Vector Calculus

2

There cannot be a language more universal and more simple, more free from errors and obscurities, that is to say more worthy to express the invariable relations of natural things [than mathematics] . . ..
. . . Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse, and discovers the hidden analogies that unite them . . .. it follows the same course in the study of all phenomena; it interprets them by the same language, as if to attest the unity and simplicity of the plan of the universe . . ..

—Jean Baptiste Joseph Fourier (1768–1830), mathematician and physicist
Introduction to the Analytic Theory of Heat, 1822 (from the 1955 Dover edition)

2.1 Introduction

Vector calculus deals with the application of calculus operations on vectors. We will often need to evaluate integrals, derivatives, and other operations that use integrals and derivatives. The rules needed for these evaluations constitute vector calculus. In particular, line, volume, and surface integration are important, as are directional derivatives.

The relations defined here are very useful in the context of electromagnetics but, even without reference to electromagnetics, we will show that the definitions given here are simple extensions to familiar concepts and they simplify a number of important aspects of calculation.

We will discuss in particular the ideas of line, surface, and volume integration, and the general ideas of gradient, divergence, and curl, as well as the divergence and Stokes' theorems. These notions are of fundamental importance for the understanding of electromagnetic fields. As with vector algebra, the number of operations and concepts we need is rather small. These are:

Integration	Vector operators	Theorems
Line or contour integral	Gradient	The divergence theorem
Surface integral	Divergence	Stokes' theorem
Volume integral	Curl	
Vector identities		

In addition, we will define the Laplacian and briefly discuss the Helmholtz theorem as a method of generalizing the definition of vector fields. These are the topics we must have as tools before we start the study of electromagnetics.

2.2 Integration of Scalar and Vector Functions

Vector functions often need to be integrated. As an example, if a force is specified, and we wish to calculate the work performed by this force, then an integration along the path of the force is required. The force is a vector and so is the path. However, the integration results in a scalar function (work). In addition, the ideas of surface and volume integrals are required for future use in evaluation of fields. The methods of setting up and evaluating these integrals will be given together

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with examples of their physical meaning. It should be remembered that the integration itself is identical to that performed in calculus. The unique nature of vector integration is in treatment of the integrand and in the physical meaning of quantities involved. Physical meaning is given to justify the definitions and to show how the various integrals will be used later. Simple applications in fluid flow, forces on bodies, and the like will be used for this purpose.

2.2.1 Line Integrals

Before defining the line integral, consider the very simple example of calculating the work performed by a force, as shown in **Figure 2.1a**. The force is assumed to be space dependent and in an arbitrary direction in the plane. To calculate the work performed by this force, it is possible to separate the force into its two components and write

$$W = \int_{x=x_1}^{x=x_2} F(x,y)\cos\alpha dx + \int_{y=y_1}^{y=y_2} F(x,y)\cos\beta dy \tag{2.1}$$

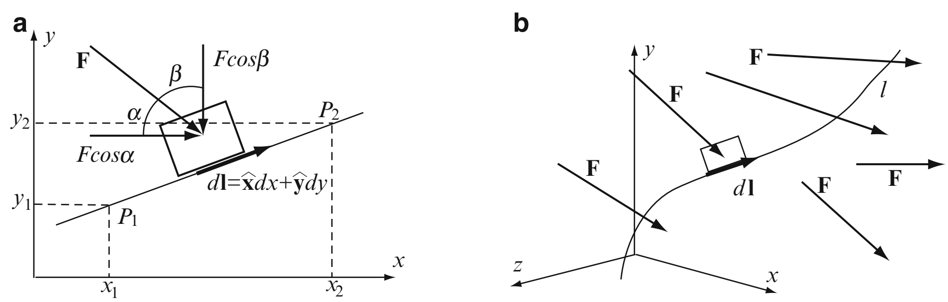


Figure 2.1 (**a**) The concept of a line integral: work performed by a force as a body moves from point P_1 to P_2. (**b**) A generalization of (**a**). Work performed in a *force field* along a general path *l*

An alternative and more general approach is to rewrite the force function in terms of a new parameter, say u, as $F(u)$ and calculate

$$W = \int_{u=u_1}^{u=u_2} F(u)du \tag{2.2}$$

We will return to the latter form, but, first, we note that the two integrands in **Eq. (2.1)** can be written as scalar products:

$$\mathbf{F}\cdot\hat{\mathbf{x}} = F(x,y)\cos\alpha \quad \text{and} \quad \mathbf{F}\cdot\hat{\mathbf{y}} = F(x,y)\cos\beta \tag{2.3}$$

This leads to the following form for the work:

$$W = \int_{x=x_1}^{x=x_2} \mathbf{F}\cdot\hat{\mathbf{x}}dx + \int_{y=y_1}^{y=y_2} \mathbf{F}\cdot\hat{\mathbf{y}}dy \tag{2.4}$$

We can now use the definition of $d\mathbf{l}$ in the $x-y$ plane as $d\mathbf{l} = \hat{\mathbf{x}}\ dx + \hat{\mathbf{y}}\ dy$ and write the work as

$$W = \int_{p_1}^{p_2} \mathbf{F}\cdot d\mathbf{l} \tag{2.5}$$

where $d\mathbf{l}$ is the differential vector in Cartesian coordinates. The path of integration may be arbitrary, as shown in **Figure 2.1b**, whereas the force may be a general force distribution in space (i.e., a force field). Of course, for a general path in space, the third term in $d\mathbf{l}$ must be included $(d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz)$.

To generalize this result even further, consider a vector field **A** as shown in **Figure 2.2a** and an arbitrary path C. The line integral of the vector **A** over the path C is written as

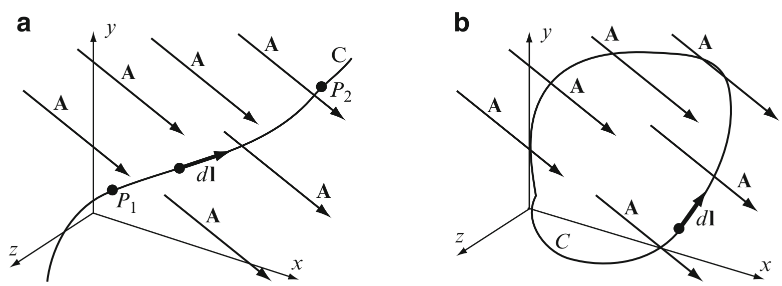
$$\boxed{Q = \int_c \mathbf{A}\cdot d\mathbf{l} = \int_c |\mathbf{A}||d\mathbf{l}|\cos\theta_{\mathbf{A},d\mathbf{l}}} \tag{2.6}$$

In this definition, we only employed the properties of the integral and that of the scalar product. In effect, we evaluate first the projection of the vector **A** onto the path and then proceed to integrate as for any scalar function. If the integration between two points is required, we write

$$\int_{p_1}^{p_2} \mathbf{A}\cdot d\mathbf{l} = \int_{p_1}^{p_2} |\mathbf{A}||d\mathbf{l}|\cos\theta_{\mathbf{A},d\mathbf{l}} \tag{2.7}$$

again, in complete accordance with the standard method of integration. As mentioned in the introduction, once the product under the integral sign is properly evaluated, the integration proceeds as in calculus.

Figure 2.2 The line integral. (**a**) Open contour integration. (**b**) Closed contour integration



Extending the analogy of calculation of work, we can calculate the work required to move an object around a closed contour. In terms of **Figure 2.2b**, this means calculating the closed path integral of the vector **A**. This form of integration is important enough for us to give it a special symbol and name. It will be called a ***closed contour integral*** or a ***loop integral*** and is denoted by a small circle superimposed on the symbol for integration:

$$\oint \mathbf{A}\cdot d\mathbf{l} = \oint |\mathbf{A}||d\mathbf{l}|\cos\theta_{\mathbf{A},d\mathbf{l}} \tag{2.8}$$

The closed contour integral of **A** is also called the ***circulation of* A** around path C. The circulation of a vector around any closed path can be zero or nonzero, depending on the vector. Both types will be important in analysis of fields; therefore, we now define the following:

(1) A vector field whose circulation around any arbitrary closed path is zero is called a ***conservative field*** or a ***restoring field***. In a force field, the line integral represents work. A conservative field in this case means that the total net work done by the field or against the field on any closed path is zero.

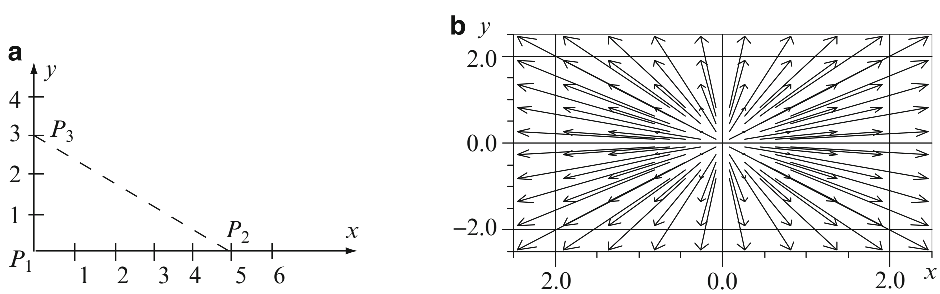
(2) A vector field whose circulation around an arbitrary closed path is nonzero is a ***nonconservative*** or ***nonrestoring field***. In terms of forces, this means that moving in a closed path requires net work to be done either by the field or against the field.

Now, we return to **Eq. (2.2)**. We are free to integrate either using **Eq. (2.4)** or **Eq. (2.5)**, but which should we use? More important, are these two integrals identical? To see this, consider the following three examples.

Example 2.1 Work in a Field A vector field is given as $\mathbf{F} = \hat{\mathbf{x}}\,2x + \hat{\mathbf{y}}\,2y$.

(a) Sketch the field in space.
(b) Assume **F** is a force. What is the work done in moving from point $P_2(5,0)$ to $P_3(0,3)$ (in **Figure 2.3a**)?
(c) Does the work depend on the path taken between P_2 and P_3?

Figure 2.3



Solution: **(b)** First, we calculate the line integral of $\mathbf{F} \cdot d\mathbf{l}$ along the path between P_2 and P_3. This is a direct path. **(c)** Then, we calculate the same integral from P_2 to P_1 and from P_1 to P_3. If the two results are the same, the closed contour integral is zero.

(a) See **Figure 2.3b**. Note that the field is zero at the origin. At any point x, y, the vector has components in the x and y directions. The magnitude depends on the location of the field (thus, the different vector lengths at different locations).

(b) From P_2 to P_3, the element of path is $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$. The integration is therefore

$$\int_{P_2}^{P_3} \mathbf{F} \cdot d\mathbf{l} = \int_{P2}^{P3} (\hat{\mathbf{x}}\, 2x + \hat{\mathbf{y}}\, 2y) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy) = \int_{P_2}^{P_3} (2xdx + 2ydy) \qquad [\mathrm{J}]$$

Since each part of the integrand is a function of a single variable, x or y, we can separate the integration into integration over each variable and write

$$\int_{P_2}^{P_3} \mathbf{F} \cdot d\mathbf{l} = \int_{x=5}^{x=0} 2xdx + \int_{y=0}^{y=3} 2ydy = x^2\Big|_5^0 + y^2\Big|_0^3 = -25 + 9 = -16 \qquad [\mathrm{J}]$$

Note: This work is negative. It decreases the potential energy of the system; that is, this work is done by the field (as, for example, in sliding on a water slide, the gravitational field performs the work and the potential energy of the slider is reduced).

(c) On paths P_2 to P_1 and P_1 to P_3, we perform separate integrations. On path P_2 to P_1, $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}\,0$ and $y = 0$. The integration is

$$\int_{P_2}^{P_1} \mathbf{F} \cdot d\mathbf{l} = \int_{P_2}^{P_1} \left(\hat{\mathbf{x}}\, 2x + \hat{\mathbf{y}}\, 2y\right) \cdot (\hat{\mathbf{x}}dx) = \int_{P_2}^{P_1} 2xdx = \int_{x=5}^{x=0} 2xdx = x^2\Big|_5^0 = -25 \qquad [\mathrm{J}]$$

Similarly, on path P_1 to P_3, $d\mathbf{l} = \hat{\mathbf{x}}\,0 + \hat{\mathbf{y}}dy$ and $x = 0$. The integration is

$$\int_{P_1}^{P_3} \mathbf{F} \cdot d\mathbf{l} = \int_{P_1}^{P_3} (\hat{\mathbf{x}}\, 2x + \hat{\mathbf{y}}\, 2y) \cdot (\hat{\mathbf{y}}dy) = \int_{P_1}^{P_3} 2ydy = \int_{y=0}^{y=3} 2ydy = y^2\Big|_0^3 = 9 \qquad [\mathrm{J}]$$

The sum of the two paths is equal to the result obtained for the direct path. This also means that the closed contour integral will yield zero. However, the fact that the closed contour integral on a particular path is zero does not necessarily mean the given field is conservative. In other words, we cannot say that this particular field is conservative unless we can show that the closed contour integral is zero for any contour. We will discuss this important aspect of fields later in this chapter.

Example 2.2 Circulation of a Vector Field Consider a vector field $\mathbf{A} = \hat{\mathbf{x}}\, xy + \hat{\mathbf{y}}\,(3x^2 + y)$. Calculate the circulation of $\mathbf{A}$ around the circle $x^2 + y^2 = 1$.

Solution: First, we must calculate the differential of path, $d\mathbf{l}$, and then evaluate $\mathbf{A} \cdot d\mathbf{l}$. This is then integrated along the circle (closed contour) to obtain the result. This problem is most easily evaluated in cylindrical coordinates (see **Exercise 2.1**), but we will solve it in Cartesian coordinates. The integration is performed in four segments: P_1 to P_2, P_2 to P_3, P_3 to P_4, and P_4 to P_1, as shown in **Figure 2.4**.

The differential of length in the $x-y$ plane is $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$. The scalar product $\mathrm{A} \cdot d\mathbf{l}$ is

$$\mathbf{A} \cdot d\mathbf{l} = \left(\hat{\mathbf{x}}\, xy + \hat{\mathbf{y}}\left(3x^2 + y\right)\right) \cdot (\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}dy) = xydx + \left(3x^2 + y\right)dy$$

The circulation is now

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \oint_L \left[xydx + \left(3x^2 + y\right)dy\right]$$

Before this can be evaluated, we must make sure that integration is over a single variable. To do so, we use the equation of the circle and write

$$x = \left(1 - y^2\right)^{1/2}, \qquad y = \left(1 - x^2\right)^{1/2}$$

By substituting the first relation into the second term and the second into the first term under the integral, we have

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \oint_L \left[x\left(1-x^2\right)^{1/2} dx + \left(3\left(1-y^2\right)+y\right) dy \right]$$

and each part of the integral is a function of a single variable. Now, we can separate these into four integrals:

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{l} + \int_{P_2}^{P_3} \mathbf{A} \cdot d\mathbf{l} + \int_{P_3}^{P_4} \mathbf{A} \cdot d\mathbf{l} + \int_{P_4}^{P_1} \mathbf{A} \cdot d\mathbf{l}$$

Evaluating each integral separately,

$$\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{l} = \int_{P_1}^{P_2} \left(x\left(1-x^2\right)^{1/2} dx + \left(3-3y^2+y\right) dy \right)$$

$$= \int_{x=1}^{x=0} x\left(1-x^2\right)^{1/2} dx + \int_{y=0}^{y=1} \left(3-3y^2+y\right) dy = -\frac{\left(1-x^2\right)^{3/2}}{3}\Bigg|_1^0 + \left(3y+\frac{y^2}{2}-y^3\right)\Bigg|_0^1 = \frac{13}{6}$$

Note that the other integrals are similar except for the limits of integration:

$$\int_{P_2}^{P_3} \mathbf{A} \cdot d\mathbf{l} = \int_{x=0}^{x=-1} x\left(1-x^2\right)^{1/2} dx + \int_{y=1}^{y=0} \left(3-3y^2+y\right) dy = -\frac{\left(1-x^2\right)^{3/2}}{3}\Bigg|_0^{-1} + \left(3y+\frac{y^2}{2}-y^3\right)\Bigg|_1^0 = -\frac{13}{6}$$

$$\int_{P_3}^{P_4} \mathbf{A} \cdot d\mathbf{l} = \int_{x=-0}^{x=0} x\left(1-x^2\right)^{1/2} dx + \int_{y=0}^{y=-1} \left(3-3y^2+y\right) dy = -\frac{\left(1-x^2\right)^{3/2}}{3}\Bigg|_{-1}^0 + \left(3y+\frac{y^2}{2}-y^3\right)\Bigg|_0^{-1} = -\frac{11}{6}$$

$$\int_{P_4}^{P_1} \mathbf{A} \cdot d\mathbf{l} = \int_{x=0}^{x=1} x\left(1-x^2\right)^{1/2} dx + \int_{y=-1}^{y=0} \left(3-3y^2+y\right) dy = -\frac{\left(1-x^2\right)^{3/2}}{3}\Bigg|_0^1 + \left(3y+\frac{y^2}{2}-y^3\right)\Bigg|_{-1}^0 = \frac{11}{6}$$

The total circulation is the sum of the four circulations above. This gives

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = 0$$

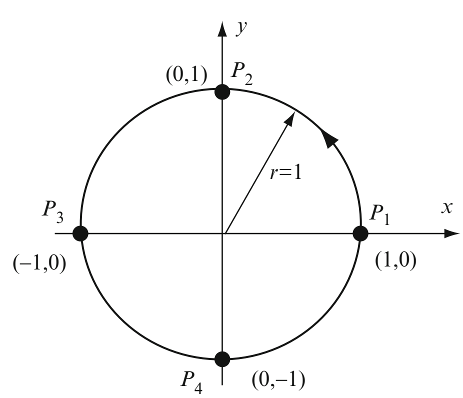


Figure 2.4 The four segments of the contour used for integration in **Example 2.2**

Exercise 2.1 Solve **Example 2.2** in cylindrical coordinates; that is, transform the vector **A** and the necessary coordinates and evaluate the integral.

Example 2.3 Line Integral: Nonconservative Field The force $\mathbf{F} = \hat{\mathbf{x}}\,(2x - y) + \hat{\mathbf{y}}\,(x + y + z) + \hat{\mathbf{z}}\,(2z - x)$ [N] is given. Calculate the total work required to move a body in a circle of radius 1 m, centered at the origin. The circle is in the x–y plane at $z = 0$.

Solution: To find the work, we first convert to the cylindrical system of coordinates. Also, since the circle is in the x–y plane ($z = 0$), we have

$$\mathbf{F}\big|_{z=0} = \hat{\mathbf{x}}\,(2x - y) + \hat{\mathbf{y}}\,(x + y) - \hat{\mathbf{z}}\,x \quad \text{and} \quad x^2 + y^2 = 1$$

Since integration is in the x–y plane, the closed contour integral is

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \oint_L (\hat{\mathbf{x}}\,(2x - y) + \hat{\mathbf{y}}\,(x + y) - \hat{\mathbf{z}}\,x) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy) = \oint_L \big((2x - y)dx + (x + y)dy$$

Conversion to cylindrical coordinates gives

$$x = r\cos\phi = 1\cos\phi, \qquad y = \sin\phi$$

Therefore,

$$\frac{dx}{d\phi} = -\sin\phi \quad \rightarrow \quad dx = -\sin\phi d\phi, \quad \text{and} \quad \frac{dy}{d\phi} = \cos\phi \quad \rightarrow \quad dy = \cos\phi d\phi$$

Substituting for x, y, dx, and dy, we get

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \int_{\phi=0}^{\phi=2\pi} \big((2\cos\phi - \sin\phi)\big(-\sin\phi d\phi\big) + \big(\cos\phi + \sin\phi\big)\cos\phi d\phi\big) = \int_{\phi=0}^{\phi=2\pi} (1 - \sin\phi\cos\phi)d\phi = 2\pi$$

This result means that integration between zero and π and between zero and $-\pi$ gives different results. The closed contour line integral is not zero and the field is clearly nonconservative.

The function in **Example 2.1** yielded identical results using two different paths, whereas the result in **Example 2.3** yielded different results. This means that, in general, we are not free to choose the path of integration as we wish. However, if the line integral is independent of path, then the closed contour integral is zero, and we are free to choose the path any way we wish.

