the plane and **C** is a general vector in the plane.

(a) Given three points $P_1(1,0,2)$, $P_2(3,1,–2)$, $P_3(2,3,2)$ in a plane, find the equation of the plane.
(b) Show that the equation of the plane may be written as

$$f(x,y,z) = n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

where n_x, n_y, and n_z are the scalar components of the normal unit vector to the plane and (x_0,y_0,z_0) are the coordinates of a point in the plane.

Solution: The three points given define two vectors (say P_1 to P_2 and P_1 to P_3). The normal to the plane is obtained through use of the vector product. The vector **A** is then defined between a general point (x,y,z) and any of the points given. The form in **(b)** is found from **(a)**:

(a) Two vectors necessary to calculate the normal vector to the plane are
P_1 to P_2: $\mathbf{A} = \hat{\mathbf{x}}(3-1) + \hat{\mathbf{y}}(1-0) + \hat{\mathbf{z}}(-2-2) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}4$
P_1 to P_3: $\mathbf{B} = \hat{\mathbf{x}}(2-1) + \hat{\mathbf{y}}(3-0) + \hat{\mathbf{z}}(2-2) = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}3$
These two vectors are in the plane. Therefore, the normal vector to the plane may be written as:

$$\mathbf{n} = \mathbf{A} \times \mathbf{B} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}4) \times (\hat{\mathbf{x}}1 + \hat{\mathbf{y}}3) = \hat{\mathbf{x}}12 - \hat{\mathbf{y}}4 + \hat{\mathbf{z}}5$$

A general vector in the plane may be written as:

$$\mathbf{C} = \hat{\mathbf{x}}(x-1) + \hat{\mathbf{y}}(y-0) + \hat{\mathbf{z}}(z-2)$$

where point P_1 was used, arbitrarily.
The equation of the plane is

$$f(x,y,z) = \mathbf{n} \cdot \mathbf{C} = (\hat{\mathbf{x}}12 - \hat{\mathbf{y}}4 + \hat{\mathbf{z}}5) \cdot (\hat{\mathbf{x}}(x-1) + \hat{\mathbf{y}}(y-0) + \hat{\mathbf{z}}(z-2)) = 0$$

or

$$f(x,y,z) = 12(x-1) - 4(y-0) + 5(z-2) = 12x - 4y + 5z - 22 = 0$$

(b) To show that the formula given produces the same result, we first calculate the normal unit vector:

$$\hat{\mathbf{n}} = \hat{\mathbf{x}}\frac{12}{\sqrt{185}} - \hat{\mathbf{y}}\frac{4}{\sqrt{185}} + \hat{\mathbf{z}}\frac{5}{\sqrt{185}}$$

The point (x_0,y_0,z_0) can be any point in the plane. Selecting P_3, for example, the equation of the plane is

$$f(x,y,z) = \frac{12}{\sqrt{185}}(x-2) - \frac{4}{\sqrt{185}}(y-3) + \frac{5}{\sqrt{185}}(z-2) = 0$$

Multiplying both sides of the equation by $\sqrt{185}$, we get

$$f(x,y,z) = 12(x-2) - 4(y-3) + 5(z-2) = 12x - 4y + 5z - 22 = 0$$

This is the same as the result obtained in **(a)**.
The formula in **(b)** is called the scalar equation of the plane whereas the form $f(x,y,z) = \mathbf{n} \cdot \mathbf{C} = 0$ is referred to as the vector equation of the plane.

1.3.3 Multiple Vector and Scalar Products

As with sums of vectors, we can define multiple products by repeatedly applying the rules of the scalar or vector product of two vectors. However, because of the particular method of defining the vector and scalar products, not all combinations of products are meaningful. For example, the result of a vector product is a vector, and therefore, it can only be obtained by scaling another vector or by a cross product with another vector. Similarly, the result of a scalar product is a scalar and cannot be used to obtain a vector. The following triple products are properly defined:

(1) The vector triple product is defined as

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})} \tag{1.39}$$

This is called a ***vector triple product*** because it involves three terms (vectors) and the result is a vector. The right-hand side can be shown to be correct by direct evaluation of the vector product (see **Exercise 1.6**). A number of properties should be noted here:

(a) The vector (double or triple) product is not associative,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}. \tag{1.40}$$

That is, the sequence in which the vector product is performed is all-important. For this reason, the brackets should always be part of the notation and should never be omitted. The product $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$ is not a properly defined product.

(b) The right-hand side of **Eq. (1.39)** is often used for evaluation of the vector triple product. Because of the combination of products, this is referred to as the *BAC-CAB* rule. It provides a means of remembering the correct sequence of products for evaluation.

(c) The vector triple product can also be evaluated using the determinant rule in **Eq. (1.38)** by applying it twice. First, the product $\mathbf{B} \times \mathbf{C}$ in **Eq. (1.39)** is evaluated. This results in a vector, say $\mathbf{D}$. Then, the product $\mathbf{A} \times \mathbf{D}$ is evaluated, resulting in the vector triple product.

(2) The following are properly defined scalar triple products:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{1.41}$$

This product is a ***scalar triple product*** since the result is a scalar. Note that the vectors in the product are cyclic permutations of each other. Any other order of the vectors in the triple product produces equal but negative results. Thus,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) \tag{1.42}$$

because of the property of the vector product. On the other hand, from the properties of the scalar product, we have

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} \tag{1.43}$$

We also note that since the vector product represents the area bounded by the two vectors $\mathbf{B}$ and $\mathbf{C}$, and the scalar product is the projection of vector $\mathbf{A}$ onto the vector product, the scalar triple product represents the volume defined by the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ (see **Figure 1.18** and **Example 1.11**). Finally, as an aid to evaluation of the scalar triple product, we mention that this can be evaluated as the determinant of a matrix as follows:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x\left(B_yC_z - B_zC_y\right) + A_y\left(B_zC_x - B_xC_z\right) + A_z\left(B_xC_y - B_yC_x\right) \tag{1.44}$$

and, as before, can be shown to be correct by direct evaluation of the product (see **Exercise 1.7**). Note, however, that this does not imply that the scalar triple product is a determinant; the determinant is merely an aid to evaluating the product.

Important The products $\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$ are not defined. Can you show why?

Other products may be defined, but these are not important in electromagnetics. For example, $\mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D}))$ is a properly defined vector product, but we will have no use for it in subsequent work.

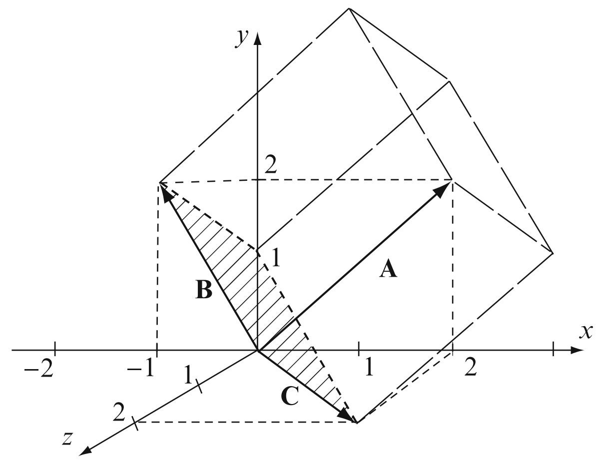


Figure 1.18 Interpretation of the scalar triple product as a volume

Example 1.11 Application: Volume of a Parallelepiped Calculate the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ defined by the vectors: $\mathbf{A} = \hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2$, $\mathbf{B} = -\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2$, and $\mathbf{C} = \hat{\mathbf{x}}\,1 + \hat{\mathbf{z}}\,2$. Show that this volume represents the volume of a parallelepiped in which the tails of the three vectors form one corner of the parallelepiped.

Solution: Consider **Figure 1.18**. The three vectors form a box as shown. The magnitude of the vector product $\mathbf{D} = \mathbf{B} \times \mathbf{C}$ represents the area of the parallelogram shown cross-hatched. Considering the vector **A** and the angle it makes with the vector **D**, the scalar product $E = \mathbf{A} \cdot \mathbf{D} = AD\cos\phi_{AD}$ gives the volume of the box since $A\cos\phi_{AD}$ gives the height of the box (projection of **A** on **D**). **Equation (1.44)** is used to evaluate the volume:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 2 & 2 & 0 \\ -2 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2(4-0) + 2(0+4) = 16 \quad [\mathrm{m}^3]$$

Example 1.12

(a) Find the vector triple product $\mathbf{B} \times (\mathbf{A} \times \mathbf{C})$ using the three vectors of the previous example.
(b) Show that the resultant vector must be in the plane formed by **A** and **C**.

Solution:

(a) The vector triple product is evaluated using the rule in **Eq. (1.39)**.
We write

$$\mathbf{D} = \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{B} \cdot \mathbf{A})$$

Note that this is the same relation as in **Eq. (1.39)** with vectors **A** and **B** interchanged. The vectors **A**, **B**, and **C** are

$$\mathbf{A} = \hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2, \quad \mathbf{B} = -\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2, \quad \mathbf{C} = \hat{\mathbf{x}}\,1 + \hat{\mathbf{z}}\,2$$

The scalar products $\mathbf{B} \cdot \mathbf{C}$ and $\mathbf{B} \cdot \mathbf{A}$ are

$$\mathbf{B} \cdot \mathbf{C} = (-\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2) \cdot \left(\hat{\mathbf{x}}\,1 + \hat{\mathbf{z}}\,2\right) = -2,$$
$$\mathbf{B} \cdot \mathbf{A} = (-\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2) \cdot \left(\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2\right) = -4 + 4 = 0$$

The vector triple product reduces to

$$\mathbf{D} = \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) = \mathbf{A}(-2) - 0 = -\hat{\mathbf{x}}\,4 - \hat{\mathbf{y}}\,4$$

(b) The simplest way to show that the vector $\mathbf{D} = \mathbf{B} \times (\mathbf{A} \times \mathbf{C})$ is in the plane formed by vectors **A** and **C** is to show that the scalar triple product $\mathbf{D} \cdot (\mathbf{A} \times \mathbf{C})$ is zero. This is to say that the box formed by vectors **D**, **A**, and **C** has zero volume.

This can only happen if the three vectors are in a plane. Substituting the scalar components of vectors **D**, **A**, and **C** in **Eq. (1.44)** gives

$$\mathbf{D}\cdot(\mathbf{A}\times\mathbf{C}) = \begin{vmatrix} -4 & -4 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -16 + 16 = 0$$

Thus, vector **D** is in the plane formed by vectors **A** and **C**.

Exercise 1.6 Show that the relation in **Eq. (1.39)** is correct by direct evaluation of the vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ using general vectors $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$, $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$, and $\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z$.

Exercise 1.7 Show that the relation in **Eq. (1.44)** is correct by direct evaluation of the scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ using general vectors $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$, $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$, and $\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z$.

1.4 Definition of Fields

A field may be defined mathematically as the function of a set of variables in a given space. This rather general definition is of little use in trying to understand properties of a field from a physical point of view. Therefore, we will use a "looser" definition of a field. For the purpose of this book, ***a field is a distribution in space of any quantity: scalar, vector, time dependent, or independent of time***. The field may be defined over the whole space or a portion of space. Thus, for example, a topographical map shows the altitude of each point in a given domain; this is an "altitude field." If we can describe the wind velocity at every point in a domain, then we have defined a "velocity field." Similarly, a gravitational force field, a temperature field, and the like may be defined. Note also that although a functional dependency always exists, a field may be postulated without these dependencies being used or, for that matter, known: The "altitude field" above is obtained by measurements and is therefore experimentally found. However, at least in principle, the functional dependency exists.

