17 Waveguides and Resonators

He alone is free who lives with free consent under the entire guidance of reason.

Baruch Spinoza (1632–1677),
philosopher

17.1 Introduction

Much of the discussion in the previous three chapters centered around lossless transmission lines. For these lines, the various relations were independent of conductivity of the materials involved. In fact, even for a distortionless transmission line, although an attenuation constant was present, all other basic properties of the line were independent of conductivity. The speed of propagation depended only on material properties of the line (ε and μ), and the phase constant and wavelength were also independent of conductivity of the line. If this is the case, are the conductors necessary? If so, what are their roles in the propagation of energy in the lossless or distortionless transmission line?

Perhaps this question is confusing since in a transmission line, we dealt with line voltage and line current. If there is a current, there must be a conductor. Or must it? What about displacement currents? These can exist in free space, and from our discussion of propagation in free space and in dielectrics, we also know that energy can be transmitted from point to point without conductors being present. Also, we saw that transmission lines may be analyzed from a field point of view rather than through voltages and currents. What then is the role of conductors in transmission lines? The answer is surprising and simple; the conductors are not necessary in general. What they do is to confine the energy being transmitted to the line itself and guide it along the line. Of course, the conductors may have other effects. If conductivity is low, there may be losses in the conductors, but these are secondary effects.

To convince yourself of this, consider the following "communication system" often used in old ships: A tube connects the bridge of the ship with the quarters below. The tube may bend and may be made of any material. The captain uses it to summon somebody to the bridge. This is an example of a guided transmission system. The tube itself is immaterial, other than to provide the path for propagation. A glass tube may work just as well as a steel or brass tube. The guiding system described here has certain advantages over an open system: the amount of energy required is smaller (no need to shout), the distance over which propagation takes place is longer, and there is less interference from and with external systems (only those listening to the tube can hear the captain).

We also saw in **Chapter 13** that a wave impinging on a conducting surface at an angle propagates along the surface of the conductor (see, for example, **Section 13.3.1** and **Example 13.7**). We concluded there that the surface of the conductor has a guiding effect on the wave in addition to generating standing waves in the direction perpendicular to the conductor.

So much for the obvious and the known. Now, we ask ourselves another question: If the material of the tube is not important, are the shape and size important? What if we tried to use a capillary tube for this purpose, or a tube 2 m in diameter? Are the two going to behave the same way? How about a square rather than a cylindrical tube? What happens if we plug the tube, or insert some material in it? How do we couple energy into and out of the tube? Pretty interesting questions from the design point of view and they all have to do with properties of waves in this environment. The same questions and considerations apply to electromagnetic guiding structures.

In this chapter, we will discuss a very special form of transmission lines: lines or structures that guide waves. In many ways, this is an extension of the results obtained for transmission lines. There are, however, major differences. In most cases,

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there will only be one apparent conductor rather than two conductors. The conductors are explicitly used for guiding the waves and treatment of line behavior will be in terms of field parameters (electric and magnetic field intensities) rather than current and voltage.

17.2 The Concept of a Waveguide

The idea of a waveguide[1] can best be explained using the optic spectrum because we can see the effects involved. Consider, first, two large, parallel mirrors facing each other as shown in **Figure** 17.1a. The light emitted from object *A* reflects repeatedly off the mirrors and eventually generates an image on the other side of the waveguide or anywhere in the waveguide. The two mirrors may be viewed as a guiding structure because the mirrors guide the waves to any location we wish. If the mirrors were flexible (for example, by coating metal surfaces and bending them), we could create bent waveguides. One particularly useful optical guiding system is the optical fiber shown in **Figure** 17.1b. For simplicity you can view the optical fiber as a long, thin glass rod, coated so that there is total reflection inside the fiber (total reflection also occurs without coating because the fiber's permittivity is higher than that of the surrounding space—see **Section** 13.4.4). Since optical fibers are flexible, waves may be guided in almost any path. Although most of the discussion here will deal with frequencies much lower than optical frequencies, the picture drawn here is useful as a background.

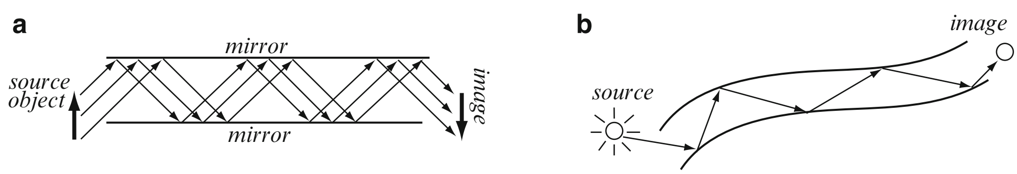


Figure 17.1 (**a**) An "optical waveguide" made of two parallel mirrors. (**b**) An optical fiber: total internal reflections at the interfaces ensure guidance of waves

Before we can look at practical waveguide structures we digress a little and look at the general properties of propagating waves in unbounded domain and in the presence of conducting surfaces since these will become very useful in our study. In particular, we define two specific types of waves, in addition to the plane waves we already saw in **Chapters** 12 and 13. These are the transverse electric (TE) waves and transverse magnetic (TM) waves. The plane waves discussed in **Chapters** 12, 13, and 14 were transverse electromagnetic (TEM) waves.

17.3 Transverse Electromagnetic, Transverse Electric, and Transverse Magnetic Waves

In **Section** 12.7, we discussed uniform plane waves as they propagate in free space and in dielectrics. One of the most important aspects of propagation was the fact that neither the electric nor the magnetic field intensity had any component in the direction of propagation. Both the electric and magnetic field intensities were always perpendicular to each other and to the direction of propagation. For this reason, we called them ***transverse electromagnetic*** (TEM) waves. There are, however, situations in which the electric or magnetic fields have components in the direction of propagation. If a wave has an electric field intensity that is entirely perpendicular to the direction of propagation, but with a component of the magnetic field intensity in the direction of propagation, this wave is called a ***transverse electric*** (TE) wave. Similarly, if the magnetic field intensity is entirely perpendicular to the direction of propagation and the electric field intensity has a component in the direction of propagation, this is a ***transverse magnetic*** (TM) wave. The differences among the three types of waves are shown in **Figure** 17.2.

[1] The first mention of a waveguide was in 1894 by Sir Oliver Joseph Lodge (1851–1940). He discovered the effect when he surrounded a spark generator, of the type used by Hertz to demonstrate propagation of waves, with a conducting tube. Three years later, Lord Rayleigh (John William Strutt (1842–1919)) developed much of the theory of guided waves. However, waveguides did not feature in electromagnetics until the early 1930s, when experiments on their properties were conducted at Bell Laboratories, first for propagation in dielectrics (water) and later in air. The main impetus for their development was the then newly developed microwave tubes and work on radar. From then on, waveguides became the basis of microwave work and are used wherever transmission of energy above about 1 GHz is required. The various optical fibers are also waveguides and their utility in communication is prevalent.

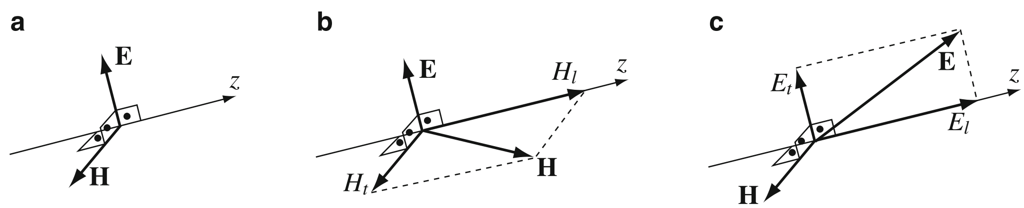


Figure 17.2 Types of waves. (**a**) TEM wave. (**b**) TE wave. (**c**) TM wave

Before proceeding with description of waves in waveguides, we need to define the conditions under which TEM, TM, and TE waves can exist. These conditions are then used to define the possible fields or modes of propagation in waveguides.

The starting point is with Maxwell's two curl equations. Each is expanded into three scalar equations by equating the vector components on both sides, as shown in **Eqs. (17.1)** through **(17.8)**. By doing so, one component of one field (electric or magnetic) is written in terms of the transverse components of the other field. **Table 17.1** summarizes these steps.

Table 17.1 Maxwell's curl equations and their transverse components

$\nabla\times\mathbf{H}=j\omega\varepsilon\mathbf{E}$ (17.1)	$\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}$ (17.5)
Components of the electric field in terms of the transverse components of the magnetic field	Components of the magnetic field in terms of the transverse components of the electric field
$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=j\omega\varepsilon E_x$ (17.2)	$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-j\omega\mu H_x$ (17.6)
$\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=j\omega\varepsilon E_y$ (17.3)	$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-j\omega\mu H_y$ (17.7)
$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=j\omega\varepsilon E_z$ (17.4)	$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu H_z$ (17.8)

The waves are assumed to propagate in the z direction, with a propagation constant $\gamma=\alpha+j\beta$. Assuming no backward-propagating waves (no reflections), the electric and magnetic field intensities for TEM waves have the following general form:

$$E_x=E_0e^{-\gamma z}\quad[\mathrm{V/m}],\qquad H_y=H_0e^{-\gamma z}\quad[\mathrm{A/m}]\tag{17.9}$$

where the $e^{j\omega t}$ variation is also implied (phasors). The propagation constant is assumed for the moment to be general, but in most of the discussion that follows, we will use lossless materials. Because all fields vary with the z parameter only, the derivatives with respect to z in **Eqs. (17.2)**, **(17.3)**, **(17.6)**, and **(17.7)** are nonzero:

$$\frac{\partial H_y}{\partial z}=-\gamma H_y,\quad \frac{\partial H_x}{\partial z}=-\gamma H_x,\quad \frac{\partial E_y}{\partial z}=-\gamma E_y,\quad \frac{\partial E_x}{\partial z}=-\gamma E_x\tag{17.10}$$

Important If we were to assume only a backward-propagating wave of the form $E_x=E_0e^{+\gamma z}$, then all terms in **Eq. (17.10)** would be positive; that is, γ is replaced by $-\gamma$. Similarly, if both a forward- and a backward-propagating wave exist, the derivatives of the total wave are the sum of the derivatives of the forward-propagating waves and those of the backward-propagating waves. At this juncture, we will assume that only forward-propagating waves exist to keep things simple, but if for any reason there are reflections in the system, the backward-propagating wave will have to be added.

Now, **Eqs. (17.2)** through **(17.4)** and **(17.6)** through **(17.8)** are

$$\frac{\partial H_z}{\partial y}+\gamma H_y=j\omega\varepsilon E_x\tag{17.11}$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{17.12}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \tag{17.13}$$

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \tag{17.14}$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{17.15}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{17.16}$$

Since the nature of the waves is defined by the longitudinal components of the field (components in the direction of propagation), it is necessary to rewrite these equations such that the transverse components of the fields (those perpendicular to the direction of propagation), in this case, E_x, E_y, H_x, and H_y, are written in terms of the longitudinal components, E_z and H_z. As an example, using **Eqs. (17.11)** and **(17.15)**, we can eliminate H_y, and write the component E_x in terms of H_z and E_z. Substitution of H_y from **Eq. (17.15)** in **Eq. (17.11)** gives

$$\frac{\partial H_z}{\partial y} + \frac{\gamma}{j\omega\mu}\left(\gamma E_x + \frac{\partial E_z}{\partial x}\right) = j\omega\varepsilon E_x \tag{17.17}$$

Multiplying both sides by $j\omega\mu$, rearranging terms, and substituting $\omega\sqrt{\mu\varepsilon} = k$, we get

$$E_x = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial E_z}{\partial x} - j\omega\mu\frac{\partial H_z}{\partial y}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.18}$$

Repeating the process for the other three transverse components (E_y, H_x, and H_y), we get

$$E_y = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial E_z}{\partial y} + j\omega\mu\frac{\partial H_z}{\partial x}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.19}$$

$$H_x = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial H_z}{\partial x} + j\omega\varepsilon\frac{\partial E_z}{\partial y}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.20}$$

$$H_y = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial H_z}{\partial y} - j\omega\varepsilon\frac{\partial E_z}{\partial x}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.21}$$

These equations are now used to define TEM, TE, and TM propagation of waves by imposing the necessary conditions for each type of propagation. The transverse components for backward-propagating waves may be obtained by replacing γ by $-\gamma$ in **Eqs. (17.18)** through **(17.21)**.

17.3.1 Transverse Electromagnetic Waves

As mentioned earlier, TEM waves require that there be no field in the direction of propagation. The condition for TEM propagation is $E_z = H_z = 0$. Substituting this into **Eqs. (17.18)** through **(17.21)** leads to the requirement that all four transverse components are zero, unless $\gamma^2 + k^2 = 0$. Thus, the constant of propagation for TEM waves must be

$$\gamma^2 = -k^2 \quad \text{or} \quad \gamma = j\omega\sqrt{\mu\varepsilon} \tag{17.22}$$

This is the condition we obtained for the propagation of uniform plane waves in lossless media. Indeed, all the properties we obtained for the propagation of plane waves in unbounded space, including the definition of intrinsic impedance (or wave impedance) and phase velocity, apply here as well. In particular, if we replace the permittivity ε with the complex permittivity $\varepsilon(1 - j\sigma/\omega\varepsilon)$, we obtain the propagation constant for a general lossy medium:

$$\boxed{\gamma_{TEM} = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}} \tag{17.23}$$

This relation was obtained in **Chapter 12 [Eq. (12.83)]** and was the basis of study of lossless, low-loss, and high-loss media.

The defining equation for transverse electromagnetic waves (plane waves) was the Helmholtz equation for the electric field:

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0} \tag{17.24}$$

In particular, if a plane wave propagates in unbounded space in the z direction and has an electric field intensity in the x direction, then the magnetic field intensity is in the y direction as required by the Poynting theorem:

$$E_x = E_0 e^{-jkz} \quad [\mathrm{V/m}] \quad \text{and} \quad H_y = \frac{E_0}{\eta_{TEM}} e^{-jkz} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.25}$$

The wave impedance in the domain in which the waves propagate was defined as the ratio between the transverse components of the electric and magnetic field intensities:

$$\boxed{\eta_{TEM} = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}} \quad [\Omega]} \tag{17.26}$$

These properties will be used to contrast transverse electromagnetic waves with transverse electric waves and transverse magnetic waves and to point out the differences. In particular, properties of TE and TM waves are often written in terms of known properties of TEM waves. For example, it is often useful to write the wave impedance of TE and TM waves in terms of the wave impedance of TEM waves.

17.3.2 Transverse Electric (TE) Waves

For TE waves to exist, E_z must be zero; that is, the only field component in the direction of propagation is a magnetic field intensity component H_z. Substituting this condition in **Eqs. (17.18)** through **(17.21)**, we get the transverse components for TE propagation:

$$\boxed{E_x = \frac{-j\omega\mu}{\gamma^2 + k^2}\frac{\partial H_z}{\partial y} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]} \tag{17.27}$$

$$\boxed{E_y = \frac{-j\omega\mu}{\gamma^2 + k^2}\frac{\partial H_z}{\partial x} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]} \tag{17.28}$$

$$H_x = \frac{-\gamma}{\gamma^2 + k^2}\frac{\partial H_z}{\partial x} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.29}$$

$$H_y = \frac{-\gamma}{\gamma^2 + k^2}\frac{\partial H_z}{\partial y} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.30}$$

Although not immediately apparent from these relations, it is possible to write these as a wave equation in H_z, which is the only longitudinal component in a TE wave. Doing so allows representation of fields and properties in terms of the solution to the wave equation. Since we have already obtained the wave equations for propagation in free space and in transmission lines, this approach will allow us to build on the existing solutions. We start by taking the derivative with respect to y of **Eq. (17.27)** and the derivative with respect to x of **Eq. (17.28)**:

$$\frac{\partial E_x}{\partial y} = \frac{-j\omega\mu}{\gamma^2 + k^2}\frac{\partial^2 H_z}{\partial y^2} \quad \text{and} \quad \frac{\partial E_y}{\partial x} = \frac{j\omega\mu}{\gamma^2 + k^2}\frac{\partial^2 H_z}{\partial x^2} \tag{17.31}$$

Subtracting the first of these relations from the second and rearranging terms gives

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\gamma^2 + k^2}{j\omega\mu}\left(-\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}\right) = 0 \tag{17.32}$$

Now, using **Eq. (17.8)** to eliminate the components of **E**, we get

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \left(\gamma^2 + k^2\right)H_z = 0 \tag{17.33}$$

This is a wave equation in H_z alone and will be taken from now on as the defining equation for TE waves whenever we need to do so.

Comparison of this equation with **Eq. (17.24)** shows that they are of the same form if we replace the term $\gamma^2 + k^2$ by a single term, which is denoted as k_c^2:

$$k_c^2 = \gamma^2 + k^2 \tag{17.34}$$

Now, we can write the propagation constant as

$$\gamma_{TE}^2 = k_c^2 - k^2 \quad \rightarrow \quad \gamma_{TE} = \sqrt{k_c^2 - k^2} = \sqrt{k_c^2 - \omega^2\mu\varepsilon} \tag{17.35}$$

that is, the propagation constant for TE waves is not the same as for TEM waves. Whereas the propagation constant for TEM waves in **Eq. (17.22)** is only zero for $\omega = 0$, the propagation constant for TE waves is zero if

$$k_c = \omega\sqrt{\mu\varepsilon} \quad [\mathrm{rad/m}] \tag{17.36}$$

Since a zero propagation constant means no propagation, this condition is quite important in propagation of TE waves. The following conditions may be distinguished:

(1) $k_c^2 = \omega^2\mu\varepsilon$. This is the condition for no propagation. k_c is called the ***cutoff wave number*** or, alternatively, we call the frequency for which this happens the ***cutoff frequency:***

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} \quad [\mathrm{Hz}] \tag{17.37}$$

and the corresponding wavelength the ***cutoff wavelength.***

(2) $\omega^2\mu\varepsilon < k_c^2$. For these values of k_c, the propagation constant γ_{TE} is real. From the definition of the propagation constant as $\gamma = \alpha + j\beta$, this condition leads to $\gamma = \alpha$; that is, there is no propagation (phase constant is zero), but there is attenuation. This type of attenuated, non-propagating wave is called an ***evanescent wave*** and occurs for $f < f_c$. Note that the attenuation here has nothing to do with losses: It occurs in lossless media as well. If we substitute this condition in **Eq. (17.35)**, we get

$$\gamma_{TE} = \sqrt{k_c^2 - \omega^2\mu\varepsilon} = \sqrt{(2\pi)^2 f^2\mu\varepsilon\left(\frac{f_c^2}{f^2} - 1\right)} = \pm\omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_c^2}{f^2} - 1}, \quad f < f_c \tag{17.38}$$

The attenuation constant for evanescent waves [taking the positive solution in **Eq. (17.38)**] is

$$\boxed{\alpha_e = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_c^2}{f^2} - 1} \quad \left[\frac{\text{Np}}{\text{m}}\right]} \tag{17.39}$$

(3) $\omega^2\mu\varepsilon > k_c^2$. Now, the propagation constant in **Eq. (17.35)** is purely imaginary and the wave propagates, without attenuation (lossless media). This occurs for any frequency above the cutoff frequency f_c.

In conclusion, for TE waves to propagate, the frequency of the waves must be above a given cutoff frequency. We will also see that this frequency depends on the conditions under which the wave propagates.

Substituting the condition for propagation in **Eq. (17.35)**, we get

$$\gamma_{TE} = \sqrt{k_c^2 - \omega^2\mu\varepsilon} = \sqrt{(-1)(2\pi)^2 f^2\mu\varepsilon\left(1 - \frac{f_c^2}{f^2}\right)} = \pm j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}, \quad f > f_c \tag{17.40}$$

that is, the phase constant ($\gamma = j\beta$ for lossless media) is

$$\boxed{\beta_{TE} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}} \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad f > f_c} \tag{17.41}$$

where, again, we took only the positive form of the phase constant. The phase constant for TE waves is smaller than that for TEM waves since the term under the square root is smaller than 1 for any frequency $f > f_c$.

By definition, the wavelength is

$$\boxed{\lambda_{TE} = \frac{2\pi}{\beta_{TE}} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}} = \frac{\lambda}{\sqrt{1 - f_c^2/f^2}} \quad [\text{m}]} \tag{17.42}$$

where λ is the wavelength for TEM waves in unbounded space. The TE wavelength is larger than that for TEM waves for any given frequency for which the two waves propagate.

From the phase constant, we can calculate the phase velocity, again by definition:

$$\boxed{v_{TE} = \frac{\omega}{\beta_{TE}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}} = \frac{v_p}{\sqrt{1 - f_c^2/f^2}} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{17.43}$$

where $v_p = 1/\sqrt{\mu\varepsilon}$ is the phase velocity for TEM waves. Therefore, the phase velocity for TE waves is always larger than the phase velocity for TEM waves.

Although we talked about the propagation and phase constants, we neglected the attenuation constant so far and, together with it, the possibility of losses in the domain in which TE waves propagate. Losses may be easily introduced by starting with the propagation constant in **Eq. (17.35)** and replacing the permittivity ε by the complex permittivity $\varepsilon_c = \varepsilon(1 - j\sigma/\omega\varepsilon)$, where σ is the conductivity of the dielectric and ε is its permittivity. This is identical to what we did in **Section 12.7** for plane

(TEM) waves. In addition, we will assume here that losses are small: high-loss TE and TM propagation is of little interest in the context of this chapter. Introducing the complex permittivity in the propagation constant in **Eq. (17.35)**, we get

$$\gamma_{TE} = \sqrt{k_c^2 - \omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \sqrt{-1\left[\omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) - k_c^2\right]} = j\sqrt{\left(\omega^2\mu\varepsilon - k_c^2\right) - j\omega\mu\sigma} \quad (17.44)$$

where $j = \sqrt{-1}$ was used to rearrange the expression. **Equation (17.44)** may be written as follows:

$$\gamma_{TE} = j\sqrt{\left(\omega^2\mu\varepsilon - k_c^2\right) - j\omega\mu\sigma} = j\sqrt{\omega^2\mu\varepsilon - k_c^2}\left(1 - \frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)^{1/2} \quad (17.45)$$

For low losses (σ small), the expression in parentheses may be expanded using the binomial expansion so that the attenuation and propagation constants may be separated. The low-loss condition now requires that the second term in the parentheses in **Eq. (17.45)** be small with respect to 1 (see **Section 12.7.2**):

$$\frac{\omega\mu\sigma}{\left(\omega^2\mu\varepsilon - k_c^2\right)} \ll 1 \quad (17.46)$$

Expansion of the term in parentheses in **Eq. (17.45)** using the binomial expansion gives

$$\left(1 - \frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)^{1/2} = 1 - \frac{1}{2}\left(\frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right) - \frac{1}{8}\left(\frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)^2 + \frac{1}{16}\left(\frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)^3 + \cdots \quad (17.47)$$

Retaining only the first two terms in the expansion gives an approximation to the propagation constant in **Eq. (17.45)** as

$$\gamma_{TE} \approx j\sqrt{\omega^2\mu\varepsilon - k_c^2}\left(1 - \frac{1}{2}\left(\frac{j\omega\mu\sigma}{\omega^2\mu\varepsilon - k_c^2}\right)\right) = \frac{\omega\mu\sigma}{2\sqrt{\omega^2\mu\varepsilon - k_c^2}} + j\sqrt{\omega^2\mu\varepsilon - k_c^2} \quad (17.48)$$

The first term on the right-hand side is the attenuation constant and the second is the phase constant. To put these in a form compatible with other expressions in this section, we can write k_c in terms of the cutoff frequency [see **Eq. (17.37)**] as $k_c^2 = (2\pi f_c)^2\mu\varepsilon = \omega_c^2\mu\varepsilon$. Substituting this in **Eq. (17.48)** gives

$$\gamma_{TE} = \alpha + j\beta \approx \frac{\omega\mu\sigma}{2\sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}} + j\sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon} = \frac{\omega\mu\sigma}{2\sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_c^2}} + j\sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_c^2}$$

$$= \frac{\omega\mu\sigma}{2\omega\sqrt{\mu\varepsilon}\sqrt{1 - (\omega_c/\omega)^2}} + j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1 - (f_c/f)^2}} + j\omega\sqrt{\mu\varepsilon}\sqrt{1 - (f_c/f)^2} \quad (17.49)$$

Separating the real and imaginary parts of the propagation constant gives the attenuation and phase constants. The phase constant is identical to that obtained in **Eq. (17.41)** for the lossless case (i.e., under the assumption of low losses and neglecting the higher-order terms in the expansion in **Eq. (17.47)**, the phase constant in the lossy case is the same as for the lossless case). The attenuation constant is

$$\boxed{\alpha_{TE} = \frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} \quad \left[\frac{\text{Np}}{\text{m}}\right]} \quad (17.50)$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the material in which the waves propagate. This relation only holds above cutoff since the condition we imposed in **Eq. (17.46)**, in effect, requires that $f > f_c$. At cutoff, there is no propagation, whereas below cutoff, the relation in **Eq. (17.39)** must be used.