2.2.2 Surface Integrals

To define the surface integral, we use a simple example of water flow. Consider first water flowing through a hose of cross section s_1 as shown in **Figure 2.5a**. If the fluid has a constant mass density ρ [kg/m^3] and flows at a fixed velocity **v**, the rate of flow of the fluid (mass per unit time) is

$$w_1 = \rho s_1 v \qquad \left[\frac{\text{kg}}{\text{s}}\right] \tag{2.9}$$

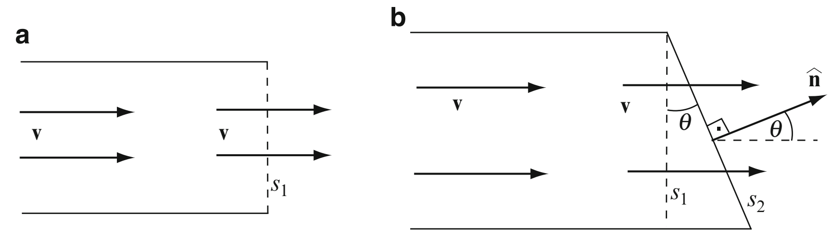


Figure 2.5 Flow through a surface. (**a**) Flow normal to surface s_1. (**b**) Flow at an angle θ to surface s_2 and the relation between the velocity vector and the normal to the surface

Now, assume that we take the same hose, but cut it at an angle as shown in **Figure 2.5b**. The cross-sectional area s_2 is larger, but the total rate of flow remains unchanged. The reason for this is that only the normal projection of the area is crossed by the fluid. In terms of area s_2, we can write

$$w_1 = \rho v s_2 \cos\theta \quad \left[\frac{\text{kg}}{\text{s}}\right] \tag{2.10}$$

Instead of using the scalar values as in **Eqs.** (**2.9**) and (**2.10**), we can use the vector nature of the velocity. Using **Figure 2.5b**, we replace the term $v s_2 \cos\theta$ by $\mathbf{v} \cdot \hat{\mathbf{n}}\, s_2$ and write

$$w_1 = \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, s_2 \quad \left[\frac{\text{kg}}{\text{s}}\right] \tag{2.11}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to surface s_2.

Now, consider **Figure 2.6** where we assumed that a hose allows water to flow with a velocity profile as shown. This is possible if the fluid is viscous. We will assume that the velocity across each small area Δs_i is constant and write the total rate of flow as

$$w_1 = \sum_{i=1}^{n} \rho \mathbf{v}_i \cdot \hat{\mathbf{n}}\, \Delta s_i \quad \left[\frac{\text{kg}}{\text{s}}\right] \tag{2.12}$$

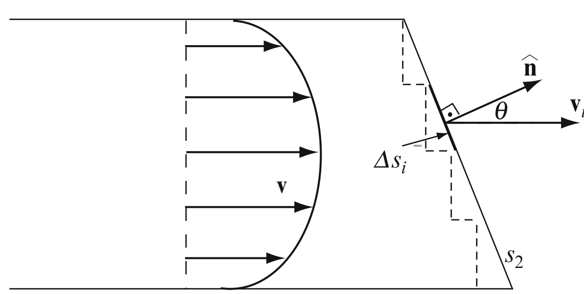


Figure 2.6 Flow with a nonuniform velocity profile

In the limit, as Δs_i tends to zero,

$$w_1 = \lim_{\Delta s \to 0} \sum_{i=1}^{\infty} \rho \mathbf{v}_i \cdot \hat{\mathbf{n}}\, \Delta s_i = \int_{s_2} \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, ds_2 \quad \left[\frac{\text{kg}}{\text{s}}\right] \tag{2.13}$$

Thus, we obtained an expression for the rate of flow for a variable velocity fluid through an arbitrary surface, provided that the velocity profile is known, and the normal to the surface can be evaluated everywhere. For purposes of this section, we now rewrite this integral in general terms by replacing $\rho\mathbf{v}$ by a general vector $\mathbf{A}$. This is the field. The rate of flow of the vector field $\mathbf{A}$ (if, indeed, the vector field $\mathbf{A}$ represents a flow) can now be written as

$$Q = \int_s \mathbf{A} \cdot \hat{\mathbf{n}}\, ds \tag{2.14}$$

This is a surface integral and, like the line integral, it results in a scalar value. However, the surface integral represents a flow-like function. In the context of electromagnetics, we call this a **flux** (fluxus = to flow in Latin). Thus, the surface integral of a vector is the flux of this vector through the surface. The surface integral is also written as

$$\boxed{Q = \int_s \mathbf{A} \cdot d\mathbf{s}} \tag{2.15}$$

where $d\mathbf{s} = \hat{\mathbf{n}}\,ds$. The latter is a convenient short-form notation that avoids repeated writing of the normal unit vector, but it should be remembered that the normal unit vector indicates the direction of positive flow. For this reason, it is important that the positive direction of $\hat{\mathbf{n}}$ is always clearly indicated. This is done as follows (see also **Section 1.5.1** and **Figure 1.24**):

(1) For a closed surface, the positive direction of the unit vector is always that direction that points out of the volume (see, for example, **Figures 2.6** and **1.24a**).

(2) For open surfaces, the defining property is the contour enclosing the surface. To define a positive direction, imagine that we travel along this contour as, for example, if we were to evaluate a line integral. Consider the example in **Figure 2.7**. In this case, the direction of travel is counterclockwise along the rim of the surface. According to the right-hand rule, if the fingers are directed in the direction of travel with the palm facing the interior of the surface, the thumb points in the direction of the positive unit vector (see also **Figure 1.24c**). This simple definition removes the ambiguity in the direction of the unit vector and, as we shall see shortly, is consistent with other properties of fields.

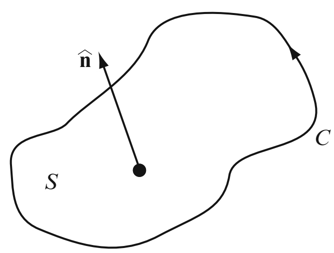


Figure 2.7 Definition of the normal to an open surface

The integration in **Eq. (2.15)** indicates the flux through a surface s. If this surface is a closed surface, we designate the integration as a ***closed surface integration***:

$$\boxed{Q = \oint_s \mathbf{A} \cdot d\mathbf{s}} \tag{2.16}$$

This is similar to the definition of closed integration over a contour. Closed surface integration gives the total or net flux through a closed surface.

Finally, we mention that since $d\mathbf{s}$ is the product of two variables, the surface integral is a double integral. The notation used in **Eq. (2.15)** or **(2.16)** is a short-form notation of this fact.

Example 2.4 Closed Surface Integral Vector $\mathbf{A} = \hat{\mathbf{x}}\,2xz + \hat{\mathbf{y}}\,2zx - \hat{\mathbf{z}}\,yz$ is given. Calculate the closed surface integral of the vector over the surface defined by a cube. The cube occupies the space between $0 \le x, y, z \le 1$.

Solution: First, we find the unit vector normal to each of the six sides of the cube. Then, we calculate the scalar product $\mathbf{A} \cdot \hat{\mathbf{n}}\,ds$, where ds is the element of surface on each side of the cube. Integrating on each side and summing up the contributions gives the net flux of $\mathbf{A}$ through the closed surface enclosing the cube.

Using **Figure 2.8**, the differentials of surface $d\mathbf{s}$ are

$$d\mathbf{s}_1 = \hat{\mathbf{x}}dydz, \quad d\mathbf{s}_2 = -\hat{\mathbf{x}}dydz$$
$$d\mathbf{s}_3 = \hat{\mathbf{z}}dxdy, \quad d\mathbf{s}_4 = -\hat{\mathbf{z}}dxdy$$
$$d\mathbf{s}_5 = \hat{\mathbf{y}}dxdz, \quad d\mathbf{s}_6 = -\hat{\mathbf{y}}dxdz$$

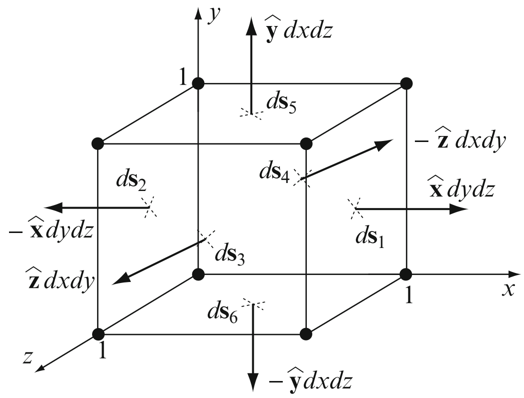


Figure 2.8 Notation used for closed surface integration in **Example 2.4**

The surface integral is now written as

$$\oint_s \mathbf{A}\cdot d\mathbf{s} = \int_{s_1}\mathbf{A}\cdot d\mathbf{s}_1 + \int_{s_2}\mathbf{A}\cdot d\mathbf{s}_2 + \int_{s_3}\mathbf{A}\cdot d\mathbf{s}_3 + \int_{s_4}\mathbf{A}\cdot d\mathbf{s}_4 + \int_{s_5}\mathbf{A}\cdot d\mathbf{s}_5 + \int_{s_6}\mathbf{A}\cdot d\mathbf{s}_6$$

Each term is evaluated separately. On side 1,

$$\int_{s_1}\mathbf{A}\cdot d\mathbf{s}_1 = \int_{s_1}(\hat{\mathbf{x}}\,2xz + \hat{\mathbf{y}}\,2zx - \hat{\mathbf{z}}\,yz)\cdot(\hat{\mathbf{x}}dydz) = \int_{s_1} 2xzdydz$$

To perform the integration, we set $x = 1$. Separating the surface integral into an integral over y and one over z, we get

$$\int_{s_1}\mathbf{A}\cdot d\mathbf{s}_1 = \int_{y=0}^{y=1}\left[\int_{z=0}^{z=1} 2z\,dz\right]dy = 2\int_{y=0}^{y=1}\left[\frac{z^2}{2}\bigg|_{z=0}^{z=1}\right]dy = \int_{y=0}^{y=1} dy = y\bigg|_{y=0}^{y=1} = 1$$

On side 2, the situation is identical, but $x = 0$ and $d\mathbf{s}_2 = -d\mathbf{s}_1$. Thus,

$$\int_{s_2}\mathbf{A}\cdot d\mathbf{s}_2 = -\int_{s_2} 2xzdydz = 0$$

On side 3, $z = 1$ and the integral is

$$\begin{aligned}\int_{s_3}\mathbf{A}\cdot d\mathbf{s}_3 &= \int_{s_3}(\hat{\mathbf{x}}\,2xz + \hat{\mathbf{y}}\,2zx - \hat{\mathbf{z}}\,yz)\cdot(\hat{\mathbf{z}}dxdy)\\ &= -\int_{s_3} yzdxdy = -\int_{x=0}^{x=1}\left[\int_{y=0}^{y=1} y\,dy\right]dx = -\int_{x=0}^{x=1}\left[\frac{y^2}{2}\bigg|_{y=0}^{y=1}\right]dx = -\int_{x=0}^{x=1}\frac{dx}{2} = -\frac{x}{2}\bigg|_{x=0}^{x=1} = -\frac{1}{2}\end{aligned}$$

On side 4, $z = 0$ and $d\mathbf{s}_4 = -d\mathbf{s}_3$. Therefore, the contribution of this side is zero:

$$\int_{s_4}\mathbf{A}\cdot d\mathbf{s}_4 = \int_{s_4} yzdxdy = 0$$

On side 5, $y = 1$:

$$\int_{s_5}\mathbf{A}\cdot d\mathbf{s}_5 = \int_{s_5}(\hat{\mathbf{x}}\,2xz + \hat{\mathbf{y}}\,2zx - \hat{\mathbf{z}}\,yz)\cdot(\hat{\mathbf{y}}dxdz) = \int_{x=0}^{1}\int_{z=0}^{1} 2zxdxdz = \frac{1}{2}$$

On side 6, $y = 0$:

$$\int_{s_6} \mathbf{A} \cdot d\mathbf{s}_6 = \int_{s_6} (\hat{\mathbf{x}}\, 2xz + \hat{\mathbf{y}}\, 2zx - \hat{\mathbf{z}}\, yz) \cdot (-\hat{\mathbf{y}} dxdz) = -\int_{x=0}^{1} \int_{z=0}^{1} 2zxdxdz = -\frac{1}{2}$$

The result is the sum of all six contributions:

$$\oint_s \mathbf{A} \cdot d\mathbf{s} = 1 + 0 - \frac{1}{2} + 0 + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$$

Example 2.5 Open Surface Integral A vector is given as $\mathbf{A} = \hat{\boldsymbol{\phi}}\, 5r$. Calculate the flux of the vector $\mathbf{A}$ through a surface defined by $0 < r < 1$ and $-3 < z < 3$, $\phi =$ constant. Assume the vector produces a positive flux through this surface.

Solution: The flux is the surface integral

$$\Phi = \int_s \mathbf{A} \cdot d\mathbf{s}$$

The surface s is in the $r-z$ plane and is therefore perpendicular to the ϕ direction, as shown in **Figure 2.9a**. Thus, the element of surface is: $d\mathbf{s}_1 = \hat{\boldsymbol{\phi}}\, drdz$ or $d\mathbf{s}_2 = -\hat{\boldsymbol{\phi}}\, drdz$. In this case, because the flux must be positive, we choose $d\mathbf{s}_1 = \hat{\boldsymbol{\phi}}\, drdz$. The flux is

$$\Phi = \int_{s_1} \left(\hat{\boldsymbol{\phi}}\, 5r\right) \cdot \left(\hat{\boldsymbol{\phi}} drdz\right) = \int_{s_1} 5rdrdz = \int_{r=0}^{r=1} \left[\int_{z=-3}^{z=+3} 5rdz\right] dr = \int_{r=0}^{r=1} 5rz\Big|_{z=-3}^{z=+3} dr = \int_{r=0}^{r=1} 30rdr = 15r^2\Big|_{r=0}^{r=1} = 15$$

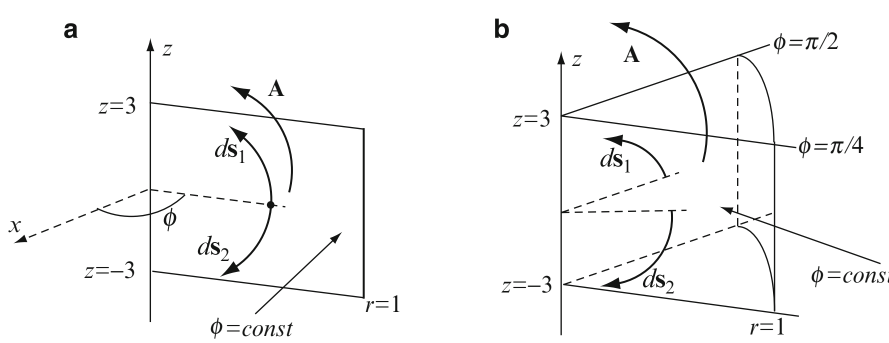


Figure 2.9 (**a**) The surface $0 \leq r \leq 1, -3 \leq z \leq 3$, $\phi =$ constant. (**b**) A wedge in cylindrical coordinates. Note that $d\mathbf{s}_1$ is in the positive ϕ direction, whereas $d\mathbf{s}_2$ is in the negative ϕ direction

Exercise 2.2 Closed Surface Integral Calculate the closed surface integral of $\mathbf{A} = \hat{\boldsymbol{\phi}}\, 5r$ over the surface of the wedge shown in **Figure 2.9b**.

Answer 0.

2.2.3 Volume Integrals

There are two types of volume integrals we may be required to evaluate. The first is of the form

$$W = \int_v wdv \tag{2.17}$$

where w is some volume density function and dv is an element of volume. For example, if w represents the volume density distribution of stored energy (i.e., energy density), then W represents the total energy stored in volume v. Thus, the volume integral has very distinct physical meaning and will often be used in this sense. We also note that for an element of volume, such as the element in **Figure 2.10**, $dv = dxdydz$ and the volume integral is actually a triple integral (over the x, y, and z variables). The volume integral as given above is a scalar.

The second type of volume integral is a vector and is written as

$$\mathbf{P} = \int_v \mathbf{p} dv \tag{2.18}$$

This is similar to the integral in **Eq. (2.17)**, but in terms of its evaluation, it is evaluated over each component independently. The only difference between this and the scalar integral in **Eq. (2.17)** is that the unit vectors may not be constant and, therefore, they may have to be resolved into Cartesian coordinates in which the unit vectors are constant and therefore may be taken outside the integral sign (see **Sections 1.5.2**, **1.5.3**, and **Example 2.7**). In Cartesian coordinates, we may write

$$\mathbf{P} = \hat{\mathbf{x}} \int_v p_x dv + \hat{\mathbf{y}} \int_v p_y dv + \hat{\mathbf{z}} \int_v p_z dv \tag{2.19}$$

This type of vector integral is often called a regular or ordinary vector integral because it is essentially a scalar integral with the unit vectors added. It occurs in other types of calculations that do not involve volumes and volume distributions, such as in evaluating velocity from acceleration (see **Problems 2.11** and **2.12**).

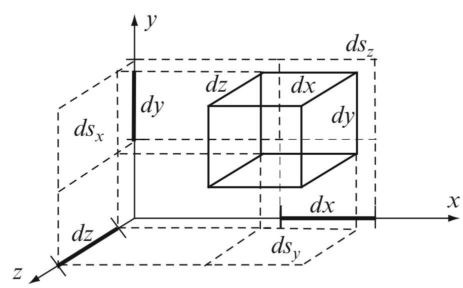


Figure 2.10 An element of volume and the corresponding projections on the axes and planes

Example 2.6 Scalar Volume Integral

(a) Calculate the volume of a section of the sphere $x^2 + y^2 + z^2 = 16$ cut by the planes $y = 0$, $z = 2$, $x = 1$, and $x = -1$.

(b) Calculate the volume of the section of the sphere cut by the planes $\theta = \pi/6$, $\theta = \pi/3$, $\phi = 0$, and $\phi = \pi/3$.

Solution: **(a)** Although, in general, the fact that integration is on a sphere may suggest the use of spherical coordinates; in this case, it is easier to evaluate the integral in Cartesian coordinates because the sphere is cut by planes parallel to the axes. The limits of integration must first be evaluated. **Figure 2.11** is used for this purpose. **(b)** Because the defining planes now are parallel to the axes in spherical coordinates, the solution is easier in spherical coordinates.

(a) The limits of integration are as follows:

(1) From the equation of the sphere, $z = \sqrt{16 - x^2 - y^2}$. From **Figure 2.11a**, the limits of integration on z are $z_1 = -\sqrt{16 - x^2 - y^2}$ and $z_2 = 2$.

(2) The limits on y are $y_1 = 0$ and $y_2 = \sqrt{16 - x^2}$ (see **Figure 2.11b**).

(3) The limits of integration on x are between $x_1 = -1$ and $x_2 = +1$ (see **Figure 2.11c**).

With the differential of volume in Cartesian coordinates, $dv = dx\,dy\,dz$, we get

$$v = \int_v dv = \int_{x=-1}^{x=1}\left\{\int_{y=0}^{y=\sqrt{16-x^2}}\left[\int_{z=-\sqrt{16-x^2-y^2}}^{z=2} dz\right]dy\right\}dx = \int_{x=-1}^{x=1}\left\{\int_{y=0}^{y=\sqrt{16-x^2}}\left[2+\sqrt{16-x^2-y^2}\right]dy\right\}dx$$

$$= \int_{x=-1}^{x=1}\left\{2y + 0.5\left[y\sqrt{16-x^2-y^2} + \left(16-x^2\right)\sin^{-1}\left(\frac{y}{\sqrt{16-x^2}}\right)\right]\right\}_{y=0}^{y=\sqrt{16-x^2}} dx$$

$$= \int_{x=-1}^{x=1}\left\{2\sqrt{16-x^2} + 0.5\left[\left(16-x^2\right)\sin^{-1}(1)\right]\right\}dx = \int_{x=-1}^{x=1} 2\sqrt{16-x^2}dx + \frac{\pi}{4}\int_{x=-1}^{x=1}\left(16-x^2\right)dx$$

$$= \left[x\sqrt{16-x^2} + 16\sin^{-1}\left(\frac{x}{4}\right)\right]_{x=-1}^{x=1} + \frac{\pi}{4}\left[16x - \frac{x^3}{3}\right]_{x=-1}^{x=1} = \sqrt{15} + \sqrt{15} + 16\sin^{-1}\left(\frac{1}{4}\right) - 16\sin^{-1}\left(-\frac{1}{4}\right) + 8\pi - \frac{\pi}{6}$$

$$= 2\sqrt{15} + 32\sin^{-1}(0.25) + \pi\left(8 - 1/6\right) = 40.44$$

Thus,

$$v = 40.44 \quad \left[\text{m}^3\right]$$

(b) The limits of integration are $0 \le R \le 4$, $\pi/6 \le \theta \le \pi/3$, and $0 \le \phi \le \pi/3$. The element of volume in spherical coordinates is $dv = R^2\sin\theta dR d\theta d\phi$. The volume of the section is therefore,

$$v = \int_v dv = \int_{R=0}^{R=4}\left\{\int_{\theta=\pi/6}^{\theta=\pi/3}\left[\int_{\phi=0}^{\phi=\pi/3} R^2\sin\theta d\phi\right]d\theta\right\}dR = \int_{R=0}^{R=4}\left\{\int_{\theta=\pi/6}^{\theta=\pi/3}\frac{\pi}{3}R^2\sin\theta d\theta\right\}dR$$

$$= \int_{R=0}^{R=4}\left[-\frac{\pi}{3}R^2\cos\theta\right]_{\pi/6}^{\pi/3} dR = \int_{R=0}^{R=4}\frac{0.366\pi}{3}R^2 dR = \left[\frac{0.366\pi}{9}R^3\right]_0^4 = \frac{64\times 0.366\pi}{9} = 8.177 \quad \left[\text{m}^3\right]$$

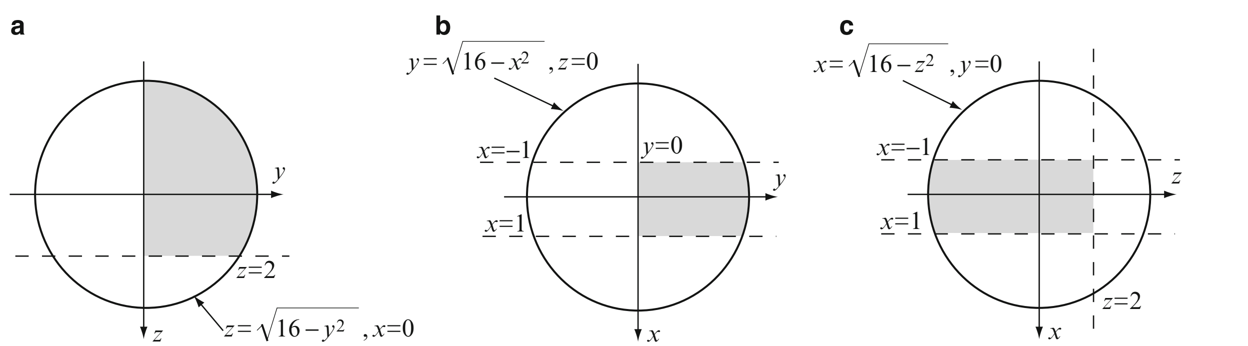


Figure 2.11 Sequence for evaluation of the volume integral in **Example 2.6**. Projections on the y–z, x–y, and x–z planes

Example 2.7 Volume Integration of a Vector Function A vector function $\mathbf{A} = \hat{\mathbf{r}}r + \hat{\mathbf{z}}3$ gives the distribution of a vector in space. This function may represent a distribution of moments or force density in volume v. Calculate the total contribution of the function in a volume defined by a cylinder of radius a and height b, centered on the z axis, above the x–y plane.