Fields are fundamental to the study of electromagnetics. In this context, we will seek to understand the properties of electromagnetic fields, which, based on the definition above, are merely the distribution of the "electric and magnetic vectors." Although we do not know at this point what these are, it is easy to conceptualize the idea that if these vectors can be defined anywhere in a given space, then their distribution in that space can also be described: This process defines the field of the corresponding vector. How these fields interact with each other and with materials is what electromagnetics is all about.

1.4.1 Scalar Fields

A ***scalar field*** is a field of scalar variables; that is, if for any point in space, say (x,y,z), we know the function $f(x,y,z)$, then f is the scalar field. This may represent a temperature distribution, potential, pressure, or any other scalar function. For example,

$$f(x,y,z) = x^2 + y^2 + 5z^2 \tag{1.45}$$

is a scalar field. An example of a scalar field is shown in **Figure 1.19**. It shows a topographical map in which contour lines show various elevations. The representation in terms of contour lines (in this case, lines of constant elevation) is a simple way of representing a scalar field. If the lines were to represent pressure, a similar map may give air pressure over a continent to provide details of a meteorological report. In some cases we will find it useful to derive physical quantities from scalar fields. For example, in the field in **Eq. (1.45)**, we could calculate first-order derivatives with respect to any of the variables. If the field is, say, an altitude field, then the first derivative describes a slope. If we were planning to build a road, then this is extremely important information to know.

Scalar fields may be time dependent or independent of time. An example of a time-dependent scalar field could be a weather map in which temperatures vary with time. Similarly, a time-dependent electric potential distribution in a block of material is a time-dependent scalar field.

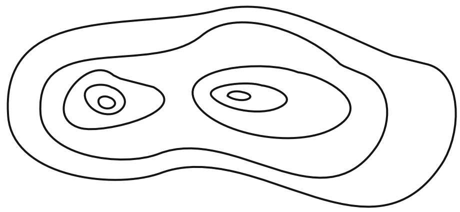


Figure 1.19 Lines of constant elevation as an example of a scalar field

1.4.2 Vector Fields

We can define a ***vector field*** as a vector function $\mathbf{F}(x,y,z,t)$. As an example,

$$\mathbf{F}(x,y,z) = \hat{\mathbf{x}} A_x(x,y,z) + \hat{\mathbf{y}} A_y(x,y,z) + \hat{\mathbf{z}} A_z(x,y,z) \tag{1.46}$$

is a static vector field. An example of a vector field is shown in **Figure 1.20**. It shows wind velocities in a hurricane. The length of the vectors indicates the magnitude of velocity and the direction gives the direction of flow.

A vector field may be obtained from a scalar field or a scalar field may be obtained from a vector field. As an example, if we were to use the scalar field in **Eq. (1.45)** and calculate the slopes with respect to x, y, and z, we obtain a vector field since a slope is only properly defined if both the magnitude and direction are defined. For example, when skiing on a mountain, the elevation is less important than the slope, and the slope is different in different directions. Starting at any given point, you may want to ski in the direction of maximum slope or maybe sideways on a less steep path, or you may want to follow a predetermined path, as in cross-country skiing. These may seem to be trivial notions, but they are exactly the operations that we need to perform in electromagnetic fields.

The properties of vector and scalar fields will be discussed extensively in this and the following chapter but, in particular, in the context of electromagnetic fields in the remainder of the book.

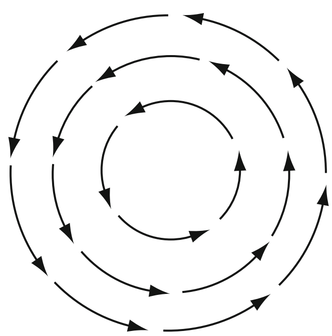


Figure 1.20 Example of a vector field: wind velocity distribution in a hurricane

Example 1.13 Graphing Scalar Fields A scalar field is given: $\psi(x,y,z) = x^2y - 3x + 3$. Obtain a graph of the field in the range $-1 < x, y < 1$.

Solution: To obtain a graph, we substitute points (x,y,z) in the expression for the field and mark the magnitude of the field on a map. In this case, the field is in the x–y plane (it does not depend on z). There are a number of methods of representation for scalar fields. One is shown in **Figure 1.21a**. For each point (x,y), the value of the function $\psi(x,y)$ is indicated. For example, $\psi(0,\ 0) = 3$, $\psi(0.5,\ 0) = 1.5$, $\psi(0.5,\ 0.5) = 1.625$, $\psi(-0.5,\ -0.5) = 4.375$, $\psi(0.75,\ 0.75) = 1.172$, $\psi(-0.75,\ -0.75) = 4.828$, and $\psi(-1,1) = 7.0$. These points are indicated on the graph. This method is simple but does not give a complete visual picture. If this equation were to represent elevation, you would be able to show the elevation at any point, but it would be hard to see what the terrain looks like.

A second representation is shown in **Figure 1.21b**. It shows the same scalar field with a large number of points, and all points of the same magnitude are connected with a line. These are contour lines as commonly used on maps. Now, the picture is easier to read. Each contour line represents a given value ψ = constant of the field.

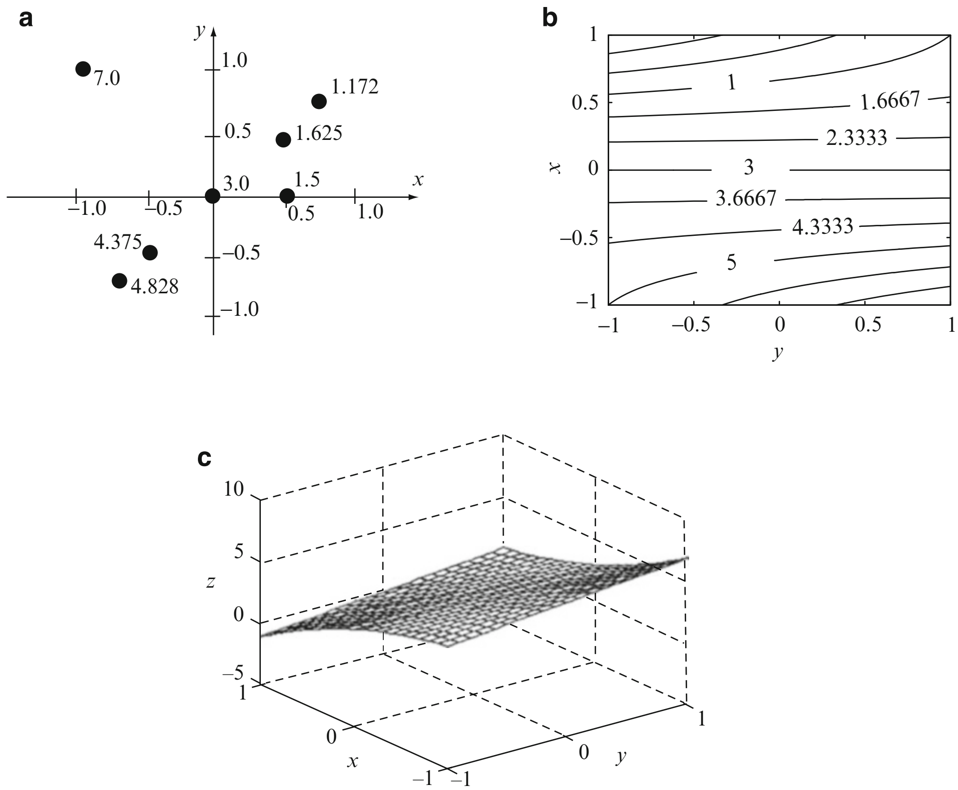


Figure 1.21 Representation of scalar fields. (**a**) Values at given coordinates. (**b**) Contours of constant value. (**c**) Values above a plane. The height at any point represents the magnitude (strength) of the scalar field

A third method is to show the magnitude of the field above a plane for all x, y. This gives a three-dimensional picture. The individual points can then be interpolated on a grid. A representation of this kind is shown in **Figure 1.21c**. This is not unlike trying to draw an actual terrain map in which the elevation is shown.

Example 1.14 Graphing Vector Fields Graph the vector field $\mathbf{A} = \hat{\mathbf{y}}x$.

Solution: For a vector field, we must show both the magnitude of the vector and its direction. Normally, this is done by locating individual points in the field and drawing an arrowed line at that point. The arrow starts at the location (point) at which the field is shown and points in the direction of the field, and the length of the arrow indicates the magnitude of the field.

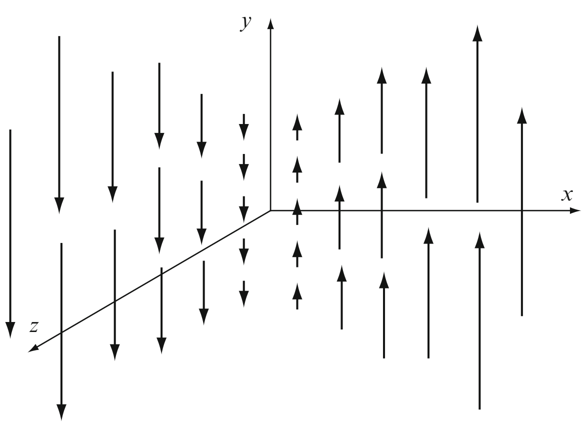


Figure 1.22 Representation of the vector field $\mathbf{A} = \hat{\mathbf{y}}x$ as *arrows*. The length of the *arrow* indicates the magnitude of the field. Only the $z = 0$ plane is shown, but this field is independent of z

In this example, the magnitude of the field is independent of the y and z directions. Thus, the magnitude is zero at $x = 0$ and increases linearly with x. The direction is in the positive y direction for $x > 0$ and in the negative y direction for $x < 0$. A simple representation of this field is shown in **Figure 1.22**.

Exercise 1.8 Graph the following vector fields in the range $-1 < x, y < 1$:

(a) $\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. **(b)** $\mathbf{A} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$.

1.5 Systems of Coordinates

A system of coordinates is a system of reference axes used to uniquely describe the various quantities needed in the study of electromagnetics (or any other discipline). In describing scalars, vectors, products, and other quantities, it is extremely important to be able to do so in a simple, unique manner. A system of coordinates forms a unique, universally understood reference by convention; that is, we can devise many systems of coordinates, but only some of these are actually useful and only a handful have been accepted universally.

Among the various systems of coordinates, the so-called orthogonal systems are the most commonly used. These are systems in which the reference axes are normal to each other. In addition, we will only use the so-called right-hand systems. Emphasis was already given to Cartesian coordinates in the previous sections. In addition, the cylindrical coordinates system and the spherical system of coordinates will be discussed. These three systems are sufficient for our purposes.

We should mention here that as a rule, when a system of coordinates is chosen over another, it is for convenience. We know intuitively that it is easier to describe a cube in a rectangular coordinate system, whereas a spherical object must be easier to describe in a spherical system. It is possible to describe a cube in a spherical system but with considerable more difficulty. For this reason, specialized systems of coordinates have been devised. A simple example is the system used to identify location of aircraft and ships: A grid, consisting of longitude and latitude lines, has been devised, measured in degrees because they are supposed to fit the spherical surface of the globe. A rectangular grid is suitable for, say, the map of a city or a small section of a country but not as a global coordinate system. This example also indicates one of the most important aspects of working with a "convenient" system of coordinates: the need to transform from one system to another. In the above example, a ship may be sailing from point A to point B for a total of x degrees latitude. However, in practical terms, more often we need to know the distance. This means that for any longitude or latitude, we should be able to convert angles to positions in terms of distances from given points or distances between points.

There are a number of other coordinate systems designed for use in three-dimensional space. These coordinate systems have been devised and used for a variety of applications and include the bipolar, prolate spheroidal, elliptic cylindrical, and ellipsoidal systems and a handful others, in addition to the Cartesian, cylindrical, and spherical systems.