Finally, we can also calculate the wave impedance by dividing the transverse electric field intensity by the transverse magnetic field intensity [E_x in **Eq. (17.27)** and H_y in **Eq. (17.30)** or E_y in **Eq. (17.28)** and H_x in **Eq. (17.29)**] and imposing the direction of propagation in the positive z direction:

$$\boxed{Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma} \quad [\Omega]} \tag{17.51}$$

Substituting the propagation constant for waves above cutoff from **Eq. (17.40)**, we get

$$\boxed{Z_{TE} = \frac{j\omega\mu}{j\omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1 - f_c^2/f^2}} = \eta\frac{1}{\sqrt{1 - f_c^2/f^2}} \quad [\Omega]} \tag{17.52}$$

The wave impedance for TE waves is frequency dependent and always larger than the wave impedance for TEM waves except at $f \to \infty$, where the two are equal. At cutoff ($f = f_c$), the wave impedance for TE propagation is infinite. This result gives another interpretation of cutoff: infinite wave impedance which, of course, means that for any finite electric field intensity at cutoff, the magnetic field intensity is zero and, therefore, there can be no propagation of power.

17.3.3 Transverse Magnetic (TM) Waves

The condition for existence of TM waves is $H_z = 0$, that is, the only field component in the direction of propagation is an electric field intensity component E_z. Substituting this condition in **Eqs. (17.18)** through **(17.21)** yields the field equations for TM waves since these waves will only have components of the magnetic field transverse to the direction of propagation as required. The steps are identical to those for the TE waves. First, we write the field components by substituting the condition $H_z = 0$ in **Eqs. (17.18)** through **(17.21)**:

$$\boxed{E_x = \frac{-\gamma}{\gamma^2 + k^2}\frac{\partial E_z}{\partial x} \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.53}$$

$$\boxed{E_y = \frac{-\gamma}{\gamma^2 + k^2}\frac{\partial E_z}{\partial y} \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.54}$$

$$\boxed{H_x = \frac{j\omega\varepsilon}{\gamma^2 + k^2}\frac{\partial E_z}{\partial y} \quad \left[\frac{\text{A}}{\text{m}}\right]} \tag{17.55}$$

$$\boxed{H_y = \frac{-j\omega\varepsilon}{\gamma^2 + k^2}\frac{\partial E_z}{\partial x} \quad \left[\frac{\text{A}}{\text{m}}\right]} \tag{17.56}$$

The wave equation equivalent to these four equations is obtained by taking the derivative of H_x with respect to y in **Eq. (17.55)** and the derivative of H_y with respect to x in **Eq. (17.56)**. Subtracting the second from the first and then using **Eq. (17.4)** to eliminate the components of **H** (see **Exercise 17.1**) gives

$$\boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(\gamma^2 + k^2\right)E_z = 0} \tag{17.57}$$

Comparison of **Eqs. (17.57)** and **(17.33)** reveals that the two equations are identical in form. Therefore, we should expect all relations in **Eqs. (17.34)** through **(17.50)** to remain unchanged since these were obtained from **Eq. (17.33)**. Also the same are the definitions of the cutoff, propagation above cutoff, and attenuation below cutoff.

However, the wave impedance is calculated from the transverse components of the electric and magnetic field and these are different, as can be seen from **Eqs.(17.53)** through **(17.56)**. For propagation in the z direction, we take the transverse components as E_x and H_y from **Eqs. (17.53)** and **(17.56)** [or E_y and $-H_x$ from **Eqs. (17.54)** and **(17.55)**] and get the wave impedance for TM waves:

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\gamma}{j\omega\varepsilon} = \frac{j\omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}}{j\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \frac{f_c^2}{f^2}} = \eta\sqrt{1 - \frac{f_c^2}{f^2}} \quad [\Omega] \tag{17.58}$$

This impedance is lower than the TEM wave impedance η, except at $f \to \infty$, where the two are equal. At cutoff ($f = f_c$), the wave impedance for TM propagation is zero. This means that for any given magnetic field intensity at cutoff, the electric field intensity is zero and, therefore, there can be no propagation of energy.

Table 17.2 summarizes the results we obtained for TE and TM waves in comparison with TEM waves, all above cutoff ($f > f_c$). The definitions of TE and TM waves in **Eqs. (17.27)** through **(17.30)** and **Eqs. (17.53)** through **(17.56)** were in completely general terms. These apply universally since no other conditions (such as boundary or interface conditions) were required. How we use these equations and how we define the interface conditions between different materials and, in particular, for conducting interfaces will define the properties of the waves. In particular, we will look next to the properties of the waves as they propagate in guiding structures which we call waveguides.

Table 17.2 Properties of TEM, TE, and TM waves

	TEM waves	TE waves	TM waves
Cutoff frequency: f_c [Hz]	$f_c = 0$	$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$	$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$
Lossless phase constant: β [rad/m]	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$	$\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$
Low-loss phase constant: γ	$j\omega\sqrt{\mu\varepsilon}$	$j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$	$j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$
Low-loss phase constant: β [rad/m]	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$	$\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_c^2}{f^2}}$
Low-loss attenuation constant: α [Np/m]	$\frac{\sigma\eta}{2}$	$\frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}}$	$\frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}}$
Low-loss propagation constant: γ	$\frac{\sigma\eta}{2} + j\omega\sqrt{\mu\varepsilon}$	$\frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} + j\omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}$	$\frac{\sigma\eta}{2\sqrt{1 - (f_c/f)^2}} + j\omega\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}$
Wavelength: λ [m]	$\frac{1}{f\sqrt{\mu\varepsilon}}$	$\frac{1}{f\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}}$	$\frac{1}{f\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}}$
Phase velocity: v_p [m/s]	$\frac{1}{\sqrt{\mu\varepsilon}}$	$\frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}}$	$\frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - f_c^2/f^2}}$
Wave impedance: Z [Ω]	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1 - f_c^2/f^2}}$	$\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - f_c^2/f^2}$

Note: $\eta = \sqrt{\mu/\varepsilon}$ is the no-loss intrinsic impedance of the medium in which the waves propagate

Exercise 17.1 Derive **Eq. (17.57)** from **Eqs. (17.53)** through **(17.56)**.

Example 17.1 An electromagnetic wave, propagating in a guiding structure, is given as follows:

$$\mathbf{E} = \hat{\mathbf{z}} jE_0 e^{-j\beta x} \quad [\mathrm{V/m}] \quad \text{and} \quad \mathbf{H} = -(\hat{\mathbf{y}} jH_0 + \hat{\mathbf{x}} H_0) e^{-j\beta x} \quad [\mathrm{A/m}]$$

(a) The wave propagates in free space, its frequency is $f = 3$ GHz, and its phase constant is $\beta = 12\pi$ [rad/m]. Determine the type of wave.
(b) Find the cutoff frequency, wave impedance, and phase velocity of the wave.
(c) Calculate the time-averaged power density in the structure.
(d) Calculate the magnitude of the electric and magnetic field intensities E_0 and H_0 if the time-averaged power density is uniform and equals 100 W/m^2.

Solution: The type of wave is determined from the longitudinal component of the wave. After determining the type of wave, its propagation properties are determined from **Table 17.2**:

(a) The wave propagates in the positive x direction and has transverse components in the y and z directions. The longitudinal component, H_x, is a magnetic field component. Therefore, this is a TE wave.
(b) The wave propagates in free space. From **Table 17.2**, we get

$$\beta = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \frac{f_c^2}{f^2}} \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right] \quad \rightarrow \quad f_c = \sqrt{f^2 - \frac{\beta^2 f^2}{\omega^2\mu_0\varepsilon_0}} = \sqrt{f^2 - \frac{\beta^2 c^2}{4\pi^2}} \quad [\mathrm{Hz}]$$

With the given values, this is

$$f_c = \sqrt{9 \times 10^{18} - \frac{144 \times \pi^2 \times 9 \times 10^{16}}{4 \times \pi^2}} = 2.4 \times 10^9 \quad [\mathrm{Hz}]$$

Any wave below this frequency will not be propagated in the given structure.
The wave impedance is given as

$$Z_{TE} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\sqrt{1 - f_c^2/f^2}} = \frac{377}{\sqrt{1 - \left(\frac{2.4 \times 10^9}{3 \times 10^9}\right)^2}} = 628 \quad [\Omega]$$

and the phase velocity is

$$v_{TE} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{1 - f_c^2/f^2}} = \frac{c}{\sqrt{1 - f_c^2/f^2}} = \frac{3 \times 10^8}{0.6} = 5 \times 10^8 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

The wave impedance is larger than the intrinsic impedance of free space, and the phase velocity is larger than the speed of light. The fact that the phase velocity can be larger than the speed of light will be discussed in the following sections and was also discussed in **Sections 12.7.4** and **13.3.1**.
(c) The time-averaged Poynting vector is

$$\mathcal{P}_{av} = \frac{\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}}{2} = \frac{\mathrm{Re}\left\{\left(\hat{\mathbf{z}} jE_0 e^{-j\beta x}\right) \times \left([\hat{\mathbf{y}} jH_0 - \hat{\mathbf{x}} H_0] e^{+j\beta x}\right)\right\}}{2} = \frac{\mathrm{Re}\{\hat{\mathbf{x}} E_0 H_0 - \hat{\mathbf{y}} jE_0 H_0\}}{2} = \hat{\mathbf{x}} \frac{E_0 H_0}{2} \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

and, as expected, the time-averaged power propagates in the x direction. The transverse component of the power density is imaginary.

(d) The magnitude of the electric and magnetic field intensities may be calculated from the wave impedance, which is equal to the ratio between the transverse electric field intensity and the transverse magnetic field intensity, and the total power per unit area:

$$z_{TE} = -\frac{E_z}{H_y} = \frac{E_0}{H_0} = 628 \quad [\Omega], \quad \mathcal{P}_{av} = \frac{E_0 H_0}{2} = 100 \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

From these, we get

$$E_0 = 354.4 \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right], \quad H_0 = 0.564 \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right].$$

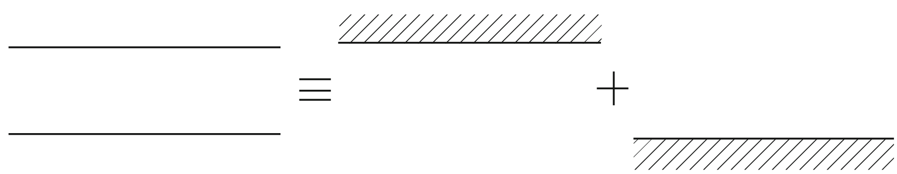
17.4 TE Propagation in Parallel Plate Waveguides

Now that we defined TE and TM waves and their general properties, it is time to define the conditions under which TE and TM waves can exist. This will lead to the definition of waveguides and to the properties of waves in waveguides.

To keep in line with a simple explanation of guided waves but also in accordance with the definition of transverse electric and transverse magnetic fields, we consider here the guiding of electromagnetic waves between two parallel conducting surfaces, as shown in **Figure 17.3**. We will discuss TE waves first, followed by TM waves but will try to give a physical feel to both the phenomenon of guided waves as well as to the properties of these waves. For this reason, we will rely less on the properties defined in the previous sections and more on physical properties of the waveguide.

The parallel plate waveguide is made of two surfaces, which may be viewed as perfect conductors. This separation into two surfaces (**Figure 17.3**) allows analysis of the reflections at each surface separately. This is convenient since we have already discussed many of the properties of reflection at conducting surfaces in **Chapter 13** and should be able to use those results.

Figure 17.3 Construction of a parallel plate waveguide



Suppose a uniform plane wave impinges on the lower surface at an angle of incidence θ_i and the wave is polarized perpendicular to the plane of incidence. The incident and reflected electric and magnetic fields are as shown in **Figure 17.4a**. The reflected wave from the lower surface then reflects off the upper surface, as shown in **Figure 17.4b**. The wave propagates between the plates by repeatedly reflecting off the conductors. To calculate the fields and the propagation properties between the plates, we calculate the fields above the lower plate using **Figure 17.4a** and then take into account the effect of the upper plate, without the need to calculate reflections off the upper plate. The incident wave propagates in the direction $\hat{\mathbf{p}}_i = -\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i$ and the reflected wave is in the direction $\hat{\mathbf{p}}_r = \hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i$. Based on these (see also **Section 13.3.1** and **Exercise 13.6**), we obtain the incident electric field as

$$\mathbf{E}_i(x,z) = \hat{\mathbf{y}}E_i e^{-j\beta(-x\cos\theta_i + z\sin\theta_i)} \quad [\mathrm{V/m}] \tag{17.59}$$

$$\mathbf{H}_i(x,z) = \frac{E_i}{\eta}(-\hat{\mathbf{x}}\sin\theta_i - \hat{\mathbf{z}}\cos\theta_i)e^{-j\beta(-x\cos\theta_i + z\sin\theta_i)} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.60}$$

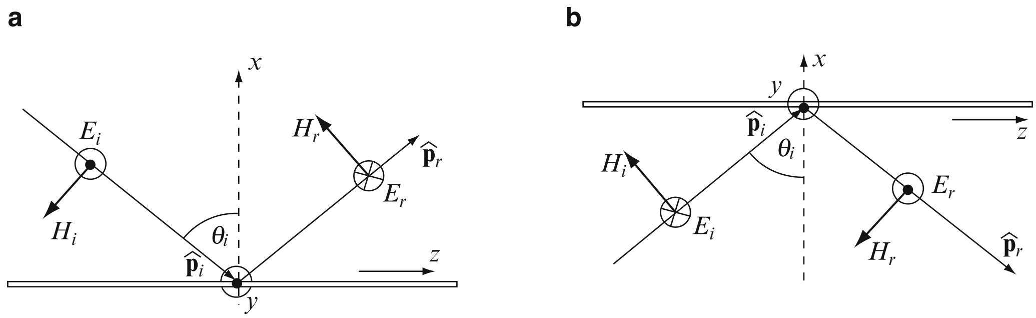


Figure 17.4 (**a**) Reflection of the incident wave off the lower plate in a parallel plate waveguide. (**b**) Reflection off the upper plate

where β and η are the phase constant and intrinsic impedance in the medium between the plates. The reflected electric and magnetic field intensities are

$$\mathbf{E}_r(x,z) = -\hat{\mathbf{y}}\,E_i e^{-j\beta(x\cos\theta_i + z\sin\theta_i)} \quad [\mathrm{V/m}] \tag{17.61}$$

$$\mathbf{H}_r(x,z) = \frac{E_i}{\eta}(\hat{\mathbf{x}}\sin\theta_i - \hat{\mathbf{z}}\cos\theta_i)e^{-j\beta(x\cos\theta_i + z\sin\theta_i)} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.62}$$

The total electric and magnetic field intensities are (after rearranging terms and using the relation $e^{j\beta x\cos\theta} - e^{-j\beta x\cos\theta} = j2\,\sin(\beta x\cos\theta)$:

$$\mathbf{E}_1(x,z) = \hat{\mathbf{y}}\,j2E_i\sin(\beta x\cos\theta_i)e^{-j\beta z\sin\theta_i} \quad [\mathrm{V/m}] \tag{17.63}$$

$$\mathbf{H}_1(x,z) = -2\frac{E_i}{\eta}[\hat{\mathbf{x}}j\sin\theta_i\sin(\beta x\cos\theta_i) + \hat{\mathbf{z}}\cos\theta_i\cos(\beta x\cos\theta_i)]e^{-j\beta z\sin\theta_i} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.64}$$

This much is almost a direct rewriting of the result in **Section 13.3.1** and has been done in detail in **Exercise 13.6**. We note that the electric field intensity has only a y component whereas the magnetic field intensity has two components: a component in the x direction and a component in the z direction. Therefore, this is a TE wave. Note also that both the incident and reflected waves are TEM waves (both **E** and **H** are perpendicular to the direction of propagation) whereas their sum is a TE wave.

The first task now is to calculate the time-averaged Poynting vector to see how power propagates. The Poynting vector will then tell us the direction of propagation of the wave:

$$\mathcal{P}_{av} = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_1 \times \mathbf{H}_1^*\} \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right] \tag{17.65}$$

We note the following:

$$\mathbf{H}_1 = -(\hat{\mathbf{z}}H_{1z} + \hat{\mathbf{x}}jH_{1x})e^{-j\beta z\sin\theta_1} \quad \Rightarrow \quad \mathbf{H}_1^* = -(\hat{\mathbf{z}}H_{1z} - \hat{\mathbf{x}}jH_{1x})e^{j\beta z\sin\theta_i} \quad [\mathrm{A/m}] \tag{17.66}$$

From **Eqs. (17.63)** and **(17.64)** and using the relation in **Eq. (17.65)**, we get

$$\mathcal{P}_{av} = \mathrm{Re}\left\{-\hat{\mathbf{x}}j\frac{2E_i^2}{\eta}\sin(2\beta x\cos\theta_i)\cos\theta_i + \hat{\mathbf{z}}\,\frac{2E_i^2}{\eta}\sin^2(\beta x\cos\theta_i)\sin\theta_i\right\} \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right] \tag{17.67}$$

where $\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}) = \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \times (-\hat{\mathbf{z}}) = -\hat{\mathbf{x}}$, $j^2 = -1$, $\sin(\beta\cos\theta_i)\cos(\beta\cos\theta_i) = (1/2)\sin(2\beta\cos\theta_i)$ were used to simplify the expression. The Poynting vector has a real part in the z direction and an imaginary part in the x direction. The real part is

$$\mathcal{P}_{av} = \hat{\mathbf{z}}\,\frac{2E_i^2}{\eta_1}\sin^2(\beta x\cos\theta_i)\sin\theta_i \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{17.68}$$

The conclusion is that the time-averaged power propagates entirely in the z direction. The x component of the Poynting vector is imaginary, and as we have seen in **Chapter 13**, this means there is no propagation in this direction. In the x direction, there are only standing waves. The presence of the conducting surface causes energy to propagate only in the z direction. The z direction is also called the ***guiding direction***.

Before continuing and discussing the properties of the waves, it is worth pausing and looking at the physical meaning of the above result. First, from **Figure 17.4a**, we note that the total wave is a superposition of two plane waves. Both plane waves are transverse electromagnetic waves. One wave, the incident wave, propagates in the direction $\hat{\mathbf{p}}_i = -\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i$, and the second wave, which we called the reflected wave, propagates in direction $\hat{\mathbf{p}}_r = \hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i$. Note, also, that the phase constant of each wave depends on the phase velocity of the wave in the material above the plate and the angle θ_i as if the conductor did not exist. From this, we can draw the following picture: The phase constants of the incident and reflected waves are the same in the direction of propagation of each wave and these depend on the phase velocity in the given material:

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \left[\frac{\text{m}}{\text{s}}\right], \quad \lambda = \frac{v_p}{f} \quad [\text{m}], \quad \beta = \frac{2\pi}{\lambda} = \frac{2\pi v_p}{f} = \frac{2\pi}{f\sqrt{\mu\varepsilon}} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{17.69}$$

These are exactly the properties we expect from a plane wave propagating in a material with properties ε and μ.

The phase velocities in the guide direction and in the direction transverse to the guide direction may be found from **Figure 17.5**. Consider the front of a plane wave propagating at an angle θ_i to the normal, as wave front A. After some time, the wave front has propagated in the direction of propagation of the wave, at a velocity v_p, and is now at wave front B. The horizontal distance the wave has traveled during the same time is the distance between A and B'. Thus, the wave front has propagated faster horizontally than in the direction of θ_i. From **Figure 17.5**, the horizontal phase velocity is

$$\boxed{v_g = \frac{v_p}{\sin\theta_i} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{17.70}$$

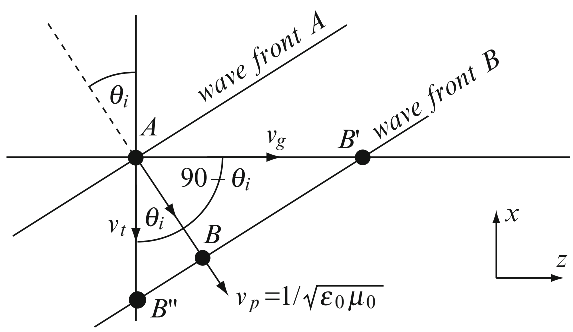


Figure 17.5 Relation between phase velocity of the incident wave and the guide and transverse phase velocities

This is called the ***guide phase velocity***.

Similarly, the vertical distance the wave traveled is the distance between point A and B''. Since this distance is traveled during the same time the wave traveled from point A to B, the phase velocity in the vertical direction is

$$\boxed{v_t = \frac{v_p}{\cos\theta_i} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{17.71}$$

The phase velocity in the x direction is called the ***transverse phase velocity***.

Note that both the guide and transverse phase velocities are larger than or equal to the phase velocity of the plane wave, v_p, for any angle between zero and $\pi/2$. At zero incidence angle, the guide phase velocity is infinite and the transverse phase velocity is v_p. At an incidence angle equal to $\pi/2$, the transverse phase velocity is infinite and the guide phase velocity is v_p.