Solution: We use cylindrical coordinates to write the integral of **A** over the volume v. The vector function is integrated as follows:

$$\mathbf{F} = \int_v \mathbf{A}dv = \int_v \hat{\mathbf{r}}A_r dv + \int_v \hat{\mathbf{z}}A_z dv = \int_v \hat{\mathbf{r}}\,r dv + \hat{\mathbf{z}}\int_v 3dv$$

where the unit vector $\hat{\mathbf{z}}$ was taken outside the integral sign (it is constant) but the unit vector $\hat{\mathbf{r}}$ cannot be taken out of the integral since it depends on ϕ. From **Eq. (1.65)**, we write $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ and, now, since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant unit vectors in Cartesian coordinates, we write (together with $dv = rdrd\phi dz$)

$$\mathbf{F} = \int_v (\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi)rdv + \hat{\mathbf{z}}\int_v 3dv = \hat{\mathbf{x}}\int_v r\cos\phi dv + \hat{\mathbf{y}}\int_v r\sin\phi dv + \hat{\mathbf{z}}\int_v 3dv$$

$$= \hat{\mathbf{x}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\left(\int_{r=0}^{r=a} r^2\cos\phi dr\right)d\phi\right]dz + \hat{\mathbf{y}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\left(\int_{r=0}^{r=a} r^2\sin\phi dr\right)d\phi\right]dz + \hat{\mathbf{z}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\left(\int_{r=0}^{r=a} 3rdr\right)d\phi\right]dz$$

$$= \hat{\mathbf{x}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\frac{a^3\cos\phi}{3}d\phi\right]dz + \hat{\mathbf{y}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\frac{a^3\sin\phi}{3}d\phi\right]dz + \hat{\mathbf{z}}\int_{z=0}^{z=b}\left[\int_{\phi=0}^{\phi=2\pi}\frac{3a^2}{2}d\phi\right]dz = \hat{\mathbf{z}}\int_{z=0}^{z=b}3\pi a^2 dz = \hat{\mathbf{z}}\,3\pi a^2 b$$

In this integration, we used the fact that $\int_0^{2\pi}\sin\phi = 0$ and $\int_0^{2\pi}\cos\phi = 0$. In summary,

$$\mathbf{F} = \hat{\mathbf{z}}\,3\pi a^2 b$$

2.3 Differentiation of Scalar and Vector Functions

As we might expect, in addition to the need to integrate scalar and vector expressions as described above, we also need to differentiate scalar and vector functions. The rules and implications of these operations are considered next. Three types of operations are defined: the ***gradient***, the ***divergence***, and the ***curl***. The first relates to scalar functions and the second and third to vector functions. These operations will be shown to be fundamental to understanding of vector fields.

2.3.1 The Gradient of a Scalar Function

Point_Charges.m

The partial spatial derivatives of a scalar function $U(x,y,z)$ with nonzero first-order partial derivatives with respect to the coordinates x, y, and z are defined at a point in space as

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}, \quad \frac{\partial U}{\partial z} \tag{2.20}$$

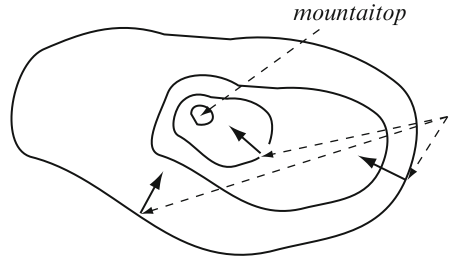
Ordinary derivatives are defined in a similar manner if the function U is a function of a single variable. Obviously, the same can be done in any system of coordinates or the above can be transformed into any system of coordinates using the formulas we obtained in the previous chapter. For this reason, we start our discussion in Cartesian coordinates.

That the derivative of a scalar function describes the slope of the function is known. Also, there is no question that this is an important aspect of the function. Now, imagine that we are standing on a mountain. The slope of the mountain at any given point is not defined, unless we qualify it with something like "slope in the northeast direction" or "slope in the direction of town" or a similar statement. Also, it is decidedly different if we describe the slope up the mountain or down the mountain. If you are designing a ski path, the slope down the mountain is most important. If you were a civil engineer, designing a road, then you might be interested in the path with minimum variation in slope. Thus, an additional aspect of the derivative has entered our considerations, and this must be satisfied. For this reason, we will define a "directional derivative" which, being a vector, gives the slope (as any other derivative) but also the direction of this slope.

In particular, at any given point on a function (say, the mountain described above), there is one direction which is unique; that is, the direction of maximum slope. This direction and the slope associated with it are extremely important, and not only in electromagnetics. However, before we continue, we immediately realize that at any point, there are actually two directions which satisfy this condition. In the example of the mountain, at any point, we might go up the mountain or down the mountain. For example, flow of water at any point is in the direction of maximum slope, but it only flows down the mountain or in the direction of decrease in potential energy. On a topological map, the maximum slope is indicated by the minimum distance between two altitude lines (see **Figure 2.12**). These properties are defined by the gradient, as follows:

"*the vector which gives both the magnitude and direction of the maximum spatial rate of change of a scalar function is called the gradient of this scalar function*."

Figure 2.12 Illustration of the gradient



The rate of change is assumed to be positive in the direction of the increase in the value of the scalar function (up the mountain). Thus, returning to our example, water always flows in the direction opposite the direction of the gradient, whereas the most difficult climb on the mountain at any point is in the direction of the gradient. **Figure 2.12** shows these considerations. The gradient on the map may indicate the direction of climbing or, if this map shows atmospheric pressure, the gradient points in the direction of increased pressure. If you were to sail in the direction of the gradient in air pressure, you will always have the wind in your face.

To define the relations involved, consider **Figure 2.13**. Two surfaces are given such that the scalar function $U(x,y,z)$ (which may represent potential energy, temperature, pressure, and the like) is constant on each surface. Assuming the value of the function to be U on the lower surface and $U + dU$ on the upper surface (but still constant on each surface), then, given the scalar function $U(x,y,z)$ with partial derivatives $\partial U/\partial x$, $\partial U/\partial y$, and $\partial U/\partial z$, we can calculate the differential of U as dU by considering points $P(x,y,z)$ and $P'(x + dx,y + dy,z + dz)$ and using the total differential:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \tag{2.21}$$

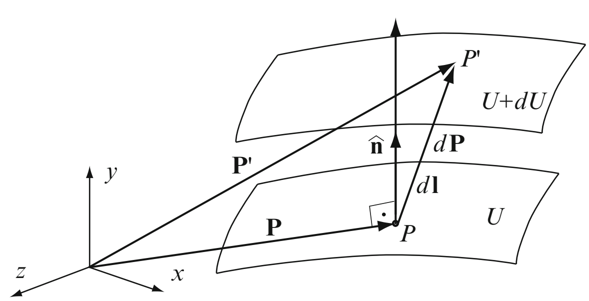


Figure 2.13 The relation between the scalar function U and its gradient

Defining the vector $d\mathbf{P} = \mathbf{P}' - \mathbf{P}$ with scalar components, dx, dy, and dz, dU can be written as the scalar product of two vectors:

$$dU = \left(\hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z}\right) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \tag{2.22}$$

We recognize the second vector in this relation as $d\mathbf{l}$ as defined in **Eq. (1.48)** and write

$$dU = \left(\hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z}\right) \cdot d\mathbf{l} \tag{2.23}$$

The vector in parentheses is now denoted as

$$\mathbf{D} = \hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z} \tag{2.24}$$

Using this notation, the total differential is

$$dU = \mathbf{D} \cdot d\mathbf{l} = |\mathbf{D}||d\mathbf{l}|\cos\theta \tag{2.25}$$

From the properties of the scalar product, we know that this product is maximum when $d\mathbf{l}$ and $\mathbf{D}$ are in the same direction ($\theta = 0$). Thus, we can write the following derivative:

$$\frac{dU}{d\mathbf{l}} = |\mathbf{D}|\cos\theta \tag{2.26}$$

This derivative depends on the direction of $d\mathbf{l}$ (in relation to $\mathbf{D}$), and, therefore, $dU/d\mathbf{l}$ is a directional derivative: the derivative of U in the direction of $d\mathbf{l}$. In formal terms, we can write the directional derivative in the direction $d\mathbf{l}$ in terms of the directional derivative in the normal ($\hat{\mathbf{n}}$) direction as

$$\frac{dU}{d\mathbf{l}} = \frac{dU}{dn}\frac{dn}{d\mathbf{l}} = \frac{dU}{dn}\cos\theta \tag{2.27}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface at the point at which the derivative is calculated. Thus, the maximum value of $dU/d\mathbf{l}$ is

$$\left.\frac{dU}{d\mathbf{l}}\right|_{\max} = \frac{dU}{dn} \tag{2.28}$$

That is, the maximum rate of change of the scalar function U is the normal derivative of the scalar function at point P. In other words, to calculate the maximum rate of change of the function, we must choose $d\mathbf{l}$ to be in the direction normal to the constant value surface. Now, returning to **Eq. (2.25)**, we get

$$\hat{\mathbf{n}}\left.\frac{dU}{d\mathbf{l}}\right|_{\max} = \hat{\mathbf{n}}\frac{dU}{dn} = \mathbf{D} = \hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z} = \text{grad}(U) \tag{2.29}$$

This result indicates not only the meaning of the gradient but also how we can calculate it from the partial derivatives of the scalar function U.

Although the form above is correct, we normally use a special notation for the gradient of a scalar function. Again, returning to the above equation, we write

$$\text{grad}\, U = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)U \tag{2.30}$$

The quantity in parentheses is a fixed operator for any scalar function we may wish to evaluate. We denote this operator in Cartesian coordinates as

$$\boxed{\nabla \equiv \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)} \tag{2.31}$$

This operator is called the ***nabla operator*** or the ***del operator***. We will use the latter name. The del operator is a vector operator by definition, and, therefore, it is not necessary to mark it is a vector.

Important Note: Although the del operator is a vector differential operator and we wrote it as a vector, it should be used with care since it is not a true vector (for instance, it does not have a magnitude). The reasons will become obvious later on but for now, the operator should only be used in the form given above. As an example, we have not defined (and, in fact, cannot define) the scalar or vector product between the del operator and other vectors or with itself. The extension of the considerations and notation given here to other coordinate systems should be avoided at this stage since all our discussion was in Cartesian coordinates. With this notation, the gradient of a scalar function is written as

$$\boxed{\mathrm{grad}\,U = \nabla U = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)U} \tag{2.32}$$

and is read as ***grad U*** or ***del U***. Either form is acceptable, although the normal use in the United States is ∇U, whereas in other countries, the form grad U is often more common. From now on, we will use the notation ∇U and the pronunciation "del U" exclusively to avoid confusion.

The del operator is a mathematical operator to which, by itself, we cannot associate any geometrical meaning. It is the interaction of the del operator with other quantities that gives it geometric significance.

On the other hand, the gradient of a scalar function has a very distinct physical meaning as was shown above. The gradient has the following general properties:

(1) It operates on a scalar function and results in a vector function.
(2) The gradient is normal to a constant value surface. This can be seen from **Eq. (2.29)**. This property will be used extensively to identify the direction of vector fields.
(3) The gradient always points in the direction of maximum change in the scalar function. In terms of potential energy, the gradient shows the direction of increase in potential energy.

Example 2.8 Application: Normal to a Surface A vector normal to a surface is $\nabla f(x,y,z)$ where $f(x,y,z)$ is the equation of the surface. Consider the plane $x + \sqrt{2}y + z = 3$. Find a normal vector to this surface and the unit vector normal to the surface.

Solution: Find the gradient of the plane. This is based on the fact that the gradient is always normal to a constant value function (for example, on a topographic map, the gradient is normal to the contour lines at any point along the contours).

We write the equation of the plane as

$$f(x,y,z) = x + \sqrt{2}y + z - 3 = 0$$

The vector normal to the plane is

$$\begin{aligned}\mathbf{A} &= \nabla f(x,y,z) = \nabla\left(x + \sqrt{2}y + z - 3\right)\\ &= \hat{\mathbf{x}}\frac{\partial}{\partial x}\left(x + \sqrt{2}y + z - 3\right) + \hat{\mathbf{y}}\frac{\partial}{\partial y}\left(x + \sqrt{2}y + z - 3\right) + \hat{\mathbf{z}}\frac{\partial}{\partial z}\left(x + \sqrt{2}y + z - 3\right) = \hat{\mathbf{x}} + \hat{\mathbf{y}}\sqrt{2} + \hat{\mathbf{z}}\end{aligned}$$

and the unit vector normal to the plane is

$$\hat{\mathbf{n}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}\sqrt{2} + \hat{\mathbf{z}}}{\sqrt{1+2+1}} = \hat{\mathbf{x}}\frac{1}{2} + \hat{\mathbf{y}}\frac{\sqrt{2}}{2} + \hat{\mathbf{z}}\frac{1}{2}$$

Note that the constant value in the equation is immaterial—it does not change the slopes in the x, y, and z directions.

Example 2.9 Application: Derivative in the Direction of a Vector Find the derivative of $xy^2 + y^2z$ at P (1,1,1) in the direction of the vector $\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}4$.

Solution: The gradient of the scalar function $V = xy^2 + y^2z$ is first calculated. This gives the directional derivative in the normal direction. Then, we evaluate the gradient at point P (1,1,1) and find the projection of this vector onto the vector $\mathbf{A} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}4$ using the scalar product between the gradient and the unit vector $\hat{\mathbf{A}}$. This gives the magnitude (or scalar component) of the directional derivative and is the derivative in the required direction. The scalar function is

$$V = xy^2 + y^2z$$

The gradient of the scalar function $V(x, y, z)$ is [using **Eq. (2.32)**]

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}(2xy + 2yz) + \hat{\mathbf{z}}y^2$$

The gradient at point (1,1,1) is

$$\nabla V(1,1,1) = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}1$$

The direction of **A** in space is given by the unit vector

$$\hat{\mathbf{A}} = \frac{\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4}{|\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4|} = \frac{\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4}{5}$$

and the projection of the gradient of V onto the direction of **A** is

$$(\nabla V)\cdot\hat{\mathbf{A}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}}4 + \hat{\mathbf{z}})\cdot\left(\frac{\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4}{5}\right) = \frac{1}{5}(3+16) = \frac{19}{5}$$

This is the derivative (or, in more practical terms, the slope) of V in the direction of **A** at P(1,1,1).

Example 2.10 Given two points $P(x,y,z)$ and $P'(x',y',z')$, calculate the gradient of the function $1/R(P, P')$ where R is the distance between the two points.

Solution: First, we find the scalar function that gives the distance between the two points. Then, we apply the gradient to this function. Because the coordinates (x,y,z) or (x',y',z') may be taken as the variables, the gradient with respect to each set of variables is calculated. In applications, one point may be fixed while the other varies, so there may not be a need to calculate the gradient with respect to both sets of variables.

The scalar function describing the distance between the two points can be written directly as (using (x,y,z) as variables and (x',y',z') as fixed)

$$R(x,y,z,x',y',z') = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad \to \quad \frac{1}{R} = \left[\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2\right]^{-1/2}$$

To calculate the gradient, we write

$$\nabla\left(\frac{1}{R}\right) = \hat{\mathbf{x}}\frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \hat{\mathbf{y}}\frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \hat{\mathbf{z}}\frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

$$= \hat{\mathbf{x}}\left(-\frac{1}{2}\frac{2(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\right) + \hat{\mathbf{y}}\left(-\frac{1}{2}\frac{2(y-y')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\right)$$

$$+\hat{\mathbf{z}}\left(-\frac{1}{2}\frac{2(z-z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\right)$$

After simplifying,

$$\nabla\left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{x}}(x-x')}{R^3} - \frac{\hat{\mathbf{y}}(y-y')}{R^3} - \frac{\hat{\mathbf{z}}(z-z')}{R^3}$$

This can also be written as

$$\nabla\left(\frac{1}{R}\right) = -\frac{1}{R^2}\left(\frac{\hat{\mathbf{x}}(x-x') + \hat{\mathbf{y}}(y-y') + \hat{\mathbf{z}}(z-z')}{R}\right) = -\frac{1}{R^2}\left(\frac{\mathbf{R}}{R}\right) = -\frac{\mathbf{R}}{R^3}$$

If we use the definition of the unit vector as $\hat{\mathbf{R}} = \mathbf{R}/R$, we get

$$\nabla\left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{R}}}{R^2}$$

Of course, the following form is equivalent:

$$\nabla\left(\frac{1}{R}\right) = -\left(\frac{\hat{\mathbf{x}}(x-x') + \hat{\mathbf{y}}(y-y') + \hat{\mathbf{z}}(z-z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\right)$$

We arbitrarily calculated the derivatives with respect to the variables (x,y,z). If we wish, we can also calculate the derivatives with respect to the variables x', y', and z'. In some cases, this might become necessary. We denote the gradient so calculated as $\nabla'(1/R)$, and, by simple inspection, we get

$$\nabla'\left(\frac{1}{R}\right) = -\nabla\left(\frac{1}{R}\right) = \frac{\hat{\mathbf{R}}}{R^2}$$

since in the evaluation of the derivatives, the inner derivatives with respect to x', y', and z' are all negative.

Exercise 2.3 Given a function $f(x,y,z)$ as the distance between a point $P(x,y,z)$ and the origin $O(0,0,0)$.

(a) Determine the gradient of this function in Cartesian coordinates.
(b) What is the magnitude of the gradient?

Answer **(a)** $\nabla f = \frac{1}{f}(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)$, where $f = \sqrt{x^2+y^2+z^2}$. **(b)** $|\nabla f| = \sqrt{\frac{1}{f^2}(x^2+y^2+z^2)} = 1$.

2.3.1.1 Gradient in Cylindrical Coordinates

To define the gradient in cylindrical coordinates, we can proceed in one of two ways:

(1) We may start with the definition of the total differential in **Eq. (2.21)**, rewrite it in cylindrical coordinates, and proceed in the same way we have done for the gradient in Cartesian coordinates, but using $d\mathbf{l}$ in cylindrical coordinates.
(2) Since the gradient is known in Cartesian coordinates and we have defined the proper transformation from Cartesian to cylindrical coordinates in **Section 1.5.2**, we may use this transformation to transform the gradient vector to cylindrical coordinates.