Our approach here is simple: We will define three systems we view as important and, within these systems, will present those quantities that are useful in the study of electromagnetic fields. These include length, surface, and volume as well as the required transformations from one system of coordinates to the others. The latter is an important step because it clearly indicates that the fundamental quantities we treat are independent of the system of coordinates. We can perform any operation in any system we wish and transform it to any other system if this is needed. Also, we will have to evaluate the various vector operations in the three systems of coordinates and then use these as the basis of analysis.

1.5.1 The Cartesian[3] Coordinate System

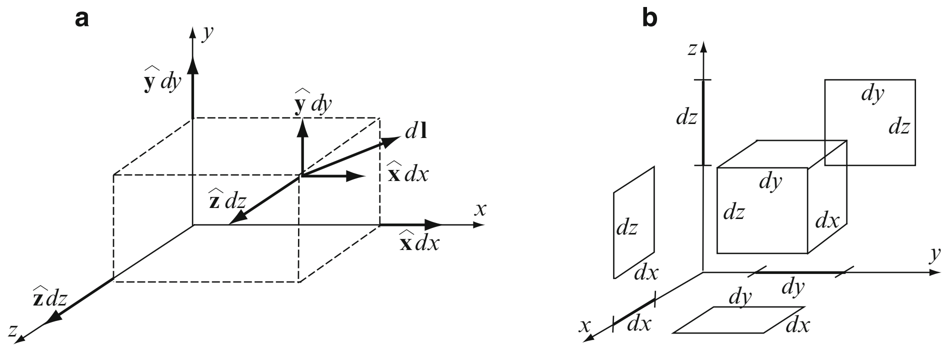
In the right-handed Cartesian system (or right-handed rectangular system), a vector **A** connecting two general points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is given as

$$\mathbf{A}(x, y, z) = \hat{\mathbf{x}}A_x(x, y, z) + \hat{\mathbf{y}}A_y(x, y, z) + \hat{\mathbf{z}}A_z(x, y, z) \tag{1.47}$$

where the components A_x, A_y, and A_z are the projections of the vector on the x, y, and z coordinates, respectively.

[3] Named after Rene Descartes (1596–1650), French philosopher and mathematician (Cartesius is his Latinized name). The philosophical system he devised held until the Newtonian system superseded it. You may be familiar with the quote "I think, therefore I am," which Descartes coined and which was a central point in his philosophical system. The Cartesian system is named after him because he is considered to be the developer of analytical geometry. He presented the system of coordinates bearing his name in "La Geometrie," a work published in 1637.

Figure 1.23 (**a**) The differential of length $d\mathbf{l}$ and its components in the Cartesian system of coordinates. (**b**) An element of volume and its surface projections on the xy, xz, and yz planes



An element of length $d\mathbf{l}$, or differential of length, is a vector with scalar components dx, dy, and dz and is shown in **Figure 1.23a**:

$$\boxed{d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz} \tag{1.48}$$

Note: The unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are constant; they point in the same direction in space at any point.

The elements of surface may be deduced from **Figure 1.23**. Each of the differential surfaces is parallel to one of the planes as shown in **Figure 1.23b**. Thus, we define three differential surfaces as

$$\boxed{d\mathbf{s}_x = dydz, \quad d\mathbf{s}_y = dxdz, \quad d\mathbf{s}_z = dxdy} \tag{1.49}$$

A differential of volume is defined as a rectangular prism with sides dx, dy, and dz, as shown in **Figure 1.23b**. The differential volume is a scalar and is written as

$$\boxed{dv = dxdydz} \tag{1.50}$$

In electromagnetics, it is often necessary to evaluate a vector function over a surface (such as integration over the surface). For this purpose, we orient the surface by defining the direction of the surface as the normal to the surface. A vector element of surface becomes a vector with magnitude equal to the element of surface [as defined in **Eq. (1.49)**] and directed in the direction of the normal unit vector to the surface. To ensure proper results, we define a positive surface vector if it points out of the volume enclosed by the surface. In a closed surface, like that shown in **Figure 1.23b**, the positive direction is easily identified. In an open surface, we must decide which side of the surface is the interior and which the exterior. **Figure 1.24** shows two surfaces. The first is positive; the second is not defined. This, however, is often overcome from physical considerations such as location of sources. For example, we may decide that the direction pointing away from the source is positive even though the surface is not closed. In general, we define a positive direction for an open surface using the right-hand rule: If the fingers of the right hand point in the direction we traverse the boundary of the open surface, with palm facing the interior of the surface, then the thumb points in the direction of positive surface. This is shown in **Figure 1.24c** but it always depends on the direction of motion on the contour. In **Figure 1.24c**, if we were to move in the opposite direction, the surface shown would be negative. Fortunately, in practical application, it is often easy to identify the positive direction. Based on this definition, an element of surface is written as

$$d\mathbf{s} = \hat{\mathbf{n}}ds \tag{1.51}$$

$d\mathbf{s}$ should always be thought of as an element of surface $d\mathbf{s}$, which is a scalar, and $\hat{\mathbf{n}}$, a normal unit vector to this surface which may be positive or negative (see **Figures 1.24** and **1.25**).

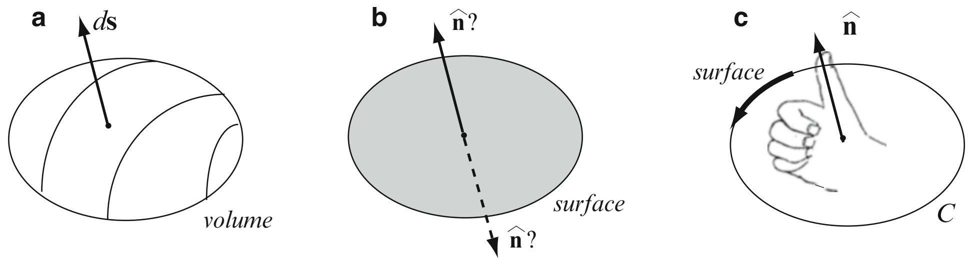


Figure 1.24 Direction of a surface. (**a**) The surface is always positive in the direction out of the volume. (**b**) In an open surface, the positive direction can be ambiguous. (**c**) The use of the right-hand rule to define direction of an open surface

Consider the small cube shown in **Figure** 1.25. The six surfaces of the cube are parallel to the planes xy, xz, and yz. The six elemental surfaces can be written as

$$\text{on the right face } d\mathbf{s}_x = \hat{\mathbf{x}}dydz \quad \text{on the left face } d\mathbf{s}_x = -\hat{\mathbf{x}}dydz$$

$$\text{on the front face } d\mathbf{s}_z = \hat{\mathbf{z}}dxdy \quad \text{on the back face } d\mathbf{s}_z = -\hat{\mathbf{z}}dxdy$$

$$\text{on the top face } d\mathbf{s}_y = \hat{\mathbf{y}}dxdz \quad \text{on the bottom face } d\mathbf{s}_y = -\hat{\mathbf{y}}dxdz$$

These can be summarized as follows:

$$d\mathbf{s}_x = \pm\hat{\mathbf{x}}dydz, \qquad d\mathbf{s}_y = \pm\hat{\mathbf{y}}dxdz, \qquad d\mathbf{s}_z = \pm\hat{\mathbf{z}}dxdy \tag{1.52}$$

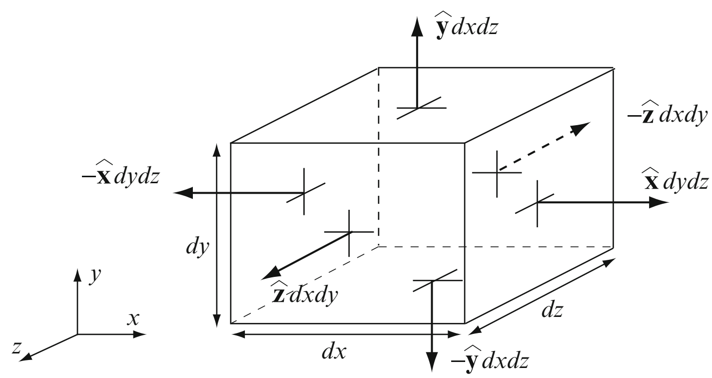


Figure 1.25 Directions of elements of surface in a cube. All surface elements are considered positive (pointing out of the volume) even though they may point in the direction of negative coordinates

Although the direction of $d\mathbf{s}$ may be in the positive or negative directions in space, the vector is considered to be positive with respect to the surface if it points out of the volume, regardless of its direction in space. Thus, all six surfaces above are considered to be positive surface vectors.

Caution The elements of surface as defined above are not vector components of an area vector: they merely define the direction normal to the surface and the differential of surface, and each element should be viewed as an independent vector.

All other aspects of use of the Cartesian coordinate system including calculation of vector and scalar components, unit vectors, and the various scalar and vector operations were discussed in **Sections** 1.2 through 1.4 and will not be repeated here. However, **Example 1.15** reviews some of the definitions involved.

Example 1.15 Three points are given in the Cartesian coordinate system: $P_1(2,–3,3)$, $P_2(1,1,5)$, and $P_3(3,–1,4)$.

(a) Find the three vectors: **A**, connecting P_1 to P_2; **B**, connecting P_1 to P_3; and **C**, connecting P_2 to P_3.
(b) Find the scalar component of vector **A** in the direction of vector **B**.
(c) Find the vector components of vector **B** in the direction of vector **C**.

Solution: **(a)** The vectors **A**, **B**, and **C** are found by calculating the components from the coordinates of the end points. For a vector connecting point (1) to point (2), the projection on each axis is the difference in the corresponding coordinates with point (2) (head) and point (1) (tail) of the vector. **(b)** The scalar component of **A** in the direction of **B** is the projection of **A**

onto the unit vector $\hat{\mathbf{B}}$ and is calculated through the scalar product. **(c)** The vector components of **B** in the direction of **C** are found as in **(b)**, but now the unit vector $\hat{\mathbf{C}}$ is scaled by the scalar component $\mathbf{B}\cdot\hat{\mathbf{C}}$:

(a)

$$\mathbf{A} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}\left(1+3\right) + \hat{\mathbf{z}}\left(5-3\right) = -\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,4 + \hat{\mathbf{z}}\,2$$
$$\mathbf{B} = \hat{\mathbf{x}}(3-2) + \hat{\mathbf{y}}\left(-1+3\right) + \hat{\mathbf{z}}\left(4-3\right) = \hat{\mathbf{x}} + \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,1$$
$$\mathbf{C} = \hat{\mathbf{x}}(3-1) + \hat{\mathbf{y}}\left(-1-1\right) + \hat{\mathbf{z}}\left(4-5\right) = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,2 - \hat{\mathbf{z}}\,1$$

(b) To find the scalar component of **A** in the direction of **B**, we first calculate the unit vector $\hat{\mathbf{B}}$ and then the scalar product $\mathbf{A}\cdot\hat{\mathbf{B}}$ (see **Example 1.4**):

$$A_B = \mathbf{A}\cdot\hat{\mathbf{B}} = \frac{\mathbf{A}\cdot\mathbf{B}}{B} = \frac{(-\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,4 + \hat{\mathbf{z}}\,2)\cdot(\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,1)}{\sqrt{1^2+2^2+1^2}} = \frac{-1+8+2}{\sqrt{6}} = \frac{9}{\sqrt{6}}.$$

The scalar component of **A** in the direction of **B** equals $9/\sqrt{6}$.