This result raises a rather interesting question. Suppose the wave propagates in free space above a conducting surface. In this case, we know the wave propagates at the speed of light ($v_p = c$). This means that the phase velocity is always larger or equal to the speed of light in any direction which does not coincide with the direction of propagation of the plane wave. You probably are distressed by this, but there is really no difficulty, since only the phase moves at this speed. No physical quantity such as power moves at this speed. Perhaps the following example may explain this apparent difficulty: Suppose an ocean wave propagates toward shore at an angle α and a constant speed v as in **Figure 17.6**. Suppose also, that we could mark two points on the wave; one point, B, is the point the wave meets the shore at time t. The second is any point on the crest of the wave. The wave propagates at a speed v and the time it takes point A to reach the shore (at point A') is $\Delta t = t = d/v$. During the same time, the point the wave meets the shore (point B) has also moved to point A'. Since the distance between B and A' is $d/\sin\alpha$, the speed of propagation of point B is $v = (d/\sin\alpha)/\Delta t = v/\sin\alpha$. For any angle $0 < \alpha < \pi/2$, the speed of point B along the shore is larger than that of point A. If $v = c$, point B travels at speeds larger than the speed of light. However, this is only an apparent speed, since nothing propagates physically from point B to A' (i.e., a surfer cannot surf along the shore at the speed point B moves!).

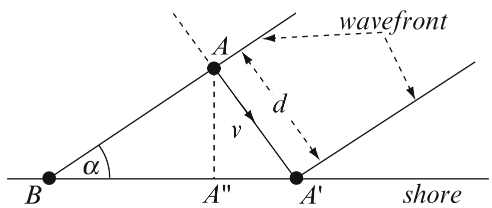


Figure 17.6 Relation between phase velocity and speed of propagation of an ocean wave impinging on the shore

Now that the phase velocities in the guiding and transverse directions are properly understood, we can also define the phase constants and wavelengths in the guiding and transverse directions. These are

$$\beta_t = \frac{\omega}{v_t} = \frac{\omega}{v_p}\cos\theta_i = \beta\cos\theta_i \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad \lambda_t = \frac{2\pi}{\beta_t} = \frac{2\pi v_p}{\omega\cos\theta_i} = \frac{v_p}{f\cos\theta_i} = \frac{1}{f\sqrt{\mu\varepsilon}\cos\theta_i} \quad [\text{m}] \tag{17.72}$$

$$\beta_g = \frac{\omega}{v_g} = \frac{\omega}{v_p}\sin\theta_i = \beta\sin\theta_i \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad \lambda_g = \frac{2\pi}{\beta_g} = \frac{2\pi v_p}{\omega\sin\theta_i} = \frac{v_p}{f\sin\theta_i} = \frac{1}{f\sqrt{\mu\varepsilon}\sin\theta_i} \quad [\text{m}] \tag{17.73}$$

The phase constants in the guiding and transverse directions are always smaller than those in the direction of propagation of the oblique wave, whereas the wavelengths are always larger than for the oblique wave.

From the imaginary part of the Poynting vector, we concluded that in addition to propagation (in the z direction), there is also a standing wave in the transverse (x) direction. To see how this standing wave behaves, we return now to the electric field intensity in **Eq. (17.63)**. We note that the field intensity has a sinusoidal variation with respect to x. Thus, the electric field intensity is zero at the conductor's surface and at any other point in space for which

$$\sin(\beta x\cos\theta_i) = 0 \quad \rightarrow \quad \beta x_m\cos\theta_i = m\pi, \quad m = 1, 2, 3, \ldots \tag{17.74}$$

where x_m are the locations of the nodes of the standing wave, a point we also made in **Chapter 13**. While the electric field propagates in the z direction, it changes its amplitude in the x direction, but the nodes of the electric field intensity remain fixed at points x_m such that

$$x_m = \frac{m\pi}{\beta\cos\theta_i} \quad [\lambda], \quad m = 1, 2, 3.... \tag{17.75}$$

Although m can be positive or negative, we will only take absolute values, since the space $x < 0$ in **Figure 17.4a** is assumed to be conducting. x_m can also be written in terms of the transverse wavelength of the wave ($\lambda_t = 2\pi/\beta_t$) or in terms of frequency of the wave ($\lambda f = v_p$). Thus, we can write

$$x_m = \frac{m\lambda}{2\cos\theta_i} = \frac{mv_p}{2f\cos\theta_i} = \frac{m}{2f\sqrt{\mu\varepsilon}\cos\theta_i} = m\frac{\lambda_t}{2} \quad [\lambda], \quad m = 1, 2, 3.... \tag{17.76}$$

where $v_p = 1/\sqrt{\mu\varepsilon}$ is the TEM phase velocity in the material above the conducting plane. Note that x_m is measured in wavelengths.

For now, we note that because the distance between two consecutive nodes of the standing wave pattern is any multiple of half-wavelengths, the distance $x\cos\theta_i$ must be a multiple of $\lambda_t/2$.

Now, suppose that we place the second conducting plane at a point x as given by **Eq. (17.76)**. Since this point is a node in the standing wave pattern, nothing in the standing wave or the propagation properties of the wave would change. There is, however, a very important difference: Waves now must be confined to the space between the two plates. We are free to place the plates at any position x as long as the above condition is satisfied. The situation described above constitutes a waveguide: the waves propagate in a given direction while they vary spatially in the transverse direction. **Figure 17.7** shows a parallel plate waveguide with a separation $d = m\lambda_t/2$. Note, also, the relation between λ, λ_t, and λ_g for the waveguide.

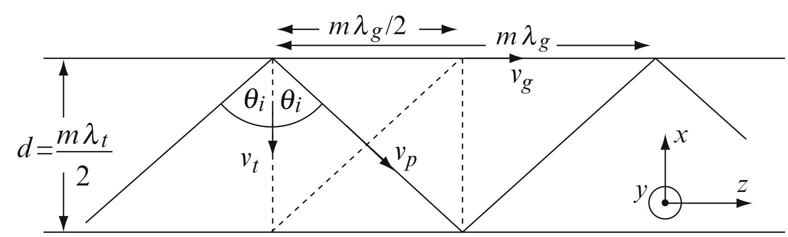


Figure 17.7 A parallel plate waveguide showing the guide and transverse phase velocities and wavelengths

In this case, we placed the conducting plates at very convenient locations: at the nodes of the standing waves. By doing so, we avoided the need to worry about interface conditions since the electric field intensity is zero at the conductors.

Suppose now we start the other way around: Instead of placing the conductors at the distance equivalent to that between two or more nodes of the standing wave, we place them at an arbitrary distance d. What happens to the wave now? We can easily answer this question from known properties: First, the electric field intensity at the conducting surfaces must be zero; that is, the two plates must still be at nodes of the standing wave pattern. Second, neither the wavelength nor, alternatively, the frequency of operation changed because of our choice of location for the plates. Inspection of **Eq. (17.75)** reveals that the only variable possible is the angle of incidence θ_i. The simple answer then is that if we were to place the plates at arbitrary locations, these locations will become nodes in the standing wave pattern (electric field intensity is zero) and only waves with an angle of incidence that satisfy this condition will propagate. It is therefore reasonable to rewrite **Eq. (17.75)** in terms of the angle as

$$\cos\theta_i = \frac{m\lambda}{2d} = \frac{mv_p}{2fd} = \frac{m}{2fd\sqrt{\mu\varepsilon}}, \quad m = 1, 2, 3 \dots \tag{17.77}$$

where d is the distance between the plates. From **Eq. (17.77)**, it appears that the larger d, the smaller the value of $\cos\theta_i$ for any given value of m. In the limit, $\cos\theta_i$ approaches zero (θ_i approaches $\pi/2$). The same effect can be observed with regard to frequency: For a given waveguide, the higher the frequency of the wave, the smaller $\cos\theta_i$ and the closer θ_i is to $\pi/2$. On the other hand, as d becomes smaller at a given frequency or as the frequency becomes lower for a given d, the angle θ_i approaches zero. From **Eq. (17.77)**, we can see that for any frequency, such that

$$2fd\sqrt{\mu\varepsilon} > m \quad \rightarrow \quad \cos\theta_i < 1, \quad m = 1, 2, 3, \dots \tag{17.78}$$

Thus, $\sin\theta_i > 0$ and the power propagated in the waveguide [based on **Eq. (17.68)**] is real and larger than zero. At

$$2fd\sqrt{\mu\varepsilon} = m \quad \rightarrow \quad \cos\theta_i = 1, \quad m = 1, 2, 3, \dots \tag{17.79}$$

At this frequency, there is no propagation of power in the waveguide since $\sin\theta_i = 0$ and the power in **Eq. (17.68)** is zero. The frequency at which a wave ceases to propagate was defined in **Section 17.3.2** as the cutoff frequency for the wave. From **Eq. (17.79)**,

$$\boxed{f_{cm} = \frac{m}{2d\sqrt{\mu\varepsilon}} \quad [\text{Hz}], \quad m = 1, 2, 3 \ldots} \tag{17.80}$$

Comparing this with the cutoff frequency, we obtained in **Eq. (17.37)**, the cutoff wave number may be written as

$$\boxed{k_{cm} = \frac{m\pi}{d} \quad \left[\frac{\text{rad}}{\text{m}}\right], \quad m = 1, 2, 3 \ldots} \tag{17.81}$$

and for each value of m, we have a different cutoff wave number. Similarly, we can write the cutoff wavelength as $\lambda_{cm} = 2\pi/k_{cm}$:

$$\boxed{\lambda_{cm} = \frac{2d}{m} \quad [\text{m}], \quad m = 1, 2, 3 \ldots} \tag{17.82}$$

Recall that m is the number of half-cycles in the standing wave pattern. We call this a ***mode of propagation*** and the cutoff frequency is specific for a particular mode. For $m = 1$, the mode of propagation is a TE_1 mode. The general mode is called a TE_m mode. Between the parallel plates (**Figure 17.7**), the field in the y direction is uniform. This results in a zero mode (no standing wave pattern) in this direction. Thus we can also call the TE_1 mode a TE_{10} mode or, in general, a TE_{m0} mode. The latter notation is more common, especially since practical waveguides are finite in both transverse directions and are therefore characterized by modes in each of the transverse directions. In a parallel plate waveguide one of the mode indices is always zero. In TE modes in parallel plate waveguides, the second index is always zero, but the first cannot be zero [$m \neq 0$: see **Eq. (17.74)**].

Now that we have the electric and magnetic fields for the parallel plate waveguide and the relations among frequency, dimensions, and modes are known, we can calculate other properties of the waveguide. One important parameter is the ***guide wavelength***, which is discussed next.

Consider a wave at any frequency $f > f_{cm}$ so that the wave propagates between two plates, separated a distance d apart, as in **Figure 17.7**. From **Eq. (17.77)**, the angle of incidence must be

$$\cos\theta_i = \frac{m\lambda}{2d} = \frac{\lambda}{\lambda_{cm}} = \frac{f_{cm}}{f}, \quad m = 1, 2, 3 \ldots \tag{17.83}$$

Note that λ_{cm} differs for each value of m for any given value of d as in **Eq. (17.82)**. Thus, again we have the familiar situation in which the closer the wavelength of the wave to the cutoff wavelength, the closer $\cos\theta_i$ is to 1 and θ_i to zero.

What about the properties of the propagating wave? Since wave properties depend on the angle of incidence, we can write for the wave of frequency $f > f_{cm}$:

$$\sin\theta_i = \sqrt{1 - \cos^2\theta_i} = \sqrt{1 - \frac{\lambda^2}{\lambda_{cm}^2}} = \sqrt{1 - \frac{f_{cm}^2}{f^2}}, \quad m = 1, 2, 3 \ldots \tag{17.84}$$

With this, the guide phase velocity **[Eq. (17.70)]**, the guide wavelength of the wave (λ_g) **[Eq. (17.73)]**, and guide phase constant are

$$\boxed{v_g = \frac{v_p}{\sin\theta_i} = \frac{v_p}{\sqrt{1 - \lambda^2/\lambda_{cm}^2}} = \frac{v_p}{\sqrt{1 - f^2/f_{cm}^2}} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{17.85}$$

$$\boxed{\lambda_g = \frac{\lambda}{\sin\theta_i} = \frac{\lambda}{\sqrt{1 - \lambda^2/\lambda_{cm}^2}} = \frac{\lambda}{\sqrt{1 - f_{cm}^2/f^2}} \quad [\text{m}]} \tag{17.86}$$

$$\beta_g = \beta\sqrt{1-\lambda^2/\lambda_{cm}^2} = \beta\sqrt{1-f_{cm}^2/f^2} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{17.87}$$

With these quantities, we go back to the equations for the electric and magnetic fields and write them in terms of the propagation constants. From **Eqs. (17.84)** and **(17.83)**, we write

$$\sin\theta_i = \sqrt{1-\lambda^2/\lambda_{cm}^2}, \quad \cos\theta_i = \frac{m\pi}{\beta d} \tag{17.88}$$

Also, we will denote the quantity $E_0 = 2E_i$ as the amplitude of the electric field intensity in **Eqs. (17.63)** and **(17.64)** and recall that the term $e^{j\omega t}$ is also present. Substituting these in **Eqs. (17.63)** and **(17.64)** gives the components of the electric and magnetic field intensities. In the time domain, these are

$$\begin{aligned}\mathbf{E}_1(x,z,t) &= \hat{\mathbf{y}}\, jE_0\sin\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \beta z\sqrt{1-\lambda^2/\lambda_{cm}^2}\right)\\ &= \hat{\mathbf{y}}\, E_0\sin\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \beta_g z + \frac{\pi}{2}\right) = -\hat{\mathbf{y}}\, E_0\sin\left(\frac{m\pi x}{d}\right)\sin\left(\omega t - \frac{2\pi}{\lambda_g}z\right) \quad \left[\frac{\text{V}}{\text{m}}\right]\end{aligned} \tag{17.89}$$

where **Eq. (17.87)** was used for β_g together with the relation $\beta_g = 2\pi/\lambda_g$, $j = e^{j\pi/2}$, and $\cos(\alpha + \pi/2) = -\sin\alpha$. Performing identical substitutions for the magnetic field intensity, we get

$$\mathbf{H}_1(x,z,t) = \hat{\mathbf{x}}\,\frac{E_0}{\eta}\sin\theta_i \sin\left(\frac{m\pi x}{d}\right)\sin\left(\omega t - \frac{2\pi}{\lambda_g}z\right) - \hat{\mathbf{z}}\,\frac{E_0}{\eta}\cos\theta_i \cos\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \frac{2\pi}{\lambda_g}z\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.90}$$

From **Eqs. (17.86)** and **(17.83)**, we write $\sin\theta_i = \lambda/\lambda_g$ and $\cos\theta_i = \lambda/\lambda_{cm}$. Substituting these into **Eq. (17.90)**,

$$\mathbf{H}_1(x,z,t) = \hat{\mathbf{x}}\,\frac{E_0}{\eta}\frac{\lambda}{\lambda_g}\sin\left(\frac{m\pi x}{d}\right)\sin\left(\omega t - \frac{2\pi}{\lambda_g}z\right) - \hat{\mathbf{z}}\,\frac{E_0}{\eta}\frac{\lambda}{\lambda_{cm}}\cos\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \frac{2\pi}{\lambda_g}z\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.91}$$

The electric and magnetic fields may also be written in the frequency domain by noting that $\sin(\omega t - 2\pi z/\lambda_g) = \cos(\omega t - 2\pi z/\lambda_g - \pi/2)$, $j = e^{j\pi/2}$, $-j = e^{-j\pi/2}$ and that $\cos(\omega t - 2\pi z/\lambda_g) = \text{Re}\{e^{j\omega t}e^{-j2\pi z/\lambda_g}\}$. Thus, the electric and magnetic field intensities in the frequency domain written with the phasor notation are

$$\mathbf{E}_1(x,z) = \hat{\mathbf{y}}\, jE_0\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.92}$$

$$\mathbf{H}_1(x,z) = -\hat{\mathbf{x}}\, j\frac{E_0}{\eta}\frac{\lambda}{\lambda_g}\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} - \hat{\mathbf{z}}\,\frac{E_0}{\eta}\frac{\lambda}{\lambda_{cm}}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.93}$$

The electric and magnetic fields vary in the transverse direction, but the electric field intensity is always zero at the conducting surfaces. The fields in waveguides are usually written as longitudinal and transverse components. Separating **Eqs. (17.92)** and **(17.93)** into their components, we get the following:

Longitudinal component:

$$H_z(x,z) = -\frac{E_0}{\eta}\frac{\lambda}{\lambda_{cm}}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.94}$$

Transverse components:

$$E_y(x,z) = jE_0 \sin\left(\frac{m\pi x}{d}\right) e^{-j2\pi z/\lambda_g} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{17.95}$$

$$H_x(x,z) = -j\frac{E_0}{\eta}\frac{\lambda}{\lambda_g} \sin\left(\frac{m\pi x}{d}\right) e^{-j2\pi z/\lambda_g} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.96}$$

The power propagated in the wave is calculated using the Poynting vector. The time-averaged power density propagated in the z direction is

$$\mathcal{P}_{av} = -\frac{1}{2}E_y(x,z)H_x^*(x,y) = \frac{E_0^2}{2\eta}\frac{\lambda}{\lambda_g}\sin^2\left(\frac{m\pi x}{d}\right) \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right] \tag{17.97}$$

Finally, since the propagation is in the z direction, we can also define the characteristic impedance of the waveguide by dividing the transverse component of the electric field intensity (E_y) by the transverse component of the magnetic field intensity (H_x):

$$Z_{TE} = \eta\frac{\lambda_g}{\lambda} = \frac{\eta}{\sqrt{1-\lambda^2/\lambda_{cm}^2}} = \frac{\eta}{\sqrt{1-f_{cm}^2/f^2}} \quad [\Omega] \tag{17.98}$$

The wave impedance is, in fact, larger than the impedance for TEM waves. Also, unlike the wave impedance η for TEM waves, the wave impedance for TE waves is frequency dependent. The wave impedance tends to infinity at cutoff ($f = f_{cm}$) and to η as the frequency approaches infinity ($f \gg f_{cm}$) (see also **Section 17.3.1** and **Table 17.2**).

Example 17.2 Application: Use of the Guiding Effects of the Ionosphere to Propagate Low-Frequency Waves The ionosphere is a region of relatively high density of charged particles produced by the solar wind. These particles act as a layer of relatively high conductivity at low frequencies. Thus, the ionosphere, which starts at about 90 km above the surface of the Earth, may be viewed as a conducting spherical surface enclosing the Earth. The surface of the Earth is also a relatively good conductor at low frequencies. These two surfaces create a parallel plate waveguide. Low-frequency waves are reflected back and forth, propagating along the surface of the Earth to long distances. The waves may even encircle the Earth and interact, causing fading of reception in radio receivers. The ionosphere is not fixed in space. It tends to be lower during the day when the supply of charged particles is high, and higher during the night when the Sun is shielded by the Earth.

Assuming the ionosphere and the surface of the Earth to be perfect conductors separated a distance of 90 km, the properties of air to be those of free space and neglecting the curvature of the Earth:

(a) Find the lowest cutoff frequency for a TE wave propagating parallel to the surface of the Earth.
(b) Find and plot the wave impedance for the lowest mode, as a function of frequency.
(c) Find the total electric field intensity at a point after the waves have encircled the globe once. Take the average radius to be 6,400 km. Assume the frequency of the wave is twice the cutoff frequency.

Solution: The propagation of waves is in a parallel plate waveguide (approximately, because the curvature of the Earth is neglected), as shown in **Figure 17.8**. The lowest cutoff frequency is the TE_1 (also called the TE_{10} mode) and it is entirely defined by the distance between the ionosphere and Earth. The total electric field intensity at a point is the sum of the incident field and the same field after it encircles the globe. Since there is no attenuation, only the phase of the electric field intensity has changed:

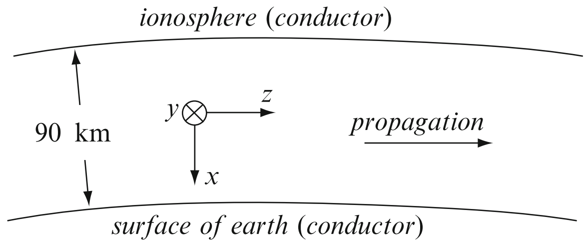


Figure 17.8 The ionosphere and Earth surfaces as a parallel plate waveguide

(a) The lowest mode is the TE_{10} mode. From **Eq. (17.80)**, with $m = 1$, and $d = 90{,}000$ m

$$f_{c10} = \frac{1}{2d\sqrt{\mu_0\varepsilon_0}} = \frac{3\times 10^8}{2\times 90{,}000} = 1667 \quad [\mathrm{H_z}]$$

where $1/\sqrt{\mu_0\varepsilon_0} = 3\times 10^8\,\mathrm{m/s}$ is the speed of light. TE waves below this frequency cannot propagate.

(b) The wave impedance is frequency dependent:

$$Z_{TE} = \frac{\eta_0}{\sqrt{1 - f_{cm}^2/f^2}} = \frac{377}{\sqrt{1-(1667)^2/f^2}} \quad [\Omega]$$

This impedance is infinite at $f = f_c$. At three times f_c, the wave impedance is 399.87 Ω. As frequency increases, the impedance decreases. As the frequency tends to infinity, the wave impedance tends to η_0. The plot of Z_{TE} with frequency is given in **Figure 17.9**.