We use the second method because it will also become useful in the following sections. To do so, we write the gradient in **Eq. (2.32)** as follows:

$$\nabla U(x,y,z) = \hat{\mathbf{x}}\frac{\partial}{\partial x}U(x,y,z) + \hat{\mathbf{y}}\frac{\partial}{\partial y}U(x,y,z) + \hat{\mathbf{z}}\frac{\partial}{\partial z}U(x,y,z) \tag{2.33}$$

To transform this into cylindrical coordinates, we must write the function $U(x,y,z)$ in cylindrical coordinates as $U(r,\phi,z)$; that is, the scalar function must be known in cylindrical coordinates. More importantly, we must transform the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ into the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ in cylindrical coordinates and the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ into their counterparts in cylindrical coordinates $\partial/\partial r$, $\partial/\partial \phi$, and $\partial/\partial z$. The transformation for the unit vectors was found in **Eq. (1.66)** as follows:

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi, \quad \hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi, \quad \hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{2.34}$$

The transformation of the partial derivatives uses the chain rule of differentiation as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial \phi}\left(\frac{\partial \phi}{\partial y}\right) \tag{2.35}$$

The derivative with respect to z remains unchanged. To evaluate the derivatives $\partial r/\partial x$, $\partial\phi/\partial x$, $\partial r/\partial y$, and $\partial\phi/\partial y$, we use the transformation for coordinates from **Eqs. (1.63)** and **(1.64)**:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z \tag{2.36}$$

$$r = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{2.37}$$

From **Eq. (2.37)**, we can write directly

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \tag{2.38}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x}{x^2+y^2} \tag{2.39}$$

Substituting for x and y from **Eq. (2.36)** and using $r = \sqrt{x^2+y^2}$ from **Eq. (2.37)**, we get

$$\frac{\partial r}{\partial x} = \cos\phi, \; \frac{\partial r}{\partial y} = \sin\phi, \; \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r}, \; \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \tag{2.40}$$

Substituting these in **Eq. (2.35)**, we get

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \quad \text{and} \quad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \tag{2.41}$$

Now substituting for $\partial/\partial x$ and $\partial/\partial y$ from **Eq. (2.41)** and for $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ from **Eq. (2.34)** into **Eq. (2.33)** and using $U(r,\phi,z)$ for the scalar function in cylindrical coordinates, we get

$$\nabla U(r,\phi,z) = [\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi]\left[\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right]U(r,\phi,z) + [\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi]\left[\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right]U(r,\phi,z) + \hat{\mathbf{z}}\frac{\partial}{\partial z}U(r,\phi,z) \tag{2.42}$$

Performing the various products and using $\sin^2\phi + \cos^2\phi = 1$, we get

$$\boxed{\nabla U = \left(\hat{\mathbf{r}}\frac{\partial U(r,\phi,z)}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial U(r,\phi,z)}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial U(r,\phi,z)}{\partial z}\right) = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)U(r,\phi,z)} \tag{2.43}$$

As a consequence, we can immediately write the del operator in cylindrical coordinates as

$$\boxed{\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}} \tag{2.44}$$

It is important to note that the del operator in cylindrical coordinates is not the same as the del operator in Cartesian coordinates. Also to be noted is that in arriving at the definition of the gradient in cylindrical coordinates, we have not used the del operator, only the gradient in Cartesian coordinates and the transformations of coordinates and unit vectors. The process is rather tedious but is straightforward. We will use this process again in future sections but without repeating the details. The main advantage of doing so is that although we use the del operator as a symbolic description or as a notation, there is no need to perform any operations on the operator itself. We avoid these operations because the del operator is not a true vector.

Example 2.11 A scalar field is given in Cartesian coordinates as $f(x,y,z) = x + 5zy^2$. Calculate the gradient of the scalar field in cylindrical coordinates.

Solution: There are two ways to obtain the solution. One is to transform the scalar field to cylindrical coordinates and then apply the gradient to the field. The second is to calculate the gradient in Cartesian coordinates and then use the transformation matrices in **Chapter 1** to transform the gradient from Cartesian to cylindrical coordinates. We show both methods.

Method A: The coordinate transformation from cylindrical to Cartesian coordinates [**Eq. (2.36)**] is

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$$

Substituting these for x, y, and z in the field gives the field in cylindrical coordinates:

$$f(r,\phi,z) = r\cos\phi + 5zr^2\sin^2\phi$$

The gradient can now be calculated directly using **Eq. (2.43)**:

$$\nabla f = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(r\cos\phi + 5zr^2\sin^2\phi) = \hat{\mathbf{r}}(\cos\phi + 10zr\sin^2\phi) + \hat{\boldsymbol{\phi}}(-\sin\phi + 10zr\cos\phi\sin\phi) + \hat{\mathbf{z}}\,5r^2\sin^2\phi$$

Method B: In this method, the gradient is calculated in Cartesian coordinates and then transformed to cylindrical coordinates as a vector. The gradient in Cartesian coordinates is

$$\nabla f = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left(x + 5zy^2\right) = \hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,10zy + \hat{\mathbf{z}}\,5y^2$$

Now, we use the transformation in **Eq. (1.71)** (see also **Example 1.16**):

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 10zy \\ 5y^2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi + 10zy\sin\phi \\ -\sin\phi + 10zy\cos\phi \\ 5y^2 \end{bmatrix} = \begin{bmatrix} \cos\phi + 10zr\sin^2\phi \\ -\sin\phi + 10zr\sin\phi\cos\phi \\ 5r^2\cos^2\phi \end{bmatrix}$$

where the coordinate transformations above were again used to replace y and z. These are the scalar components of the gradient in cylindrical coordinates. If we write the vector, we get

$$\nabla f = \hat{\mathbf{r}}\left(\cos\phi + 10zr\sin^2\phi\right) + \hat{\boldsymbol{\phi}}\left(-\sin\phi + 10zr\sin\phi\cos\phi\right) + \hat{\mathbf{z}}\,5r^2\sin^2\phi$$

This is identical to the result obtained by **Method A**.

Example 2.12 Application: Slope of a Scalar Field A scalar field is given as $f(r,\phi,z) = r\phi + 3\phi z$.

(a) Calculate the slope of the scalar field in the direction of the vector $\mathbf{A} = \hat{\mathbf{r}}\,2 + \hat{\mathbf{z}}$.
(b) What is the slope of the field at a point $P(2,90°,1)$ in the direction of vector **A**?

Solution: The gradient of the scalar field is calculated first. This gives the derivative in the direction of maximum change in field. Find the projection of the gradient onto the direction of vector **A** using the scalar product. The direction of the slope is that of **A**. In **(b)**, the coordinates of P are substituted into the vector obtained in **(a)** to obtain the scalar component of the gradient in the direction of **A** at point P (slope).

(a) First, we calculate the gradient of the function $f(r,\phi,z)$ in cylindrical coordinates using **Eq. (2.43)**:

$$\nabla f = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(r\phi + 3\phi z) = \hat{\mathbf{r}}\,\phi + \hat{\boldsymbol{\phi}}\left(1 + \frac{3z}{r}\right) + \hat{\mathbf{z}}\,3\phi$$

Next, we need to calculate the unit vector in the direction of **A**. This is

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{r}}\,2 + \hat{\mathbf{z}}}{\sqrt{2^2 + 1}} = \hat{\mathbf{r}}\frac{2}{\sqrt{5}} + \hat{\mathbf{z}}\frac{1}{\sqrt{5}}$$

The projection of ∇f in the direction of **A** is the scalar product between ∇f and $\hat{\mathbf{A}}$:

$$(\nabla f)\cdot\hat{\mathbf{A}} = \left(\hat{\mathbf{r}}\,\phi + \hat{\boldsymbol{\phi}}\left(1 + \frac{3z}{r}\right) + \hat{\mathbf{z}}\,3\phi\right)\cdot\left(\hat{\mathbf{r}}\frac{2}{\sqrt{5}} + \hat{\mathbf{z}}\frac{1}{\sqrt{5}}\right) = \frac{2\phi}{\sqrt{5}} + \frac{3\phi}{\sqrt{5}} = \sqrt{5}\phi$$

This is the scalar component of the gradient in the direction of vector **A**.

(b) The gradient gives the slope of the scalar field at any point in space. To find the slope at a particular point, we substitute the coordinates of the point in the general expression of the projection of the gradient in the direction of **A**. Since the projection is independent of r and z and $\phi = \pi/2$ at P, we get

$$(\nabla f)\cdot\hat{\mathbf{A}}\,\big|_p = \frac{\sqrt{5}\pi}{2}$$

The slope at $P(2,90°,1)$ is $\sqrt{5}\pi/2$.

Exercise 2.4 Given the configuration of **Exercise 2.3**, calculate the gradient in cylindrical coordinates. Use the direct approach or the transformation from Cartesian to cylindrical coordinates.

Answer $\nabla f(r,\phi,z)=\frac{\mathbf{f}}{f}=\frac{\hat{\mathbf{r}}r+\hat{\mathbf{z}}z}{\sqrt{r^2+z^2}}$

2.3.1.2 Gradient in Spherical Coordinates

The gradient in spherical coordinates is defined analogously to the gradient in cylindrical coordinates; that is, we start with the gradient in Cartesian coordinates [**Eq. (2.33)**] and transform the partial derivatives, unit vectors, and variables from Cartesian to spherical coordinates. Although we will not perform all details of the derivation here (see **Exercise 2.5**), the important steps are as follows:

Step 1: We first write a general scalar function in spherical coordinates as $U(R,\phi,\theta)$.

Step 2: The unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are transformed into spherical coordinates using **Eq. (1.88)**:

$$\begin{aligned}\hat{\mathbf{x}} &= \hat{\mathbf{R}}\sin\theta\cos\phi+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi,\\ \hat{\mathbf{y}} &= \hat{\mathbf{R}}\sin\theta\sin\phi+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi \quad \hat{\mathbf{z}}=\hat{\mathbf{R}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\end{aligned} \tag{2.45}$$

Step 3: The derivatives $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ are transformed into their counterparts in spherical coordinates $\partial/\partial R$, $\partial/\partial\theta$, and $\partial/\partial\phi$. To do so, we use the chain rule of differentiation, but unlike the transformation into cylindrical coordinates, now all three coordinates change and we have

$$\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial}{\partial R}\left(\frac{\partial R}{\partial x}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial\phi}\left(\frac{\partial\phi}{\partial x}\right),\\ \frac{\partial}{\partial y}&=\frac{\partial}{\partial R}\left(\frac{\partial R}{\partial y}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial\phi}\left(\frac{\partial\phi}{\partial y}\right),\\ \frac{\partial}{\partial z}&=\frac{\partial}{\partial R}\left(\frac{\partial R}{\partial z}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\theta}{\partial z}\right)+\frac{\partial}{\partial\phi}\left(\frac{\partial\phi}{\partial z}\right)\end{aligned} \tag{2.46}$$

Step 4: Transformation of variables from Cartesian to spherical coordinates. These are listed in **Eqs. (1.82)** and **(1.81)** and are used to evaluate the partial derivatives in **Eq. (2.46)**:

$$x=R\sin\theta\cos\phi,\quad y=R\sin\theta\sin\phi,\quad z=R\cos\theta \tag{2.47}$$

$$R=\sqrt{x^2+y^2+z^2},\quad \theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right),\quad \phi=\tan^{-1}\left(\frac{y}{x}\right) \tag{2.48}$$

Although the evaluation of the various derivatives in **Eq. (2.46)** is clearly lengthier than for cylindrical coordinates, it follows identical steps, which, for the sake of brevity, we do not show. With these derivatives and substitution of these and the terms in **Eq. (2.45)** into **Eq. (2.33)**, we get the gradient in spherical coordinates as

$$\boxed{\nabla U(R,\theta,\phi)=\hat{\mathbf{R}}\frac{\partial U(R,\theta,\phi)}{\partial R}+\hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial U(R,\theta,\phi)}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial U(R,\theta,\phi)}{\partial\phi}} \tag{2.49}$$

From this, we can write the del operator in spherical coordinates as

$$\boxed{\nabla\equiv\hat{\mathbf{R}}\frac{\partial}{\partial R}+\hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}} \tag{2.50}$$

The del operator in spherical coordinates is different than the del operator in Cartesian or cylindrical coordinates.

Example 2.13 A sphere of radius a is given.

(a) Find the normal unit vector to the sphere at point $P(a,90°,30°)$.
(b) Find the normal unit vector at $P(a,90°,30°)$ in Cartesian coordinates.

Solution: First we write the equation of the sphere as a scalar function in spherical coordinates. Then the gradient of the scalar function is calculated. This gives the vector normal to the sphere's surface. The unit vector is found by dividing the gradient by the magnitude of the gradient. Substitution of the coordinates of P gives the unit vector at the given point.

(a) The sphere may be described in spherical coordinates as $f(R,\theta,\phi) = R$
The gradient is therefore

$$\nabla f(R,\theta,\phi) = \hat{\mathbf{R}}\frac{\partial f(R,\theta,\phi)}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial f(R,\theta,\phi)}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial f(R,\theta,\phi)}{\partial \theta} = \hat{\mathbf{R}}\frac{\partial R}{\partial R} = \hat{\mathbf{R}}$$

and the unit vector is

$$\hat{\mathbf{n}} = \nabla f_P(R,\theta,\phi) = \hat{\mathbf{R}}$$

The unit vector is independent of the location on the sphere or its radius.

(b) The unit vector $\hat{\mathbf{R}}$ is normal to the surface of the sphere and may be written in Cartesian coordinates as [see **Eq. (1.90)** or (**2.45**)]

$$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$$

And, clearly, the normal unit vector varies from point to point. At $P(a,90°,30°)$, the normal unit vector is

$$\hat{\mathbf{n}} = \hat{\mathbf{x}}\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) + \hat{\mathbf{y}}\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) + \hat{\mathbf{z}}\cos\left(\frac{\pi}{2}\right) = \hat{\mathbf{x}}\frac{\sqrt{3}}{2} + \hat{\mathbf{y}}\frac{1}{2}$$

Note that this is independent of the radius of the sphere.

Exercise 2.5 Derive the gradient in spherical coordinates using the steps outlined in **Eqs. (2.45)** through (**2.48**). Verify that **Eq. (2.49)** is obtained.

Reminder The gradient is only defined for a scalar function.

2.3.2 The Divergence of a Vector Field

After defining the gradient of a scalar function, we wish now to define the spatial derivatives of a vector function. This will lead to two relations. One is the divergence of a vector, while the other is the curl of a vector. The divergence is defined first.

To understand the ideas involved, we first look at some physical quantities with which we are familiar and which lead to the definition of the divergence.

Consider first the two vector fields shown in **Figure 2.14**. In **Figure 2.14a**, the magnitude of the vector field is constant, and the direction does not vary. For example, this may represent flow of water in a channel or current in a conductor. If we

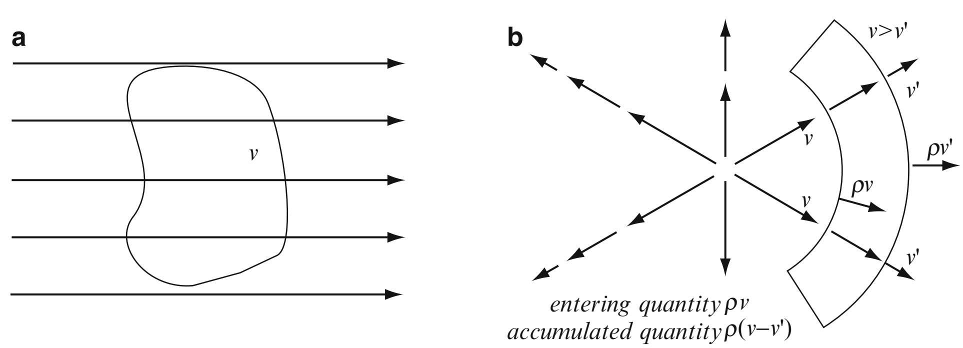


Figure 2.14 Flow through a volume. (**a**) Field is uniform and the total quantity entering volume v equals the quantity leaving the volume. (**b**) Nonuniform flow. There is an accumulation in volume v

draw any volume in the flow, the total net flow out of the volume is zero; that is, the total amount of water or current flowing into the volume v is equal to the total flow out of the volume.

In **Figure 2.14b**, the flow is radial from the center and the vector changes in magnitude as the flow progresses. This is indicated by the fact that the length of the vectors is reduced. A physical situation akin to this is a spherical can in which holes were made and the assembly is connected to a water hose. Water squirts in radial directions and water velocity is reduced with distance from the can. Now, if we were to draw a volume (an imaginary can), the total amount of water entering the volume is larger than the amount of water leaving the volume since water velocity changes and the amount of water is directly dependent on velocity. This fact can be stated in another way: There is a net flow of water into the volume through the surface enclosing the volume of the can where it accumulates. The latter statement is what we wish to use since it links the surface of the volume to the net flow out of the volume. In the example in **Figure 2.14b**, the net outward flow is negative. The total flux out of the volume is given by the closed surface integral of the vector **A** [see **Eq. (2.16)**]:

$$Q = \oint_s \mathbf{A} \cdot d\mathbf{s} \tag{2.51}$$

where the closed surface integral must be used since flow (into or out of the volume) occurs everywhere on the surface. Although this amount is written as a surface integral, the quantity Q clearly depends on the volume we choose. Thus, it makes sense to define the flow through the surface of a clearly defined volume such as a unit volume. If we do so, the quantity Q is the flow per unit volume. Our choice here is to do exactly that, but to define the flow through the surface, per measure of volume and then allow this volume to tend to zero. In the limit, this will give us the net outward flow at a point. Thus, we define a quantity that we will call the divergence of the vector **A** as

$$\boxed{\mathrm{Div}\,\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta v}} \tag{2.52}$$

That is,

"*the divergence of vector **A** is the net flow of the flux of vector **A** out of a small volume, through the closed surface enclosing the volume, as the volume tends to zero.*"

The meaning of the term divergence can be at least partially understood from **Figure 2.15a** where the source in **Figure 2.14** is shown again, but now we take a small volume around the source itself. Again using the analogy of water, the flow is outward only. This indicates that there is a net flow out of the volume through the closed surface. Moreover, the flow "diverges" from the point outward. We must, however, be careful with this description because divergence does not necessarily imply as clear a picture as this. The flow in **Figure 2.15b** has nonzero divergence as well even though it does not "look" divergent. A simple visual picture of divergence is a jet engine. Enclosing the engine by an imaginary surface indicates a net flow outward.

A second important point is that in both examples given above, nonzero divergence implies either accumulation in the volume (in this case of fluid) or flow out of the volume. In the latter case, we must conclude that if the divergence is nonzero, there must be a ***source*** of flow at the point, whereas in the former case, a negative source or ***sink*** must exist. We, therefore, have an important interpretation and use for the divergence: a measure of the (scalar) source of the vector field. From **Eqs. (2.51)** and **(2.52)**, this source is clearly a volume density. We also must emphasize here that the divergence is a point value: a differential quantity defined at a point.