(c) The vector component of **B** in the direction of **C** is calculated (see **Example 1.4**) as

$$\hat{\mathbf{C}}B_C = \hat{\mathbf{C}}\left(\mathbf{B}\cdot\hat{\mathbf{C}}\right) = \frac{\mathbf{C}(\mathbf{B}\cdot\mathbf{C})}{C^2} = \frac{(\hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,2 - \hat{\mathbf{z}}\,1)[(\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,1)\cdot(\hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,2 - \hat{\mathbf{z}}\,1)]}{C_x^2 + C_y^2 + C_z^2}$$
$$= \frac{(\hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,2 - \hat{\mathbf{z}}\,1)[2-4-1]}{4+4+1} = \frac{-\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,1}{3}$$

The vector components of **B** in the direction of **C** are $-\hat{\mathbf{x}}\,2/3, \hat{\mathbf{y}}\,2/3$, and $\hat{\mathbf{z}}\,1/3$.

1.5.2 The Cylindrical Coordinate System

The need for a cylindrical coordinate system should be apparent from **Figure 1.26a**, where the cylindrical surface (such as a pipe) is located along the z axis in the Cartesian system of coordinates. To describe a point P_1 on the surface, we must give the three coordinates $P_1(x_1,y_1,z_1)$. A second point on the cylindrical surface, $P_2(x_2,y_2,z_2)$ is shown and a vector connects the two points. The vector can be immediately written as

$$\mathbf{A} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) \tag{1.53}$$

On the other hand, we observe that points P_1 and P_2 are at a constant distance from the z axis, equal to the radius of the cylinder. To draw this cylinder all we need is to draw a circle at a constant radius and repeat the process for each value of z. In doing so, a point at radius r is rotated an angle of 2π to describe the circle. This suggests that the process above is easiest to describe in terms of a radial distance (radius of the point), an angle of rotation, and length of the cylinder in the z direction. The result is the cylindrical system of coordinates. It is also called a circular-cylindrical system to distinguish it from the polar-cylindrical system, but we will use the short form "cylindrical" throughout this book.

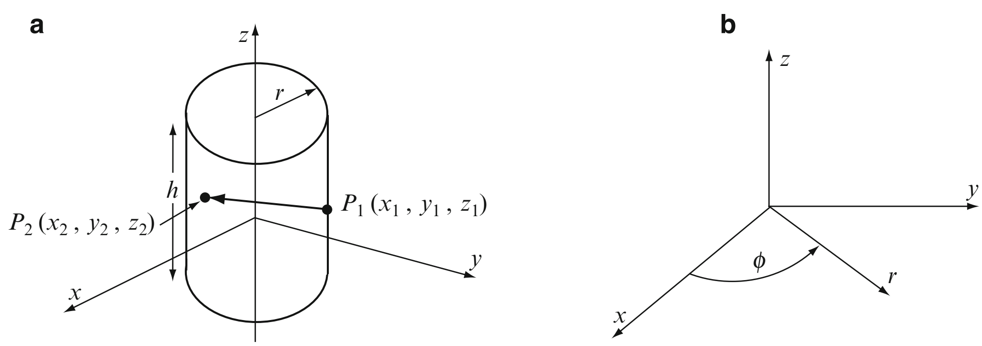


Figure 1.26 (**a**) Two points on a cylindrical surface described in terms of their Cartesian coordinates. (**b**) The cylindrical coordinate system and its relation to the Cartesian system

The cylindrical system is shown in **Figure 1.26b**. The axes are orthogonal and the ϕ axis is positive in the counterclockwise direction when viewed from the positive z axis. Similarly, r is positive in the direction away from the z axis whereas the z axis is the same as for the Cartesian coordinate system and serves as the axis of the cylinder. The r axis extends from zero to $+\infty$, the z axis extends from $-\infty$ to $+\infty$, and the ϕ axis varies from 0 to 2π. The ϕ angle is also called the ***azimuthal angle*** and is given with reference to the x axis of a superposed Cartesian system. This reference also allows transformation between the two systems of coordinates.

A general vector in the cylindrical coordinate system is given as

$$\mathbf{A} = \hat{\mathbf{r}} A_r(r,\phi,z) + \hat{\boldsymbol{\phi}} A_\phi(r,\phi,z) + \hat{\mathbf{z}} A_z(r,\phi,z) \tag{1.54}$$

All other aspects of vector algebra that we have defined are preserved. The unit vector, the magnitude of the vector, as well as vector and scalar products are evaluated in an identical fashion although the fact that one of the coordinates is an angle must be taken into account, as we shall see shortly. Since the three coordinates are orthogonal to each other, the scalar and vector products of the unit vectors are

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1 \tag{1.55}$$

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{z}} \cdot \hat{\boldsymbol{\phi}} = 0 \tag{1.56}$$

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \tag{1.57}$$

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = -\hat{\mathbf{z}}, \quad \hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\hat{\boldsymbol{\phi}} \tag{1.58}$$

Next, we need to define the differentials of length, surface, and volume in the cylindrical system. This is shown in **Figure 1.27**. The differential lengths in the r and z directions are $\hat{\mathbf{r}}\,dr$ and $\hat{\mathbf{z}}\,dz$, correspondingly. In the ϕ direction, the differential of length is an arc of length $rd\phi$, as shown in **Figure 1.27**. Thus, the differential length in cylindrical coordinates is

$$\boxed{d\mathbf{l} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}rd\phi + \hat{\mathbf{z}}dz} \tag{1.59}$$

The differentials of area are

$$\boxed{d\mathbf{s}_r = rd\phi\,dz, \quad d\mathbf{s}_\phi = drdz, \quad d\mathbf{s}_z = rd\phi\,dr} \tag{1.60}$$

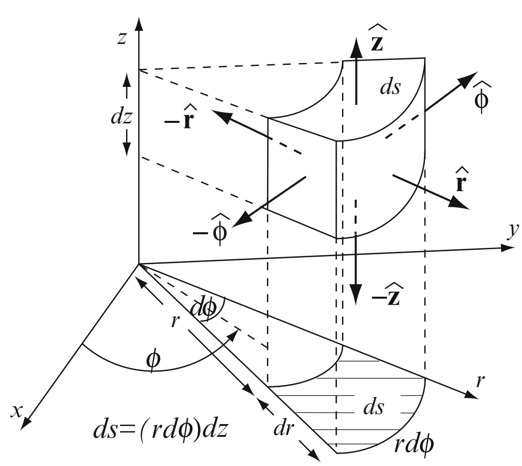


Figure 1.27 Differentials of length, surface, and volume in cylindrical coordinates

The differential volume is therefore

$$\boxed{dv = dr(rd\phi)dz = r\,drd\phi dz} \tag{1.61}$$

The oriented differentials of surface are defined as for the Cartesian coordinate system in terms of unit vectors normal to the surface of a cylindrical object (see **Figure 1.27**):

$$\boxed{d\mathbf{s}_r = \pm\hat{\mathbf{r}}\, rd\phi dz, \quad d\mathbf{s}_\phi = \pm\hat{\boldsymbol{\phi}} drdz, \quad d\mathbf{s}_z = \pm\hat{\mathbf{z}}\, rd\phi\, dr} \tag{1.62}$$

Of course, the surface vector for a general surface will vary, but it must be normal to the surface and is considered positive if it points out of the volume enclosed by the surface.

The fundamental principle that we followed in defining systems of coordinates is that the fields are independent of the system of coordinates. This also means that we can transform from one system of coordinates to another at will. All we need is to identify the transformation necessary and ensure that this transformation is unique. To find the transformation between the cylindrical and Cartesian systems, we superimpose the two systems on each other so that the z axes of the systems coincide as in **Figure 1.28a**. In the Cartesian system, the point P has coordinates (x,y,z). In the cylindrical system, the coordinates are r, ϕ, and z.

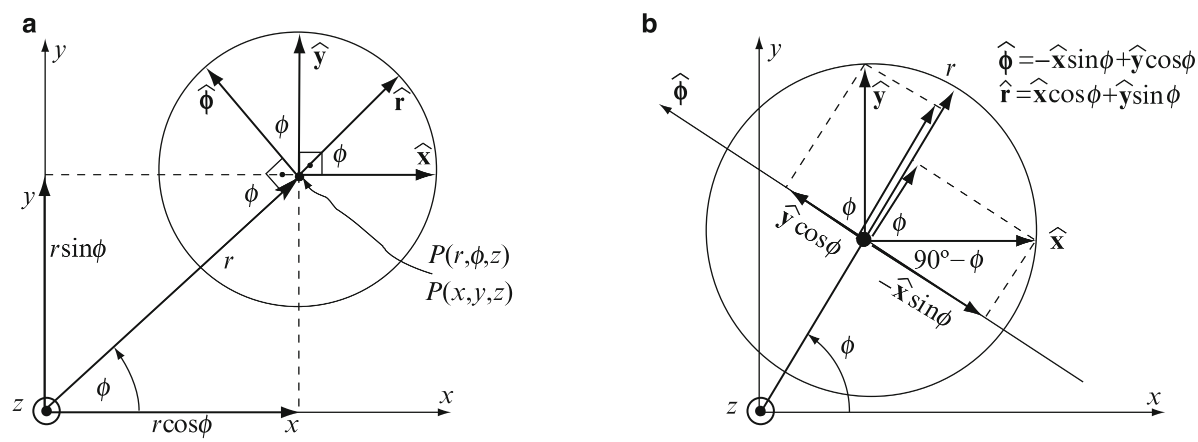


Figure 1.28 (**a**) Relation between unit vectors in the Cartesian and cylindrical systems. The circle has unit radius. (**b**) Calculation of unit vectors in cylindrical coordinates as projections of the unit vectors in Cartesian coordinates

If we assume (r,ϕ,z) are known, we can transform these into the Cartesian coordinates using the relations in **Figure 1.28a**:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z \tag{1.63}$$

Similarly, we can write for the inverse transformation (assuming x, y, and z are known) either directly from **Figure 1.28a** or from **Eq. (1.63)**:

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{1.64}$$

These are the transformations for a single point. We also need to transform the unit vectors and, finally, the vectors from one system to the other. To transform the unit vectors, we use **Figure 1.28b**. First, we note that the unit vector in the z direction remains unchanged as expected. To calculate the unit vectors in the r and ϕ directions, we resolve the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ onto the r–ϕ plane by calculating their projections in the r and ϕ directions. Since all unit vectors are of unit length, they all fall on the unit circle shown. Thus,

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi, \quad \hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \quad \hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{1.65}$$

The inverse transformation is also obtained from **Figure 1.28b** by an identical process:

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi, \quad \hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi, \quad \hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{1.66}$$

Important Note: Unlike in Cartesian coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ (in cylindrical coordinates) are not constant; both depend on ϕ (whereas $\hat{\mathbf{z}}$ is constant). Therefore, whenever they are used, such as in integration, this fact must be taken into account. It will often become necessary to transform the unit vectors into Cartesian coordinates using **Eq. (1.66)** to avoid this difficulty.