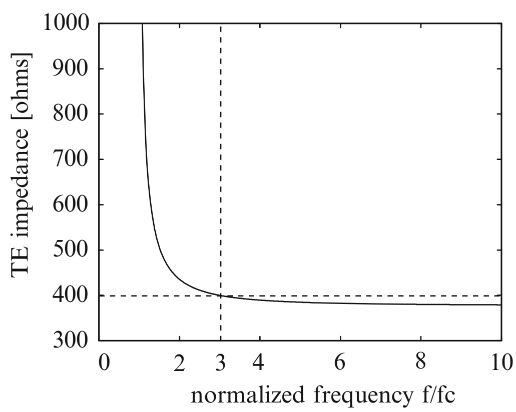


Figure 17.9 Wave impedance for the TE_{10} mode propagating in the ionosphere–Earth waveguide

(c) Assuming the electric field intensity to be as in **Eq. (17.96)**, we get

$$E_y(x,z) = jE_0 \sin\left(\frac{m\pi x}{d}\right) e^{-j2\pi z/\lambda_g} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

At the given frequency, the guide wavelength is

$$\lambda_g = \frac{\lambda}{\sqrt{1 - f_{cm}^2/f^2}} = \frac{c}{f\sqrt{1 - f_{cm}^2/f^2}} = \frac{3 \times 10^8}{2 \times 1667\sqrt{1 - (0.5)^2}} = 103,902 \quad [\mathrm{m}]$$

Note that this is larger than the wavelength in free space, which is 89,982 m. Substituting λ_g in the expression for the electric field, we get

$$E_y(x,z) = jE_0 \sin\left(\frac{\pi x}{90,000}\right) e^{-j2\pi z/103,902} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

The phase constant of the wave is therefore

$$\beta_g = 2\pi/103,902 \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right]$$

In encircling the globe, the phase changes by

$$\theta = \beta_g z = 2\pi R \beta_g = 2 \times \pi \times 6400 \times 10^3 \times 2 \times \pi/103,902 = 2,432 \quad [\mathrm{rad}]$$

Taking the point at $z = 0$ as a reference ($\theta(z = 0) = 0$), the electric field intensity after encircling the globe is

$$E_y(x,0) = jE_0 \sin\left(\frac{\pi x}{90,000}\right) e^{-j2,432} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

This adds to the electric field intensity that already exists at $z = 0$, and the total electric field intensity is

$$E_y(x,0) = jE_0 \sin\left(\frac{\pi x}{90,000}\right)\left(1 + e^{-j2,432}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right].$$

17.5 TM Propagation in Parallel Plate Waveguides

Propagation of TM modes can be obtained as for TE modes by starting with a wave impinging obliquely on a surface, but now we will assume parallel polarization; that is, the electric field is parallel to the plane of incidence. Since we have done this in detail for TE waves and since oblique incidence on a conductor for parallel polarization was discussed in **Section 13.3.2**, we will take the electric and magnetic fields obtained in **Exercise 13.8** as the given fields above the conducting surface (in medium (1)). The configuration is shown in **Figure 17.10**. The total electric and magnetic fields are

$$\mathbf{E_1}(x,z) = 2E_i[\hat{\mathbf{x}}\sin\theta_i \cos(\beta x \cos\theta_i) + \hat{\mathbf{z}}\, j\cos\theta_i \sin(\beta x\cos\theta_i)]e^{-j\beta z \sin\theta_i} \quad [\mathrm{V/m}] \tag{17.99}$$

$$\mathbf{H}_1(x,z) = \hat{\mathbf{y}}\, 2\frac{E_i}{\eta}\cos(\beta x\cos\theta_i)e^{-j\beta z\sin\theta_i} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.100}$$

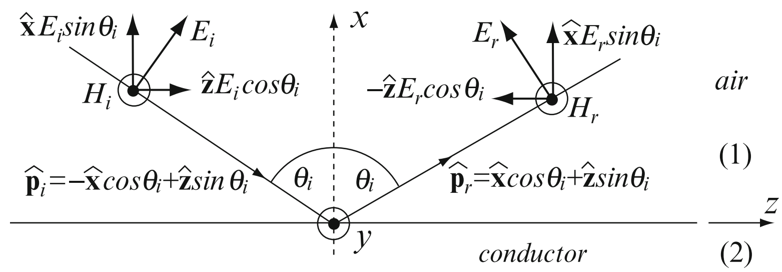


Figure 17.10 One plate of a parallel plate waveguide and the incident and reflected waves for parallel polarization

The similarity between these equations and **Eqs. (17.63)** and **(17.64)** is easy to see. For the purpose of this discussion, the following properties are important:

(1) From the calculation of the time-averaged Poynting vector (see **Exercise 17.3**), the wave propagates in the positive z direction.
(2) The waves in the x direction are standing waves only.
(3) The magnetic field intensity is in the y direction and is entirely perpendicular to the direction of propagation.
(4) The electric field intensity has components in the x and z directions. Thus, the fields in **Eqs. (17.99)** and **(17.100)** are the fields of a transverse magnetic (TM) wave.
(5) All other relations, including the angle of incidence, the propagation constants, wavelengths, and phase constants remain as defined for TE waves. A comparison between **Eqs. (17.99)** and **(17.64)** shows that the x component of the electric field intensity for TM propagation varies in the same way as the z component of the magnetic field intensity for TE propagation. Similar comparison can be made on the z component of the electric field intensity and the y component of the magnetic field intensity. Thus, we can use the relations obtained for TE waves directly. Since the zeros in the standing wave pattern are the same as for the TE wave, the parallel plate waveguide looks like the waveguide in **Figure 17.7**; that is, the distance between the plates is d and the relations in **Eqs. (17.77)** through **(17.88)** apply here as well. In particular, we use the following:

$$\sin\theta_i = \frac{\lambda}{\lambda_g}, \quad \cos\theta_i = \frac{\lambda}{\lambda_{cm}}, \quad \beta\cos\theta_i = \frac{m\pi}{d}, \quad \beta\sin\theta_i = \beta_g = \frac{2\pi}{\lambda_g} \tag{17.101}$$

Now, we substitute these in **Eqs. (17.99)** and **(17.100)** and obtain

$$\boxed{\mathbf{E}_1(x,z) = \hat{\mathbf{x}}E_0\frac{\lambda}{\lambda_g}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} + \hat{\mathbf{z}}\,jE_0\frac{\lambda}{\lambda_{cm}}\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.102}$$

$$\boxed{\mathbf{H}_1(x,z) = \hat{\mathbf{y}}\frac{E_0}{\eta}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{A}}{\text{m}}\right]} \tag{17.103}$$

where $E_0 = 2E_i$ is the amplitude of the electric field intensity. Rewriting these as longitudinal and transverse components, we get

Longitudinal component:

$$\boxed{E_z(x,z) = jE_0\frac{\lambda}{\lambda_{cm}}\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.104}$$

Transverse components:

$$\boxed{E_x(x,z) = E_0\frac{\lambda}{\lambda_g}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.105}$$

$$H_y(x,z) = \frac{E_0}{\eta}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.106}$$

The time domain expressions are obtained using $\text{Re}\{-je^{-j2\pi z/\lambda g}e^{j\omega t}\} = \sin(\omega t - 2\pi z/\lambda_g)$ and $\text{Re}\{e^{-j2\pi z/\lambda g}e^{j\omega t}\} = \cos(\omega t - 2\pi z/\lambda_g)$:

$$E_1(x,z,t) = \hat{\mathbf{x}}E_0\frac{\lambda}{\lambda_g}\cos\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \frac{2\pi z}{\lambda_g}\right) - \hat{\mathbf{z}}E_0\frac{\lambda}{\lambda_{cm}}\sin\left(\frac{m\pi x}{d}\right)\sin\left(\omega t - \frac{2\pi z}{\lambda_g}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.107}$$

$$\text{H}_1(x,z,t) = \hat{\mathbf{y}}\frac{E_0}{\eta}\cos\left(\frac{m\pi x}{d}\right)\cos\left(\omega t - \frac{2\pi z}{\lambda_g}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.108}$$

The time-averaged Poynting vector may be calculated from the transverse components in **Eqs. (17.105)** and **(17.106)**. Its magnitude is

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}\left\{E_x(x,z)H_y^*(x,z)\right\} = \frac{E_0^2\lambda}{2\eta\lambda_g}\cos^2\left(\frac{m\pi x}{d}\right) \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{17.109}$$

We can also calculate the wave impedance of the waveguide as

$$Z_{TM} = \frac{E_x}{H_y} = \eta\frac{\lambda}{\lambda_g} = \eta\sqrt{1 - \lambda^2/\lambda_{cm}^2} = \eta\sqrt{1 - f_{cm}^2/f^2} \quad [\Omega] \tag{17.110}$$

where **Eq. (17.86)** was used to obtain the expressions in terms of the mode cutoff wavelength (λ_{cm}) or mode cutoff frequency (f_{cm}). Note that the wave impedance for TM waves is always smaller than the intrinsic impedance η, whereas at cutoff, $Z_{TM} = 0$ (see also **Section 17.3.3**).

Exercise 17.2 Derive **Eqs. (17.102)** and **(17.103)** for a conducting surface at $x = 0$. Assume the electric and magnetic fields are as in **Figure 17.10**. Show first that given the incident electric field intensity $E_i(x,z)$, the incident magnetic field intensity and reflected electric and magnetic fields must be as shown. Then, evaluate the total fields $E_1(x,z)$ and $H_1(x,z)$ above the conducting surface.

Exercise 17.3 Calculate the time-averaged Poynting vector using the fields in **Eqs. (17.99)** and **(17.100)**. Show that real power propagates in the z direction and reactive power exists in the x direction.

Example 17.3 Application: TM Propagation in Microstrip Waveguides An integrated circuit waveguide is made of a very thin layer of silicon with two strips of aluminum deposited as shown in **Figure 17.11** (hence the name microstrip or stripline waveguide). The thickness of the silicon is 0.1 mm and the width of the strips is 2 mm. The structure is used to couple energy between two devices (not shown). TM waves are used in the second TM mode.

Assume lossless propagation, with permittivity of silicon equal to $12\varepsilon_0$ [F/m] and permeability equal to μ_0 [H/m]. Also, assume there is no fringing at the edges of the stripline (all energy is contained between the plates):

(a) What is the lowest frequency at which the waveguide can be used and still maintain the second TM mode of propagation?

(b) What is the maximum time-averaged power the waveguide can propagate at 900 GHz without causing breakdown in the silicon? Breakdown in silicon occurs at 32 kV/mm.

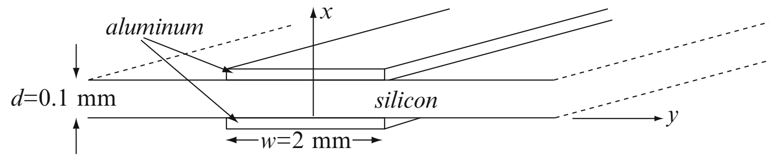


Figure 17.11 Construction and dimensions of a microstrip waveguide

Solution: The cutoff frequency of the second mode is given in **Eq. (17.80)** with $d = 0.1$ mm and $m = 2$. The peak electric field intensity in the waveguide cannot exceed 32,000 V/mm anywhere, else breakdown will occur. Thus, the peak electric field intensity is 3.2×10^7 V/m. The allowed power density is now evaluated from **Eq. (17.109)** using the peak value for E_0. The power density is then integrated over the cross-sectional area of the waveguide to calculate the total power:

(a) The cutoff frequency for the TM_{20} mode is

$$f_{c20} = \frac{2}{2d\sqrt{\mu\varepsilon}} = \frac{2 \times 3 \times 10^8}{0.0002 \times \sqrt{12}} = 866 \quad [\text{GHz}]$$

This waveguide operates well into the millimeter-wave range (see **Section 12.4**).

(b) The time-averaged power density is

$$\mathcal{P}_{av} = \frac{E_0^2 \lambda}{2\eta\lambda_g} \cos^2\left(\frac{m\pi x}{d}\right) \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

where $\lambda = v_p/f$ is the wavelength, v_p is the phase velocity, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance in silicon, and λ_g is the guide wavelength given in **Eq. (17.86)**, all calculated in silicon. These values are

$$v_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{12}} = 8.66 \times 10^7 \quad \left[\frac{\text{m}}{\text{s}}\right]$$

$$\lambda = \frac{v_p}{f} = \frac{8 \cdot 66 \times 10^7}{9 \times 10^{11}} = 9.622 \times 10^{-5} \quad [\text{m}]$$

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{12}} = 108.83 \quad [\Omega]$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} = \frac{9 \cdot 622 \times 10^{-5}}{\sqrt{1 - (866/900)^2}} = 3.53 \times 10^{-4} \quad [\text{m}]$$

The time-averaged power density is

$$\mathcal{P}_{av} = \frac{(3.2 \times 10^7)^2 \times 9.622 \times 10^{-5}}{2 \times 108.83 \times 3.53 \times 10^{-4}} \cos^2\left(\frac{2\pi x}{0.0001}\right) = 1.282 \times 10^{12} \cos^2(20,000\pi x) \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

This power density is independent of y but varies with x. Taking a strip parallel to the width of the structure as wdx, multiplying by the power density above, and integrating over x gives the total power flowing through the cross section of the waveguide:

$$P = \int_{x=0}^{x=0.0001} \mathcal{P}_{av}\omega dx = 0.002\int_{x=0}^{x=0.0001} 1.282 \times 10^{12} \cos^2(20{,}000\pi x)dx$$

$$= 2.564 \times 10^9 \left[\frac{x}{2} + \frac{0.0001}{8\pi}\sin^2(40{,}000\pi x)\right]_{x=0}^{x=0.0001} = 128.2 \quad [\text{kW}]$$

This is a considerable amount of power for a waveguide this small, but then the electric field intensity is also extremely large. Normally, the maximum electric field intensity is kept well below breakdown. Even so, waveguides can transfer relatively large amounts of power.

Example 17.4 Application: Discontinuities in Waveguides Two striplines are connected as shown in **Figure 17.12**. Both striplines are 30 mm wide. Assuming TM propagation, calculate:

(a) The lowest frequency that can be propagated if the source is connected on side *A* of the structure.
(b) The lowest frequency that can be propagated if the source is connected on side *B* of the structure.
(c) Does it make any difference in (**a**) and (**b**), if the waves are TE waves? Why?
(d) Calculate the wave impedance of the two striplines for TE and TM waves at a frequency twice the structure's cutoff frequency.
(e) Calculate the reflection and transmission coefficients for waves propagating from *A* to *B* and for waves propagating from *B* to *A* at the frequency in **(d)** for TE and TM propagation.

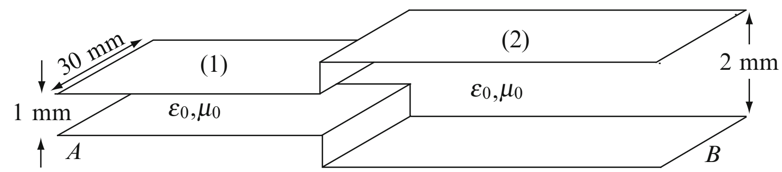


Figure 17.12 Two microstrip waveguides of different spacings connected together

Solution: The cutoff frequency of the combined structure is that below which a wave cannot propagate in the structure even though it may propagate in one of the two waveguides. To solve the problem, we calculate the individual guide's cutoff frequencies and compare to see which of the modes can be propagated in both waveguides:

(a) The cutoff frequencies for TM_{10} mode in the two sections are:
In the small guide:

$$f_{c1}^{(1)} = \frac{1}{2d\sqrt{\mu_0\varepsilon_0}} = \frac{3 \times 10^8}{2 \times 0.001} = 150 \quad [\text{GHz}]$$

In the large guide:

$$f_{c1}^{(2)} = \frac{1}{2d\sqrt{\mu_0\varepsilon_0}} = \frac{3 \times 10^8}{2 \times 0.002} = 75 \quad [\text{GHz}]$$

Since the small guide can only propagate above 150 GHz and the large guide above 75 GHz, the combined structure can only propagate above 150 GHz.

(b) The individual cutoff frequencies are the same as in **(a)**. The large guide can propagate above 75 GHz, but the small guide cannot propagate between 75 GHz and 150 GHz. The minimum frequency that the structure can propagate is 150 GHz.

(c) No, it makes no difference, since the cutoff frequencies for TE_{m0} and TM_{m0} modes are the same.
Note: This structure is a mismatched structure and causes reflections at the continuity. In practice, this type of connection should be avoided.

(d) The cutoff frequency of each stripline is different. Therefore, their wave impedances must also be different. The cutoff frequency for the composite structure is 150 GHz. Therefore, the wave impedances for TE and TM modes at 300 GHz, for the two striplines, are [from **Eqs.** (**17.98**) and (**17.110**)]

$$Z_{TM1} = \eta\sqrt{1 - f_{cm1}^2/f^2} = 377\sqrt{1 - (150/300)^2} = 326.5 \quad [\Omega]$$
$$Z_{TM2} = \eta\sqrt{1 - f_{cm2}^2/f^2} = 377\sqrt{1 - (75/300)^2} = 365 \quad [\Omega]$$
$$Z_{TE1} = \frac{\eta}{\sqrt{1 - f_{cm1}^2/f^2}} = \frac{377}{\sqrt{1 - (150/300)^2}} = 435.35 \quad [\Omega]$$
$$Z_{TE2} = \frac{\eta}{\sqrt{1 - f_{cm^2}^2/f^2}} = \frac{377}{\sqrt{1 - (75/300)^2}} = 389 \quad [\Omega]$$

Note: $Z_{TEM} = \eta_0 = 377\ \Omega$ and $Z_{TM} \leq Z_{TEM} \leq Z_{TE}$.

(e) The discontinuity caused by the connection of the two striplines is due to the differences in wave impedances of the two sections. Therefore, the connection may be viewed as the interface between two line sections with different properties. In propagating from A to B:
For TE propagation:

$$\Gamma_{12} = \frac{Z_{TE2} - Z_{TE1}}{Z_{TE2} + Z_{TE1}} = \frac{389 - 435.32}{389 + 435.32} = -0.056$$

For TM propagation:

$$\Gamma_{12} = \frac{Z_{TM2} - Z_{TM1}}{Z_{TM2} + Z_{TM1}} = \frac{365 - 326.5}{365 + 326.5} = 0.0557$$

In propagating from B to A:
For TE propagation:

$$\Gamma_{21} = -\Gamma_{12} = 0.0557$$

For TM propagation:

$$\Gamma_{21} = -\Gamma_{12} = -0.0557.$$

17.6 TEM Waves in Parallel Plate Waveguides

In developing the relations for TE and TM waves in parallel plate waveguides, we relied on the oblique incidence of a TEM wave on the conducting surfaces of the waveguide. This caused reflections at the conducting surfaces, and the sum of the incident and reflected waves produced either a TE or a TM wave, depending on the initial polarization of the TEM wave. We also mentioned that if the incident TEM wave propagates parallel to the surface of the conductors, the conductors do not affect the wave or any of its properties. Therefore, if a TEM wave, such as a plane wave, propagates such that it does not reflect off the conducting surfaces (i.e., if the angle of incidence is $\pi/2$) a TEM rather than a TE or TM wave will propagate in parallel plate waveguides. This possibility was discussed at length in **Chapter** 14, particularly in **Section** 14.6. There is little that needs to be added here except to indicate that TEM waves can indeed exist in parallel plate waveguides and, when they do, the properties in column 1 of **Table** 17.1 apply. These are the same properties we used for parallel plate transmission lines and for plane waves in the unbounded domain. However, one point needs to be mentioned again: The cutoff frequency of any TEM wave is zero; TEM waves of any frequency may propagate on the line, as we have seen for parallel plate transmission lines. Also, unlike TE and TM waves, the phase velocity and wave impedance are independent of frequency (for lossless dielectrics). This also means that the lowest possible mode of propagation is a TEM mode. For any waveguide, the lowest possible mode of propagation is called a ***dominant mode***. In parallel plate waveguides, this is the TEM mode (with cutoff at zero frequency).

TEM waves can only propagate in waveguides made of two conductors. In single, conductor waveguides only TE and/or TM modes may exist.

17.7 Rectangular Waveguides

Waves.m

In the parallel plate waveguide in the previous sections, the fields only varied in one transverse direction, whereas propagation was along the plates. Because of that, analysis of the fields was simple. True parallel plate waveguides are not practical since all dimensions must be finite. Although structures resembling the parallel plate waveguide can be built (such as the striplines in **Examples 17.3** and **17.4**) and are quite common in microwave integrated circuits, most waveguides are closed structures. A rectangular or cylindrical tube or some other type of enclosed conductor may be used. In the most general sense, the conductor is not a condition of existence of guided waves; only total reflection from a boundary is required. However, to simplify the discussion, we will restrict ourselves to waveguides defined by highly conducting surfaces.

One of the most common and simple waveguide structures is the rectangular waveguide. You can imagine a rectangular waveguide as the intersection of two pairs of parallel plate waveguides, one lying horizontally and one vertically as shown in **Figure 17.13**. This view has the advantage of defining the waveguide in terms of the parallel plate waveguides we have already discussed. We will not take this approach here since now the structure is two dimensional and the definition of angles of incidence is not as easy to visualize. The actual calculation of fields in the waveguide will be done based on the TE and TM waves described in **Eqs. (17.27)** through **(17.30)** and **Eqs. (17.53)** through **(17.56)**. However, it helps to view the rectangular waveguide as being made of two sets of parallel plates because, then, we can argue that the transverse variation of fields in the rectangular waveguide is a combined variation of the vertical and horizontal plates since both transverse components of the field (if both components exist) are standing waves. The results we obtain here will show these variations.

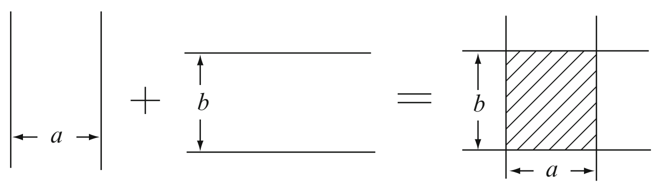


Figure 17.13 A rectangular waveguide (shown in cross section) as a combination of two parallel plate waveguides

A rectangular waveguide is shown in **Figure 17.14**. The dimensions of the waveguide are the internal dimensions and the walls are assumed to be perfectly conducting. To see how the electric and magnetic field intensities in a waveguide of the type shown in **Figure 17.14** behave, we will solve **Eq. (17.33)** for TE waves or **Eq. (17.57)** for TM waves, subject to boundary conditions on the conducting boundaries. We start with TM waves because the boundary conditions are straightforward.

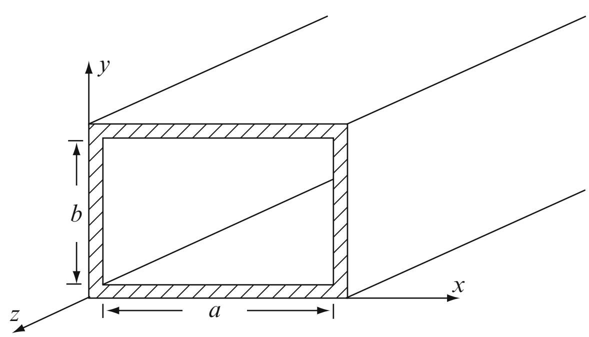


Figure 17.14 Structure and dimensions of a rectangular waveguide

17.7.1 TM Modes in Rectangular Waveguides

The TM modes are defined by the longitudinal (z component) of the electric field intensity through the following equation **[Eq. (17.57)]**:

$$\frac{\partial^2 E_Z}{\partial x^2} + \frac{\partial^2 E_Z}{\partial y^2} + \left(\gamma^2 + k^2\right) E_z = 0 \tag{17.111}$$

subject to the condition that the electric field intensity must vanish on the conducting boundaries. These conditions (shown in **Figure 17.15**) are

$$E_z(0, y) = 0 \quad \text{and} \quad E_z(a, y) = 0 \tag{17.112}$$

$$E_z(x, 0) = 0 \quad \text{and} \quad E_z(x, b) = 0 \tag{17.113}$$

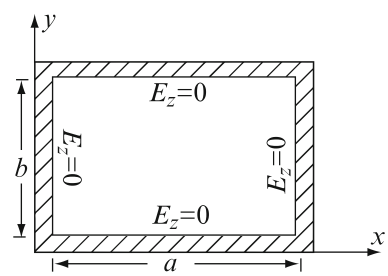


Figure 17.15 Boundary conditions for TM propagation in a rectangular waveguide

Equation (17.111) can be solved directly using separation of variables. This leads to two second-order ordinary differential equations, the solutions to which are sinusoidal functions. The solution process is much the same as for Laplace's equation in **Chapter 5**. To solve this equation using separation of variables, we first use **Eq. (17.34)** to replace the term $\gamma_2 + k_2$ by a single term we denoted k_c^2. Then, we assume a separable solution for E_z in the form

$$E_z(x, y) = X(x)Y(y) \tag{17.114}$$

where $X(x)$ only depends on the x variable and $Y(y)$ only on the y variable. This gives the x and y variations of the field. The z variation is known from the propagation constant. Thus, once we obtain $E_z(x,y)$, we can write

$$E_z(x, y, z) = E_z(x, y)e^{-\gamma z} \tag{17.115}$$

where we assume forward-propagating waves. If backward-propagating waves also exist due to reflections, these must be added, and as required, they must propagate in the negative z direction. Also, for simplicity, we will assume lossless dielectrics in the waveguide ($\gamma = j\beta$) but will retain the general form for the propagation constant to show that, in general, losses exist in the waveguide.