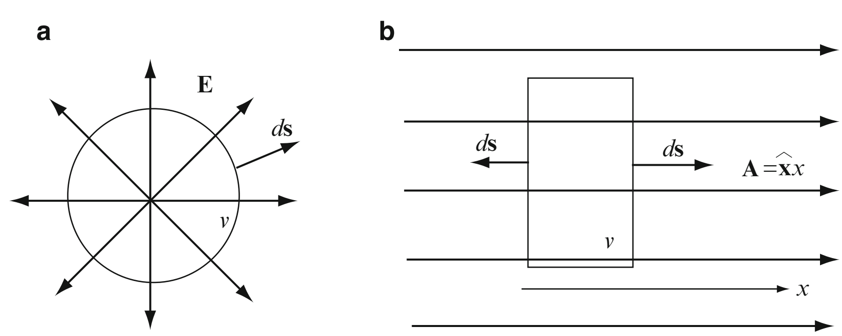


Figure 2.15 Net outward flow from a volume v. (**a**) For a radial field. (**b**) For a field varying with the coordinate x. Both fields have nonzero divergence

2.3.2.1 Divergence in Cartesian Coordinates

The definition in **Eq. (2.52)**, while certainly physically meaningful, is very inconvenient for practical applications. It would be rather tedious to evaluate the surface integral and then let the volume tend to zero every time the divergence is needed. For this reason, we seek a simpler, more easily evaluated expression to replace the definition for practical applications. This is done by considering a general vector and a convenient but general element of volume Δv as shown in **Figure 2.16**. First, we evaluate the surface integral over the volume, then divide by the volume, and let the volume tend to zero to find the divergence at point P. To find the closed surface integral, we evaluate the open surface integration of the vector **A** over the six sides of the volume and add them. Noting the directions of the vectors $d\mathbf{s}$ on all surfaces, we can write

$$\oint_S \mathbf{A}\cdot d\mathbf{s} = \int_{s_{fr}} \mathbf{A}\cdot d\mathbf{s}_{fr} + \int_{s_{bk}} \mathbf{A}\cdot d\mathbf{s}_{bk} + \int_{s_{tp}} \mathbf{A}\cdot d\mathbf{s}_{tp} + \int_{s_{bt}} \mathbf{A}\cdot d\mathbf{s}_{bt} + \int_{s_{rt}} \mathbf{A}\cdot d\mathbf{s}_{rt} + \int_{s_{lt}} \mathbf{A}\cdot d\mathbf{s}_{lt} \quad (2.53)$$

where fr = front surface, bk = back surface, tp = top surface, bt = bottom surface, rt = right surface, and lt = left surface. Each integral is evaluated separately, and because we chose the six surfaces such that they are parallel to coordinates, their evaluation is straightforward. To do so, we will also assume the vector **A** to be constant over each surface, an assumption which is justified from the fact that these surfaces tend to zero in the limit. Since the divergence will be calculated at point $P(x,y,z)$, we take the coordinates of this point as reference at the center of the volume as shown in **Figure 2.16b**. The front surface is located at $x + \Delta x/2$, whereas the back surface is at $x - \Delta x/2$. Similarly, the top surface is at $z + \Delta z/2$ and the bottom surface at $z - \Delta z/2$, whereas the right and left surfaces are at $y + \Delta y/2$ and $y - \Delta y/2$, respectively. With these definitions in mind, we can start evaluating the six integrals. On the front surface,

$$\int_{s_{fr}} \mathbf{A}\cdot d\mathbf{s}_{fr} = \mathbf{A}_{fr}\cdot \Delta\mathbf{s}_{fr} \quad (2.54)$$

where $\mathbf{A}_{fr}$ is that component of the vector **A** perpendicular to the front surface. From the definition of the scalar product, this vector component is in the x direction, and its scalar component is equal to

$$\left|\mathbf{A}_{fr}\right| = \hat{\mathbf{x}}\cdot\mathbf{A} = A_x\left(x + \frac{\Delta x}{2}, y, z\right) \quad (2.55)$$

The latter expression requires that we evaluate the x component of **A** at a point $(x + \Delta x/2, y, z)$. To do so, it is useful to use the Taylor series expansion of $f(x + \Delta x)$ around point x:

$$f(x+\Delta x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}(\Delta x)^k = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2}f''(x) + \frac{(\Delta x)^3}{6}f'''(x) + \cdots \quad (2.56)$$

Anticipating truncation of the expansion after the first two terms and replacing Δx with $\Delta x/2$, $f(x)$ with $A_x(x,y,z)$, $f(x + \Delta x)$ with $A_x(x + \Delta x/2,y,z)$, $f(x - \Delta x)$ with $A_x(x - \Delta x/2,y,z)$ and $f'(x)$ with $\partial A_x(x,y,z)/\partial x$, we get

$$A_x\left(x + \frac{\Delta x}{2}, y, z\right) \approx A_x(x,y,z) + \frac{\Delta x}{2}\frac{\partial A_x(x,y,z)}{\partial x} \quad (2.57)$$

and

$$A_x\left(x - \frac{\Delta x}{2}, y, z\right) \approx A_x(x,y,z) - \frac{\Delta x}{2}\frac{\partial A_x(x,y,z)}{\partial x} \quad (2.58)$$

Neglecting the higher-order terms is justified because in the calculations that follow, we will let Δx go to zero. Rather than keeping the higher-order terms, the forms in **Eqs. (2.57)** and **(2.58)** will be used and, then, after obtaining the final result, we will return to justify neglecting the higher-order terms.

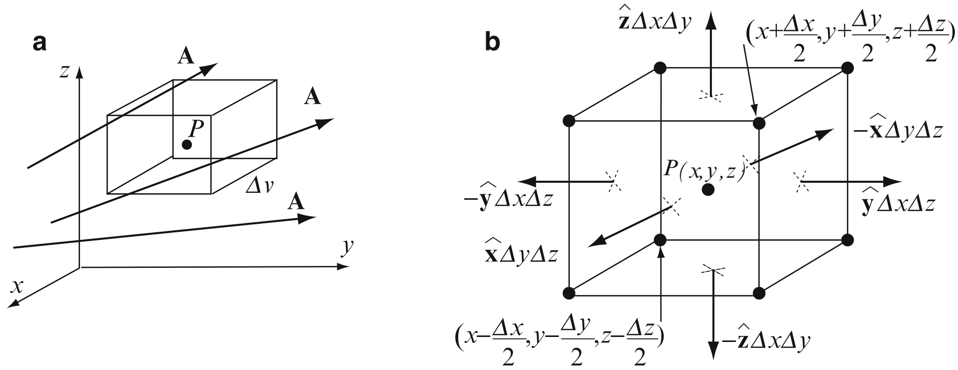


Figure 2.16 Evaluation of a closed surface integral over an element of volume. (**a**) The volume and its relation to the axes. (**b**) The elements of surface and coordinates

An element of surface on the front face is

$$\Delta \mathbf{s}_{fr} = \hat{\mathbf{n}}\,\Delta s_{fr} = \hat{\mathbf{x}}\,\Delta s_{fr} = \hat{\mathbf{x}}\,\Delta y \Delta z \tag{2.59}$$

Substitution of this and **Eq. (2.57)** into **Eq. (2.54)** gives the surface integral as

$$\int_{s_{fr}} \mathbf{A}_{fr} \cdot d\mathbf{s}_{fr} \approx \hat{\mathbf{x}}\left(A_x(x,y,z) + \frac{\Delta x}{2}\frac{\partial A_x(x,y,z)}{\partial x}\right) \cdot \hat{\mathbf{x}}\,\Delta y \Delta z = \Delta y \Delta z A_x(x,y,z) + \frac{\Delta x \Delta y \Delta z}{2}\frac{\partial A_x(x,y,z)}{\partial x} \tag{2.60}$$

Since **A** has the same direction on the back surface but $d\mathbf{s}$ is in the opposite direction compared with the front surface, we get for the back surface

$$\mathbf{A}_{bk} = \hat{\mathbf{x}}\,\mathbf{A}_x\left(x - \frac{\Delta x}{2}, y, z\right), \quad d\mathbf{s}_{bk} = -\hat{\mathbf{x}}\,dydz \tag{2.61}$$

With these and replacing x by $-x$ and $\Delta x/2$ by $-\Delta x/2$ in **Eq. (2.60)**, we have for the back surface

$$\int_{s_{bk}} \mathbf{A}_{bk} \cdot d\mathbf{s}_{bk} \approx -\Delta y \Delta z A_x(x,y,z) + \frac{\Delta x \Delta y \Delta z}{2}\frac{\partial A_x(x,y,z)}{\partial x} \tag{2.62}$$

Summing the terms in **Eqs. (2.60)** and **(2.62)** gives for the front and back surfaces

$$\int_{s_{fr}} \mathbf{A} \cdot d\mathbf{s}_{fr} + \int_{s_{bk}} \mathbf{A} \cdot d\mathbf{s}_{bk} \approx \Delta x \Delta y \Delta z \frac{\partial A_x(x,y,z)}{\partial x} \tag{2.63}$$

The result was obtained for the front and back surfaces, but there is nothing special about these two surfaces. In fact, if we were to rotate the volume in space such that the front and back surfaces are perpendicular to the y axis, the only difference is that the component of **A** in this expression must be taken as the y component. Although you should convince yourself that this is the case by repeating the steps in **Eqs. (2.54)** through **(2.63)** for the left and right surfaces, the following can be written directly simply because of this symmetry in calculations:

$$\int_{s_{lt}} \mathbf{A} \cdot d\mathbf{s}_{lt} + \int_{s_{rt}} \mathbf{A} \cdot d\mathbf{s}_{rt} \approx \Delta x \Delta y \Delta z \frac{\partial A_y(x,y,z)}{\partial y} \tag{2.64}$$

Similarly, for the top and bottom surfaces

$$\int_{s_{tp}} \mathbf{A} \cdot d\mathbf{s}_{tp} + \int_{s_{bt}} \mathbf{A} \cdot d\mathbf{s}_{bt} \approx \Delta x \Delta y \Delta z \frac{\partial A_z(x,y,z)}{\partial z} \tag{2.65}$$

The total surface integral is the sum of the surface integrals in **Eqs. (2.63), (2.64), and (2.65)**:

$$\oint_s \mathbf{A} \cdot d\mathbf{s} = \Delta v \frac{\partial A_x(x,y,z)}{\partial x} + \Delta v \frac{\partial A_y(x,y,z)}{\partial y} + \Delta v \frac{\partial A_z(x,y,z)}{\partial z} + (\text{higher-order terms}) \tag{2.66}$$

where the higher-order terms are those neglected in the Taylor series expansion and $\Delta v = \Delta x \Delta y \Delta z$. Now, we can return to the definition of the divergence in **Eq. (2.52)**:

$$\text{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta v} = \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\oint_s \mathbf{A} \cdot d\mathbf{s}}{\Delta x \Delta y \Delta z} = \frac{\partial A_x(x,y,z)}{\partial x} + \frac{\partial A_y(x,y,z)}{\partial y} + \frac{\partial A_z(x,y,z)}{\partial z} \tag{2.67}$$

In the process, we neglected all higher-order terms indicated in **Eq. (2.66)**. It is relatively easy to show that these terms tend to zero as Δx, Δy, and Δz tend to zero. As an example, consider the remainder of the expansion in **Eq. (2.56)**:

$$R = \frac{(\Delta x)^2}{4} \frac{\partial^2 A_x}{\partial x^2} + \frac{(\Delta x)^3}{12} \frac{\partial^3 A_x}{\partial x^3} + \cdots \tag{2.68}$$

Integrating this over the front and back surface, in a manner analogous to **Eq. (2.63)** gives

$$\int_{s_{fr}+s_{bk}} R ds = \frac{\Delta x \Delta v}{2} \frac{\partial^2 A_x}{\partial x^2} + \frac{(\Delta x)^3 \Delta v}{6} \frac{\partial^3 A_x}{\partial x^3} + \cdots \tag{2.69}$$

As we apply the limit in **Eq. (2.67)** to this remainder term, it is clear that the terms are multiplied by Δx, $(\Delta x)^2$, etc., and, therefore, all tend to zero in the limit $\Delta x \to 0$. Similar arguments apply to the y and z components of **A**, justifying the result in **Eq. (2.67)**.

It is customary to write **Eq. (2.67)** in a short-form notation as

$$\text{div}\,\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2.70}$$

since this applies at any point in space.

The calculation of the divergence of a vector **A** is therefore very simple since all that are required are the spatial derivatives of the scalar components of the vector. The divergence is a scalar as required and may have any magnitude, including zero. The result in **Eq. (2.70)** well justifies the two pages of algebra that were needed to obtain it because now we have a simple, systematic way of evaluating the divergence. For historical reasons, the notation for divergence is $\nabla \cdot \mathbf{A}$ (read: del dot **A**).[1] The divergence of vector **A** is written as follows:

$$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \tag{2.71}$$

However, it must be pointed out that a scalar product between the del operator and the vector **A** is not implied and should never be attempted. The symbolic notation $\nabla \cdot \mathbf{A}$ is just that: A notation to the right-hand side of **Eq. (2.71)**. Whenever we need to calculate the divergence of a vector **A**, the right-hand side of **Eq. (2.71)** is calculated, never a scalar product. Note also that calculation of divergence using the definition in **Eq. (2.52)** is independent of the system of coordinates. The actual evaluation of the surface integrals is obviously coordinate dependent.

[1] The notation used here is due to Josiah Willard Gibbs (1839–1903), who, however, never indicated or implied the notation to mean a scalar product. The implication of a scalar product between ∇ and **A** is a common error in vector calculus and for that reason alone should be avoided.

2.3.2.2 Divergence in Cylindrical and Spherical Coordinates

The divergence in cylindrical and spherical coordinates may be obtained in an analogous manner: We define a small volume with sides parallel to the required system of coordinates and evaluate **Eq. (2.52)** as we have done for the Cartesian system in **Section 2.3.2.1**. The method is rather lengthy but is straightforward (see **Exercise 2.6**). An alternative is to start with **Eq. (2.71)** and transform it into cylindrical or spherical coordinates in a manner similar to **Section 2.3.1**. This method is outlined next.

For cylindrical coordinates, we use **Eq. (2.41)**, which defines the transformations of the operators ∂/x and ∂/y while ∂/z remains unchanged. Then, from the transformations of the scalar components of a general vector from Cartesian to cylindrical coordinates given in **Eqs. (1.68)** and **(1.69)**, we get

$$A_x = A_r \cos\phi - A_\phi \sin\phi, \qquad A_y = A_r \sin\phi + A_\phi \cos\phi, \qquad A_z = A_z \tag{2.72}$$

Substitution of these and the relations in **Eq. (2.41)** into **Eq. (2.71)** gives

$$\nabla \cdot \mathbf{A}(r,\phi,z) = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)(A_r\cos\phi - A_\phi\sin\phi) + \left(\cos\phi\frac{\partial}{\partial r} - \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)(A_r\sin\phi + A_\phi\cos\phi) + \frac{\partial A_z}{\partial z} \tag{2.73}$$

Expanding this expression and evaluating the derivatives (see **Exercise 2.7**) gives the divergence in cylindrical coordinates:

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{r}\left(\frac{\partial(rA_r)}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial A_\phi}{\partial\phi}\right) + \frac{\partial A_z}{\partial z}} \tag{2.74}$$

Similar steps may be followed to obtain the divergence in spherical coordinates. Although we do not show the steps here, the process starts again with **Eq. (2.71)**. The transformations for the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ from Cartesian to spherical coordinates are obtained from the expressions in **Eqs. (2.46)** through **(2.48)**, whereas the transformations of the scalar components A_x, A_y, and A_z from Cartesian to spherical coordinates are given in **Eq. (1.88)**. Substituting these into **Eq. (2.71)** and carrying out the derivatives (see **Exercise 2.8**) gives the following expression for the divergence in spherical coordinates:

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A\phi}{\partial\phi}} \tag{2.75}$$

Reminder The notation $\nabla \cdot \mathbf{A}$ in **Eqs. (2.74)** and **(2.75)** should always be viewed as a notation only. It should never be taken as implying a scalar product.

Exercise 2.6

(a) Find the divergence in cylindrical coordinates using the method in **Section 2.3.2.1** by defining an elementary volume in cylindrical coordinates.
(b) Find the divergence in spherical coordinates using the method in **Section 2.3.2.1** by defining an elementary volume in spherical coordinates.

Exercise 2.7 Carry out the detailed operations outlined in **Section 2.3.2.2** needed to obtain **Eq. (2.74)**.

Exercise 2.8 Carry out the detailed operations outlined in **Section 2.3.2.2** needed to obtain **Eq. (2.75)**.

Example 2.14 A vector field $\mathbf{F} = \hat{\mathbf{x}}\,3y + \hat{\mathbf{y}}\,(5 - 2x) + \hat{\mathbf{z}}\,(z^2 - 2)$ is given. Find the divergence of **F**.

Solution: The divergence in **Eq. (2.71)** can be applied directly:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial (3y)}{\partial x} + \frac{\partial (5-2x)}{\partial y} + \frac{\partial (z^2-2)}{\partial z} = 2z$$

The divergence of the vector field varies in the z direction only.

Example 2.15 Find $\nabla \cdot \mathbf{A}$ at ($R = 2$, $\theta = 30°$, $\phi = 90°$) for the vector field

$$\mathbf{A} = \hat{\mathbf{R}}\,0.2R^{\mathbf{3}}\phi \sin^2\theta + \hat{\boldsymbol{\theta}}\,0.2R^3\phi\sin^2\theta + \hat{\boldsymbol{\phi}}\,0.2R^3\phi\sin^2\theta.$$

Solution: We apply the divergence in spherical coordinates using **Eq. (2.75)**:

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial\left(0.2R^5\phi\sin^2\theta\right)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial\left(0.2R^3\phi\sin^3\theta\right)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial\left(0.2R^3\phi\sin^2\theta\right)}{\partial\phi}$$
$$= R^2\phi\sin^2\theta + 0.6R^2\phi\sin\theta\cos\theta + 0.2R^2\sin\theta$$

At (2,30°,90°),

$$\nabla \cdot \mathbf{A} = 4 \times \left(\frac{\pi}{2}\right) \times \left(\frac{1}{4}\right) + 0.6 \times 4 \times \left(\frac{\pi}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{\sqrt{3}}{2}\right) + 0.2 \times 4 \times \left(\frac{1}{2}\right) = 3.6032$$

The scalar source of the vector field **A** is equal to 3.6032 at the given point.

2.3.3 The Divergence Theorem

Consider the surface of a rectangular box whose sides are dx, dy, and dz and are parallel to the xy, xz, and yz planes, respectively, as shown in **Figure 2.17**. The surface of the lower face $PQRS$ is $dxdy$, and $d\mathbf{s}$ is in the negative z direction:

$$d\mathbf{s}_l = -\hat{\mathbf{z}}dxdy \tag{2.76}$$

The flux of $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ crossing this surface is

$$d\Phi_{ls} = \mathbf{A} \cdot d\mathbf{s}_l = \hat{\mathbf{z}}A_z \cdot (-\hat{\mathbf{z}}dxdy) = -A_z dxdy \tag{2.77}$$

On the upper surface $P'Q'R'S'$, the normal to the surface is in the positive z direction and the component A_z of the vector **A** changes by an amount dA_z. Therefore, A_z on the upper face is

$$A_z + dA_z = A_z + \frac{\partial A_z}{\partial z}dz \tag{2.78}$$

The flux on the upper surface is found by multiplying by the area of the surface $dxdy$:

$$d\Phi_{us} = A_z dxdy + \frac{\partial A_z}{\partial z}dxdydz \tag{2.79}$$

The sum of the fluxes on the upper and lower surfaces gives

$$d\Phi_z = d\Phi_{ls} + d\Phi_{us} = \frac{\partial A_z}{\partial z}dv \tag{2.80}$$

where the index z denotes that this is the total flux on the two surfaces perpendicular to the z axis and $dv = dxdydz$ is the volume of the rectangular box.

Using the same rationale on the other two pairs of parallel surfaces and summing the three contributions yields the expression

$$d\Phi = \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dv \tag{2.81}$$

for the total flux through the box. The expression in brackets is the divergence of the vector **A**. The expression for the total flux through the small box becomes [see **Eq. (2.71)**]

$$d\Phi = (\nabla \cdot \mathbf{A}) dv \tag{2.82}$$

Now, consider an arbitrary volume v, enclosed by a surface s. Since $d\Phi$ through a differential volume is known, integration of this $d\Phi$ over the whole volume v gives the total flux passing through the volume

$$\Phi = \int_v d\Phi = \int_v (\nabla \cdot \mathbf{A}) dv \tag{2.83}$$

In **Eq. (2.51)**, the flux was evaluated by integrating over the whole surface s, which encloses the volume v. This also gives the total flux through the volume v:

$$\Phi = \oint_s \mathbf{A} \cdot d\mathbf{s} \tag{2.84}$$

Since the total flux through the volume or through the surface enclosing the volume must be the same, we can equate **Eqs. (2.83)** and **(2.84)** to get

$$\boxed{\int_v (\nabla \cdot \mathbf{A}) dv = \oint_s \mathbf{A} \cdot d\mathbf{s}} \tag{2.85}$$

This equality between the two integrals means that the flux of the vector **A** through the closed surface s is equal to the volume integral of the divergence of **A** over the volume enclosed by the surface s. We call this the ***divergence theorem***.

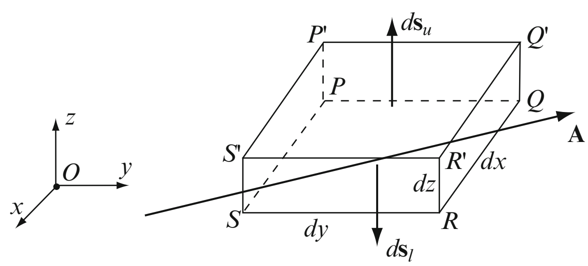


Figure 2.17

Its most important use is the conversion of volume integrals of the divergence of a vector field into closed surface integrals. This theorem is often invoked to simplify expressions or to rewrite them in more convenient alternative forms.

Example 2.16 The vector field $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}z^2$ is given. Verify the divergence theorem for this vector over a cube 1 m on the side. Assume the cube occupies the space $0 \le x, y, z \le 1$.

Solution: First, we find the product $\mathbf{A} \cdot d\mathbf{s}$ and integrate it over the surface of the volume. Then, we integrate $\nabla \cdot \mathbf{A}$ over the whole volume of the cube of side 1 with four of its vertices at (0,0,0), (0,0,1), (0,1,0), and (1,0,0) (see **Figure 2.18**). The two results should be the same.