To obtain the transformation necessary for a vector, we use the properties of the scalar product to find the scalar components of the vector in one system of coordinates in the directions of the unit vectors of the other system. For a vector **A** given in the cylindrical system, we can write

$$A_x = \hat{\mathbf{x}} \cdot \mathbf{A} = \hat{\mathbf{x}} \cdot \left(\hat{\mathbf{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z\right) = (\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})A_r + \left(\hat{\mathbf{x}} \cdot \hat{\boldsymbol{\phi}}\right)A_\phi + (\hat{\mathbf{x}} \cdot \hat{\mathbf{z}})A_z \quad (1.67)$$

From **Figure 1.28b**, $\hat{\mathbf{x}} \cdot \hat{\mathbf{r}} = \cos\phi$, $\hat{\mathbf{x}} \cdot \hat{\boldsymbol{\phi}} = -\sin\phi$, and $\hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$. Therefore,

$$A_x = A_r \cos\phi - A_\phi \sin\phi \quad (1.68)$$

Similarly, calculating the products $A_y = \hat{\mathbf{y}} \cdot \mathbf{A}$ and $A_z = \hat{\mathbf{z}} \cdot \mathbf{A}$, we get

$$A_y = A_r \sin\phi + A_\phi \cos\phi \quad \text{and} \quad A_z = A_z \quad (1.69)$$

To find the inverse transformation, we can write vector **A** in the Cartesian coordinate system and repeat the process above by finding its projections on the r, ϕ, and z axes. Alternatively, we calculate the inverse transformation by first writing **Eqs. (1.68)** and **(1.69)** as a system of equations:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \quad (1.70)$$

Calculating the inverse of this system, we get

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (1.71)$$

Now, for a general vector given in the cylindrical coordinate system, we can write the same vector in the Cartesian system by using the scalar components from **Eq. (1.70)** and adding the unit vectors. Similarly, if a vector in the Cartesian system must be transformed into the cylindrical system, we use **Eq. (1.71)** to evaluate its components.

Example 1.16 Two points in cylindrical coordinates are given as $P_1(r_1,\phi_1,z_1)$ and $P_2(r_2,\phi_2,z_2)$. Find the expression of the vector pointing from P_1 to P_2:

(a) In Cartesian coordinates.
(b) In cylindrical coordinates.
(c) Calculate the length of the vector.

Solution: The cylindrical coordinates are first converted into Cartesian coordinates. After the components of the vector are found, these are transformed back into cylindrical coordinates. Calculation of the length of the vector must be done in Cartesian coordinates in which each coordinate represents the same quantity (for example, length):

(a) The coordinates of points P_1 and P_2 in Cartesian coordinates are [**Eq. (1.63)**]

$$x_1 = r_1\cos\phi_1, \quad y_1 = r_1\sin\phi_1, \quad z_1 = z_1$$
$$x_2 = r_2\cos\phi_2, \quad y_2 = r_2\sin\phi_2, \quad z_2 = z_2$$

Thus, the vector **A** in Cartesian coordinates is

$$\mathbf{A} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}\left(y_2 - y_1\right) + \hat{\mathbf{z}}\left(z_2 - z_1\right) = \hat{\mathbf{x}}\left(r_2\cos\phi_2 - r_1\cos\phi_1\right) + \hat{\mathbf{y}}\left(r_2\sin\phi_2 - r_1\sin\phi_1\right) + \hat{\mathbf{z}}\left(z_2 - z_1\right)$$

(b) To find the components of the vector in cylindrical coordinates, we write from **Eq. (1.71)**

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_2\cos\phi_2 - r_1\cos\phi_1 \\ r_2\sin\phi_2 - r_1\sin\phi_1 \\ z_2 - z_1 \end{bmatrix}$$

Expanding, we get:

$$A_r = (r_2\cos\phi_2 - r_1\cos\phi_1)\cos\phi + (r_2\sin\phi_2 - r_1\sin\phi_1)\sin\phi$$

$$A_\phi = -(r_2\cos\phi_2 - r_1\cos\phi_1)\sin\phi + (r_2\sin\phi_2 - r_1\sin\phi_1)\cos\phi$$

$$A_z = z_2 - z_1$$

The vector in cylindrical coordinates is

$$\begin{aligned} \mathbf{A}(r,\phi,z) = \hat{\mathbf{r}}\,&[(r_2\cos\phi_2 - r_1\cos\phi_1)\cos\phi + (r_2\sin\phi_2 - r_1\sin\phi_1)\sin\phi] \\ &+ \hat{\boldsymbol{\phi}}\,[-(r_2\cos\phi_2 - r_1\cos\phi_1)\sin\phi + (r_2\sin\phi_2 - r_1\sin\phi_1)\cos\phi] + \hat{\mathbf{z}}\,[z_2 - z_1] \end{aligned}$$

Note that the r and ϕ components of the vector in cylindrical coordinates are not constant (they depend on the angle ϕ in addition to the coordinates of points P_1 and P_2). In Cartesian coordinates, the components only depend on the two end points. Because of this, it is often necessary to transform from cylindrical to Cartesian coordinates, especially when the magnitudes of vectors need to be evaluated.

(c) To calculate the length of the vector, we use the representation in Cartesian coordinates because it is easier to evaluate. The length of the vector is

$$\begin{aligned} |\mathbf{A}| &= \sqrt{A_x^2 + A_y^2 + A_z^2} \\ &= \sqrt{(r_2\cos\phi_2 - r_1\cos\phi_1)^2 + (r_2\sin\phi_2 - r_1\sin\phi_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{r_2^2 + r_1^2 - 2r_2r_1(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1) + (z_2 - z_1)^2} \end{aligned}$$

With $\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1 = \cos(\phi_2 - \phi_1)$, we get

$$|\mathbf{A}| = \sqrt{r_2^2 + r_1^2 - 2r_2r_1\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

This expression may be used to calculate the magnitude of a vector when two points are given in cylindrical coordinates without first transforming into Cartesian coordinates.

Example 1.17 A vector is given in cylindrical coordinates as: $\mathbf{A} = \hat{\mathbf{r}}\,2 + \hat{\boldsymbol{\phi}}\,3 - \hat{\mathbf{z}}\,1$. Describe this vector in Cartesian coordinates.

Solution: The vector in Cartesian coordinates is written directly from **Eqs. (1.70), (1.63),** and **(1.64)**:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\phi - 3\sin\phi \\ 2\sin\phi + 3\cos\phi \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{2x}{\sqrt{x^2+y^2}} - \dfrac{3x}{\sqrt{x^2+y^2}} \\ \dfrac{2y}{\sqrt{x^2+y^2}} + \dfrac{3_x}{\sqrt{x^2+y^2}} \\ -1 \end{bmatrix}$$

where the following substitutions were made using **Eq. (1.63)**:

$$\cos\phi = \frac{x}{r}, \quad \sin\phi = \frac{y}{r}, \quad r = \sqrt{x^2 + y^2}$$

Thus, vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}\left(\frac{2x}{\sqrt{x^2+y^2}} - \frac{3y}{\sqrt{x^2+y^2}}\right) + \hat{\mathbf{y}}\left(\frac{2y}{\sqrt{x^2+y^2}} + \frac{3x}{\sqrt{x^2+y^2}}\right) - \hat{\mathbf{z}}\,1$$

Exercise 1.9 Transform $\mathbf{A} = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,5 + \hat{\mathbf{z}}\,3$ into cylindrical coordinates at point ($x = -2$, $y = 3$, $z = 1$).

Answer $\mathbf{A} = \hat{\mathbf{r}}\,5.27 - \hat{\boldsymbol{\phi}}\,1.11 + \hat{\mathbf{z}}\,3$

1.5.3 The Spherical Coordinate System

The spherical coordinate system is defined following the same basic ideas used for the cylindrical system. First, we observe that a point on a spherical surface is at a constant distance from the center of the sphere. To draw a sphere, we can follow the process in **Figure 1.29**. First, a point $P(x,y,z)$ is defined in the Cartesian coordinate system at a distance R from the origin as in **Figure 1.29a**. Suppose this point is rotated around the origin to form a half-circle as shown in **Figure 1.29b**. Now, we can rotate the half-circle in **Figure 1.29c** on a 2π angle to obtain a sphere. This process indicates that a point on a sphere is best described in terms of its radial location and two angles, provided proper references can be identified. To do so, we use the Cartesian reference system in **Figure 1.30a**. The z axis serves as reference for the first angle, θ, while the x axis is used for the

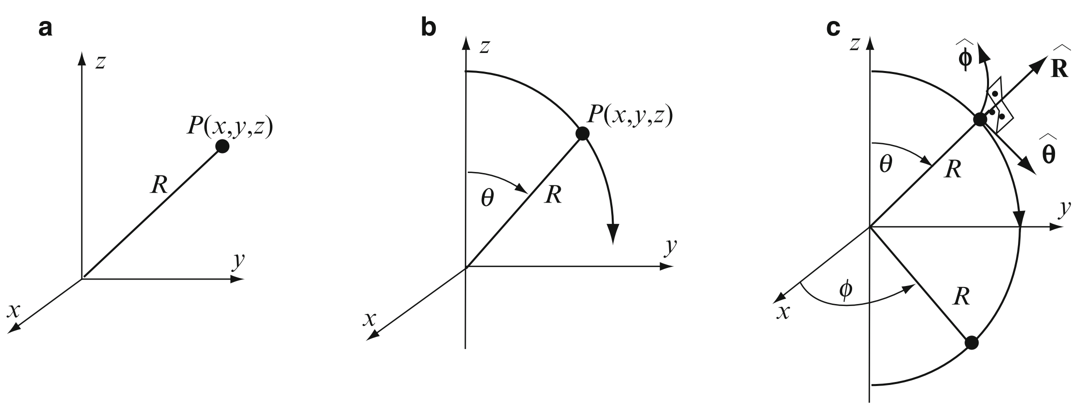


Figure 1.29 Drawing a sphere: (**a**) A point at distance R from the origin. (**b**) Keeping the distance R constant, we move away from the z axis describing an angle θ. (**c**) The half-circle is rotated at an angle ϕ. If $\phi = 2\pi$, a sphere is generated

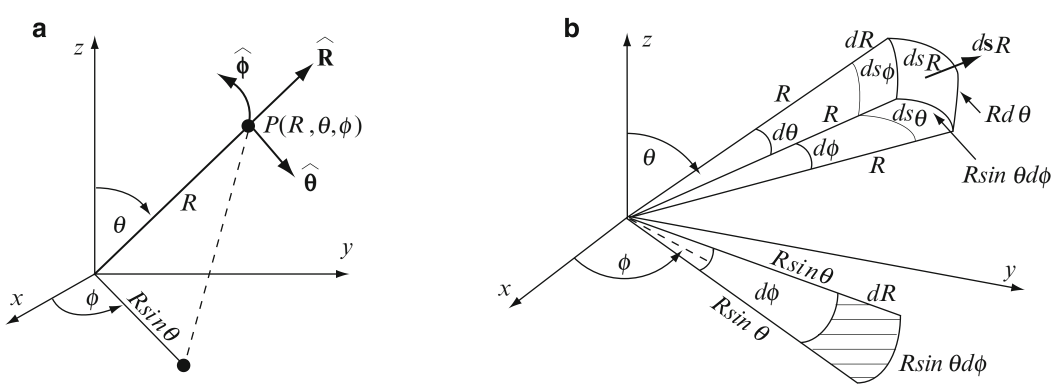


Figure 1.30 (**a**) A point in spherical coordinates and the relationship between the spherical and Cartesian coordinate systems. (**b**) Differentials of length, area, and volume in spherical coordinates

second angle, ϕ. The three unit vectors at point P are also shown and these form an orthogonal, right-hand system of coordinates. The range of the angle θ is between zero and π and that of ϕ between zero and 2π, whereas the range of R is between 0 and ∞. The point P is now described as $P(R,\theta,\phi)$ and a general vector is written as

$$\mathbf{A} = \hat{\mathbf{R}}\,A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi \tag{1.72}$$

Since the axes are orthogonal, we have

$$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1 \tag{1.73}$$

$$\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\mathbf{R}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\theta}} = 0 \tag{1.74}$$

$$\hat{\mathbf{R}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\phi}} = 0 \tag{1.75}$$

$$\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}},\quad \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}},\quad \hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}},\quad \hat{\boldsymbol{\theta}}\times\hat{\mathbf{R}} = -\hat{\boldsymbol{\phi}},\quad \hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\theta}} = -\hat{\mathbf{R}},\quad \hat{\mathbf{R}}\times\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}} \tag{1.76}$$