Substituting **Eq. (17.114)** into **Eq. (17.111)** and dividing both sides of the equation by E_z, we get

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + k_c^2 = 0 \tag{17.116}$$

For this to be satisfied, the first and second terms must each be equal to a constant which we will take as $-k_x^2$ and $-k_y^2$; that is, the following conditions must be satisfied:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -k_x^2 \quad \rightarrow \quad \frac{\partial^2 X(x)}{\partial x^2} + k_x^2 X(x) = 0 \tag{17.117}$$

$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -k_y^2 \quad \rightarrow \quad \frac{\partial^2 Y(y)}{\partial y^2} + k_y^2 Y(y) = 0 \tag{17.118}$$

The separation constants must also satisfy the following conditions:

$$-k_x^2 - k_y^2 + k_c^2 = 0 \tag{17.119}$$

Equations (**17.117**) and (**17.118**) have the following solutions (see **Section 5.4.4.1**):

$$X(x) = A_1 \sin k_x x + B_1 \cos k_x x \tag{17.120}$$

$$Y(y) = A_2 \sin k_y y + B_2 \cos k_y y \tag{17.121}$$

Substituting these into **Eq.** (**17.114**), we obtain the general solution

$$E_z(x,y) = (A_1 \sin k_x x + B_1 \cos k_x x)(A_2 \sin k_y y + B_2 \cos k_y y) \tag{17.122}$$

The constants are now evaluated from the boundary conditions in **Eqs.** (**17.112**) and (**17.113**):

$$E_z(0,y) = (A_1 \sin k_x 0 + B_1 \cos k_x 0)(A_2 \sin k_y y + B_2 \cos k_y y) = 0 \quad \rightarrow \quad B_1 = 0 \tag{17.123}$$

$$E_z(x,0) = A_1 \sin k_x x (A_2 \sin k_y 0 + B_2 \cos k_y 0) = 0 \quad \rightarrow \quad B_2 = 0 \tag{17.124}$$

$$E_z(a,y) = A \sin(k_x a)\sin(k_y y) = 0 \quad \rightarrow \quad k_x = \frac{m\pi}{a} \tag{17.125}$$

$$E_z(x,b) = A \sin\left(\frac{m\pi}{a}x\right)\sin(k_y b) = 0 \quad \rightarrow \quad k_y = \frac{n\pi}{b} \tag{17.126}$$

The amplitude of the electric field intensity is arbitrary; it does not affect the form of the solution and we will denote it by E_0. Therefore, the solution for the longitudinal component of the electric field intensity is

$$E_z(x,y) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{17.127}$$

The general solution also includes the z variation. From **Eq.** (**17.115**), we get

$$\boxed{E_z(x,y,z) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma z} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]} \tag{17.128}$$

Before proceeding with the evaluation of the transverse components, we note the following from **Eq.** (**17.119**):

$$\boxed{k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{17.129}$$

Also, from **Eq.** (**17.34**), we get

$$\boxed{\gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} \tag{17.130}$$

Now, we can calculate the transverse components of the electric and magnetic fields by substituting the general solution [**Eq.** (**17.128**)] into **Eqs.** (**17.53**) through (**17.56**). This gives

$$E_x(x,y,z) = \frac{-\gamma}{\gamma^2+k^2} E_0 \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.131}$$

$$E_y(x,y,z) = \frac{-\gamma}{\gamma^2+k^2} E_0 \frac{n\pi}{ba} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.132}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{\gamma^2+k^2} E_0 \frac{n\pi}{b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.133}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon}{\gamma^2+k^2} E_0 \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.134}$$

These, together with **Eqs. (17.129)** and **(17.130)**, define the transverse fields in the waveguide. We can easily write the time domain form of the equations (see **Exercise 17.4**) by adding the $e^{j\omega t}$ term and writing $e^{j\pi/2}$ for j and $e^{-j\pi/2}$ for $-j$. Also, the longitudinal and transverse components for a backward-propagating wave can be written directly from **Eqs. (17.128)** and **(17.131)** through **(17.134)** by replacing γ by $-\gamma$ wherever these occur (see **Exercise 17.5** and **Problems 17.22** and **17.23**).

Equations (17.128) and **(17.131)** through **(17.134)** are written for the general modes (mn). Since we solved for TM modes, the general mode is a TM_{mn} mode where m, n are any integers, including zero. An infinite number of modes are possible, but usually only the first few modes are used in practice. Also, some modes are not useful. For example, if $m = 0$ and $n = 0$, all components of the field, including the longitudinal component, are zero. This is clearly not a useful mode. Also, if $m = 0$ and $n \neq 0$, or $m \neq 0$ and $n = 0$, all field components become zero as can be seen by substitution in **Eq. (17.128)**. Thus, for TM modes, neither m nor n may be zero. The lowest possible TM mode is the TM_{11} mode. The longitudinal components of the TM_{12} and TM_{22} modes are shown in **Figure 17.16**. Similar plots may be obtained for the transverse components of the field.

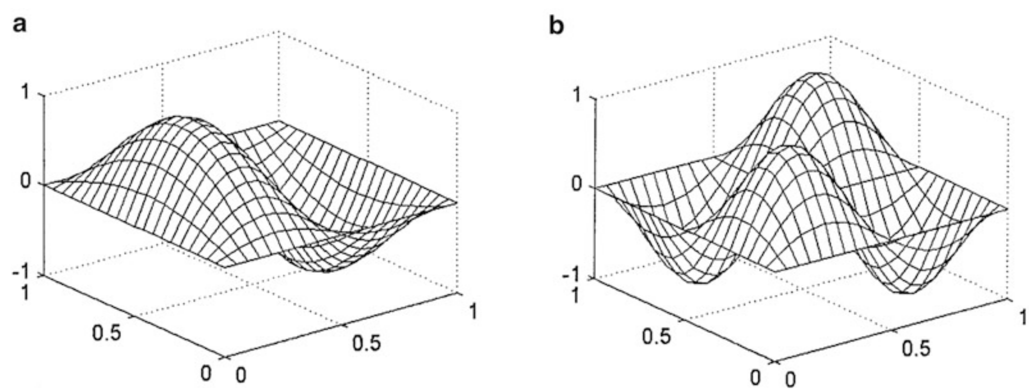


Figure 17.16 The longitudinal electric field distribution in a waveguide. (**a**) For TM_{12} mode. (**b**) For TM_{22} mode

The propagation properties in the waveguide are obtained from **Eqs. (17.129)** and **(17.130)**. The propagation constant for lossless propagation is

$$\boxed{\gamma = j\beta_g = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \tag{17.135}$$

where β_g is the guide phase constant:

$$\boxed{\beta_g = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \left[\frac{\text{rad}}{\text{m}}\right]} \tag{17.136}$$

The cutoff wave number is given in **Eq. (17.129)**:

$$\boxed{k_{cmn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad \left[\frac{\text{rad}}{\text{m}}\right]} \tag{17.137}$$

The cutoff frequency may be obtained from **Eq. (17.137)** by using $k_{cmn} = 2\pi f_{cmn}\sqrt{\mu\varepsilon}$ or by setting the propagation constant in **Eq. (17.130)** to zero:

$$\boxed{f_{cmn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad [\text{Hz}]} \tag{17.138}$$

Propagation in the waveguide can only occur above this frequency. As with the parallel plate waveguide, below this frequency there is rapid attenuation of the wave (evanescent wave).

The cutoff wavelength is then

$$\boxed{\lambda_{cmn} = \frac{v_p}{f_{cmn}} = \frac{1}{\sqrt{(m/2a)^2 + (n/2b)^2}} \quad [\text{m}]} \tag{17.139}$$

From **Eq. (17.136)**, we can write

$$\beta_g = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{1}{4\pi^2 f^2 \mu\varepsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{17.140}$$

or, using **Eq. (17.138)**,

$$\boxed{\beta_g = \beta\sqrt{1 - \frac{f_{cmn}^2}{f^2}} \quad \left[\frac{\text{rad}}{\text{m}}\right]} \tag{17.141}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ is the phase constant in the material filling the waveguide as if propagation were in infinite space. Note that this expression is the same as **Eq. (17.87)** for TM propagation in parallel plate waveguides and **Eq. (17.41)** for TM propagation in general. We called β_g the guide phase constant to distinguish it from β, which is the phase constant in unbounded space.

From β_g, we can also calculate the guide phase velocity v_g and the guide wavelength λ_g as

$$\boxed{v_g = \frac{\omega}{\beta_g} = \frac{v_p}{\sqrt{1 - \lambda^2/\lambda_{cmn}^2}} = \frac{v_p}{\sqrt{1 - f_{cmn}^2/f^2}} \quad \left[\frac{\text{m}}{\text{s}}\right]} \tag{17.142}$$

where $v_p = 1/\sqrt{\mu\varepsilon}$ is the phase velocity in the unbounded space (with properties of the space identical to those in the waveguide), and

$$\boxed{\lambda_g = \frac{2\pi}{\beta_g} = \frac{\lambda}{\sqrt{1 - \lambda^2/\lambda_{cmn}^2}} = \frac{\lambda}{\sqrt{1 - f_{cmn}^2/f^2}} \quad [\text{m}]} \tag{17.143}$$

where $\lambda = 2\pi/\beta$ is the wavelength in unbounded space.

The wave impedance is obtained by taking the ratio between the transverse components of the electric and magnetic field intensities. Since the wave must propagate in the z direction (i.e., the Poynting vector must be in the z direction), we can take either the ratio between E_x and H_y or the negative ratio of E_y and H_x:

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\gamma}{j\omega\varepsilon} \quad [\Omega] \tag{17.144}$$

Or, using **Eq. (17.135)** to replace γ, we can write for lossless propagation

$$\boxed{Z_{TM} = \frac{\beta_g}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \frac{f_{cmn}^2}{f^2}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \frac{\lambda^2}{\lambda_{cmn}^2}} = \eta\frac{\lambda}{\lambda_g} \quad [\Omega]} \tag{17.145}$$

Also, from **Eqs. (17.130), (17.129),** and **(17.135)**, we have for lossless propagation:

$$k_{cmn}^2 = \gamma^2 + k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \text{and} \quad \gamma = j\beta_g \tag{17.146}$$

With these, we can write the transverse components in **Eqs. (17.131)** through **(17.134)** in terms of the phase constant β_g and the cutoff wave number k_{cmn} instead of the propagation constant γ. Substituting from **Eq. (17.146)** into **Eqs. (17.131)** through **(17.134)** gives

$$E_x(x,y,z) = \frac{-j\beta_g}{k_{cmn}^2}E_0\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.147}$$

$$E_y(x,y,z) = \frac{-j\beta_g}{k_{cmn}^2}E_0\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.148}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{k_{cmn}^2}E_0\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.149}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon}{k_{cmn}^2}E_0\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_g z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.150}$$

The longitudinal component in **Eq. (17.128)** and the transverse components in **Eqs. (17.131)** through **(17.134)** or in **Eqs. (17.147)** through **(17.150)** may be written in the time domain by multiplying the expressions by $e^{j\omega t}$ and taking the real part of the expression (see **Exercise 17.4** and **Problem 17.21**).

The purpose of a waveguide is to guide waves from a source to a load. Therefore, it must propagate power. To see that this is the case, we calculate the time-averaged power density in the cross section of the waveguide (i.e., for any value of z). This power density must be real and must propagate in the positive z direction. From **Eqs. (17.147)** through **(17.150)**, we see that there are two sets of transverse components of **E** and **H**. Each pair produces average power given by $E_xH_y^*/2$ and $-E_yH_x^*/2$, where the negative sign comes from the fact that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ and $\hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}$. The total power density is the sum of these two terms:

$$\boldsymbol{\mathcal{P}}_{av}(x,y) = \hat{\mathbf{z}}\frac{1}{2}\text{Re}\left\{E_x(x,y)H_y^*(x,y) - E_y(x,y)H_x^*(x,y)\right\} \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{17.151}$$

Substituting for E_x, E_y, H_x, and H_y from **Eqs. (17.147)** through **(17.150)** gives

$$\mathcal{P}_{av}(x,y) = \hat{\mathbf{z}}\,\frac{\omega\varepsilon\beta_g E_0^2}{2k_{cmn}^4}\left[\left(\frac{m\pi}{a}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{n\pi}{b}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right] \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{17.152}$$

The total power in the waveguide cross section is the power density, integrated over the waveguide cross section. Performing the integration over $\boldsymbol{\mathcal{P}}_{av} \cdot d\mathbf{s}$, with $d\mathbf{s} = \hat{\mathbf{z}}\,dxdy$, gives

$$P = \frac{\omega\varepsilon\beta_g E_0^2}{2k_{cmm}^4}\int_{x=0}^{x=a}\int_{y=0}^{y=b}\left[\left(\frac{m\pi}{a}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{n\pi}{b}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{\eta\pi y}{b}\right)\right]dxdy$$
$$= \frac{\omega\varepsilon\beta_g E_0^2}{k_{cmn}^4}\frac{ab}{8}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right] = \frac{\omega\varepsilon\beta_g E_0^2 ab}{8k_{cmn}^2} \quad [\mathrm{W}] \tag{17.153}$$

where $k_{cmn}^2 = (m\pi/a)^2 + (n\pi/b)^2$. The total power is directly proportional to the cross-sectional area of the waveguide (ab). In any given waveguide, the total power may be increased by using larger fields (electric and magnetic) or increasing the physical dimensions of the waveguide. Also, the power is proportional to frequency and the dielectric constant in the waveguide. Most waveguides use air as the dielectric, but increasing the frequency is feasible up to certain limits, imposed by the circuits used to generate the fields.

Exercise 17.4 Write the time domain expression for the longitudinal component of the electric field intensity for TM propagation in a lossless rectangular waveguide.

Answer $E_z(x, y, z, t) = E_0\sin(m\pi x/a)\sin(n\pi y/b)\cos(\omega t - \beta_g z)$ [V/m].

Exercise 17.5

(a) Find the longitudinal component $E_z(x,y,z)$ for a backward-propagating wave in a general lossy rectangular waveguide. Assume the backward-propagating wave propagates in the negative z direction and the amplitude of the wave is E_0^-.

(b) Find the total longitudinal field in a waveguide if both a forward-propagating wave of amplitude E_0^+ and a backward-propagating wave of amplitude E_0^- exist.

Answer

(a) $E_z^-(x, y, z) = E_0^-\sin(m\pi x/a)\sin(n\pi y/b)e^{\gamma z}$. **(b)** $E_z(x, y, z) = \sin(m\pi x/a)\sin(n\pi y/b)(E_0^+e^{-\gamma z} + E_0^-e^{\gamma z})$.

Note: If the magnitude of the backward-propagating wave equals that of the forward-propagating wave ($|E_0^-| = |E_0^+|$), the term in the last parentheses in **(b)** can be written as sine or cosine functions using the exponential forms. We will use this property in **Section 17.9**.

Example 17.5 A standard rectangular waveguide, designated as EIA WR75, has internal dimensions $a = 19.05$ mm and $b = 9.53$ mm. The waveguide is air filled and propagates waves at 18 GHz:

(a) Calculate the lowest possible TM mode at which the wave may be excited.

(b) For the mode in **(a)**, calculate the guide wavelength, guide phase constant, guide phase velocity, and the wave impedance for TM propagation (at 18 GHz).

(c) Calculate the maximum time-averaged power transmitted through the waveguide at the mode calculated in **(a)** if the electric field intensity is not to exceed the breakdown level in air (3×10^6 V/m).

Solution: The possible modes are all the modes with cutoff frequencies below 18 $\mathrm{GH_z}$. The propagated time-averaged power is given in **Eq. (17.153)**:

(a) The cutoff frequency for the TM modes is given as

$$f_{cmn} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{\mathrm{b}}\right)^2} = 1.5 \times 10^8\sqrt{\left(\frac{m}{0.01905}\right)^2 + \left(\frac{n}{0.00953}\right)^2} \quad [\mathrm{Hz}]$$

The possible cutoff frequencies below 18 GHz are:

$$f_{c10} = 1.5 \times 10^8\sqrt{\left(\frac{1}{0.01905}\right)^2 + \left(\frac{0}{0.00953}\right)^2} = 7.874 \quad [\mathrm{GHz}]$$

$$f_{c01} = 1.5 \times 10^8\sqrt{\left(\frac{0}{0.01905}^2\right) + \left(\frac{1}{0.00953}\right)^2} = 15.74 \quad [\mathrm{GHz}]$$

$$f_{c11} = 1.5 \times 10^8\sqrt{\left(\frac{1}{0.01905}\right)^2 + \left(\frac{1}{0.00953}\right)^2} = 17.60 \quad [\mathrm{GHz}]$$

$$f_{c20} = 1.5 \times 10^8\sqrt{\left(\frac{2}{0.01905}\right)^2 + \left(\frac{0}{0.00953}\right)^2} = 15.75 \quad [\mathrm{GHz}]$$

All other cutoff frequencies are above 18 GHz. Since, in TM modes, m or n cannot be zero, only the third of these, namely, $f_{c11} = 17.60$ GHz, corresponds to a possible TM mode. Thus, the only possible mode is the TM_{11} mode.

(b) At 18 GHz, the waves in the waveguide are TM_{11} from **Eqs. (17.141)** through **(17.145)**, we get

$$\beta_g = \beta\sqrt{1 - \frac{f_{cmn}^2}{f^2}} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \frac{f_{cmn}^2}{f^2}} = \frac{2 \times \pi \times 18 \times 10^9}{3 \times 10^8}\sqrt{1 - \left(\frac{17.60}{18}\right)^2} = 79 \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right]$$

$$v_g = \frac{\omega}{\beta_g} = \frac{2 \times \pi \times 18 \times 10^9}{79} = 1.43 \times 10^9 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

$$\lambda_g = \frac{2\pi}{\beta_g} = \frac{2 \times \pi}{79} = 0.0795 \quad [\mathrm{m}]$$

$$Z_{TM} = \frac{\beta_g}{\omega\varepsilon_0} = \frac{79}{2 \times \pi \times 18 \times 10^9 \times 8.854 \times 10^{-12}} = 78.89 \quad [\Omega].$$

(c) To calculate the power density inside the waveguide, we use the expressions in **Eqs. (17.153)** and **(17.137)** with $m = 1$, $n = 1$, $a = 0.01905$ m, $b = 0.00953$ m, $E_0 = 3 \times 10^6$ V/m, $f = 18$ GHz, and $\varepsilon = \varepsilon_0$. Also, we have $\lambda_c(11) = \mathrm{v}_{p/fc11} = 3 \times 10^8/17.6 \times 10^9 = 0.017$ m. With these, we first calculate the cutoff wave number k_{c11} from **Eq. (17.137)**:

$$k_{c11} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \sqrt{\left(\frac{\pi}{0.01905}\right)^2 + \left(\frac{\pi}{0.00953}\right)^2} = 368.6 \quad [\mathrm{rad/m}]$$

The maximum total power is

$$P = \frac{\omega\varepsilon\beta_g E_0^2 ab}{8k_c^2} = \frac{2 \times \pi \times 18 \times 10^9 \times 8.854 \times 10^{-12} \times 79 \times 9 \times 10^{12} \times 0.01905 \times 0.00953}{8 \times (368.6)^2} = 118{,}918 \quad [\mathrm{W}]$$

This is almost 119 kW of power for a tube of cross-sectional area of less than 2 cm^2 (181.55 mm^2). In practice however, the fields chosen are much lower than the maximum. Since the power is proportional to the square of the field, the power decreases rapidly.

Exercise 17.6 A rectangular waveguide has dimensions $a = 0.015$ m and $b = 0.01$ m. Find the first 10 cutoff frequencies for TM modes.

Answer (Frequencies given in GHz, in increasing frequency order)

Mode	TM_{11}	TM_{21}	TM_{12}	TM_{31}	TM_{22}	TM_{32}	TM_{41}	TM_{42}	TM_{51}	TM_{33}
f_{cmn}	18.027	25.0	31.622	33.541	36.055	42.426	42.720	50	52.201	54.08

Note: Lower modes may have higher frequencies than higher modes, depending on dimensions.

Exercise 17.7 Two standard waveguides are given: WR 3, with dimensions $a = 0.86$ mm and $b = 0.43$ mm, operating at 400 GHz, and WR 2300, with dimensions $a = 0.5842$ m and $b = 0.2921$ m operating at 600 MHz. Both waveguides are air filled (these two waveguides represent the smallest and largest standard waveguides):

(a) Find the TM_{11} cutoff frequencies of the two waveguides.
(b) Find the ratio between the total power carried by the two waveguides for a given electric field intensity E.

Answer

(a) $f_{c11}(\text{WR } 3) = f_{c1} = 390$ GHz. $f_{c11}(\text{WR } 2300) = f_{c2} = 574.14$ MHz. **(b)** $p = \dfrac{P_1}{P_2} = \dfrac{f_1\beta_{g1}a_1b_1k_{c2}^2}{f_2\beta_{g2}a_2b_2k_{c1}^2} = 1.597 \times 10^{-6}$.

(Index 1 stands for the WR 3 waveguide and 2 for the WR 2300 waveguide.)