(a) Use the flux of **A** through the surface enclosing the volume:

$$\oint_s \mathbf{A}\cdot d\mathbf{s} = \int_{s_1}\mathbf{A}\cdot d\mathbf{s}_1 + \int_{s_2}\mathbf{A}\cdot d\mathbf{s}_2 + \int_{s_3}\mathbf{A}\cdot d\mathbf{s}_3 + \int_{s_4}\mathbf{A}\cdot d\mathbf{s}_4 + \int_{s_5}\mathbf{A}\cdot d\mathbf{s}_5 + \int_{s_6}\mathbf{A}\cdot d\mathbf{s}_6$$

where, from **Figure 2.18**

$$d\mathbf{s_1} = \hat{\mathbf{x}}dydz,\quad d\mathbf{s}_2 = \hat{\mathbf{x}}dydz,\quad d\mathbf{s}_3 = \hat{\mathbf{y}}dxdz$$
$$d\mathbf{s}_4 = \hat{\mathbf{y}}dxdz,\quad d\mathbf{s}_5 = \hat{\mathbf{z}}dxdy,\quad d\mathbf{s}_6 = -\hat{\mathbf{z}}dxdy$$

Perform each surface integral separately:

(1) At $x = 1: \int_{s_1}\mathbf{A}\cdot d\mathbf{s}_1 = \int_{s_1}\left(\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,y^2 + \hat{\mathbf{z}}\,z^2\right)\cdot(\hat{\mathbf{x}}dydz) = \int_{y=0}^{y=1}\int_{z=0}^{z=1} dy\,dz = 1.$

(2) At $x = 0: \int_{s_2}\mathbf{A}\cdot d\mathbf{s}_2 = \int_{s_2}\left(\hat{\mathbf{y}}\,y^2 + \hat{\mathbf{z}}\,z^2\right)\cdot(-\hat{\mathbf{x}}dydz) = 0.$

(3) At $y = 1: \int_{s_3}\mathbf{A}\cdot d\mathbf{s}_3 = \int_{s_3}\left(\hat{\mathbf{x}}\,x^2 + \hat{\mathbf{y}}\,1 + \hat{\mathbf{z}}\,z^2\right)\cdot(\hat{\mathbf{y}}dxdz) = \int_{x=0}^{x=1}\int_{z=0}^{z=1} dx\,dz = 1.$

(4) At $y = 0: \int_{s_4}\mathbf{A}\cdot d\mathbf{s}_4 = \int_{s_4}\left(\hat{\mathbf{x}}\,x^2 + \hat{\mathbf{z}}\,z^2\right)\cdot(-\hat{\mathbf{y}}dxdz) = 0.$

(5) At $z = 1: \int_{s_5}\mathbf{A}\cdot d\mathbf{s}_5 = \int_{s_5}\left(\hat{\mathbf{x}}\,x^2 + \hat{\mathbf{y}}\,y^2 + \hat{\mathbf{z}}\,1\right)\cdot(\hat{\mathbf{z}}dxdy) = \int_{x=0}^{x=1}\int_{y=0}^{y=1} dx\,dy = 1.$

(6) At $z = 0: \int_{s_6}\mathbf{A}\cdot d\mathbf{s}_6 = \int_{s_6}\left(\hat{\mathbf{x}}\,x^2 + \hat{\mathbf{y}}\,y^2\right)\cdot(-\hat{\mathbf{z}}dxdy) = 0.$

The sum of all six integrals is 3.

(b) Use the divergence of **A** in the volume. The divergence of **A** is $\nabla\cdot\mathbf{A} = 2x + 2y + 2z$
Integration of $\nabla\cdot\mathbf{A}$ over the volume of the cube gives

$$\int_v(\nabla\cdot\mathbf{A})dv = \int_{z=0}^{z=1}\int_{y=0}^{y=1}\int_{x=0}^{x=1}(2x + 2y + 2z)dxdydz = \int_{z=0}^{z=1}\int_{y=0}^{y=1}\left(x^2 + 2xy + 2xz\right)dydz\Big|_{x=0}^{x=1}$$
$$= \int_{z=0}^{z=1}\int_{y=0}^{y=1}(1 + 2y + 2z)dydz = \int_{z=0}^{z=1}\left(y + y^2 + 2yz\right)dz\Big|_{y=0}^{y=1} = \int_{z=0}^{z=1}(1 + 1 + 2z)dz = \left(2z + z^2\right)\Big|_{z=0}^{z=1} = 3$$

Since the result in **(a)** and **(b)** are equal, the divergence theorem is verified for the given vector and volume.

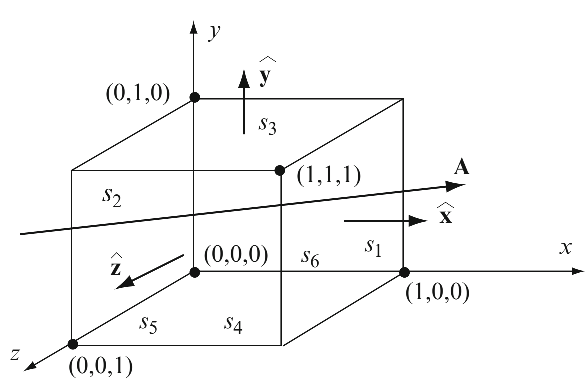


Figure 2.18

2.3.4 Circulation of a Vector and the Curl

We defined the gradient of a scalar and the divergence of a vector in the previous two sections. Both of these have physical meaning, and some applications of the two were shown in examples. In particular, the divergence of a vector was shown to be an indication of the strength of the scalar source of the vector. The question now is the following: If a vector can be generated by a scalar source (for example, a water spring is a scalar source, but it gives rise to a vector flow which has both direction and magnitude), is it also possible that a vector source gives rise to a vector field? The answer is clearly yes. Consider again the flow of a river; the flow is never uniform; it is faster toward the center of the river and slower at the banks. If you were to toss a stick into the river, perpendicular to the flow, the stick, in addition to drifting with the flow, will rotate and align itself with the direction of the flow. This rotation is caused by the variation in flow velocity: One end of the stick is dragged down the river at higher velocity than the other as shown in **Figure 2.19**. The important point here is that we cannot explain this rotation using the scalar source of the field. To explain this behavior, and others, we introduce the ***curl*** of a vector. The curl is related to circulation and spatial variations in the vector field. To define the curl, we first define the circulation of a vector. In the process, we will also try to look at the meaning of the curl and its utility.

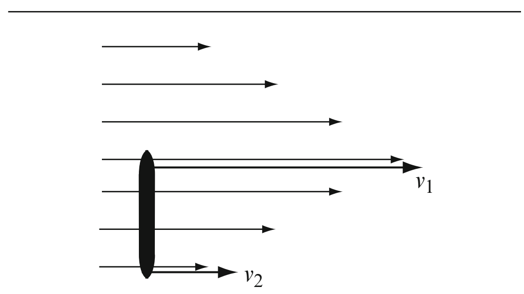


Figure 2.19 Illustration of circulation. The stick shown will rotate clockwise as it moves downstream

2.3.4.1 Circulation of a Vector Field

The closed contour integral of a vector field **A** was introduced in **Eq. (2.8)** and was called the circulation of the vector field around the contour:

$$C = \oint_L \mathbf{A} \cdot d\mathbf{l} \tag{2.86}$$

where $d\mathbf{l}$ is a differential length vector along the contour L. Why do we call this a circulation? To understand this, consider first a circular flow such as a hurricane (the wind path is circular). If **A** represents force, then the circulation represents work or energy expended. This energy increases with the circulation. If we take this as a measure for a hurricane, then measuring the circulation (if we could) would be a good measurement of the strength of the hurricane. If **A** and $d\mathbf{l}$ are parallel, as in **Figure 2.20a**, the circulation is largest. However, if **A** and $d\mathbf{l}$ are perpendicular to each other everywhere along the contour, the circulation is zero (**Figure 2.20b**). For example, an airplane, flying straight toward the eye of the hurricane, flies perpendicular to the wind and experiences no circulation. There is plenty of buffeting force but no circulation. This picture should be kept in mind since it shows that circulation as meant here does not necessarily mean geometric circulation. In other words, a vector may rotate around along a contour and its circulation may still be zero, whereas a vector that does not rotate (for example, the flow in **Figure 2.19**) may have nonzero circulation. All that circulation implies is the line integral of a vector field along a closed contour. This circulation may or may not be zero, depending on the vector field, the contour, and the relation between the two.

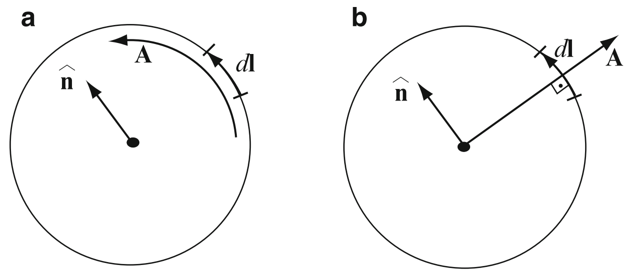


Figure 2.20 Circulation. (**a**) Maximum circulation. (**b**) Zero circulation

Although the foregoing explanation and the use of **Eq. (2.86)** as a measure of circulation are easy to understand physically, measuring the circulation in this fashion is not very useful. For one thing, **Eq. (2.86)** gives an integrated value over the contour. This tends to smooth local variations, which, in fact, may be the most important aspects of the field. Second, if we want to physically measure any quantity associated with the flow, we can only do this locally. A measuring device for wind velocity, force, etc., is a small device and the measurement may be regarded as a point measurement. Thus, we need to calculate or measure circulation in a small area. In addition to this, circulation also has a spatial meaning. In the case of a hurricane, the rotation may be regarded to be in a plane parallel to the surface of the ocean, but rotation can also be in other planes. For example, a gyroscope may rotate in any direction in space. Thus, when measuring rotation, the direction and plane of rotation are also important. These considerations lead directly to the definition of the curl. The curl is a vector measure of circulation which gives both the circulation of a vector and the direction of circulation per unit area of the field. More accurately, we define the curl using the following relation:

$$\text{curl } \mathbf{A} \equiv \lim_{\Delta s \to 0} \frac{\hat{\mathbf{n}} \oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta s} \tag{2.87}$$

"*The curl of* **A** *is the circulation of the vector* **A** *per unit area, as this area tends to zero and is in the direction normal to the area when the area is oriented such that the circulation is maximum.*"

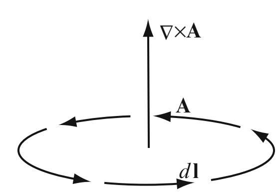
The curl of a vector field is, therefore, a vector field, defined at any point in space.

From the definition of contour integration, the normal to a surface enclosed by a contour is given by the right-hand rule as shown in **Figure 2.21** which also gives the direction of the curl. The definition in **Eq. (2.87)** has one drawback: It looks hopeless as far as using it to calculate the curl of a vector. Certainly, it is not practical to calculate the circulation and then use the limit to evaluate the curl every time a need arises. To find a simpler, more systematic way of evaluating the curl, we observe that curl **A** is a vector with components in the directions of the coordinates. In the Cartesian system, for example, the vector $\mathbf{B} = \text{curl } \mathbf{A}$ can be written as

$$\mathbf{B} = \text{curl } \mathbf{A} = \hat{\mathbf{x}}(\text{curl } A)_x + \hat{\mathbf{y}}(\text{curl } A)_y + \hat{\mathbf{z}}(\text{curl } A)_z \tag{2.88}$$

where the indices x, y, and z indicate the corresponding scalar component of the vector. For example, $(\text{curl } A)_x$ is the scalar x component of curl **A**. This notation shows that curl **A** is the sum of three components, each a curl, one in the x direction, one in the y direction, and one in the z direction. To better understand this, consider a small general loop with projections on the x–y, y–z, and x–z planes as shown in **Figure 2.22**. The magnitudes of the curls of the three projections are the scalar components B_x, B_y, and B_z in **Eq. (2.88)**. Calculation of these components and summation in **Eq. (2.88)** will provide the appropriate method for calculation of the curl. Now, consider an arbitrary vector **A** with scalar components A_x, A_y, and A_z. For simplicity in derivation, we assume all three components of **A** to be positive. Consider **Figure 2.23**, which shows the projection of a small loop on the x–y plane (from **Figure 2.22**). The circulation along the closed contour *abcda* is calculated as follows:

Figure 2.21 Relation between vector **A** and its curl



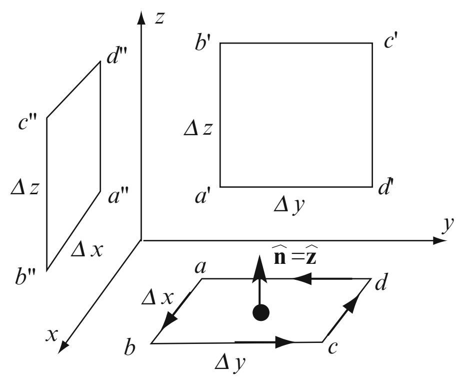
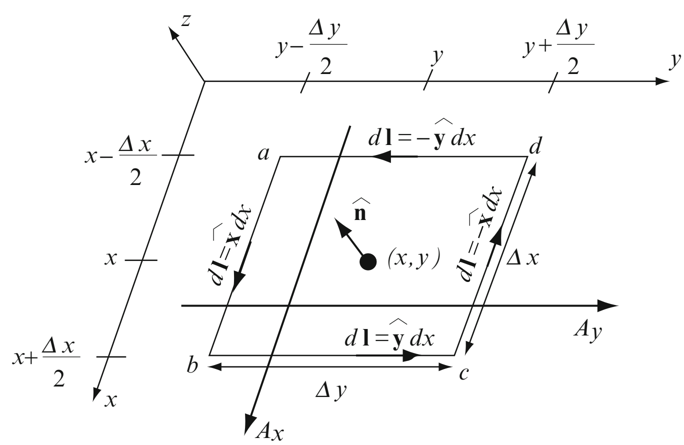


Figure 2.22 Projections of a general loop onto the x–y, x–z, and y–z planes

Figure 2.23



The projection of the vector **A** onto the x–y plane has x and y components: $\mathbf{A}_{xy} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y$. Along ab, $d\mathbf{l} = \hat{\mathbf{x}}dx$ and A_x remains constant (because Δx is very small). The circulation along this segment is

$$\int_a^b \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y\right) \cdot \hat{\mathbf{x}}dx = \int_a^b A_x\,dx = A_x\left(x, y - \frac{\Delta y}{2}, 0\right)\Delta x \approx \left(A_x(x,y,0) - \frac{\Delta y}{2}\frac{\partial A_x(x,y,0)}{\partial y}\right)\Delta x \tag{2.89}$$

The approximation in the parentheses on the right-hand side is the truncated Taylor series expansion of $A_x(x,y - \Delta y/2,0)$ around the point $P(x,y,0)$, as described in **Eqs.** (**2.56**) through (**2.58**).

Along segment bc, $d\mathbf{l} = \hat{\mathbf{y}}dy$ and we assume A_y remains constant. The circulation along this segment is

$$\int_b^c \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y\right) \cdot (\hat{\mathbf{y}}dy) = \int_b^c A_y dy = A_y\left(x + \frac{\Delta x}{2}, y, 0\right)\Delta y \approx \left(A_y(x,y,0) + \frac{\Delta x}{2}\frac{\partial A_y(x,y,0)}{\partial x}\right)\Delta y \tag{2.90}$$

Along segment cd, $d\mathbf{l} = -\hat{\mathbf{x}}dx$ and we get

$$\int_c^d \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y\right)\cdot(-\hat{\mathbf{x}}dx) = -\int_c^d A_x dx = -A_x\left(x, y+\frac{\Delta y}{2}, 0\right)\Delta x \approx -\left(A_x(x,y,0) + \frac{\Delta y}{2}\frac{\partial A_x(x,y,0)}{\partial y}\right)\Delta x \qquad (2.91)$$

Finally, along segment da, $d\mathbf{l} = -\hat{\mathbf{y}}dy$ and we get

$$\int_d^a \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y\right)\cdot(-\hat{\mathbf{y}}dy) = -\int_d^a A_y dy = -A_y\left(x-\frac{\Delta x}{2}, y, 0\right)\Delta y \approx -\left(A_y(x,y,0) - \frac{\Delta x}{2}\frac{\partial A_y(x,y,0)}{\partial x}\right)\Delta y \qquad (2.92)$$

The total circulation is the sum of the four segments calculated above:

$$\oint_{abcda} \mathbf{A}\cdot d\mathbf{l} \approx -\Delta x\Delta y\frac{\partial A_x(x,y,0)}{\partial y} + \Delta x\Delta y\frac{\partial A_y(x,y,0)}{\partial x} \qquad (2.93)$$

If we now take the limit in **Eq. (2.87)** but only on the surface $\Delta x\Delta y$, we get the component of the curl perpendicular to the x–y plane. Dividing **Eq. (2.93)** by $\Delta x\Delta y$ and taking the limit $\Delta x\Delta y \to 0$ gives

$$(\text{curl}\,\mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad (2.94)$$

As indicated above, this is the scalar component of the curl in the z direction since the normal $\hat{\mathbf{n}}$ to $\Delta x\Delta y$ is in the positive z direction.

The other two components are obtained in exactly the same manner. We give them here without repeating the process (see **Exercise 2.9**). The scalar component of the curl in the x direction is obtained by finding the total circulation around the loop $a'd'c'b'a'$ in the y–z plane in **Figure 2.22** and then taking the limit in **Eq. (2.87)**:

$$(\text{curl}\,\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad (2.95)$$

Similarly, the scalar component of the curl in the y direction is found by calculating the circulation around loop $a''d''c''b''a''$ in the x–z plane in **Figure 2.22**, and then taking the limit in **Eq. (2.87)**:

$$(\text{curl}\,\mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad (2.96)$$

The curl of the vector **A** in Cartesian coordinates can now be written from **Eqs. (2.94)** through **(2.96)** and **Eq. (2.88)** as follows:

$$\text{curl}\,\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad (2.97)$$

The common notation for the curl of a vector **A** is $\nabla \times \mathbf{A}$ (read: del cross **A**), and we write

$$\boxed{\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)} \qquad (2.98)$$

As with divergence, this does not imply a vector product,[2] only a notation to the operation on the right-hand side of **Eq. (2.98)**. Because of the form in **Eq. (2.98)**, the curl can be written as a determinant. The purpose in doing so is to avoid the need of remembering the expression in **Eq. (2.98)**. In this form, we write:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{2.99}$$

The latter is particularly useful as a quick way of writing the curl. Again, it should be remembered that the curl is not a determinant: only that the determinant in **Eq. (2.99)** may be used to write the expression in **Eq. (2.98)**.

The curl can also be evaluated in exactly the same manner in cylindrical and spherical coordinates. We will not do so but merely list the expressions.

In cylindrical coordinates:

$$\boxed{\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)} \tag{2.100}$$

In spherical coordinates:

$$\boxed{\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{R^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} R & \hat{\boldsymbol{\phi}} R\sin\theta \\ \partial/\partial R & \partial/\partial \theta & \partial/\partial \phi \\ A_R & RA_\theta & R\sin\theta A_\phi \end{vmatrix} \\ &= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right) + \hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial R}\right) + \hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right) \end{aligned}} \tag{2.101}$$

Now that we have proper definitions of the curl and the methods of evaluating it, we must return to the physical meaning of the curl. First, we note the following properties of the curl:

(1) The curl of a vector field is a vector field.
(2) The magnitude of the curl gives the maximum circulation of the vector per unit area at a point.
(3) The direction of the curl is along the normal to the area of maximum circulation at a point.
(4) The curl has the general properties of the vector product: it is distributive but not associative

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \quad \text{and} \quad \nabla \times (\mathbf{A} + \mathbf{B}) \neq (\nabla \times \mathbf{A}) \times \mathbf{B} \tag{2.102}$$

(5) The divergence of the curl of any vector function is identically zero:

$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0 \tag{2.103}$$

(6) The curl of the gradient of a scalar function is also identically zero for any scalar:

$$\nabla \times (\nabla V) \equiv 0 \tag{2.104}$$

The latter two can be shown to be correct by direct evaluation of the products involved (see **Exercises 2.10** and **2.11**). These two identities play a very important role in electromagnetics and we will return to them later on in this chapter.

[2] In Cartesian coordinates, the curl is equal to the cross product between the ∇ operator and the vector **A**, but this is not true in other systems of coordinates (see also footnote 1 on page 71).

To summarize the discussion up to this point, you may view the curl as an indication of the rotation or circulation of the vector field calculated at any point. Zero curl indicates no rotation and the vector field can be generated by a scalar source alone. A general vector field with nonzero curl may only be generated by a scalar source (the divergence of the field) and a vector source (the curl of the field). Some vector fields may have zero divergence and nonzero curl. Thus, in this sense, the curl of a vector field is also an indication of the source of the field, but this source is a vector source. In the context of fluid flow, a curl is an indicator of nonuniform flow, whereas the divergence of the field only shows the scalar distribution of its sources. However, you should be careful with the idea of rotation. Rotation in the field does not necessarily mean that the field itself is circular; it only means that the field causes a circulation. The example of the stick thrown into the river given above explains this point. The following examples also dwell on this and other physical points associated with the curl.

Example 2.17 Vector $\mathbf{A} = \hat{\mathbf{R}}\,2\cos\theta - \hat{\boldsymbol{\theta}}\,3R\sin\theta$ is given. Find the curl of $\mathbf{A}$.