The differential length, volume, and surface are defined with the aid of **Figure** 1.30b, but, now, the lengths in both θ and ϕ directions are arc lengths. These are given as

$$\boxed{d\mathbf{l} = \hat{\mathbf{R}}dR + \hat{\boldsymbol{\theta}}Rd\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi} \tag{1.77}$$

Differentials of surface are defined in a manner similar to the cylindrical coordinate system. These are

$$\boxed{ds_R = R^2 sin\theta\,d\theta d\phi,\quad ds_\theta = R\sin\theta\,dR\,d\phi,\quad ds_\phi = R\,dR\,d\theta} \tag{1.78}$$

The differential volume is

$$\boxed{dv = dR(Rd\theta)(R\sin\theta\,d\phi) = R^2\sin\theta\,dR\,d\theta\,d\phi} \tag{1.79}$$

The three basic surface vectors in the directions perpendicular to the three planes $\theta\phi$, $R\phi$, and $R\theta$ are

$$\boxed{d\mathbf{s}_R = \pm\hat{\mathbf{R}}R^2\sin\theta d\theta d\phi,\quad d\mathbf{s}_\theta = \pm\hat{\boldsymbol{\theta}}R\sin\theta dRd\phi,\quad d\mathbf{s}_\phi = \pm\hat{\boldsymbol{\phi}}RdRd\theta} \tag{1.80}$$

To define the coordinate transformation between spherical and Cartesian coordinates, we again use **Figure** 1.30a. The basic geometrical relations between the coordinates in the two systems are

$$R = \sqrt{x^2+y^2+z^2},\quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right),\quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.81}$$

Similarly, the inverse transformation is

$$x = R\sin\theta\cos\phi,\quad y = R\sin\theta\sin\phi,\quad z = R\cos\theta \tag{1.82}$$

Transformation of a general vector from spherical to Cartesian coordinates is performed by calculation of the scalar components through the scalar product:

$$A_x = \hat{\mathbf{x}}\cdot\mathbf{A} = \hat{\mathbf{x}}\cdot\left(\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi\right) = (\hat{\mathbf{x}}\cdot\hat{\mathbf{R}})A_R + \left(\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\theta}}\right)A_\theta + (\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\phi}})A_\phi \tag{1.83}$$

From **Figure** 1.30a,

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{R}} = \sin\theta\cos\phi,\quad \hat{\mathbf{x}}\cdot\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\quad \hat{\mathbf{x}}\cdot\hat{\boldsymbol{\phi}} = -\sin\phi \tag{1.84}$$

The scalar components of **A** are

$$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi \tag{1.85}$$

$$A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi \tag{1.86}$$

$$A_z = A_R \cos\theta - A_\theta \sin\theta \tag{1.87}$$

Again making use of the matrix notation,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} \tag{1.88}$$

To obtain the inverse transformation, we invert the system:

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{1.89}$$

Equation (**1.89**) may also be used to obtain the unit vectors $\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ in terms of the unit vectors in Cartesian coordinates. These transformations are

$$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta\cos\phi + \hat{\mathbf{y}} \sin\theta\sin\phi + \hat{\mathbf{z}} \cos\theta, \quad \hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta\cos\phi + \hat{\mathbf{y}} \cos\theta\sin\phi - \hat{\mathbf{z}} \sin\theta, \\ \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \tag{1.90}$$

Important Note: Clearly, $\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are not constant unit vectors in space; $\hat{\mathbf{R}}$ and $\hat{\boldsymbol{\theta}}$ depend on θ and ϕ, whereas $\hat{\boldsymbol{\phi}}$ depends on ϕ. Whenever unit vectors in spherical coordinates occur inside integrals, they must be resolved into Cartesian coordinates. The Cartesian unit vectors can then be taken outside the integral sign.

1.5.4 Transformation from Cylindrical to Spherical Coordinates

On occasion, there will also be a need to transform vectors or points from cylindrical to spherical coordinates and vice versa. We list the transformation below without details of the derivation (see **Exercise 1.11**).

The spherical coordinates R, θ, and ϕ are obtained from the cylindrical coordinates r, ϕ, and z as

$$R = \sqrt{r^2 + z^2}, \quad \theta = \tan^{-1}(r/z), \quad \phi = \phi \tag{1.91}$$

The scalar components of a vector **A** in spherical coordinates can be obtained from the scalar components of the vector in cylindrical coordinates as

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \tag{1.92}$$

The cylindrical coordinates r, ϕ, and z are obtained from the spherical coordinates R, θ, and ϕ as

$$r = R\sin\theta, \quad \phi = \phi, \quad z = R\cos\theta \tag{1.93}$$

The scalar components of the vector **A** in cylindrical coordinates can be obtained from the scalar components of the vector in spherical coordinates as

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} \tag{1.94}$$

The scalar components of the unit vectors are obtained by replacing the scalar components in **Eq.** (**1.92**) or (**1.94**) with the appropriate unit vector components (see **Exercise 1.11**).

Example 1.18 Two points are given in spherical coordinates as $P_1(R_1,\theta_1,\phi_1)$ and $P_2(R_2,\theta_2,\phi_2)$:

(a) Write the vector connecting P_1 (tail) to P_2 (head) in Cartesian coordinates.
(b) Calculate the length of the vector (distance between P_1 and P_2).

Solution: The coordinates of P_1 and P_2 are first transformed into Cartesian coordinates, followed by evaluation of the magnitude of the vector connecting P_1 and P_2:

(a) The transformation from spherical to Cartesian coordinates [**Eq. (1.82)**] for points P_1 and P_2 gives

$$x_1 = R_1\sin\theta_1\cos\phi_1,\qquad y_1 = R_1\sin\theta_1\sin\phi_1,\qquad z_1 = R_1\cos\theta_1$$
$$x_2 = R_2\sin\theta_2\cos\phi_2,\qquad y_2 = R_2\sin\theta_2\sin\phi_2,\qquad z_2 = R_2\cos\theta_2$$

The vector in Cartesian coordinates is

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}(x_2-x_1)+\hat{\mathbf{y}}(y_2-y_1)+\hat{\mathbf{z}}(z_2-z_1)\\ &= \hat{\mathbf{x}}(R_2\sin\theta_2\cos\phi_2-R_1\sin\theta_1\cos\phi_1)+\hat{\mathbf{y}}(R_2\sin\theta_2\sin\phi_2-R_1\sin\theta_1\sin\phi_1)+\hat{\mathbf{z}}(R_2\cos\theta_2-R_1\cos\theta_1)\end{aligned}$$

(b) The length of the vector in terms of spherical components can be written from **(a)**:

$$\begin{aligned}|\mathbf{A}|^2 &= (R_2\sin\theta_2\cos\phi_2-R_1\sin\theta_1\cos\phi_1)^2+(R_2\sin\theta_2\sin\phi_2-R_1\sin\theta_1\sin\phi_1)^2+(R_2\cos\theta_2-R_1\cos\theta_1)^2\\ &= R_2^2\sin^2\theta_2\cos^2\phi_2+R_1^2\sin^2\theta_1\cos^2\phi_1-2R_1R_2\sin\theta_1\sin\theta_2\cos\phi_1\cos\phi_2+R_2^2\sin^2\theta_2\sin^2\phi_2\\ &\quad +R_1^2\sin^2\theta_1\sin^2\phi_1-2R_1R_2\sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2+R_2^2\cos^2\theta_2+R_1^2\cos^2\theta_1-2R_1R_2\cos\theta_2\cos\theta_1\end{aligned}$$

Rearranging terms and using the relation $\sin^2\alpha+\cos^2\alpha=1$, we get

$$|\mathbf{A}|^2 = R_2^2+R_1^2-2R_1R_2\sin\theta_1\sin\theta_2[\cos\phi_2\cos\phi_1+\sin\phi_2\sin\phi_1]-2R_1R_2\cos\theta_2\cos\theta_1$$

Using $\cos\alpha\cos\beta=(\cos(\alpha-\beta)+\cos(\alpha+\beta))/2$ and $\sin\alpha\sin\beta=(\cos(\alpha-\beta)-\cos(\alpha+\beta))/2$, after taking the square root of the expression, we get

$$|\mathbf{A}| = \sqrt{R_2^2+R_1^2-2R_1R_2\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)-2R_1R_2\cos\theta_2\cos\theta_1}$$

This is a convenient general formula for the calculation of the distance between two points in spherical coordinates, without the need to first convert the points to Cartesian coordinates.

Example 1.19 Two points are given in Cartesian coordinates as $P_1(0,0,1)$ and $P_2(2,1,3)$ and a vector **A** connects P_1 (tail) to P_2 (head). Find the unit vector in the direction of **A** in spherical and cylindrical coordinates.

Solution: The vector **A** connecting P_1 (tail) and P_2 (head) is found first, followed by the unit vector in the direction of **A**. The unit vector is then transformed into spherical and cylindrical coordinates using **Eqs. (1.89)** and **(1.71)**. The vector connecting P_1 (tail) and P_2 (head) is

$$\mathbf{A} = \hat{\mathbf{x}}(x_2-x_1)+\hat{\mathbf{y}}(y_2-y_1)+\hat{\mathbf{z}}(z_2-z_1) = \hat{\mathbf{x}}(2-0)+\hat{\mathbf{y}}(1-0)+\hat{\mathbf{z}}(3-1) = \hat{\mathbf{x}}2+\hat{\mathbf{y}}1+\hat{\mathbf{z}}2$$

The unit vector in the direction of **A** is

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}1 + \hat{\mathbf{z}}2}{3} = \hat{\mathbf{x}}\frac{2}{3} + \hat{\mathbf{y}}\frac{1}{3} + \hat{\mathbf{z}}\frac{2}{3}$$

Taking this as a regular vector in Cartesian coordinates, the transformation of its components into spherical coordinates is

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta) \\ \frac{1}{3}(2\cos\theta\cos\phi + \cos\theta\sin\phi - 2\sin\theta) \\ -\frac{1}{3}(2\sin\phi - \cos\phi) \end{bmatrix}$$

The unit vector in spherical coordinates is

$$\hat{\mathbf{A}} = \hat{\mathbf{R}}\frac{1}{3}(2\sin\theta\cos\phi + \sin\theta\sin\phi + 2\cos\theta) + \hat{\boldsymbol{\theta}}\frac{1}{3}(2\cos\theta\cos\phi + \cos\theta\sin\phi - 2\sin\theta) - \hat{\boldsymbol{\phi}}\frac{1}{3}(2\sin\phi - \cos\phi)$$

Although we do not show that the magnitude of the unit vector equals 1, the transformation cannot modify a vector in any way other than describing it in different coordinates. You are urged to verify the magnitude of $\hat{\mathbf{A}}$. Similarly, the transformation of the components into cylindrical coordinates is [from **Eq. (1.71)**]

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2\cos\phi + \sin\phi) \\ -\frac{1}{3}(2\sin\phi - \cos\phi) \\ \frac{2}{3} \end{bmatrix}$$

and the unit vector in cylindrical coordinates is

$$\hat{\mathbf{A}} = \hat{\mathbf{r}}\frac{1}{3}(2\cos\phi + \sin\phi) - \hat{\boldsymbol{\phi}}\frac{1}{3}(2\sin\phi - \cos\phi) + \hat{\mathbf{z}}\frac{2}{3}$$

Exercise 1.10 Repeat **Example 1.19**, but first transform the vector **A** into spherical and cylindrical coordinates and then divide each vector by its magnitude to find the unit vector.