17.7.2 TE Modes in Rectangular Waveguides

For TE waves to exist, E_z must be zero. The wave equation to solve is now **Eq. (17.33)**:

$$\frac{\partial_2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (\gamma_2 + k_2)H_z = 0 \tag{17.154}$$

subject to the condition that the x and y components of the electric field intensity in **Eqs. (17.27)** and **(17.28)** must vanish on the conducting boundaries. These conditions are shown in **Figure 17.17**:

$$\frac{\partial H_z}{\partial x}(0, y) = 0 \quad \text{and} \quad \frac{\partial H_z}{\partial x}(a, y) = 0 \tag{17.155}$$

$$\frac{\partial H_z}{\partial y}(x, 0) = 0 \quad \text{and} \quad \frac{\partial H_z}{\partial y}(x, b) = 0 \tag{17.156}$$

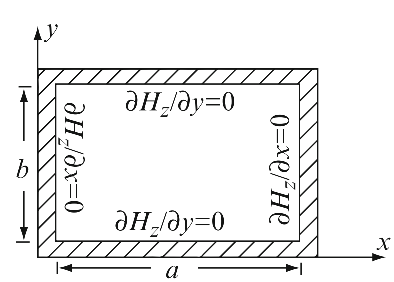


Figure 17.17 Boundary conditions on the magnetic field for TE modes in a rectangular waveguide

Since **Eq. (17.154)** is identical in form to **Eq. (17.111)**, the general solution must also be of the same form as the solution in **Eq. (17.122)**:

$$H_z(x,y) = (A_1 \sin k_x x + B_1 \cos k_x x)\left(A_2 \sin k_y y + B_2 \cos k_y y\right) \quad (17.157)$$

To find the constants, we write from the four boundary conditions in **Eqs. (17.155)** and **(17.156)**

$$\left.\frac{\partial H_z(x,y)}{\partial x}\right|_{x=0} = (k_x A_1 \cos 0 - k_x B_1 \sin 0)\left(A_2 \sin k_y y + B_2 \cos k_y y\right) = 0 \quad \rightarrow \quad A_1 = 0 \quad (17.158)$$

$$\left.\frac{\partial H_z(x,y)}{\partial y}\right|_{y=0} = B_1 \sin k_x x\left(k_y A_2 \cos 0 - k_y B_2 \sin 0\right) \quad \rightarrow \quad A_2 = 0 \quad (17.159)$$

At this point, the solution is

$$H_z(x,y) = B\cos(k_x x)\cos\left(k_y y\right) \quad (17.160)$$

where $B = B_1 B_2$. From the remaining boundary conditions

$$\left.\frac{\partial H_z(x,y)}{\partial x}\right|_{x=a} = -Bk_x \sin(k_x a)\sin\left(k_y y\right) = 0 \quad \rightarrow \quad k_x = \frac{m\pi}{a}, m = 0, 1, \ldots \quad (17.161)$$

$$\left.\frac{\partial H_z(x,y)}{\partial y}\right|_{y=b} = -B\cos\left(\frac{m\pi x}{a}\right)k_y \sin\left(k_y b\right) = 0 \quad \rightarrow \quad k_y = \frac{n\pi}{b}, n = 0, 1, \ldots \quad (17.162)$$

The solution is obtained by substituting k_x and k_y from **Eqs. (17.161)** and **(17.162)** into **Eq. (17.160)**. Taking the amplitude B in **Eq. (17.160)** as H_0, we get

$$H_z(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \quad (17.163)$$

If we now add the variation in z (assuming only a forward-propagating wave), we get the general form of the longitudinal component of the magnetic field:

$$\boxed{H_z(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right]} \quad (17.164)$$

This is now substituted into the general expressions for TE waves in **Eqs. (17.27)** through **(17.30)** to obtain the transverse components of the electric and magnetic field intensities. Performing the derivatives of H_z with respect to x and y as required in **Eqs. (17.27)** through **(17.30)** gives

$$E_x(x,y,z) = \frac{j\omega\mu}{\gamma^2 + k^2} H_0 \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right] \quad (17.165)$$

$$E_y(x,y,z) = \frac{-j\omega\mu}{\gamma^2 + k^2} H_0 \frac{m\pi}{a} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right] \quad (17.166)$$

$$H_x(x,y,z) = \frac{\gamma}{\gamma^2 + k^2} H_0 \frac{m\pi}{a} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right] \quad (17.167)$$

$$H_y(x,y,z) = \frac{\gamma}{\gamma^2 + k^2} H_0 \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right] \quad (17.168)$$

The propagation properties for TE modes in the waveguide are identical to those for TM modes except for the wave impedance. This can be seen either from **Table 17.2** or by direct calculation of the various properties (f_{cmn}, β_g, v_g, etc.). We will not reevaluate these here and simply use the properties in **Eqs. (17.135)** through **(17.143)** and **(17.146)** as given. The wave impedance, however, is different for TE modes and is given by the ratio of the transverse components of the electric and magnetic fields as follows:

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma} \quad [\Omega] \tag{17.169}$$

or, using **Eq. (17.135)** to replace γ by $j\beta_g$, we can write for lossless propagation

$$\boxed{Z_{TE} = \frac{\omega\mu}{\beta_g} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1 - f_{cmn}^2/f^2}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1 - \lambda^2/\lambda_{cmn}^2}} = \eta\frac{\lambda_g}{\lambda} \quad [\Omega]} \tag{17.170}$$

Using the relations $k_{cmn}^2 = \gamma^2 + k^2$ and $\gamma = j\beta_g$, we can rewrite the transverse components of the electric and magnetic field in **Eqs. (17.165)** through **(17.168)** as

$$E_x(x, y, z) = \frac{j\omega\mu}{k_{cmn}^2} H_0 \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.171}$$

$$E_y(x, y, z) = \frac{-j\omega\mu}{k_{cmn}^2} H_0 \frac{m\pi}{b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.172}$$

$$H_x(x, y, z) = \frac{j\beta_g}{k_{cmn}^2} H_0 \frac{m\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.173}$$

$$H_y(x, y, z) = \frac{j\beta_g}{k_{cmn}^2} H_0 \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.174}$$

TE_{mn} modes are obtained for all possible pairs of the integers m and n, except for $m = 0$ and $n = 0$. Unlike TM modes, in TE modes either m or n can be zero but not both. This indicates that the lowest propagating mode is a TE_{0n} or TE_{m0}, depending on the dimensions a and b of the waveguide. If $a > b$, the lowest cutoff frequency is for a TE_{10} mode. Also to be noted is that TM and TE modes with the same indices have the same cutoff frequency, as can be seen from **Eq. (17.138)**. The magnetic field intensity (H_z) distribution in a waveguide with $a = 2b$ for the TE_{11} and TE_{30} modes is shown in **Figure 17.18**.

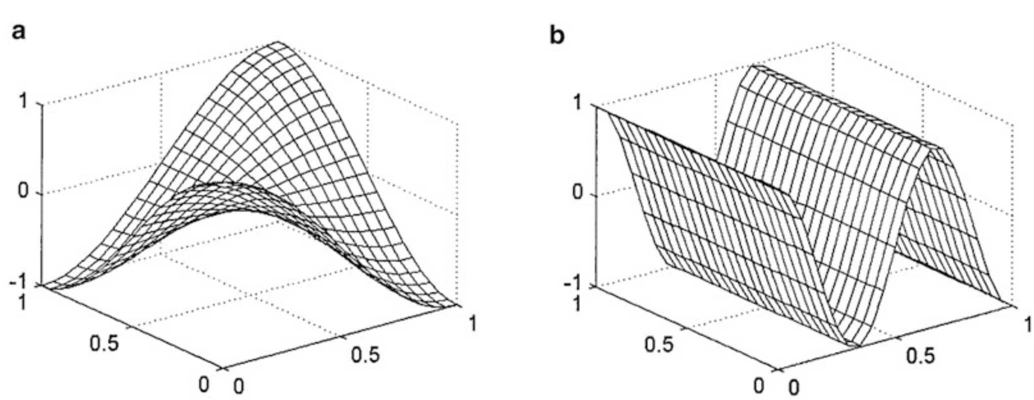


Figure 17.18 The longitudinal magnetic field distribution in a waveguide. (**a**) For the TE_{11} mode. (**b**) For the TE_{30} mode

The power density in a waveguide propagating a TE mode may be calculated using steps identical to those for TM modes, using the Poynting vector for the transverse components of **E** and **H**. Each pair of transverse components produces a time-averaged power density propagating in the z direction given by $E_x H_y^*/2$ and $-E_y H_x^*/2$. The total power density is the sum of the two terms as in **Eq. (17.151)**. Substituting for E_x, E_y, H_x, and H_y from **Eqs. (17.171)** through **(17.174)** into **Eq. (17.151)** gives

$$\mathcal{P}_{av}(x,y) = \hat{\mathbf{z}}\frac{\omega\mu\beta_g H_0^2}{2k_{cmn}^4}\left[\left(\frac{n\pi}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m\pi}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\quad\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right] \tag{17.175}$$

The total power in a rectangular waveguide of dimensions a and b is

$$P = \frac{\omega\mu\beta_g H_0^2}{2k_{cmn}^4}\int_{x=0}^{x=a}\int_{y=0}^{y=a}\left[\left(\frac{m\pi}{a}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{n\pi}{b}\right)^2\sin^2\left(\frac{m\pi x}{b}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]dxdy = \frac{\omega\mu\beta_g H_0^2 ab}{8k_{cmn}^2}\quad[\mathrm{W}] \tag{17.176}$$

In particular, for the TE_{10} mode, we get

$$\boxed{P(\mathrm{TE}_{10}) = \frac{\omega\mu\beta_g H_0^2 ab}{8k_{cmn}^2} = \frac{\omega\mu\beta_g H_0^2 ab}{8(\pi/a)^2} = \frac{\omega\mu\beta_g H_0^2 a^3 b}{8\pi^2}\quad[\mathrm{W}]} \tag{17.177}$$

The discussion on waveguides was limited here to rectangular cross-sectional waveguides. Other shapes can be analyzed in a similar fashion. However, since the analysis entails the solution of the wave equation, only simple shapes can be solved exactly. In particular, cylindrical waveguides can be analyzed using steps similar to those presented above, but the solution is in terms of Bessel rather than harmonic functions.

The lowest cutoff frequency mode in any waveguide is called a ***dominant mode***. In rectangular waveguides, this is a TE mode and, usually, the TE_{10} mode. Different modes which have the same cutoff frequency are called ***degenerate modes***.

Example 17.6 Application: The TE_{10} Mode The TE_{10} mode is the most important mode of propagation in rectangular waveguides. One reason for this is because it is possible to design waveguides with the largest possible mode separation between the TE_{10} and the other possible modes. This gives the largest possible bandwidth for propagation in a waveguide, as well as the lowest cutoff frequency for a given waveguide. Consider the following example.

The WR34 waveguide has dimensions $a = 8.64$ mm and $b = 4.32$ mm, is air filled, and may be excited in any mode:

(a) Calculate the lowest possible cutoff frequency.
(b) Calculate the next cutoff frequencies and identify the modes. Decide based on bandwidth (mode separation) which mode is most suitable for general purpose use.
(c) Repeat **(a)** and **(b)** if the waveguide is filled with a perfect dielectric with relative permittivity of 4.

Solution: The cutoff frequencies of the waveguide are calculated from **Eq. (17.138)** for all modes, including TM modes. The only difference between the various modes is that in TM modes, $m = 0$ or $n = 0$ is not allowed, whereas in TE modes they are. Also, TM_{mn} and TE_{mn} modes are always degenerate (have the same cutoff frequencies). From **Eq. (17.138)**

$$f_{cmn} = 1.5\times 10^8\sqrt{\left(\frac{m}{0.00864}\right)^2+\left(\frac{n}{0.00432}\right)^2}\quad[\mathrm{Hz}]$$

(a) The lowest possible mode must have $m = 1$ and $n = 0$. Since in TM modes, $n = 0$ results in all zero fields, this mode is the TE_{10} mode. Therefore, for any waveguide, with $a > b$, the dominant mode is the TE_{10} mode. The TE_{10} cutoff frequency is

$$f_{c10} = 1.5 \times 10^8 \sqrt{\left(\frac{1}{0.00864}\right)^2 + \left(\frac{0}{0.00432}\right)^2} = 17.361 \quad \text{[GHz]}.$$

(b) The remaining modes are calculated similarly. All modes with one index zero are TE modes. All modes that have both indices nonzero are both TE and TM modes. Using the appropriate indices, the remaining (higher) cutoff frequencies are calculated and listed in the first row in **Table 17.3** in ascending order of cutoff frequencies.

The largest mode separation is between the TE_{10} and TE_{01} modes. Although separation between TE_{20} and TE_{02} is larger, the TE_{11}, TM_{11}, TE_{30}, TE_{21}, and TM_{21} modes are also in this range and, therefore, it is not suitable for single-mode operation. The TE_{10} mode can be propagated between 17.361 GHz and 34.722 GHz. Normally, the operation will be about 25 % above 17.361 GHz and below 34.722 GHz so as not to be too close to the cutoff frequencies. The recommended range for this waveguide in the TE_{10} mode is 21.7–33 GHz.

(c) If the waveguide is filled with a dielectric, the cutoff frequencies are reduced by a factor of $\sqrt{\varepsilon_r}$. For $\varepsilon_r = 4$, the cutoff frequencies are reduced by a factor of 2. The new cutoff frequencies (in GHz) are shown in the second row in **Table 17.3**, again in ascending order.

Table 17.3 First 10 cutoff frequencies for empty and dielectric filled waveguide in Example 17.6

	TE_{10}	TE_{01}	TE_{11}	TE_{21}	TE_{30}	TE_{31}	TE_{40}	TE_{12}	TE_{22}	TE_{32}
		TE_{20}	TM_{11}	TM_{21}		TM_{31}	TE_{02}	TM_{12}	TE_{41}	TE_{50}
									TM_{22}	TM_{32}
									TM_{41}	
$\varepsilon = \varepsilon_0$	17.361	34.722	38.82	49.1	52.083	62.596	69.44	71.58	77.64	86.806
$\varepsilon = 4\varepsilon_0$	8.68	17.361	19.41	24.55	26.04	31.298	34.72	35.79	38.82	43.403

Example 17.7 Application: The Practical Rectangular Waveguide A practical rectangular waveguide is a tube of any rectangular cross section made of a high-conductivity material. The dimensions of the waveguide are arbitrary but are normally chosen such that the cutoff frequencies of the various modes are not degenerate as much as possible. Every waveguide is designed for a lowest, dominant mode, with the next mode defining the operating range or bandwidth of the waveguide. Normally, the dominant mode is the TE_{10} mode. Thus, for example, the EIA WR90 waveguide has internal dimensions of 22.86 mm and 10.16 mm. The TE_{10} cutoff frequency is at 6.562 GHz and the normal operating range (recommended) for the TE_{10} mode is 8.2–12.5 GHz (*X*-band). Similarly, the EIA WR5 waveguide has dimensions of 1.3 mm and 0.66 mm (internal) and a cutoff frequency of 115.385 GHz. The recommended operating frequency for the TE_{10} mode is 145–220 GHz. Although waveguides normally operate in the TE_{10} mode, they may also operate in any other mode, including TM modes.

It is required to design a rectangular waveguide with lowest cutoff frequency at 10 GHz. Two designs are proposed. One has both dimensions a and b equal. The second is designed such that $a = 2b$:

(a) Find the cutoff frequencies of the first 10 TM and first 10 TE modes in order of ascending frequencies and compare the two waveguides.
(b) What is the dominant mode and which modes are degenerate?
(c) Which waveguide is better suited for use in general purpose applications in terms of mode separation?

Solution: First, we find the dimensions of the two waveguides from the given cutoff frequency. Then, the cutoff frequencies of the remaining modes are found:

(a) The required lowest cutoff frequency is 10 GHz. For the square waveguide we write $b = a$ and substitute in **Eq. (17.138)**:

$$f_{c10} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{a}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2} \quad \rightarrow \quad a = \frac{c}{2f_{c10}} = \frac{3 \times 10^8}{2 \times 10^{10}} = 0.015 \quad \text{[m]}$$

The required square waveguide is 15 mm by 15 mm in internal dimensions and the required rectangular waveguide is 15 mm by 7.5 mm in internal dimensions. Substituting these dimensions in the general expression for cutoff frequencies we get for the square waveguide ($b = a = 0.015$ m):

$$f_{cmn} = 1.5 \times 10^8 \sqrt{\left(\frac{m}{0.015}\right)^2 + \left(\frac{n}{0.015}\right)^2} \quad [\text{Hz}]$$

For the rectangular waveguide ($a = 0.015$ and $b = 0.0075$ m):

$$f_{cmn} = 1.5 \times 10^8 \sqrt{\left(\frac{m}{0.015}\right)^2 + \left(\frac{n}{0.0075}\right)^2} \quad [\text{Hz}]$$

Substituting the indices for the first ten modes, we get the required TM and TE modes (listed in ascending mode order):

	TM_{11}	TM_{21}	TM_{12}	TM_{22}	TM_{31}	TM_{13}	TM_{32}	TM_{23}	TM_{33}	TM_{41}
$a = b$	14.142	22.360	22.360	28.284	31.622	31.622	36.055	36.055	42.426	41.231
$a = 2b$	22.360	28.284	41.231	44.721	36.055	60.827	50.000	63.245	67.082	44.721

	TE_{10}	TE_{01}	TE_{11}	TE_{20}	TE_{02}	TE_{21}	TE_{12}	TE_{22}	TE_{30}	TE_{03}
$a = b$	10.0	10.0	14.142	20.0	20.0	22.360	22.360	28.284	30.0	30.0
$a = 2b$	10.0	20.0	22.360	20.0	40.0	28.284	41.231	44.721	30.0	60.0

(b) In the square waveguide, the TM_{12} and TM_{21} are degenerate modes as are any of the TM_{mn} and TM_{nm} or TE_{mn} and TE_{nm}. On the other hand, the rectangular waveguide has fewer degenerate modes in the range shown and the cutoff frequencies are much better spaced. For example, in the range between 22.360 and 28.284 GHz, only the TM_{11} mode can propagate. The rectangular waveguide is therefore much better suited for general purpose use than the square waveguide. This is one reason, standard rectangular waveguides have dimensions which are either exactly a ratio of one to two or very close to this ratio. Most waveguides only propagate the dominant mode and therefore separation of this mode is very important.

Example 17.8 A waveguide is given with dimensions $a = 12.95$ mm and $b = 6.48$ mm. The waveguide is required to propagate at 30 GHz. Suppose we are free to choose any mode with cutoff frequency below 30 GHz. Find the ratio between the powers propagated:

(a) In the TE_{10} and TE_{01} modes.
(b) In the TE_{10} and TM_{11} modes.

Solution: First, we must find the cutoff frequencies, cutoff wavelengths, and the guide propagation constants for the three modes. Then, we use **Eq. (17.176)** to find the power propagated for the TE modes and **Eq. (17.153)** for the TM modes. The amplitude of the magnetic field intensity is assumed because when calculating the ratio between powers, it cancels out.

(a) The cutoff frequencies for the TE_{10}, TE_{01}, and TM_{11} modes are given in **Eq. (17.138)**. For the waveguide given,

$$f_{c10} = 11.583\,\text{GHz}, \quad f_{c01} = 23.148\,\text{GHz}, \quad f_{c11} = 25.884\,\text{GHz}$$

All these modes are below 30 GHz and, therefore, appropriate modes for the required wave. The guide propagation constant β_g is

$$\beta_{g10} = \beta\sqrt{1 - \frac{f_{c10}^2}{f^2}} = 2\pi f\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \left(\frac{11.583}{30}\right)^2} = 579.6 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

$$\beta_{g01} = \beta\sqrt{1 - \frac{f_{c01}^2}{f^2}} = 2\pi f\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \left(\frac{23.148}{30}\right)^2} = 399.68 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

$$\beta_{g11} = \beta\sqrt{1 - \frac{f_{c11}^2}{f^2}} = 2\pi f\sqrt{\mu_0\varepsilon_0}\sqrt{1 - \left(\frac{25.884}{30}\right)^2} = 317.64 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

Similarly, the cutoff wave numbers are [from **Eq. (17.137)**]

$$k_{c10}^2 = \frac{\pi^2}{a^2} = \frac{\pi^2}{(0.01295)^2} = 5.885 \times 10^4 \quad \left[\frac{\text{rad}^2}{\text{m}^2}\right]$$

$$k_{c01}^2 = \frac{\pi^2}{b^2} = \frac{\pi^2}{(0.00648)^2} = 2.35 \times 10^5 \quad \left[\frac{\text{rad}^2}{\text{m}^2}\right]$$

$$k_{c11}^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} = \frac{\pi^2}{(0.01295)^2} + \frac{\pi^2}{(0.00648)^2} = 2.939 \times 10^5 \quad \left[\frac{\text{rad}^2}{\text{m}^2}\right]$$

The ratio between the powers in the TE_{10} and TE_{01} mode is

$$p = \frac{P_{10}}{P_{01}} = \frac{\omega\mu_0\beta_{g10}H_0^2 ab8k_{c01}^2}{\omega\mu_0\beta_{g01}H_0^2 ab8k_{c10}^2} = \frac{\beta_{g10}k_{c01}^2}{\beta_{g01}k_{c10}^2} = \frac{579.6 \times 2.35 \times 10^5}{399.68 \times 5.885 \times 10^4} = 5.79.$$

(b) To calculate the ratio between the powers propagated in the TE_{10} mode and TM_{11} mode, we use **Eq. (17.176)** for the TE mode and **Eq. (17.153)** for the TM mode. Also, from **Eq. (17.144)**, the electric field intensity for the TM wave may be written as

$$E_0 = \frac{\gamma H_0}{j\omega\varepsilon} = \frac{\beta_g H_0}{\omega\varepsilon} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

because $\gamma = j\beta$ (no losses in this case). With this, the ratio between the TE_{10} and TM_{11} modes is

$$p = \frac{P_{10}}{P_{11}} = \frac{\omega\mu_0\beta_{g10}H_0^2 ab8k_{c11}^2}{\omega\varepsilon_0\beta_{g11}E_0^2 ab8k_{c10}^2} = \frac{\omega^2\mu_0\varepsilon_0\beta_{g10}k_{c11}^2}{\beta_{g11}^3 k_{c10}^2} = \frac{\omega^2\beta_{g10}k_{c11}^2}{c^2\beta_{g11}^3 k_{c10}^2}$$

$$= \frac{4 \times \pi^2 \times 900 \times 10^{18} \times 579.6 \times 2.939 \times 10^5}{9 \times 10^{16} \times (317.4)^3 \times 5.885 \times 10^4} = 35.74$$

In either case, the TE_{10} mode carries more power for a given electric or magnetic field intensity, at a given frequency.

Exercise 17.8

(a) Find the longitudinal component $H_z(x, y, z)$ for a backward-propagating TE wave in a rectangular waveguide. Assume lossy propagation in the negative z direction and the amplitude of the wave is H_1.

(b) Find the total longitudinal TE fields in a waveguide if both a forward-propagating wave with amplitude H_0 and a backward-propagating wave with amplitude H_1 exist.

Answer

(a) $H_z^-(x,y,z) = H_1 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right].$

(b) $H_z(x,y,z) = \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\left(H_0 e^{-\gamma z} + H_1 e^{\gamma z}\right) \quad \left[\frac{\text{A}}{\text{m}}\right].$

Note: If the magnitude of the backward-propagating wave equals that of the forward-propagating wave ($|H_1| = |H_0|$), the term in the last set of parentheses in each component in (**b**) can be written as sine or cosine functions using the exponential forms. This is particularly simple if propagation is in a lossless material ($\gamma = j\beta_g$).

17.7.3 Attenuation and Losses in Rectangular Waveguides

So far in our discussion we have avoided attenuation and losses in waveguides except for the use of the general propagation constant γ in deriving the equations. However, no system can operate without losses. The mechanism for losses in waveguides is the same as in any other transmission line and consists of two parts: (1) losses in the dielectric and (2) losses in the imperfect conductors or wall losses. In addition, below cutoff, the attenuation constant is very high even for perfect dielectrics in the guide and perfectly conducting walls. These losses, the resulting attenuation constants, and their influence on the power relations in the waveguide are discussed next.