Solution: We apply the curl in spherical coordinates using **Eq. (2.101)**. In this case, we perform the calculation for each scalar component separately:

$$(\nabla\times\mathbf{A})_R = \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{1}{R\sin\theta}\frac{\partial A_\theta}{\partial\phi} = \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0) - \frac{1}{R\sin\theta}\frac{\partial(-3R\sin\theta)}{\partial\phi} = 0$$

$$(\nabla\times\mathbf{A})_\theta = \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right) = \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial(2R\cos\theta)}{\partial\phi} - \frac{\partial(R(0))}{\partial R}\right) = 0$$

$$(\nabla\times\mathbf{A})_\phi = \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right) = \frac{1}{R}\frac{\partial}{\partial R}(-3R^2\sin\theta) - \frac{1}{R}\frac{\partial}{\partial\theta}(2R\cos\theta) = -6\sin\theta + 2\sin\theta = -4\sin\theta$$

Combining the components, the curl of $\mathbf{A}$ is

$$\nabla\times\mathbf{A} = -\hat{\boldsymbol{\phi}}\,4\sin\theta.$$

Example 2.18 Application: Nonuniform Flow A fluid flows in a channel of width $2d$ with a velocity profile given by $\mathbf{v} = \hat{\mathbf{y}}\,v_0(d - |x|)$.

(a) Calculate the curl of the velocity.
(b) How can you explain the fact that circulation of the flow is nonzero while the water itself flows in a straight line (see **Figure 2.24**)?
(c) What is the direction of the curl? What does this imply for an object floating on the water (such as a long stick)? Explain.

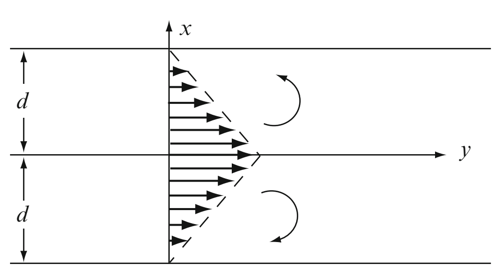


Figure 2.24 A vector field with nonzero curl. If this were a flow, a short stick placed perpendicular to the flow would rotate as shown

Solution: We calculate the curl of $\mathbf{v}$ using **Eq.** (**2.98**). Even though there is only one component of the vector, this component depends on another variable. This means the curl is nonzero.

(a) Since the velocity depends on the absolute value of x, we separate the problem into three parts: one describes the solution for $x > 0$, the second for $x = 0$, the third for $x < 0$:

$$\begin{aligned} \mathbf{v} &= \hat{\mathbf{y}}\, v_0(d+x), \quad && x < 0, \\ \mathbf{v} &= \hat{\mathbf{y}}\, v_0(d-x), \quad && x > 0, \\ \mathbf{v} &= \hat{\mathbf{y}}\, v_0 d, && x = 0 \end{aligned}$$

For $x < 0$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & v_0(d+x) & 0 \end{vmatrix} = \hat{\mathbf{z}} \left(\frac{\partial(v_0(d+x))}{\partial x} \right) = \hat{\mathbf{z}}\, v_0$$

For $x > 0$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & v_0(d-x) & 0 \end{vmatrix} = \hat{\mathbf{z}} \left(\frac{\partial(v_0(d-x))}{\partial x} \right) = -\hat{\mathbf{z}}\, v_0$$

and for $x = 0$, $\nabla \times \mathbf{v} = 0$. Thus,

$$\nabla \times \mathbf{v} = \begin{cases} \hat{\mathbf{z}}\, v_0 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ -\hat{\mathbf{z}}\, v_0 & \text{for} \quad x > 0. \end{cases}$$

(b) The curl implies neither multiple components nor a rotating vector, only that the vector varies in space. If the flow velocity were constant, the curl would be zero.

(c) This particular flow is unique in that the curl changes direction at $x = 0$. It is in the positive z direction for $x < 0$ and in the negative z direction for $x > 0$. Thus, if we were to place a stick anywhere in the positive part of the x axis, the stick will turn counterclockwise until it aligns itself with the flow (assuming a very thin stick). If the stick is placed in the negative part of the x axis, it will turn clockwise to align with the flow (see **Figure 2.24**).

Exercise 2.9 Following the steps in **Eqs.** (**2.87**) through (**2.94**), derive the terms (curl $\mathbf{A}$)$_x$ and (curl $\mathbf{A}$)$_y$ as defined in **Eq.** (**2.88**).

Exercise 2.10 Show by direct evaluation that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ [**Eq.** (**2.103**)] for any general vector **A.** Use Cartesian coordinates.

Exercise 2.11 Show by direct evaluation that $\nabla \times (\nabla V) = 0$ [**Eq.** (**2.104**)] for any general scalar function V. Use Cartesian coordinates.

2.3.5 Stokes'[3] Theorem

Stokes' theorem is the second theorem in vector algebra we introduce. It is in a way similar to the divergence theorem but relates to the curl of a vector. Stokes' theorem is given as

$$\boxed{\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_L \mathbf{A} \cdot d\mathbf{l}} \tag{2.105}$$

It relates the open surface integral of the curl of vector **A** over a surface s to the closed contour integral of the vector **A** over the contour enclosing the surface s. To show that this relation is correct, we will use the relations derived from the curl and recall that curl is circulation per unit area.

Consider again the components of the curl in **Eqs. (2.94)** through **(2.96)**. These were derived for the rectangular loops in **Figure 2.22**. Now, we argue as follows: The total circulation of the vector **A** around a general loop *ABCDA* is the sum of the circulations over its projections on the x–y, x–z, and y–z planes as was shown in **Figure 2.23**. That this is correct follows from the fact that the circulation is calculated from a scalar product. Thus, we can write the total circulation around the elementary loops of **Figure 2.22** using **Eq. (2.105)** as

$$\begin{aligned} C_{ABCDA} &= C_{abcda} + C_{a'd'c'b'a'} + C_{a''d''c''b''a''} \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\Delta y \Delta z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\Delta x \Delta z + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x \Delta y \end{aligned} \tag{2.106}$$

or

$$C_{ABCDA} = (\nabla \times \mathbf{A})_x \Delta s_x + (\nabla \times \mathbf{A})_y \Delta s_y + (\nabla \times \mathbf{A})_z \Delta s_z \tag{2.107}$$

where the indices x, y, and z indicate the scalar components of the vectors $\nabla \times \mathbf{A}$ and $\Delta\mathbf{s}$. The use of $\Delta\mathbf{s}$ in this fashion is permissible since $\Delta y \Delta z$ is perpendicular to the x coordinate and, therefore, can be written as a vector component: $\hat{\mathbf{x}}\,\Delta y \Delta z$, similarly for the other two projections. Thus, we can write the circulation around a loop of area Δs (assuming $\nabla \times \mathbf{A}$ is constant over Δs) as

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = (\nabla \times \mathbf{A}) \cdot \Delta\mathbf{s} \tag{2.108}$$

Now, suppose that we need to calculate the circulation around a closed contour L enclosing an area s as shown in **Figure 2.25a**. To do so, we divide the area into small square loops, each of area Δs as shown in **Figure 2.25b**. As can be seen, every two neighboring contours have circulations in opposite directions on the connecting sides. This means that the circulations on each two connected sides must cancel. The only remaining, nonzero terms in the circulations are due to the

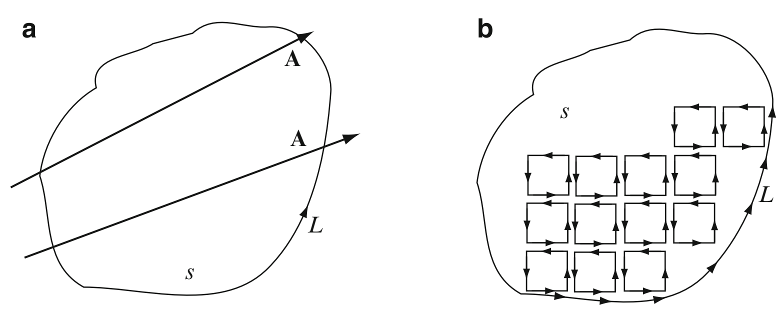


Figure 2.25 Stokes' theorem. (**a**) Vector field **A** and an open surface s. (**b**) The only components of the contour integrals on the small loops that do not cancel are along the outer contour L

[3] After Sir George Gabriel Stokes (1819–1903). Stokes was one of the great mathematical physicists of the nineteenth century. His work spanned many disciplines including propagation of waves in materials, water waves, optics, polarization of light, luminescence, and many others. The theorem bearing his name is one of the more useful relations in electromagnetics.

outer contour. Letting the area Δs be a differential area ds (i.e., let Δs tend to zero), the total circulation is

$$\sum_{i=1}^{\infty} \oint_{L_i} \mathbf{A} \cdot d\mathbf{l}_i = \oint_L \mathbf{A} \cdot d\mathbf{l} \tag{2.109}$$

The right-hand side of **Eq. (2.108)** becomes

$$\lim_{\left|\Delta s_i\right| \to 0} \sum_{i=1}^{\infty} (\nabla \times \mathbf{A})_i \cdot \Delta \mathbf{s}_i = \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} \tag{2.110}$$

Equating **Eqs. (2.109)** and **(2.110)** gives Stokes' theorem in **Eq. (2.105)**.

Example 2.19 Verify Stokes' theorem for the vector field $\mathbf{A} = \hat{\mathbf{x}}\,(2x - y) - \hat{\mathbf{y}}\,2yz^2 - \hat{\mathbf{z}}\,2zy^2$ on the upper half-surface of the sphere $x^2 + y^2 + z^2 = 4$ (above the x–y plane), where the contour C is its boundary (rim of surface in the x–y plane).

Solution: To verify the theorem, we perform surface integration of the curl of **A** on the surface and closed contour integration of $\mathbf{A} \cdot d\mathbf{l}$ along C and show they are the same.

From **Eq. (2.98)**, with $A_x\,(2x - y)$, $A_y = -2yz^2$, and $A_z = -2zy^2$,

$$\begin{aligned}\nabla \times \mathbf{A} &= \hat{\mathbf{x}} \left(\frac{\partial(-2zy^2)}{\partial y} - \frac{\partial(-2yz^2)}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial(2x - y)}{\partial z} - \frac{\partial(-2zy^2)}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial(-2zy^2)}{\partial x} - \frac{\partial(2x - y)}{\partial y} \right) \\ &= \hat{\mathbf{x}}\,(-4yz + 4yz) + \hat{\mathbf{y}}\,(0 - 0) + \hat{\mathbf{z}}\,(0 + 1) = \hat{\mathbf{z}}\,1\end{aligned}$$

Surface Integral: We write the differential of surface on the sphere as $d\mathbf{s} = \hat{\mathbf{R}}\,R^2 \sin\theta\, d\theta\, d\phi$:

$$\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_s \hat{\mathbf{z}} \cdot \hat{\mathbf{R}}\,R^2 \sin\theta d\theta d\phi = R^2 \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} \cos\theta \sin\theta d\theta d\phi$$

$$= 2\pi(2)^2 \int_{\theta=0}^{\theta=\pi/2} \cos\theta \sin\theta d\theta = 8\pi \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2} \sin 2\theta d\theta = 8\pi \left[-\frac{\cos 2\theta}{4} \right]_0^{\pi/2} = 4\pi$$

where $\hat{\mathbf{z}} \cdot \hat{\mathbf{R}} = \cos\theta$ [from **Eq. (2.45)**], $R = 2$ and $\sin\theta\cos\theta = (1/2)sin2\theta$.

Contour Integral: $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$, and using $z = 0$ on C ($x - y$ plane),

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C [\hat{\mathbf{x}}\,(2x - y)] \cdot [\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy]$$

Using cylindrical coordinates,

$$x = r\cos\phi = 2\cos\phi, \quad y = r\sin\phi = 2\sin\phi$$

and

$$\frac{dx}{d\phi} = -r\sin\phi \quad \rightarrow \quad dx = -2\sin\phi\, d\phi$$

Thus,

$$\begin{aligned}\oint_C \mathbf{A} \cdot d\mathbf{l} &= \oint_C (2x - y)dx = \int_0^{2\pi} \big(2(2\cos\phi) - 2\sin\phi\big)\big(-2\sin\phi\, d\phi\big) \\ &= -8\int_0^{2\pi} \cos\phi \sin\phi\, d\phi + 4\int_0^{2\pi} \sin^2\phi d\phi = 0 + 4\pi = 4\pi\end{aligned}$$

and Stokes' theorem is verified since

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}}\, d\mathbf{s} = 4\pi$$

2.4 Conservative and Nonconservative Fields

A vector field is said to be conservative if the closed contour integral for any contour L in the field is zero (see also **Section 2.2.1**). It also follows from Stokes' theorem that the required condition is that the curl of the field must be zero:

$$\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_L \mathbf{A} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad \nabla \times \mathbf{A} = 0 \tag{2.111}$$

To see if a field is conservative, we can either show that the closed contour integral on any contour is zero or that its curl is zero. The latter is often easier to accomplish. Since the curl can be shown to be zero or nonzero in general (unlike a contour integral), the curl is the only true measure of the conservative property of the field.

Example 2.20 Two vectors $\mathbf{F}_1 = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}z^2 - \hat{\mathbf{z}}2(zy+1)$ and $\mathbf{F}_2 = \hat{\mathbf{x}}x^2y - \hat{\mathbf{y}}z^2 - \hat{\mathbf{z}}2(zy+1)$ are given. Show that $\mathbf{F}_1$ is conservative and $\mathbf{F}_2$ is nonconservative.

Solution: To show that a vector field $\mathbf{F}$ is conservative, it is enough to show that its curl is zero. Similarly, for a vector field to be nonconservative, its curl must be nonzero.

The curls of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\nabla \times \mathbf{F}_1 = \hat{\mathbf{x}}\left(\frac{\partial F_{1z}}{\partial y} - \frac{\partial F_{1y}}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial F_{1x}}{\partial z} - \frac{\partial F_{1z}}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial F_{1y}}{\partial x} - \frac{\partial F_{1x}}{\partial y}\right) = \hat{\mathbf{x}}(-2z+2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) = 0$$

$$\nabla \times \mathbf{F}_2 = \hat{\mathbf{x}}(-2z+2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-x^2) = -\hat{\mathbf{z}}x^2$$

Thus, $\mathbf{F}_1$ is a conservative vector field, whereas $\mathbf{F}_2$ is clearly nonconservative.

2.5 Null Vector Identities and Classification of Vector Fields

After discussing most properties of vector fields and reviewing vector relations, we are now in a position to define broad classes of vector fields. This, again, is done in preparation of discussion of electromagnetic fields. This classification of vector fields is based on the curl and divergence of the fields and is described by the Helmholtz theorem. Before doing so, we wish to discuss here two particular vector identities because these are needed to define the Helmholtz theorem and because they are fundamental to understanding of electromagnetics. These are

$$\boxed{\nabla \times (\nabla V) \equiv 0} \tag{2.112}$$

$$\boxed{\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0} \tag{2.113}$$

Both identities were mentioned in **Section 2.3.4.1** in the context of properties of the curl of a vector field and are sometimes called the ***null identities***. These can be shown to be correct in any system of coordinates by direct evaluation and performing the prescribed operations (see **Exercises 2.10** through **2.12**). The first of these indicates that the curl of the gradient of any scalar field is identically zero. This may be written as

$$\nabla \times (\nabla V) = \nabla \times \mathbf{C} \equiv 0 \tag{2.114}$$

In other words, if a vector **C** is equal to the gradient of a scalar, its curl is always zero. The converse is also true, if the curl of a vector field is zero, it can be written as the gradient of a scalar field:

$$\boxed{\text{If } \nabla \times \mathbf{C} = 0 \quad \rightarrow \quad \mathbf{C} = \nabla V \quad \text{or} \quad \mathbf{C} = -\nabla V} \tag{2.115}$$

Not all vector fields have zero curl, but if the curl of a vector field happens to be zero, then the above form can be used because $\nabla \times \mathbf{C}$ is zero. This type of field is called a ***curl-free field*** or an ***irrotational field***. Thus, we say that an irrotational field can always be written as the gradient of a scalar field. In the context of electromagnetics, we will use the second form in **Eq. (2.115)** by convention.

To understand the meaning of an irrotational field, consider the Stokes' theorem for the irrotational vector field **C** defined in **Eq. (2.115)**:

$$\int_s (\nabla \times \mathbf{C}) \cdot d\mathbf{s} = \oint_L \mathbf{C} \cdot d\mathbf{l} = 0 \tag{2.116}$$

This means that the closed contour integral of an irrotational field is identically zero; that is, an irrotational field is a conservative field. A simple example of this type of field is the gravitational field: if you were to drop a weight down the stairs and lift it back up the stairs to its original location, the weight would travel a closed contour. Although you may have performed strenuous work, the potential energy of the weight remains unchanged and this is independent of the path you take.

The second identity states that the divergence of the curl of any vector field is identically zero. Since the curl of a vector is a vector, we may substitute $\nabla \times \mathbf{A} = \mathbf{B}$ in **Eq. (2.113)** and write

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot (\mathbf{B}) \equiv 0 \tag{2.117}$$

This can also be stated as follows: If the divergence of a vector field **B** is zero, this vector field can be written as the curl of another vector field **A**:

$$\boxed{\text{If } \quad \nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}} \tag{2.118}$$

The vector field **B** is a special field: It has zero divergence. For this reason we call it a ***divergence-free*** or ***divergenceless field***. This type of vector field is also called ***solenoidal***.[4] We will not try to explain this term at this point; the source of the name is rooted in electromagnetic theory. We will eventually understand its meaning, but for now we simply take this as a name for divergence-free fields.

The foregoing can also be stated mathematically by using the divergence theorem:

$$\int_v (\nabla \cdot \mathbf{B}) dv = \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \tag{2.119}$$

This means that the total flux of the vector **B** through any closed surface is zero or, alternatively, that the net outward flux in any volume is zero or that the inward flux is equal to the outward flux, indicating that there are no net sources or sinks inside any arbitrary volume in the field.

Exercise 2.12 Using cylindrical coordinates, show by direct evaluation that for any scalar function Ψ and vector **A**,

$$\nabla \times (\nabla \Psi) = 0 \text{ and } \nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

[4] The term solenoid was coined by Andre Marie Ampere from the Greek *solen* = channel and *eidos* = form. When he built the first magnetic coil, in 1820, he gave it the name solenoid because the spiral wires in the coil reminded him of channels.

2.5.1 The Helmholtz[5] Theorem

After defining the properties of vector fields, we can now summarize these properties and draw some conclusions. In the process, we will also classify vector fields into groups, using the Helmholtz theorem which is based on the divergence and curl of the vector fields.

The Helmholtz theorem states:

"*A vector field is uniquely defined (within an additive constant) by specifying its divergence and its curl.*"

That this must be so follows from the fact that, in general, specification of the sources of a field should be sufficient to specify the vector field. Although we could go into a mathematical proof of this theorem (which also requires imposition of conditions on the vector such as continuity of derivatives and the requirement that the vector vanishes at infinity), we will accept this theorem and look at its meaning. The Helmholtz theorem is normally written as

$$\boxed{\mathbf{B} = -\nabla U + \nabla \times \mathbf{A}} \tag{2.120}$$

where U is a scalar field and $\mathbf{A}$ is a vector field. That is, any vector field can be decomposed into two terms; one is the gradient of a scalar function and the other is the curl of a vector function. The vector $\mathbf{B}$ must be defined in terms of its curl and divergence. The divergence of $\mathbf{B}$ is given as

$$\nabla \cdot \mathbf{B} = \nabla \cdot (-\nabla U) + \nabla \cdot (\nabla \times \mathbf{A}) \tag{2.121}$$

The second term on the right-hand side is zero from the identity in **Eq. (2.113)**. The first term is, in general, a nonzero scalar density function and we may denote it as ρ:

$$\boxed{\nabla \cdot \mathbf{B} = \rho} \tag{2.122}$$

Because $\nabla \cdot \mathbf{B} \neq 0$, this is a nonsolenoidal field.

The curl of the vector $\mathbf{B}$ is

$$\nabla \times \mathbf{B} = \nabla \times (-\nabla U) + \nabla \times (\nabla \times \mathbf{A}) \tag{2.123}$$

Now, the first term is zero from the identity in **Eq. (2.112)**. The second term is a nonzero vector that will be denoted here as a general vector $\mathbf{J}$.

$$\boxed{\nabla \times \mathbf{B} = \mathbf{J}} \tag{2.124}$$

$\mathbf{J}$ may be regarded as the strength of the vector source. Since $\nabla \times \mathbf{B} \neq 0$, this vector field is a rotational field.