Example 1.20 Given the points $P_1(2,2,-5)$ in Cartesian coordinates and $P_2(3,\pi,-2)$ in cylindrical coordinates, find:

(a) The spherical coordinates of P_1.
(b) The spherical coordinates of P_2.
(c) The magnitude of the vector connecting P_1 (tail) to P_2 (head).

Solution:

(a) Because P_1 is given in Cartesian coordinates, it is necessary to transform the point to spherical coordinates. The required transformation is given in **Eq. (1.81)**:

$$R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} = \sqrt{4 + 4 + 25} = 5.745$$

$$\theta_1 = \tan^{-1}\left(\frac{\sqrt{x_1^2 + y_1^2}}{z_1}\right) = \tan^{-1}\left(\frac{\sqrt{4+4}}{-5}\right) = \tan^{-1}(-0.56568) = -29^\circ 30'$$

Because θ only varies between zero and π, we add π to get

$$\theta_1 = -29^\circ 30' + 180^\circ = 150^\circ 30' \quad \text{and} \quad \phi_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{2}{2}\right) = 45^\circ$$

P_1 in spherical coordinates is, therefore, $P_1(5.745, 150^\circ 30', 45^\circ)$.

(b) P_2 is given in cylindrical coordinates with $r = 3$, $\phi = \pi$, $z = -2$. To convert it into spherical coordinates, we use the point transformation in **Eq. (1.91)**:

$$R_2 = \sqrt{r^2 + z^2} = \sqrt{3^2 + (-2)^2} = 3.6$$

$$\theta_2 = \tan^{-1}\left(\frac{r}{z}\right) = \tan^{-1}\left(\frac{3}{-2}\right) = \tan^{-1}(-1.5) = -56^\circ 18'$$

Adding 180° to get θ_2 between zero and π gives

$$\theta_2 = -56^\circ 18' + 180 = 123^\circ 42' \quad \text{and} \quad \phi_2 = 180^\circ$$

P_2 in spherical coordinates is $P_2(3.6, 123^\circ 42', 180^\circ)$.

(c) To calculate the distance, we convert P_2 from cylindrical to Cartesian coordinates using the transformation in **Eq. (1.63)**:

$$x_2 = r\cos\phi = 3\cos\pi = -3, \quad y_2 = 3\sin\pi = 0, \quad z_2 = -2$$

The two points are

$$P_1(2, 2, -5), \qquad P_2(-3, 0, -2)$$

The vector connecting P_1 to P_2 is

$$\mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z = \hat{\mathbf{x}}(-3-2) + \hat{\mathbf{y}}(0-2) + \hat{\mathbf{z}}(-2-(-5)) = -\hat{\mathbf{x}}\,5 - \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3$$

The magnitude of **A** is

$$|\mathbf{A}| = \sqrt{25 + 4 + 9} = \sqrt{38} = 6.164$$

The distance between P_1 and P_2 is 6.164.

Exercise 1.11 Derive the transformation matrices from cylindrical to spherical and spherical to cylindrical coordinates, that is, find the coefficients in **Eqs. (1.92)** and **(1.94)** by direct application of the scalar product.

Exercise 1.12 Write the vector **A** connecting points $P_1(R_1, \theta_1, \phi_1)$ and $P_2(R_2, \theta_2, \phi_2)$, in cylindrical coordinates.

Answer

$$\begin{aligned}\mathbf{A} = \hat{\mathbf{r}}\,&\left[(R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1)\cos\phi + (R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1)\sin\phi\right]\\ &+\hat{\boldsymbol{\phi}}\left[-(R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1)\sin\phi + (R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1)\cos\phi\right]\\ &+\hat{\mathbf{z}}\left[R_2\cos\theta_2 - R_1\cos\theta_1\right]\end{aligned}$$

Exercise 1.13 Write the vector **A** connecting points $P_1(R_1,\theta_1,\phi_1)$ and $P_2(R_2,\theta_2,\phi_2)$ in spherical coordinates.

Answer

$$\begin{aligned}\mathbf{A} = \hat{\mathbf{R}}\,&\left[R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1)\sin\theta\cos\phi + (R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1)\sin\theta\sin\phi + (R_2\cos\theta_2 - R_1\cos\theta_1)\cos\theta\right]\\ &+\hat{\boldsymbol{\theta}}\left[(R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1)\cos\theta\cos\phi + (R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1)\cos\theta\sin\phi - (R_2\cos\theta_2 - R_1\cos\theta_1)\sin\theta\right]\\ &+\hat{\boldsymbol{\phi}}\left[-(R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1)\sin\phi + (R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1)\cos\phi\right]\end{aligned}$$

1.6 Position Vectors

A ***position vector*** is defined as the vector connecting a reference point and a location or position (point) in space. The position vector always points to the position it identifies, as shown in **Figure 1.31a**. The advantage of using position vectors lies in the choice of the reference point. Normally, this point will be chosen as the origin of the system of coordinates.

A vector can always be represented by two position vectors: one pointing to its head and one to its tail as shown in **Figure 1.31b**. Vector **A** can now be written as

$$\mathbf{A} = \mathbf{P}_2 - \mathbf{P}_1 \tag{1.95}$$

This form of describing a vector will be used often because it provides easy reference to the vector **A**.

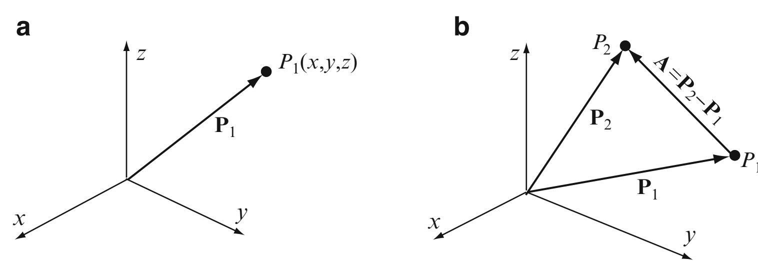


Figure 1.31 (**a**) Position vector of point P_1. (**b**) Vector **A** described in terms of the position vectors of its end points

Example 1.21 Two points are given in the Cartesian coordinate system as $P_1(1,1,3)$ and $P_2(3,1,3)$. Find the vector connecting point P_2 (tail) to point P_1 (head):

(a) As a regular vector.
(b) In terms of the position vectors connecting the origin of the system to points P_1 and P_2.
(c) Show that the two representations are the same.

Solution: The vector connecting points P_2 and P_1 is found as in **Example 1.19**.

The position vectors of points P_1 and P_2 are found as the vectors that connect the origin with these points. The vector **A** connecting P_2 (tail) and P_1 (head) is $\mathbf{A} = \mathbf{R}_1 - \mathbf{R}_2$ (see **Figure 1.32**).

(a) From **Example 1.19**, we write

$$\mathbf{A} = \hat{\mathbf{x}}\,(x_1 - x_2) + \hat{\mathbf{y}}\,(y_1 - y_2) + \hat{\mathbf{z}}\,(z_1 - z_2) = \hat{\mathbf{x}}\,(1 - 3) + \hat{\mathbf{y}}\,(1 - 1) + \hat{\mathbf{z}}\,(3 - 3) = -\hat{\mathbf{x}}\,2$$

(b) The two position vectors are

$$\mathbf{R}_1 = \hat{\mathbf{x}}\,(x_1 - 0) + \hat{\mathbf{y}}\,(y_1 - 0) + \hat{\mathbf{z}}\,(z_1 - 0) = \hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,3$$
$$\mathbf{R}_2 = \hat{\mathbf{x}}\,(x_2 - 0) + \hat{\mathbf{y}}\,(y_2 - 0) + \hat{\mathbf{z}}\,(z_2 - 0) = \hat{\mathbf{x}}\,3 + \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,3$$

(c) To show that $\mathbf{A} = \mathbf{R}_1 - \mathbf{R}_2$, we evaluate the expression explicitly:

$$\mathbf{A} = \mathbf{R}_1 - \mathbf{R}_2 = (\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3) - (\hat{\mathbf{x}}\,3 + \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,3) = \hat{\mathbf{x}}\,(1 - 3) + \hat{\mathbf{y}}\,(1 - 1) + \hat{\mathbf{z}}\,(3 - 3) = -\hat{\mathbf{x}}\,2$$

and this is identical to vector **A** above.

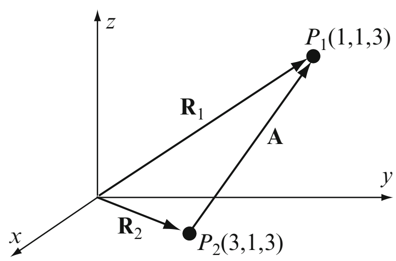


Figure 1.32 Vector **A** and its relationship with position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ and end points P_1 and P_2

Example 1.22 Two points are given in cylindrical coordinates as $P_1(1,30°,1)$ and $P_2(2,0°,2)$. Calculate the vector connecting P_1 and P_2 in terms of the position vectors of points P_1 and P_2.

Solution: First, we calculate the vectors connecting the origin with points P_1 and P_2. These are the position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$. The vector connecting P_1 to P_2 is then $\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$.

To calculate R_1 we take the tail of the vector at $P_0(0,0,0)$ and the head at $P_1(1,30°,1)$. The expression for a general vector in cylindrical coordinates was found in **Example 1.16**. With $r_0 = 0$, $\phi_0 = 0$, $z_0 = 0$,

$$\mathbf{R}_1(r,\phi,z) = \hat{\mathbf{r}}\left[(r_1\cos\phi_1 - 0)\cos\phi + (r_1\sin\phi_1 - 0)\sin\phi\right]$$
$$+\hat{\boldsymbol{\phi}}\left[-(r_1\cos\phi_1 - 0)\sin\phi + (r_1\sin\phi_1 - 0)\cos\phi\right] + \hat{\mathbf{z}}\,[z_1 - 0]$$

or

$$\mathbf{R}_1 = \hat{\mathbf{r}}\left[\left(\sqrt{3}/2\right)\cos\phi + 0.5\sin\phi\right] + \hat{\boldsymbol{\phi}}\left[-\left(\sqrt{3}/2\right)\sin\phi + 0.5\cos\phi\right] + \hat{\mathbf{z}}\,1$$

Using similar steps, the position vector $\mathbf{R}_2$ is

$$\mathbf{R}_2 = \hat{\mathbf{r}}\left[(2\cos 0 - 0)\cos\phi + (2\sin 0 - 0)\sin\phi\right] + \hat{\boldsymbol{\phi}}\left[-(2\cos 0 - 0)\sin\phi + (2\sin 0 - 0)\cos\phi\right] + \hat{\mathbf{z}}\left[2 - 0\right] = \hat{\mathbf{r}}\,2\cos\phi - \hat{\boldsymbol{\phi}}\,2\sin\phi + \hat{\mathbf{z}}\,2$$

The vector $\mathbf{R}_2 - \mathbf{R}_1$ is

$$\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 = \hat{\mathbf{r}}\left[\frac{4-\sqrt{3}}{2}\cos\phi - \frac{1}{2}\sin\phi\right] + \hat{\boldsymbol{\phi}}\left[\frac{\sqrt{3}-4}{2}\sin\phi - \frac{1}{2}\cos\phi\right] + \hat{\mathbf{z}}\,1$$

Exercise 1.14 Write the position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ in **Example 1.22** in Cartesian coordinates and write the vector pointing from P_1 to P_2.