17.7.3.1 Dielectric Losses

The medium in waveguides is normally a low-loss dielectric such as air. Therefore, it is safe to assume that the low-loss approximation used in **Sections 17.3.2** and **17.3.3** for TE and TM propagation applies here. The attenuation constant is the same for TM and TE propagation (see **Table 17.2**). Replacing f_c by f_{cmn} to indicate that each mode has a different cutoff frequency and therefore a different attenuation constant and using an index d to indicate that this attenuation constant is due to dielectric losses, we get

$$\boxed{\alpha_{dTE} = \alpha_{dTE} = \frac{\sigma_d \eta_d}{2\sqrt{1 - f_{cmn}^2/f^2}} \quad \left[\frac{\text{Np}}{\text{m}}\right]} \tag{17.178}$$

Normally, the attenuation due to the dielectric for air-filled conducting-wall waveguides is rather small and is normally much smaller than the attenuation caused by losses in the walls of the waveguide. On the other hand, for dielectric waveguides such as optical waveguides, almost all losses are due to dielectric losses.

17.7.3.2 Wall Losses

To calculate the wall losses, we start with the expression for the time-averaged power density in **Eq. (17.151)**, which applies to both TE and TM waves. The only difference is in the expressions for the transverse components. The total power in the waveguide cross section is found by integrating this power density over the cross-sectional area of the waveguide and is given in **Eq. (17.153)** for TM modes and in **Eq. (17.176)** for TE modes.

Now, we assume there are no losses in the dielectric, but there are losses in the wall. The propagation constant is $\gamma = \alpha_\omega + j\beta$. Substituting this in **Eq. (17.151)**, and integrating gives the following expression for the total power in the cross section of the waveguide for TM modes:

$$P = \frac{\omega\varepsilon\beta_g E_0^2 ab}{8k_{cmn}^2} e^{-2\alpha_\omega z} \quad [\text{W}] \tag{17.179}$$

Similarly, for TE modes

$$P = \frac{\omega\mu\beta_g H_0^2 ab}{8k_{cmn}^2} e^{-2\alpha_\omega z} \quad [\text{W}] \tag{17.180}$$

In general, we can write

$$P = P_0 e^{-2\alpha_\omega z} \quad [\mathrm{W}] \tag{17.181}$$

In other words, as the wave propagates, starting with some power P_0 (which depends on the mode used), the power is attenuated continuously with the distance z.

We can calculate the attenuation constant as the attenuation per unit length by taking a 1 m length of the waveguide and calculating the power loss in this section. From the Poynting theorem, the rate of decrease in the time-averaged power equals the time-averaged power lost in this section of the waveguide. For $z = 1$ m, we get

$$-\frac{d(P)}{dz} = P_{loss} = 2\alpha_\omega P_0 e^{-2\alpha_\omega 1} \quad \rightarrow \quad \alpha_\omega = \frac{P_{loss}}{2P_0 e^{-2\alpha_\omega}} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right] \tag{17.182}$$

Assuming losses are low, $e^{-2\alpha_\omega} \approx 1$ and we get

$$\boxed{\alpha_\omega = \frac{P_{loss}}{2P_{av}} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]} \tag{17.183}$$

Now, all we have to do is calculate the total power loss in the walls per unit length of the waveguide and the time-averaged power in the waveguide. However, both power loss and time-averaged power in the waveguide are mode dependent. The general method is as follows: We calculate the total time-averaged power density in the waveguide using **Eqs. (17.152)** or **(17.175)**, which depend on the mode. The power in the waveguide may be calculated at any point z, but for simplicity we choose $z = 0$. Power loss occurs only in the wall and is calculated as for any conducting material using the relation $P_{loss} = I^2R/2$. The walls are highly conducting; the only current density in the walls is on and near the surface. Therefore, we assume a surface current density on each of the walls equal to $J_s = |H_t|$, where H_t is the tangential magnetic field intensity at the wall. The total current is the current density, integrated over the width of the wall. The resistance of the wall is the surface resistance. We have already calculated this resistance for lossy parallel plate transmission lines. The surface resistance **[see Eq. (14.7)]** is

$$R_s = \frac{1}{\sigma_c \delta} \quad [\Omega] \tag{17.184}$$

where σ_c is the conductivity of the wall material and $\delta = 1/\sqrt{\pi f \mu \sigma_c}$ its skin depth (see **Section 12.7.3**). The total power loss in the wall is therefore

$$P_{loss} = \frac{R_s}{2}\int_s J_s^2 ds = \frac{R_s}{2}\int_s |H_t|^2 ds \quad [\mathrm{W}] \tag{17.185}$$

where s is the total interior surface of the waveguide of length 1 m. This expression looks simple, but the integration must be done on each wall of the waveguide separately, after calculating the tangential component of the magnetic field intensity at the wall. We perform these calculations in **Example 17.9** for the TE_{10} mode. The attenuation constant due to wall losses is therefore

$$\boxed{\alpha_\omega = \frac{R_s}{4P_{av}}\int_s |H_t|^2 ds \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]} \tag{17.186}$$

If both dielectric and wall losses exist, the attenuation constant in the waveguide is the sum of the attenuation due to the dielectric and the attenuation due to the wall losses:

$$\boxed{\alpha = \alpha_d + \alpha_\omega \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]} \tag{17.187}$$

Attenuation in waveguides depends on a variety of parameters including the mode index, frequency, type of mode, conductivity of walls, and the dielectric in the waveguide. Typical values for waveguides in the microwave region are between about 0.1 dB/m and 0.5 dB/m. On the other hand, optical waveguides have losses that are between 0.5 dB/km to 10 dB/km. The much lower losses in optical waveguides is one of the most important reasons for their widespread use in communication.

17.7.3.3 Attenuation Below Cutoff

Any wave propagating below cutoff will be attenuated because below cutoff, the propagation constant is real. Consider the propagation constant for a mode above cutoff, propagating in a lossless medium ($\alpha_d = 0$), as written in terms of the guide phase constant β_g in **Eq. (17.141)**:

$$\gamma = j\beta_g = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f_{cmn}^2}{f^2}}, \quad f > f_{cmn} \tag{17.188}$$

If $f < f_{cmn}$, the term under the square root sign becomes negative and the propagation constant becomes

$$\gamma = -\omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_{cmn}^2}{f^2} - 1}, \quad f < f_{cmn} \tag{17.189}$$

The propagation constant is real and consists of an attenuation constant and zero propagation constant ($\gamma = \alpha + j0$). Taking the positive solution for α (a negative value of α would imply fields increase in magnitude as they propagate which is physically impossible) gives the attenuation below cutoff as

$$\boxed{\alpha_{bc} = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_{cmn}^2}{f^2} - 1}, \quad f < f_{cmn} \quad \left[\frac{\text{Np}}{\text{m}}\right]} \tag{17.190}$$

This attenuation constant is very high and for this reason, we say that waves do not propagate. As in parallel plate waveguides, waves below cutoff are evanescent waves.

Example 17.9 An air-filled waveguide has dimensions $a = 2.850$ cm and $b = 1.262$ cm (WR-112 waveguide) and is gold coated ($\sigma_c = 4.7 \times 10^7$ S/m). Conductivity of air is $\sigma_d = 10^{-5}$ S/m and permittivity and permeability of air may be taken as those of free space. The peak magnetic field intensity in the waveguide is 1 A/m and the waveguide operates in the TE_{10} mode at 9 GHz:

(a) Find the surface current densities in the walls of the waveguide.
(b) Find the attenuation constant in the waveguide.
(c) If the waveguide is 100 m long, what must be the total power required from the generator to transfer 1 W to the load? Assume both generator and load are matched.

Solution: In the TE_{10} mode, only the E_y, H_x, and H_z components exist. From these, we calculate the tangential components on each of the walls of the waveguide. The tangential component of the magnetic field intensity is equal in magnitude to the current density in the wall because of the interface conditions. The attenuation constant is the sum of the attenuation constant due to dielectric losses **[Eq. (17.178)]** and the attenuation constant due to wall losses **[Eq. (17.186)]**:

(a) The TE_{10} fields are found by setting $m = 1$, $n = 0$ in **Eqs. (17.164)** through **(17.168)**:

$$H_z(x,y,z) = H_0\cos\left(\frac{\pi x}{a}\right)e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

$$E_y(x,y,z) = \frac{-j\omega\mu a}{\pi}H_0\sin\left(\frac{\pi x}{a}\right)e^{-\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$H_x(x,y,z) = \frac{-j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

where $\gamma^2 + k^2 = k_c^2 = (\pi/a)^2$ was used to simplify the expressions. Also, in **Eq. (17.167)** we assumed that $\gamma \approx j\beta_g$ (low-loss approximation). All other field components are zero.

At the walls parallel to the y axis, both E_y and H_x are zero. The only tangential component on these walls is H_z. However, on the walls parallel to the x axis, both H_x and H_z are nonzero. At $x = 0$ and $x = a$, the tangential magnetic field intensity is (see, for example, **Figure 17.14**):

$$\mathbf{H}_t(0,y,z) = -\mathbf{H}_t(a,y,z) = \hat{\mathbf{z}} H_0 e^{-\gamma z} \quad [\text{A/m}]$$

At $y = 0$ and $y = b$, the tangential magnetic field intensity is

$$\mathbf{H}_t(x,0,z) = \mathbf{H}_t(x,b,z) = \hat{\mathbf{x}} \frac{j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} + \hat{\mathbf{z}} H_0 \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

To calculate the current on the surfaces, we recall that the relation between current and magnetic field intensity is a curl relation. Therefore, the current density on any surface is $\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}$ (see **Chapter 11**), where $\hat{\mathbf{n}}$ is the normal unit vector to the surface. Since by definition the normal to a surface points out of the surface (i.e., into the waveguide), the normal to the surface at $x = 0$ is $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, that at $x = a$ is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, that on the plate at $y = 0$ is $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, and that on the plate at $y = b$ is $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$. Performing the products $\hat{\mathbf{n}} \times \mathbf{H}$ on the four surfaces, we get

$$\mathbf{J}_s(0,y,z) = -\hat{\mathbf{y}} H_0 e^{-\gamma z}, \quad \mathbf{J}_s(a,y,z) = -\hat{\mathbf{y}} H_0 e^{-\gamma z} \quad [\text{A/m}]$$

$$\mathbf{J}_s(x,0,z) = \hat{\mathbf{y}} \times \left(\hat{\mathbf{x}} \frac{j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} + \hat{\mathbf{z}} H_0 \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z}\right) = \hat{\mathbf{x}} H_0 \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z} - \hat{\mathbf{z}} \frac{j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

$$\mathbf{J}_s(x,b,z) = -\hat{\mathbf{y}} \times \left(\hat{\mathbf{x}} \frac{j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} + \hat{\mathbf{z}} H_0 \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z}\right) = -\hat{\mathbf{x}} H_0 \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z} - \hat{\mathbf{z}} \frac{j\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} \quad \left[\frac{\text{A}}{\text{m}}\right].$$

(b) To find the attenuation constant, we first calculate the time-averaged power through the cross section of the waveguide. This was calculated in **Eq. (17.177)**. With $k_c^2 = (\pi/a)^2$, we get, at $z = 0$,

$$P_{av} = \frac{\omega\mu_0\beta_g H_0^2 a^3 b}{8\pi^2} \quad [\text{W}]$$

The losses in the wall are now calculated from **Eqs. (17.184)** and **(17.185)**, but first we need to calculate the skin depth δ:

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma_c}} = \frac{1}{\sqrt{\pi \times 9 \times 10^9 \times 4 \times \pi \times 10^{-7} \times 4.7 \times 10^7}} = 7.7384 \times 10^{-7} \quad [\text{m}]$$

The surface resistance is therefore **[Eq. (17.184)]**

$$R_s = \frac{1}{\sigma_c \delta} = \frac{1}{4.7 \times 10^7 \times 7.7384 \times 10^{-7}} = 2.75 \times 10^{-2} \quad [\Omega]$$

The losses are calculated on each surface separately, but since the current density on each two parallel plates is the same, we only need to calculate one of each and multiply the result by 2. We assume a section of the waveguide, 1 m long, and also assume that $\alpha z \ll 1$ so that $e^{-\alpha z} \approx 1$. At $x = 0$, we have

$$P_{Lav}(x=0) = \frac{I^2 R_s}{2} = \frac{R_s}{2}\int_{y=0}^{y=b} |J_s(0,y,z)|^2 dy = \frac{R_s}{2}\int_{y=0}^{y=b} H_0^2 dy = \frac{R_s H_0^2 b}{2} \quad [\mathrm{W}]$$

At $y = 0$

$$P_{Lav}(y=0) = \frac{R_s}{2}\int_{x=0}^{x=a} \left(|J_{sx}(x,0,z)|^2 + |J_{sz}(x,0,z)|^2\right) dx$$
$$= \frac{R_s}{2}\int_{x=0}^{x=a} \left\{ \left[\frac{\beta_g a}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right)\right]^2 + \left[H_0 \cos\left(\frac{\pi x}{a}\right)\right]^2 \right\} dx = \frac{R_s H_0^2 a}{4}\left(1 + \frac{\beta_g^2 a^2}{\pi^2}\right) \quad [\mathrm{W}]$$

The total power loss is

$$P_{Lav} = 2P_{Lav}(y=0) + 2P_{Lav}(x=0) = R_s H_0^2 \left[b + \frac{a}{2}\left(1 + \frac{\beta_g^2 a^2}{\pi^2}\right)\right] \quad [\mathrm{W}]$$

Dividing this by twice the time-averaged power entering the section at $z = 0$ gives

$$\alpha_\omega = \frac{P_{Lav}}{2P_{av}} = \frac{4\pi^2 R_s}{\omega \mu_0 \beta_g a^3 b}\left[b + \frac{a}{2}\left(1 + \frac{\beta_g^2 a^2}{\pi^2}\right)\right] \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]$$

The attenuation constant due to the dielectric is given in **Eq. (17.178)**

$$\alpha_{dTE_{10}} = \frac{\sigma_d \eta_0}{2\sqrt{1 - f_{c10}^2/f^2}} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]$$

With the cutoff frequency for the TE_{10} mode equal to $f_{c10} = (c/2)/0.0285 = 5.263$ GHz and $\beta_g = \omega\sqrt{\mu_0 \varepsilon_0} \times \sqrt{1 - f_{c10}^2/f^2} = 152.9$ rad/m, the total attenuation constant is

$$\alpha = \alpha_\omega + \alpha_d = \frac{4\pi^2 R_s}{\omega \mu_0 \beta_g a^3 b}\left[b + \frac{a}{2}\left(1 + \frac{\beta_g^2 a^2}{\pi^2}\right)\right] + \frac{\sigma_d \eta_0}{2\sqrt{1 - f_{c10}^2/f^2}}$$
$$= \frac{4 \times \pi^2 \times 2.75 \times 10^{-2}}{2 \times \pi \times 9 \times 10^9 \times 4 \times \pi \times 10^{-7} \times 152.9 \times (0.0285)^3 \times 0.01262}$$
$$\times \left[0.01262 + \frac{0.0285}{2}\left(1 + \frac{(152.9)^2 \times (0.0285)^2}{\pi^2}\right)\right]$$
$$+ \frac{10^{-5} \times 377}{2\sqrt{1 - \left(\frac{5.263}{9}\right)^2}} = 0.01856 + 0.00232 = 0.02088 \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]$$

The attenuation constant is 0.02088 Np/m or $0.02088 \times 8.69 = 0.18$ dB/m, a relatively low attenuation.

(c) For a 100 m long waveguide, the input power required to deliver 1 W to the load is

$$P_{Load} = 1 \;\; \text{W} = P_0 e^{-2\alpha z} = P_0 e^{-200\times 0.02088} \quad \rightarrow \quad P_0 = \frac{1}{e^{-4.176}} = 65.1 \quad [\text{W}]$$

That is, over 98 % of the input power is lost (dissipated) in the waveguide itself over the 100 m distance.

Example 17.10 Application: Operation of Waveguides Below Cutoff: Use of Waveguides as High-Pass Filters A waveguide operates at 8 GHz. The cutoff frequency for the mode propagated is 10 GHz. Calculate the attenuation constant in the waveguide. Assume an air-filled waveguide.

Solution: From **Eq. (17.190)**:

$$\alpha_{bc} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\frac{f_{cmn}^2}{f^2} - 1} = \frac{2\times\pi\times 8\times 10^9}{3\times 10^8}\sqrt{\left(\frac{10}{8}\right)^2 - 1} = 125.66 \quad \left[\frac{\text{Np}}{\text{m}}\right]$$

To get a better idea of this attenuation, the attenuation in [dB] is 125.66 × 8.69 = 1,092 dB/m. Thus, waveguides are excellent high-pass filters, blocking any waves below cutoff.

Exercise 17.9 An air-filled waveguide operates below cutoff. The operating frequency is 1 GHz and the cutoff frequency is 1.5 GHz. The power P is supplied to the waveguide at a point $z = 0$. How far does the power propagates before it is attenuated to $10^{-12}P$?

Answer 0.59 m

17.8 Other Waveguides

We discussed in this chapter only conducting, rectangular waveguides. However, we also mentioned that the basic requirement is that waves must be totally reflected at the boundaries of the guiding structure. Therefore, any structure that provides this facility may be used as a waveguide. In particular, cylindrical waveguides are very common, but other structures exist. For example, elliptical waveguides are sometimes used because their cross-sectional area remains constant when bent.

The analysis of any waveguide is, in principle, the same as that performed for the rectangular waveguides discussed here: the wave equation is solved in a convenient system of coordinates, and then the boundary conditions of the waveguide are satisfied.

Unfortunately, the solution is relatively simple only for rectangular waveguides. Cylindrical waveguides may also be analyzed relatively easily through use of cylindrical coordinates and solution to the Bessel equation. Other, rather complicated-shaped waveguides exist and are used for specialized applications. Their analysis is often much more involved than the rectangular waveguides presented here and may require approximate analysis, including the use of computational techniques.

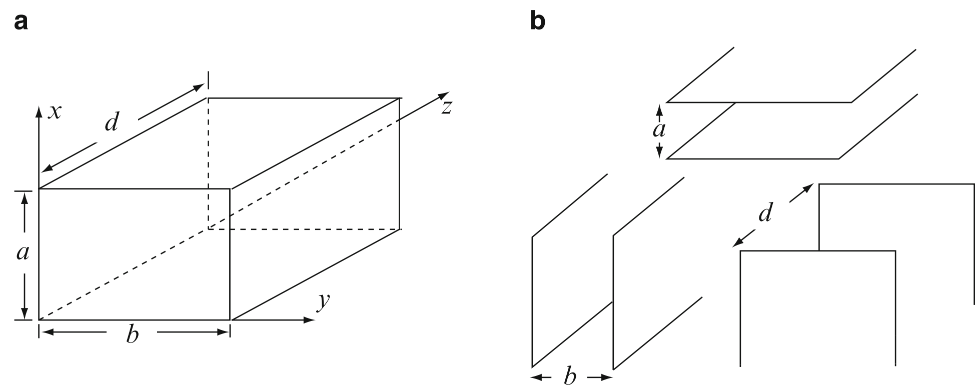
17.9 Cavity Resonators

Waves.m

A rectangular cavity resonator is built out of a rectangular waveguide by adding two conducting walls at $z = 0$ and $z = d$, as shown in **Figure 17.19a**. The cavity resonator may be viewed as being made of three parallel plate waveguides, as shown in **Figure 17.19b**. The cavity is a modified waveguide, in which there are standing waves in the z direction as well as in the x

and y directions. The main difference between cavities and waveguides is that in cavities, the z direction imposes additional boundary conditions and there is no propagation of waves as in waveguides. The cavity acts as a resonant structure in which there is exchange of energy between the electric and magnetic field at given (resonant) frequencies. This is equivalent to resonant LC circuits in the case of lossless cavities and to RLC circuits in the case of lossy cavities.

Figure 17.19 (**a**) Structure and dimensions of a rectangular cavity resonator. (**b**) Construction of the cavity resonator as the intersection of three parallel plate waveguides



The analysis of fields in a cavity requires the solution of the full three-dimensional wave equation with the required boundary conditions. The procedure here will be to take the TM and TE waves we have already defined and to modify them to satisfy the additional boundary conditions imposed by the additional conducting walls. However, the TE and TM equations given in **Eqs. (17.33)** and **(17.57)** cannot be used directly. The main reason is that in waveguides, we assumed explicitly that the wave propagates in the z direction and that the transverse directions are the directions perpendicular to the direction of propagation (z direction). In cavities, there is no clear direction we can take as a transverse direction. The approach here is to take the z direction (usually the long dimension of the cavity) as a reference direction, allow the waves to propagate along this direction, and calculate the total waves as the sum of the forward- and backward-propagating waves reflected off the shorting walls. We define the TE and TM modes and waves as:

(1) A TM wave in a cavity resonator is any wave which has no magnetic field component in the z direction of the cavity.
(2) A TE wave in a cavity resonator is any wave which has no electric field component in the z direction of the cavity.

By direct extension of **Eq. (17.57)**, the TM fields satisfy the following equation:

$$\boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0} \tag{17.191}$$

From **Eq. (17.33)**, the TE fields satisfy

$$\boxed{\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0} \tag{17.192}$$

Comparing these with **Eqs. (17.111)** and **(17.154)**, it is relatively easy to see that the fields E_z and H_z are those in **Eqs. (17.122)** and **(17.157)** multiplied by an additional product due to the z dependence of the field. However, we will find the solutions by formal application of the above ideas to the waveguide fields found in the previous sections.

17.9.1 TM Modes in Cavity Resonators

To find the TM and TE modes in cavity resonators, we can proceed in two ways. One is to start with the wave equations in **Eqs. (17.191)** and **(17.192)** and solve them subject to the boundary conditions on all eight walls of the cavity resonator. This is what we have done for waveguides. Another is to use the waves obtained for the rectangular waveguides and calculate the reflected waves caused by the introduction of the two additional conducting walls. This causes backward-propagating waves, and the required boundary conditions on the conducting walls at $z = 0$ and $z = d$ are used to calculate the reflected waves.

Then, the sum of the forward and backward waves gives the correct fields inside the cavity. We will use the first method to evaluate the longitudinal components of the electric and magnetic field intensities, and the second method to evaluate the transverse components for TE and TM modes, to demonstrate the various techniques involved.