A general field will have both nonzero curl and nonzero divergence; that is, the field is both rotational and nonsolenoidal. There are, however, fields in which the curl or the divergence or both are zero. In all, there are four types of fields that can be defined:

(1) A nonsolenoidal, rotational vector field. $\nabla \cdot \mathbf{B} = \rho$ and $\nabla \times \mathbf{B} = \mathbf{J}$. This is the most general vector field possible. The field has both a scalar and a vector source.
(2) A nonsolenoidal, irrotational vector field. $\nabla \cdot \mathbf{B} = \rho$ and $\nabla \times \mathbf{B} = 0$. The vector field has only a scalar source.
(3) A solenoidal, rotational vector field. $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mathbf{J}$. The vector field has only a vector source.
(4) A solenoidal, irrotational vector field. $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. The vector field has no scalar or vector sources.

[5] Hermann Ludwig Ferdinand von Helmholtz (1821–1894). Helmholtz was one of the most prolific of the scientists of the nineteenth century. His work encompasses almost every aspect of science as well as philosophy. Perhaps his best known contribution is his statement of the law of conservation of energy. However, he is also the inventor of the ophthalmoscope—an instrument used to this day in testing eyesight. He contributed considerably to optics and physiology of vision and hearing. His work *On the Sensation of Tone* defines tone in terms of harmonics. In addition, he worked on mechanics, hydrodynamics, as well as electromagnetics. In particular, he was the person to suggest to his student Heinrich Hertz the experiments that led to the discovery of the propagation of electromagnetic waves, and started the age of communication.

The study of electromagnetics will be essentially that of defining the conditions and properties of the foregoing four types of fields. We start in **Chapter 3** with the static electric field, which is a nonsolenoidal, irrotational field (type 2 above). These properties, the curl and the divergence of the vector field, will be the basis of study of all fields.

2.5.2 Second-Order Operators

The del operator as well as the gradient, divergence, and curl are first-order operators; the result is first-order partial derivatives of the scalar or vector functions. It is possible to combine two first-order operators operating on scalar function U and vector function $\mathbf{A}$. By doing so, we obtain second-order expressions, some of which are very useful. The valid combinations are

$$\nabla \cdot (\nabla U) \quad \text{(divergence of the gradient of } U) \tag{2.125}$$

$$\nabla \times (\nabla U) \quad \text{(curl of the gradient of U)} \tag{2.126}$$

$$\nabla(\nabla \cdot \mathbf{A}) \quad \text{(gradient of the divergence of } \mathbf{A}) \tag{2.127}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) \quad \text{(divergence of the curl of } \mathbf{A}) \tag{2.128}$$

$$\nabla \times (\nabla \times \mathbf{A}) \quad \text{(curl of the curl of } \mathbf{A}) \tag{2.129}$$

The scalar product $\nabla \cdot (\nabla U)$ [**Eq. (2.125)**] can be calculated by direct derivation using the gradient of the scalar function U. In Cartesian coordinates, the gradient is given in **Eq. (2.32)**:

$$\nabla U(x, y, z) = \hat{\mathbf{x}} \frac{\partial U(x, y, z)}{\partial x} + \hat{\mathbf{y}} \frac{\partial U(x, y, z)}{\partial y} + \hat{\mathbf{z}} \frac{\partial U(x, y, z)}{\partial z} \tag{2.130}$$

The divergence of ΔU (x, y, z) is now written using **Eq. (2.71)**:

$$\nabla \cdot (\nabla U) = \frac{\partial(\nabla U(x, y, z))_x}{\partial x} + \frac{\partial(\nabla U(x, y, z))_y}{\partial y} + \frac{\partial(\nabla U(x, y, z))_z}{\partial z} = \frac{\partial^2 U(x, y, z)}{\partial x^2} + \frac{\partial^2 U(x, y, z)}{\partial y^2} + \frac{\partial^2 U(x, y, z)}{\partial z^2} \tag{2.131}$$

or, in short-form notation

$$\boxed{\nabla \cdot (\nabla U) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}} \tag{2.132}$$

From this, we can define the scalar ***Laplace operator*** (or, in short, the ***Laplacian***) as

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \tag{2.133}$$

In cylindrical and spherical coordinates, we must start with the components of the vector ∇U in the corresponding system and calculate the divergence of the vector as in **Section 2.3.2** (see **Exercise 2.4** and **Examples 2.11** and **2.15**). The result is as follows:

In cylindrical coordinates:

$$\boxed{\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}} \tag{2.134}$$

In spherical coordinates:

$$\boxed{\nabla^2 U = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial U}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}} \tag{2.135}$$

The expressions $\nabla \times (\nabla U)$ and $\nabla \cdot (\nabla \times \mathbf{A})$ are the null identities discussed in **Section 2.5**. Finally, **Eqs. (2.127)** and **(2.129)** are often used together using the vector identity:

$$\boxed{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})} \tag{2.136}$$

where $\nabla^2 \mathbf{A}$ is called the ***vector Laplacian*** of **A**. This is written in Cartesian coordinates as

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}}\, \nabla^2 A_x + \hat{\mathbf{y}}\, \nabla^2 A_y + \hat{\mathbf{z}}\, \nabla^2 A_z \tag{2.137}$$

and can be obtained by direct application of the scalar Laplacian operator to each of the scalar components of the vector **A**. The scalar components of the vector Laplacian are

$$\begin{aligned} \nabla^2 A_x &= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}, \\ \nabla^2 A_y &= \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}, \\ \nabla^2 A_z &= \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \end{aligned} \tag{2.138}$$

The second-order operators define second-order partial differential equations and constitute a very important area in mathematics and physics. We will use the second-order operators described here throughout this book.

The scalar and vector Laplacians as well as other vector quantities and identities in Cartesian, cylindrical, and spherical coordinate systems are listed in the appendix for easy reference.

Exercise 2.13

(a) Show that in Cartesian coordinates, the following is correct:

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}}\, \nabla^2 A_x + \hat{\mathbf{y}}\, \nabla^2 A_y + \hat{\mathbf{z}}\, \nabla^2 A_z.$$

(b) Show that in any other coordinate system, this relation is not correct. Use the cylindrical system as an example; that is, show that $\nabla^2 \mathbf{A} \neq \hat{\mathbf{r}}\, \nabla^2 A_r + \hat{\boldsymbol{\phi}}\, \nabla^2 A_\phi + \hat{\mathbf{z}}\, \nabla^2 A_z$.

2.5.3 Other Vector Identities

If U and Q are scalar functions and **A** and **B** are vector functions, all dependent on the three variables (for example, x, y, and z), we can show that

$$\nabla(UQ) = U(\nabla Q) + Q(\nabla U) \tag{2.139}$$

$$\nabla \cdot (U\mathbf{A}) = U(\nabla \cdot \mathbf{A}) + (\nabla U) \cdot \mathbf{A} \tag{2.140}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{A} \cdot (\nabla \times \mathbf{B}) + (\nabla \times \mathbf{A}) \cdot \mathbf{B} \tag{2.141}$$

$$\nabla \times (U\mathbf{A}) = U(\nabla \times \mathbf{A}) + (\nabla U) \times \mathbf{A} \tag{2.142}$$

Problems

2.1 A force is described in cylindrical coordinates as $\mathbf{F} = \hat{\boldsymbol{\phi}}/r$. Find the work performed by the force along the following paths:

(a) From $P(a,0,0)$ to $P(a,b,c)$.
(b) From $P(a,0,0)$ to $P(a,b,0)$, and then from $P(a,b,0)$ to $P(a,b,c)$.

2.2 Determine whether $\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{l}$ between points p_1 (0,0,0) and p_2(1,1,1) is path dependent for $\mathbf{A} = \hat{\mathbf{x}}\,y^2 + \hat{\mathbf{y}}\,2x + \hat{\mathbf{z}}$.

2.3 A body is moved along the path shown in **Figure 2.26** by a force $\mathbf{A} = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,5$. The path between point a and point b is a parabola described by $y = 2x^2$.

(a) Calculate the work necessary to move the body from point a to point b along the parabola.
(b) Calculate the work necessary to move the body from point a to point c and then to point b.
(c) Compare the results in **(a)** and **(b)**.

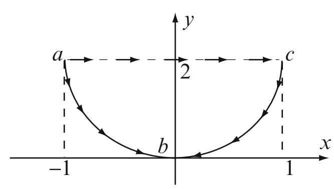


Figure 2.26

Surface Integrals (Closed and Open)

2.4 A volume is defined in cylindrical coordinates as $1 \le r \le 2$, $\pi/6 \le \phi \le \pi/3$, $1 \le z \le 2$. Calculate the flux of the vector $\mathbf{A} = \hat{\mathbf{r}}\,4z$ through the surface enclosing the given volume.

2.5 Given a surface $S = S_1 + S_2$ defined in spherical coordinates with S_1 defined as $0 \le R \le 1$; $\theta = \pi/6$; $0 \le \phi \le 2\pi$ and S_2 defined as $R = 1$; $0 \le \theta \le \pi/6$; $0 \le \phi \le 2\pi$. Vector $\mathbf{A} = \hat{\mathbf{R}}\,1 + \hat{\boldsymbol{\theta}}\,\theta$ is given. Find the integral of $\mathbf{A} \cdot d\mathbf{s}$ over the surface S.

2.6 Given $\mathbf{A} = \hat{\mathbf{x}}\,x^2 + \hat{\mathbf{y}}\,y^2 + \hat{\mathbf{z}}\,z^2$, integrate $\mathbf{A} \cdot d\mathbf{s}$ over the surface of the cube of side 1 with four of its vertices at (0,0,0), (0,0,1), (0,1,0), and (1,0,0).

2.7 The axis of a disk of radius a is in the direction of the vector $\mathbf{k} = \hat{\mathbf{z}}\,3$. Vector field $\mathbf{A} = \hat{\mathbf{r}}\,5 + \hat{\mathbf{z}}\,3$ is given. Find the total flux of $\mathbf{A}$ through the disk.

Volume Integrals

2.8 A mass density in space is given by $\rho(r, z) = r(r + a) + z(z + d)$ kg/m^3 (in cylindrical coordinates).

(a) Calculate the total mass of a cylinder of length d, radius a, centered at the origin with its axis along the z axis.
(b) Calculate the total mass of a sphere of radius a centered at the origin.

2.9 A right circular cone is cut off at height h_0. The radius of the small base is a and that of the large base is b (**Figure 2.27**). The cone is filled with particles in a nonuniform distribution: $n(r, h) = 10^5 r^3 + 10^3 r(h - h_0)^2$. Find the total number of particles contained in the cone.

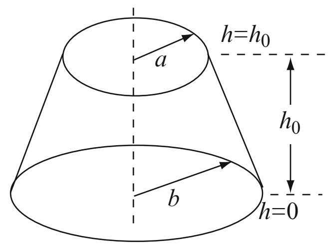


Figure 2.27

2.10 Vector field $\mathbf{f} = \hat{\mathbf{x}}\,2xy + \hat{\mathbf{y}}\,z + \hat{\mathbf{z}}\,y^2$ is defined as a volume force density (in N/m^3) in a sufficiently large region in space. This force acts on every particle of any body placed in the field (similar to a gravitational force).

(a) A cubic body $2 \times 2 \times 2$ m^3 in dimensions is placed in the field with its center at the origin and with its sides parallel to the system of coordinates. Calculate the total force acting on the body.
(b) The same cube as in **(a)** is placed in the first quadrant with one corner at the origin and with its sides parallel to the system of coordinates. Calculate the total force acting on the body.

Other Regular Integrals

2.11 The acceleration of a body is given as $\mathbf{a} = \hat{\mathbf{x}}\,(t^2 - 2t) + \hat{\mathbf{y}}\,3t$ [m/s^2]. Find the velocity of the body after 5 s.

2.12 Evaluate the integral $\int_C r^2\, d\mathbf{l}$, where $r^2 = x^2 + y^2$, from the origin to the point $P(1,3)$ along the straight line connecting the origin to $P(1,3)$. $d\mathbf{l}$ is the differential vector in Cartesian coordinates.

The Gradient

2.13 Find the derivative of $xy^2 + yz$ at (1,1,2) in the direction of the vector $\hat{\mathbf{x}}\,2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}\,2$.

2.14 An atmospheric pressure field is given as $P(x,y,z) = (x-2)^2 + (y-2)^2 + (z+1)^2$, where the x–y plane is parallel to the surface of the ocean and the z direction is vertical. Find:

(a) The magnitude and direction of the pressure gradient.
(b) The derivative of the pressure in the vertical direction.
(c) The derivative of pressure in the direction parallel to the surface, at 45° between the positive x and y axes.

2.15 The scalar field $f(r,\phi,z) = r\cos^2\phi + z\sin\phi$ is given. Calculate:

(a) The gradient of $f(r,\phi,z)$ in cylindrical coordinates.
(b) The gradient of $f(r,\phi,z)$ in Cartesian coordinates.
(c) The gradient of $f(r,\phi,z)$ in spherical coordinates.

2.16 Find the unit vector normal to the following planes:

(a) $z = -5x - 3y$.
(b) $4x - 3y + z + 5 = 0$.
(c) $z = ax + by$.

Show by explicit derivation that the result obtained is in fact normal to the plane.

2.17 Find the unit vector normal to the following surfaces:

(a) $z = -3xy - yz$.
(b) $x = z^2 + y^2$.
(c) $z^2 + y^2 + x^2 = 8$.

The Divergence

2.18 Calculate the divergence of the following vector fields:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}y^2$.
(b) $\mathbf{B} = \hat{\mathbf{r}}2z^2 + \hat{\boldsymbol{\phi}}5r - \hat{\mathbf{z}}3r^2$.
(c) $\mathbf{C} = \hat{\mathbf{x}}\sqrt{x^2 + z^2} + \hat{\mathbf{y}}\sqrt{x^2 + y^2}$.

2.19 Find the divergence of $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}z^2$ at (1,–1,2).

2.20 Find the divergence of $\mathbf{A} = \hat{\mathbf{r}}2r\cos\phi - \hat{\boldsymbol{\phi}}r\sin\phi + \hat{\mathbf{z}}4z$ at $(2,90°,1)$.

2.21 Find the divergence of $\mathbf{A} = 0.2R^3\phi\sin^2\theta\left(\hat{\mathbf{R}} + \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\right)$ at (2,30°,90).

The Divergence Theorem

2.22 Verify the divergence theorem for $\mathbf{A} = \hat{\mathbf{x}}4z - \hat{\mathbf{y}}2y^2 - \hat{\mathbf{z}}2z^2$ for the region bounded by $x^2 + y^2 = 9$ and $z = -2$, $z = 2$ by evaluating the volume and surface integrals.

2.23 A vector field is given as $\mathbf{A}(\mathbf{R}) = \mathbf{R}$, where $\mathbf{R}$ is the position vector of a point in space. Show that the divergence theorem applies to the vector $\mathbf{A}$ for a sphere of radius a.

2.24 Given $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}z^2$:

(a) Integrate $\mathbf{A} \cdot d\mathbf{s}$ over the surface of the cube of side 1 with four of its vertices at (0,0,0), (0,0,1), (0,1,0), and (1,0,0) (see **Problem 2.6**).
(b) Integrate $\nabla \cdot \mathbf{A}$ over the volume of the cube in **(a)** and show that the two results are the same.

The Curl

2.25 Calculate the curl of the following three vectors:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}y^2$.
(b) $\mathbf{B} = \hat{\mathbf{r}}2z^2 + \hat{\boldsymbol{\phi}}5r - \hat{\mathbf{z}}3r^2$.
(c) $\mathbf{C} = \hat{\mathbf{x}}\sqrt{x^2 + z^2} + \hat{\mathbf{y}}\sqrt{x^2 + y^2}$.

2.26 A fluid flows in a circular pattern with the velocity vector $\mathbf{v} = \hat{\boldsymbol{\phi}}/r$.

(a) Sketch the vector field $\mathbf{v}$.
(b) Calculate the curl of the vector field.

2.27 A vector field $\mathbf{A} = \hat{\mathbf{y}}3x\cos(\omega t + 50z)$ is given.

(a) What is the curl of $\mathbf{A}$?
(b) Is this a conservative field?

2.28 Find $\nabla \times \mathbf{A}$ for:

(a) $\mathbf{A} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}2(x+1)yz - \hat{\mathbf{z}}(x+1)z^2$.
(b) $\mathbf{A} = \hat{\mathbf{r}}2r\cos\phi - \hat{\boldsymbol{\phi}}4r\sin\phi + \hat{\mathbf{z}}3$.

Stokes' Theorem

2.29 Verify Stokes' theorem for $\mathbf{A} = \hat{\mathbf{x}}(2x - y) - \hat{\mathbf{y}}2yz^2 - \hat{\mathbf{z}}2zy^2$ on the upper half-surface of the sphere $x^2 + y^2 + z^2 = 4$ above the xy plane. The contour bounding the surface is the rim of the half-sphere.

2.30 Vector field $\mathbf{F} = \hat{\mathbf{x}}3y + \hat{\mathbf{y}}(5 - 2x) + \hat{\mathbf{z}}(z^2 - 2)$ is given. Find:

(a) The divergence of **F**.
(b) The curl of **F**.
(c) The surface integral of the normal component of the curl of **F** over the open hemisphere $x^2 + y^2 + z^2 = 4$ above the x–y plane.

2.31 Vector field $\mathbf{F} = \hat{\mathbf{x}}\, y + \hat{\mathbf{y}}\, z + \hat{\mathbf{z}}\, x$ is given. Find the total flux of $\nabla \times \mathbf{F}$ through a triangular surface given by three points $P_1(a,0,0)$, $P_2(0,0,b)$, and $P_3(0,c,0)$.

The Helmholtz Theorem and Vector Identities

2.32 The following vector operations are given:

1. $\nabla \cdot (\nabla \phi)$	2. $(\nabla \cdot \nabla)\phi$
3. $(\nabla \times \nabla)\, \phi$	4. $\nabla \times (\nabla \phi)$
5. $\nabla \cdot (\nabla \times \mathbf{A})$	6. $(\nabla \cdot \nabla) \times \mathbf{A}$
7. $(\nabla \times \nabla) \times \mathbf{A}$	8. $\nabla \times (\nabla \times \mathbf{A})$
9. $\nabla \cdot (\phi\, \nabla \times \mathbf{A})$	10. $\phi\, (\nabla \times \mathbf{A})$
11. $\nabla\, (\nabla \times \mathbf{A})$	12. $\nabla \times (\nabla \cdot \mathbf{A})$

where **A** is an arbitrary vector field and ϕ an arbitrary scalar field.

(a) Which of the operations are valid?
(b) Evaluate explicitly those that are valid (in Cartesian coordinates).

2.33 Calculate the Laplacian for the following scalar fields:

(a) $p = (x-2)^2(y-2)^2(z+1)^2$.
(b) $p = 5r\cos\phi + 3zr^2$.

2.34 Calculate the Laplacian for the following vector fields:

(a) $\mathbf{A} = \hat{\mathbf{x}}\, 3y + \hat{\mathbf{y}}\, (5 - 2x) + \hat{\mathbf{z}}\, (z^2 - 2)$.
(b) $\mathbf{A} = \hat{\mathbf{r}}\, 2r\cos\phi - \hat{\boldsymbol{\phi}}\, 4r\sin\phi + \hat{\mathbf{z}}\, 3$.

2.35 Show that if **F** is a conservative field, then $\nabla^2 \mathrm{F} = \nabla(\nabla \cdot \mathrm{F})$. Use cylindrical coordinates.

2.36 Given the scalar field $f(x,y,z) = 2x^2 + y$ and the vector field $\mathbf{R} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y + \hat{\mathbf{z}}\, z$, find:

(a) The gradient of f.
(b) The divergence of $f\,\mathbf{R}$.
(c) The Laplacian of f.
(d) The vector Laplacian of **R**.
(e) The curl of $f\mathbf{R}$.

2.37 A vector field $\mathbf{A} = \hat{\mathbf{x}}\, 5x + \hat{\mathbf{y}}\, 2y + \hat{\mathbf{z}}\, 1$. What type of field is this according to the Helmholtz theorem?

2.38 A vector field $\mathbf{A} = \hat{\mathbf{R}}\, \phi R^2 + \hat{\boldsymbol{\theta}}\, R\sin\theta$ is given in spherical coordinates. What type of field is this according to the Helmholtz theorem?

2.39 The following vector fields are given:

(1) $\mathbf{A} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y$.
(2) $\mathbf{B} = \hat{\boldsymbol{\phi}}\, \cos\phi + \hat{\mathbf{r}}\, \cos\phi$.
(3) $\mathbf{C} = \hat{\mathbf{x}}\, y + \hat{\mathbf{z}}\, y$.
(4) $\mathbf{D} = \hat{\mathbf{R}}\, \sin\theta + \hat{\boldsymbol{\theta}}\, 5R + \hat{\boldsymbol{\phi}}\, R\sin\theta$.
(5) $\mathbf{E} = \hat{\mathbf{R}}\, k$.

(a) Which of the fields are solenoidal?
(b) Which of the fields are irrotational?
(c) Classify these fields according to the Helmholtz theorem.

2.40 Show by direct derivation of the products that the following holds:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$