Answer

$$\mathbf{R}_1 = \hat{\mathbf{x}}\frac{\sqrt{3}}{2} + \hat{\mathbf{y}}\frac{1}{2} + \hat{\mathbf{z}}\,1, \quad \mathbf{R}_2 = \hat{\mathbf{x}}\,2 + \hat{\mathbf{z}}\,2,$$

$$\mathbf{R}_2 - \mathbf{R}_1 = \hat{\mathbf{x}}\left(\frac{4-\sqrt{3}}{2}\right) - \hat{\mathbf{y}}\frac{1}{2} + \hat{\mathbf{z}}\,1$$

Problems

Vectors and Scalars

1.1 Two points $P_1(1,0,1)$ and $P_2(6,-3,0)$ are given. Calculate:

(a) The scalar components of the vector pointing from P_1 to P_2.
(b) The scalar components of the vector pointing from the origin to P_1.
(c) The magnitude of the vector pointing from P_1 to P_2.

1.2 A ship is sailing in a north–east direction at a speed of 50 km/h. The destination of the ship is on a meridian 3,000 km east of the starting point. Note that speed is the absolute value of velocity:

(a) What is the velocity vector of the ship?
(b) How long does it take the ship to reach its destination?
(c) What is the total distance traveled from the starting point to its destination?

Addition and Subtraction of Vectors

1.3 An aircraft flies from London to New York at a speed of 800 km/h. Assume New York is straight west of London at a distance of 5,000 km. Use a Cartesian system of coordinates, centered in London, with New York in the negative x direction. At the altitude the airplane flies, there is a wind, blowing horizontally from north to south (negative y direction) at a speed of 100 km/h:

(a) What must be the direction of flight if the airplane is to arrive in New York?
(b) What is the speed in the London–New York direction?
(c) How long does it take to cover the distance from London to New York?

1.4 Vectors **A** and **B** are given: $\mathbf{A} = \hat{\mathbf{x}}\,5 + \hat{\mathbf{y}}\,3 - \hat{\mathbf{z}}$ and $\mathbf{B} = -\hat{\mathbf{x}}\,3 + \hat{\mathbf{y}}\,5 - \hat{\mathbf{z}}\,2$. Calculate:

(a) $|\mathbf{A}|$.
(b) $\mathbf{A} + \mathbf{B}$.
(c) $\mathbf{A} - \mathbf{B}$.
(d) $\mathbf{B} - \mathbf{A}$.
(e) Unit vector in the direction of $\mathbf{B} - \mathbf{A}$.

Sums and Scaling of Vectors

1.5 Three vectors are given as: $\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 + \hat{\mathbf{z}}3$, $\mathbf{B} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3$ and $\mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$:

(a) Calculate the sums **A** + **B** + **C**, **A** + **B** – **C**, **A** – **B** – **C**, **A** – **B** + **C**, **A** + (**B** – **C**), and (**A** + **B**) – **C** using one of the geometric methods.
(b) Calculate the same sums using direct summation of the vectors.
(c) Comment on the two methods in terms of ease of solution and physical interpretation of results.

1.6 A satellite rotates around the Earth in the equatorial plane at 16,000 km/h moving in the direction of rotation of the planet. To reenter into the atmosphere, the speed is reduced by 1,000 km/h by firing a small rocket in the direction opposite that of the satellite's motion:

(a) What are the velocity vectors of the satellite before and immediately after firing the rocket?
(b) Find the scaling factor of the original velocity vector required to get the satellite to its new speed.

1.7 A particle moves with a velocity $\mathbf{v} = \hat{\mathbf{x}}300 + \hat{\mathbf{y}}50 - \hat{\mathbf{z}}100$. Now the velocity is reduced by a factor of 2:

(a) Calculate the direction of motion of the particle.
(b) What is the speed of the particle?

Scalar and Vector Products

1.8 Calculate the unit vector normal to the plane $3x + 4y + z = 0$.

1.9 Two vectors $\mathbf{v}_1 = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}2$ [m/s] and $\mathbf{v}_2 = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3$ [m/s] describe the velocities of two objects in space:

(a) Calculate the angle between the trajectories of the two objects.
(b) If the ground coincides with the x–y plane, calculate the ground velocities of each object.
(c) What is the angle between the ground velocities?

1.10 Find a unit vector normal to the following planes, at the given point:

(a) $z = -x - y$, at point $P(0,0,0)$.
(b) $4x - 3y + z + 5 = 0$, at point $P(0,0,-5)$.
(c) $z = ax + by$, at point $P(0,0,0)$.

1.11 A force is given as $\mathbf{F} = \hat{\mathbf{x}}/r$. Calculate the vector component of the force **F** in the direction of the vector $\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}$. Note: solution is not unique.

1.12 Calculate the area of a general triangle defined by three points: $P_1(a, b, c)$, $P_2(a', b', c')$, and $P_3(a'', b'', c'')$. From the result here, write a general explicit expression for the area of any triangle if its three vertices are known.

1.13 Show using vector algebra that the law of sines holds in the triangle in **Figure 1.33**, where A, B, and C are the lengths of the corresponding sides; that is, show that the following is correct:

$$\frac{A}{\sin\phi_{BC}} = \frac{B}{\sin\phi_{AC}} = \frac{C}{\sin\phi_{AB}}.$$

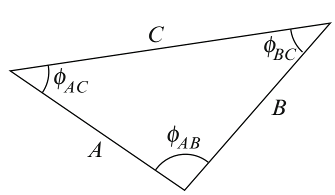


Figure 1.33

1.14 A vector is given as $\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}2$:

(a) Find the angle between **A** and the positive *z* axis.
(b) Find a vector perpendicular to **A** and a unit vector in the direction of the positive *z* axis.

1.15 A force is given as $\mathbf{F} = \hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$. Calculate the magnitude of the force in the direction of the vector $\mathbf{A} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2$.

1.16 Three vertices of a parallelogram are given as $P_1(7,3,1)$, $P_2(2,1,0)$, and $P_3(2,2,5)$:

(a) Find the area of a parallelogram with these vertices.
(b) Is the answer in **(a)** unique: that is, is there only one parallelogram that can be defined by these points? If not, what are the other possible solutions?

1.17 **The equation of a plane.** A plane through the origin of the system of coordinates is defined by two points: $P_1(1,2,-1)$ and $P_2(5,3,2)$. Find the equation of the plane.

Multiple Products

1.18 To define the volume of a parallelepiped, we need to define a corner of the parallelepiped and three vectors emanating from this point (see **Example 1.11**). Four corners of a parallelepiped are known as $P_1(0,0,0)$, $P_2(a,0,1)$, $P_3(a,2,c)$, and $P_4(1,b,1)$:

(a) Show that there are six vectors that can be defined using these nodes, but only three vectors, emanating from a node, are necessary to define a parallelepiped.
(b) Show that there are four possible parallelepipeds that can be defined using these four nodes, depending on which node is taken as the root node.
(c) Calculate the volumes of the four parallelepipeds.

1.19 Which of the following vector products yield zero and why? **A, B,** and **C** are vectors and $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

(a) $\mathbf{A} \times (\mathbf{B} \times (\mathbf{A} \times \mathbf{B}))$
(b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) \times \mathbf{B}$
(c) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{A} \times \mathbf{B})$
(d) $((\mathbf{A} \times \mathbf{B}) \times \mathbf{A}) \times \mathbf{B}$
(e) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$
(f) $(\mathbf{A} \times \mathbf{A}) \times \mathbf{B}$
(g) $\mathbf{A} \times \mathbf{C}$
(h) $(\mathbf{C} \times \mathbf{A}) \times \mathbf{B}$

1.20 Which of the following products are properly defined and which are not? Explain why. (*a* and *b* are scalars, and **A**,**B**, and **C** are vectors.)

(a) $ab\mathbf{A} \times \mathbf{C}$
(b) $\mathbf{A} \times \mathbf{C} \times \mathbf{B}$
(c) $\mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$
(d) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}$
(e) $a\mathbf{B} \cdot \mathbf{C}$
(f) $(a\mathbf{B} \times b\mathbf{A})$

1.21 Which of the following products are meaningful? Explain. **A**, **B**, and **C** are vectors; *c* is a scalar.

(a) $\mathbf{A} \cdot (\mathbf{A} \times (\mathbf{B} \times \mathbf{C}))$
(b) $c\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$
(c) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$
(d) $(\mathbf{A} \cdot \mathbf{B}) \times (\mathbf{A} \times \mathbf{B})$
(e) $(\mathbf{A} \cdot \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}$
(f) $\mathbf{A} \cdot (\mathbf{A} \times (\mathbf{A} \times \mathbf{B}))$
(g) $\mathbf{A} \cdot (\mathbf{A} \times (\mathbf{A} \times \mathbf{A}))$
(h) $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{A})$

1.22 Vectors $\mathbf{A} = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}1 + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}1 + \hat{\mathbf{z}}2$, and $\mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3$ are given. Find the height of the parallelepiped defined by the three vectors:

(a) If **A** and **B** form the base.
(b) If **A** and **C** form the base.
(c) If **B** and **C** form the base.

Definition of Scalar and Vector Fields

1.23 A pressure field is given as $P = x(x-1)(y-2) + 1$:

(a) Sketch the scalar field in the domain $0 < x,y < 1$.
(b) Find the point(s) at which the slope of the field is zero.

1.24 Sketch the scalar fields

$$A = x + y, \quad B = x - y, \quad c = \frac{x+y}{\sqrt{x^2+y^2}}.$$

1.25 Sketch the vector fields

$$\mathbf{A} = \hat{\mathbf{x}}\, y + \hat{\mathbf{y}}\, x, \quad \mathbf{B} = \hat{\mathbf{x}}\, y - \hat{\mathbf{y}}\, x, \quad \mathbf{C} = \frac{\hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y}{\sqrt{x^2+y^2}}.$$

Systems of Coordinates

1.26 Three points are given in Cartesian coordinates: $P_1(1,1,1)$, $P_2(1,1,0)$, and $P_3(0,1,1)$:

(a) Find the points P_1, P_2, and P_3 in cylindrical and spherical coordinates.
(b) Find the equation of a plane through these three points in Cartesian coordinates.
(c) Find the equation of the plane in cylindrical coordinates.
(d) Find the equation of the plane in spherical coordinates.

1.27 Write the equation of a sphere of radius a:

(a) In Cartesian coordinates.
(b) In cylindrical coordinates.
(c) In spherical coordinates.

1.28 A sphere of radius a is given. Choose any point on the sphere:

(a) Describe this point in Cartesian coordinates.
(b) Describe this point in spherical coordinates.

1.29 Transform the vector $\mathbf{A} = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,5 + \hat{\mathbf{z}}\,3$ into spherical coordinates at ($x = -2$, $y = 3$, $z = 1$). That is, find the general transformation of the vector and then substitute the coordinates of the point to obtain the transformation at the specific point.

1.30 Vector $\mathbf{A} = \hat{\mathbf{r}}\,3\cos\phi - \hat{\boldsymbol{\phi}}\,2r^{1/2} + \hat{\mathbf{z}}\,r\phi$ is given:

(a) Transform the vector to Cartesian coordinates.
(b) Find the scalar components of the vector in spherical coordinates.

Position Vectors

1.31 Points $P_1(a,b,c)$ and $P_2(a',b',c')$ are given:

(a) Calculate the position vector $\mathbf{r}_1$ of point P_1.
(b) Calculate the position vector $\mathbf{r}_2$ of P_2.
(c) Calculate the vector $\mathbf{R}$ connecting P_1 (tail) to P_2 (head).
(d) Show that the vector $\mathbf{R}$ can be written as $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$.

1.32 Two points on a sphere of radius 3 are given as $P_1(3,0°,30°)$ and $P_2(3,45°,45°)$:

(a) Find the position vectors of P_1 and P_2.
(b) Find the vector connecting P_1 (tail) to P_2 (head).
(c) Find the position vectors and the vector $\mathbf{P}_1\mathbf{P}_2$ in cylindrical and Cartesian coordinates.

1.33 Given the position vectors $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$, find the equation of the plane they form.