For TM modes in a waveguide, we imposed the condition that the tangential components of the electric field intensity must be zero on the conducting walls. These conditions still apply here for the transverse fields. However, the z component of the field is normal to the two walls perpendicular to the z axis. Thus, we cannot impose the zero tangential electric field condition on these boundaries for E_z. On the other hand, since E_z is normal to the surfaces at $z = 0$ and $z = d$, the following conditions apply:

$$\left.\frac{\partial E_z(x,y,z)}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial E_z(x,y,z)}{\partial z}\right|_{z=d} = 0 \tag{17.193}$$

Following the solution process as for the TM modes in waveguides [**Eqs. (17.111)** through **(17.128)**], we write the general solution as

$$E_z(x,y,z) = X(x)Y(y)Z(z) \tag{17.194}$$

Substituting this in **Eq. (17.191)** and dividing by $E(x,y,z)$, we get

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} + k^2 = 0 \tag{17.195}$$

where $k^2 = \omega^2\mu\varepsilon$, as for the waveguide solution. This equation may be separated as follows:

$$\frac{\partial^2 X(x)}{\partial x^2} + k_x^2 X(x) = 0 \tag{17.196}$$

$$\frac{\partial^2 Y(y)}{\partial y^2} + k_y^2 Y(y) = 0 \tag{17.197}$$

$$\frac{\partial^2 Z(z)}{\partial z^2} + k_z^2 Z(z) = 0 \tag{17.198}$$

with the additional condition that

$$-k_x^2 - k_y^2 - k_z^2 + k^2 = 0 \tag{17.199}$$

Note that we have not used here the notation of cutoff wave number k_c as in **Eq. (17.119)** because cutoff has no meaning in cavity resonators; that is, since waves do not propagate, the concept of cutoff does not exist. From the discussion in **Sections 17.7.1** and **17.7.2**, the general solution for E_z is

$$E_z(x,y,z) = (A_1\sin k_x x + B_1\cos k_x x)\left(A_2\sin k_y y + B_2\cos k_y y\right)(A_3\sin k_z z + B_3\cos k_z z) \quad [\text{V/m}] \tag{17.200}$$

We already found in **Eqs. (17.123)** through **(17.126)** that

$$B_1 = B_2 = 0 \quad \text{and} \quad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \tag{17.201}$$

With these, we have

$$E_z(x,y,z) = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)(A_3\sin k_z z + B_3\cos k_z z) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.202}$$

Now, we apply the additional boundary conditions in **Eq. (17.193)**:

$$\left.\frac{\partial E_z(x,y,z)}{\partial z}\right|_{z=0} = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)k_z(A_3K_z\cos 0 - B_3k_z\sin 0) = 0 \quad \rightarrow \quad A_3 = 0 \tag{17.203}$$

$$\left.\frac{\partial E_z(x,y,z)}{\partial z}\right|_{z=d} = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)k_z(-\sin k_z d) = 0 \quad \rightarrow \quad k_z = \frac{p\pi}{d} \tag{17.204}$$

where the constant B_3 was absorbed into the general constant A. The solution for E_z is therefore

$$\boxed{E_z(x,y,z) = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{V}}{\text{m}}\right]} \tag{17.205}$$

The transverse components in the cavity resonator may be found from those of the waveguide using the following argument: Considering the cavity resonator to be a waveguide in which shorts were introduced at two locations along the length of the waveguide (see **Figure 17.19a**), we can argue that the transverse components of the waves propagating along the guide will be reflected at these shorts, causing, in addition to the forward-propagating wave, a backward-propagating wave. The total electric field intensity, which is the sum of the forward- and backward-propagating waves, must then vanish at the conducting surfaces at $z = 0$ and $z = d$.

For TM waves, we start with **Eqs. (17.131)** through **(17.134)**. To see how this is accomplished we now perform the steps necessary to obtain the x component of the electric field intensity in the cavity resonator. **Equation (17.131)** is a forward-propagating wave in the waveguide:

$$E_x^+(x,y,z) = \frac{-\gamma}{\gamma^2+k^2}E_0^+\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{-\gamma z} \tag{17.206}$$

where we added the notation E_0^+ to show explicitly that this is the forward-propagating wave. The backward-propagating wave is obtained by simply replacing z by $-z$ (see **Sections 17.3** and **Exercise 17.5**):

$$E_x^-(x,y,z) = \frac{-\gamma}{\gamma^2+k^2}E_0^-\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{\gamma z} \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.207}$$

The total wave is the sum of the forward- and backward-propagating waves:

$$E_x(x,y,z) = E_x^+(x,y,z) + E_x^-(x,y,z) = \frac{-\gamma}{\gamma^2+k^2}\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(E_0^+e^{-\gamma z}+E_0^-e^{\gamma z}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.208}$$

Because the x component of the electric field intensity is zero on the conducting planes at $z = 0$ and $z = d$, these become the boundary conditions from which both the z variation of E_x and the magnitude of the backward-propagating wave E_0^- are found:

$$E_x(x,y,0) = \frac{-\gamma}{\gamma^2+k^2}\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(E_0^+ + E_0^-\right) = 0 \quad \rightarrow \quad E_0^- = -E_0^+) \tag{17.209}$$

$$E_x(x,y,d) = \frac{-\gamma}{\gamma^2+k^2}\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(E_0^+e^{-\gamma d}+E_0^-e^{\gamma d}\right) = 0 \tag{17.210}$$

Writing $E_0^- = -E_0^+$ and $\gamma = jk_z$, **Eq. (17.210)** gives

$$e^{-\gamma d} - e^{\gamma d} = 0 \quad \rightarrow \quad -2\sinh(jk_z d) = 0 \tag{17.211}$$

where the relation $(e^{\gamma d} - e^{-\gamma d})/2 = \sinh(\gamma d) = \sinh(jk_z d)$ was used. Using the relation $\sinh(jk_z d) = j\,\sin(k_z d)$, we get

$$2\sinh(jk_z d) = 2j\sin(k_z d) = 0 \quad \rightarrow \quad k_z d = p\pi \quad \rightarrow \quad k_z = \frac{p\pi}{d} \tag{17.212}$$

This condition was already obtained in **Eq. (17.204)** for the longitudinal component, but it is worth repeating here to show that the two methods of evaluating the fields are equivalent. Substituting these conditions (i.e., $k_z = p\pi/d$, $\gamma = jk_z$, and $e^{-\gamma z} - e^{\gamma z} = -j2\sin(k_z z) = -j2\sin(p\pi z/d)$ in **Eq. (17.208)** gives

$$E_x(x,y,z) = \frac{-1}{\gamma^2 + k^2}\frac{p\pi}{d}\frac{m\pi}{a}E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{17.213}$$

where $E_0 = 2E_0^+$ is the amplitude of the field. Similar steps for the remaining components of the electric field and magnetic field intensities (see **Exercise 17.5**) lead to the following:

$$E_y(x,y,z) = \frac{-1}{\gamma^2 + k^2}E_0\frac{n\pi}{b}\frac{p\pi}{d}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{17.214}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{\gamma^2 + k^2}E_0\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.215}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon}{\gamma^2 + k^2}E_0\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.216}$$

From **Eq. (17.199)** we can write the wave number as

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 = \omega^2\mu\varepsilon \tag{17.217}$$

or the resonant frequency as

$$\boxed{f_{mnp} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \quad [\mathrm{Hz}]} \tag{17.218}$$

where the indices m, n, and p indicate the mode in which the cavity resonates. In resonant cavities, the concept of cutoff is different than in waveguides. Since there is no propagation in a cavity, these are called ***resonant frequencies*** or ***resonant modes*** rather than cutoff frequencies.

Any combination of mode indices m, n, and p results in a resonant frequency of the cavity except for those with $m = 0$ or $n = 0$ [for which the longitudinal component of the field in **Eq. (17.205)** becomes zero]. If m or n or both are zero, all field components become zero. However, p can be zero. The lowest TM resonant mode (assuming $a > b > c$) is TM_{110}.

17.9.2 TE Modes in Cavity Resonators

To find the TE modes in a cavity resonator, we must solve **Eq. (17.192)**, subject to the appropriate boundary conditions on the surfaces of the resonator. For TE modes in a waveguide, we imposed the condition that the normal components of the magnetic field intensity must be zero on the conducting walls. These conditions also apply here for the transverse components of the magnetic field. In the z direction, the magnetic field component, H_z, is normal to the surfaces perpendicular to the z axis. Therefore, the additional condition required for the cavity resonator is for H_z to vanish on the surfaces at $z = 0$ and $z = d$:

$$H_z(x,y,z)\big|_{z=0} = 0, \quad H_z(x,y,z)\big|_{z=d} = 0 \tag{17.219}$$

Following the solution process as for the TM modes in cavity resonators and since **Eq. (17.192)** for TE waves is identical in form to **Eq. (17.191)** for TM waves, the general solution is also identical in form. That is, the magnetic field intensity for the TM waves has the same form as the electric field intensity for the TE waves given in **Eq. (17.200)**:

$$H_z(x,y,z) = (A_1\sin k_x x + B_1\cos k_x x)(A_2\sin k_y y + B_2\cos k_y y)(A_3\sin k_z z + B_3\cos k_z z) \quad [\text{A/m}] \tag{17.220}$$

To obtain TE modes in a cavity resonators, we start with the results obtained for the TE modes in a waveguide in **Eqs. (17.158)** through **(17.162)**:

$$A_1 = A_2 = 0 \quad \text{and} \quad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \tag{17.221}$$

With these, the solution is [see **Eq. (17.164)**]

$$H_z(x,y,z) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)(A_3\sin k_z z + B_3\sin k_z z) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.222}$$

Now, we apply the boundary conditions in **Eq. (17.219)**:

$$H_z(x,y,z)\big|_{z=0} = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)(A_3\sin 0 + B_3\cos 0) = 0 \quad \rightarrow \quad B_3 = 0 \tag{17.223}$$

and

$$H_z(x,y,z)\big|_{z=d} = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin k_z d = 0 \quad \rightarrow \quad k_z = \frac{p\pi}{d} \tag{17.224}$$

where the constant A_3 was absorbed into A. The longitudinal component of the magnetic field intensity is therefore

$$H_z(x,y,z) = H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.225}$$

where H_0 is the amplitude of the magnetic field intensity. To obtain the transverse components, we use the same sequence as in the previous section; that is, we write the forward- and backward-propagating TE waves in a shorted waveguide, sum the two waves up, and set the total transverse electric field intensity to zero or the derivatives with respect to z of the total transverse magnetic field intensity to vanish ($\partial Hz(x,y,z)/\partial z = 0$) on the shorting walls at $z = 0$ and $z = d$. This gives (see **Exercises 17.8** and **17.10**)

$$E_x(x,y,z) = \frac{j\omega\mu}{\gamma^2 + k^2} H_0 \frac{n\pi}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.226}$$

$$E_y(x,y,z) = \frac{-j\omega\mu}{\gamma^2 + k^2} H_0 \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.227}$$

$$H_x(x,y,z) = -\frac{1}{\gamma^2 + k^2} H_0 \frac{m\pi}{a}\frac{p\pi}{d}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.228}$$

$$H_y(x,y,z) = -\frac{1}{\gamma^2 + k^2} H_0 \frac{n\pi}{b}\frac{p\pi}{d}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.229}$$

From the fields in **Eqs. (17.225)** through **(17.229)**, we see that for TE modes, either m or n can be zero (but not both) while p must be nonzero (otherwise the longitudinal component of the field is zero). For $p = 0$ or for $m = n = 0$, all components of the field are zero. The lowest resonant mode is therefore either the TE_{101} or TE_{011}, depending on the dimensions a, b, and c.

The resonant frequencies for TE modes are the same as for the TM modes:

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \quad [\text{Hz}] \tag{17.230}$$

Some of the modes may have the same resonant frequency even though they are different modes. As an example, for a cubic cavity ($a = b = d$), TE_{011} and TE_{101} have the same frequency. These are called degenerate modes, as in waveguides.

Exercise 17.10 Starting with the TE transverse components in a rectangular waveguide **[Eqs. (17.165)** through **(17.168)]**, derive **Eqs. (17.226)** through **(17.229)** by first writing the backward-propagating waves in the waveguide, summing the forward- and backward-propagating waves, and then applying the appropriate conditions at $z = 0$ and $z = d$.

Example 17.11 A cavity resonator is made in the form of a cubic box, 100 mm on the side. The cavity is air filled and is made of a perfect conductor:

(a) Find the first 15 possible resonant modes of the cavity.
(b) Separate the TE and TM modes.
(c) Which resonant frequency is the dominant mode of the cavity and which modes are degenerate?

Solution: The resonant frequencies of the cavity are calculated from **Eq. (17.230)**. Any combination of the integers m, n, and p may be considered to be a resonant frequency, except, of course, for $m = 0$, $n = 0$, and $p = 0$. Other combinations may also be inappropriate combinations in the sense that they result in zero fields. We first calculate the first 15 possible resonant frequencies, including those that may not correspond to physical modes and then identify those modes that may exist in the cavity:

(a) Using **Eq. (17.230)**, we get the possible resonant frequencies in **Table 17.4**.
(b) To identify which of the calculated frequencies correspond to TE and TM modes, we use the properties of the modes:
TE modes: m or n can be zero, while p must be nonzero.
TM modes: m and n must be nonzero, while p can be zero.
This means that the combinations ($00p$), ($0n0$), and ($m00$), where m, n, p are nonzero, are not physical resonant frequencies; that is, the combinations 100, 010, 001, 200, 020, and 002 lead to zero fields and are therefore not resonant modes. The combinations ($mn0$) and (mnp), $m, n, p \neq 0$, are TM modes. These are (110), (111), (210), and (120). The combinations ($0np$), ($m0p$), and (mnp), for $m, n, p \neq 0$, are TE modes. These are (011), (021), (101), (102), (201), and (111). The physical resonant frequencies and their designation are shown in **Table 17.5**.
(c) The dominant mode is the mode with lowest resonant frequency. In this case, there are three modes with lowest frequency. Any of the TM_{110}, TE_{101}, and TE_{011} modes is the dominant mode.

Modes TM_{110}, TE_{101}, and TE_{011} are degenerate modes as are the TE_{111} and TM_{111} and TE_{210}, TE_{201}, TE_{021}, TE_{102}, and TM_{120} modes. This high-order degeneracy is one reason why cubic cavity resonators are seldom used. Rectangular cavities are usually preferred because they have better mode separation.

Table 17.4 The first 15 possible modes in a square resonant cavity (frequencies given in GHz)

100	010	001	110	101	011	111	200	020	002	210	201	021	102	120
1.5	1.5	1.5	2.121	2.121	2.121	2.598	3.0	3.0	3.0	3.354	3.354	3.354	3.354	3.354

Table 17.5 Possible modes in a cubic cavity resonator (frequencies given in GHz)

TM_{110}	TE_{101}	TE_{011}	TM_{111}, TE_{111}	TM_{210}	TE_{201}	TE_{021}	TE_{102}	TM_{120}
2.121	2.121	2.121	2.598	3.354	3.354	3.354	3.354	3.354

17.10 Energy Relations in a Cavity Resonator

Power and energy relations in a cavity are defined by the Poynting theorem. Since there is a certain amount of energy stored in the fields of a cavity, the calculation of this energy is an important aspect of analysis. This is particularly obvious if we recall that in a resonant device, these relations change dramatically at or near resonance. This was true with resonant circuits and is certainly true with resonant cavities. The stored energy and dissipated power in a cavity define the basic qualities of the cavity. A lossless cavity is not practically realizable; therefore, we also define a quantity called quality factor of the cavity, which is a measure of losses in the cavity. A shift in the resonant frequency of the cavity can also be described in terms of energy. These relations can then be used to characterize a cavity and for measurements in the cavity.

To define the energy relations in the cavity, we need to calculate the Poynting vector ($\mathcal{P} = \mathbf{E} \times \mathbf{H}$) in the cavity. From **Eqs. (17.213)** through **(17.216)** or **Eqs. (17.226)** through **(17.229)**, we note that the Poynting vector is purely imaginary; that is, the time-averaged power density in the cavity is zero:

$$\mathcal{P}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) = 0 \tag{17.231}$$

This means that there is no real power transferred in or out of the cavity, but there is stored energy in the magnetic and electric fields inside the cavity. From the complex Poynting vector **[Eq. (12.75)]**, we have

$$S = j2\omega \int_v \left(\frac{1}{4}\varepsilon \mathbf{E} \cdot \mathbf{E}^* - \frac{1}{4}\mu \mathbf{H} \cdot \mathbf{H}^* \right) dv \quad [\mathrm{W}] \tag{17.232}$$

The total time-averaged stored electric and magnetic energy in the cavity can now be written as

$$W_0 = \int_v \left(\frac{\varepsilon \mathbf{E} \cdot \mathbf{E}^*}{4} - \frac{\mu \mathbf{H} \cdot \mathbf{H}^*}{4} \right) dv \quad [\mathrm{J}] \tag{17.233}$$

where **E** and **H** are the fields in the cavity and v the volume of the cavity. This relation is correct at any frequency regardless of resonance.

If the cavity also has wall losses, the time-averaged dissipated power in the cavity walls is

$$P_{loss} = \frac{R_s}{2} \int_s J_s^2 ds = \frac{R_s}{2} \int_s |H_t|^2 ds \quad [\mathrm{W}] \tag{17.234}$$

where R_s is the surface resistance of the cavity walls, H_t is the tangential magnetic field intensity at the walls surface, and s is the internal surface of the cavity walls. This relation is the same as that obtained for waveguides in **Eq. (17.185)**. The calculation of the wall losses is the same as for the waveguide (see **Example 17.9**). In addition, there may also be losses due to the dielectric inside the cavity and these must be added to **Eq. (17.234)**.

17.11 Quality Factor of a Cavity Resonator

The ***quality factor*** of the cavity resonator is defined as the ratio between the stored energy in the cavity and the dissipated power per cycle of the wave:

$$Q = 2\pi \frac{\text{time-averaged stored energy}}{\text{energy loss in one cycle}} = \frac{2\pi W_0}{P_{loss} T} = \frac{\omega_0 W_0}{P_{loss}} \quad [\text{dimensionless}] \tag{17.235}$$

where T is the period of the wave and ω_0 is the resonant frequency. Since the higher the Q factor, the more selective the cavity is, Q is a measure of the bandwidth of the cavity. It also defines, indirectly, the amount of energy needed to couple into the cavity to maintain an energy balance. Ideal cavities have an infinite quality factor. The calculation of both stored energy W_0 and dissipated power P_{loss} is tedious but straightforward operations [see **Eqs. (17.233)** and **(17.234)**]. The stored energy is calculated by direct integration over the volume of the cavity using either the TE or TM fields and the dissipated power in the walls of the cavity is calculated using the method in **Section 17.7.3** by first finding the current densities in the six walls of the cavity resonator and then integrating **Eq. (17.234)** over the walls.

There are two sources of losses in a cavity. One is the wall loss in conductors, the other is the loss in dielectrics (see **Section 17.7.3**). It is possible to separate the quality factor into a quality factor due to the dielectric, Q_d, and a quality factor due to conductors, Q_c. If this is done, then the quality factor of the cavity may be written as

$$Q = \frac{Q_c Q_d}{Q_c + Q_d} \quad \text{[dimensionless]} \tag{17.236}$$

A small advantage in this separation is that the quality factor due to the dielectric can be calculated in terms of the loss tangent [see **Eq. (12.80)**] or in terms of the complex permittivity [see **Eq. (12.79)**]:

$$Q_d = \frac{\varepsilon'}{\varepsilon''} = \frac{1}{\tan\theta_{loss}} \quad \text{[dimensionless]} \tag{17.237}$$

where ε' is the real part of permittivity and ε'' its imaginary part. In some instances one or the other quality factor may dominate. If, for example, dielectric losses are negligible, Q_d tends to infinity and the quality factor of the cavity is dominated by wall losses.

17.12 Applications

Application: The Slotline A very useful measuring device in waveguide applications is the slotline. The slotline is a section of a waveguide, with a lengthwise slot that allows a probe to measure the electric field intensity in the waveguide as shown in **Figure 17.20a**. The probe can be adjusted with a micrometer over a considerable length and measure the electric field intensity in the waveguide. The basic measurement is that of the maximum and minimum electric fields, indicating the standing wave ratio. However, other measurements may be performed. For example, the slotline may be connected to a waveguide section, and the waveguide caused to reflect some energy back into the slotline, as shown in **Figure 17.20b** by means of a short, an open, or any dielectric material in the cavity. Then, measuring two minima in the standing wave pattern (minima are preferred because they are sharper than maxima), the wavelength and, therefore, the frequency may be measured. The measurement proceeds by identifying one minimum and then moving the probe to the next minimum. The distance between the two minima is always $\lambda/2$. The slotline must be identical in dimensions to the waveguide to which it is connected if reflections due to the connections are to be avoided.

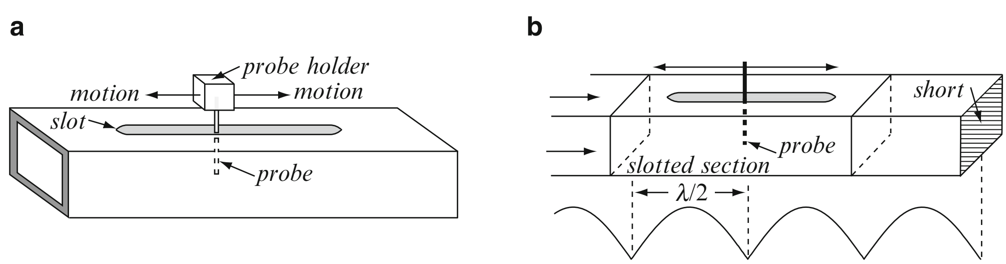


Figure 17.20 (**a**) The slotline. (**b**) Frequency measurement using the slotline

Application: Optical Waveguides Optical waveguides are somewhat different than most other electromagnetic waveguides in that a conductor is not used. The guide is a dielectric guide which relies on total internal reflection for confinement of waves within the waveguide. The most common optical waveguides are made of thin silica fibers (thus, the common name of glass fiber or optical fiber) or plastic. The fiber is normally coated with an opaque dielectric material for protection, even though it is not absolutely necessary for operation. However, the coating or cladding must have lower dielectric constant than the core fiber to ensure total internal reflection. In some cases, the fiber may be coated with metal such as aluminum or nickel. There are a number of types of optical fibers, depending on their constructions. The best optical fibers are the so-called single-mode fibers. These are very thin (1–8 μm) and, as their name implies, allow a single mode to propagate. Propagation is almost entirely parallel to the fiber and because of that, there is little dispersion in the fiber and the attenuation is also low. However, these fibers are difficult to make and are quite expensive. In addition, connection to the fibers is complicated and requires a laser as the source. More common are the multimode fibers and multimode graded-index fibers. The first are about 125–400 μm thick and are made of a uniform material (the index of refraction is constant throughout the cross section of the fiber). This is the simplest fiber but also the worst in terms of dispersion and attenuation. Dispersion in optical fibers refers to the delay in transmission because different modes travel at different speeds depending on the angle of reflection in the fiber. Typical values are 15–30 ns/km meaning that the difference in time of arrival between the slowest/fastest waves is 15–30 ns per km length of the fiber. The second type of fiber has a graded index of refraction which is high at the center and lower toward the outer surface. This reduces dispersion but is more difficult to fabricate.

The optical fiber is commonly used as a waveguide for communication purposes because it has low attenuation, has very high bandwidth, has no interference from other signals, is thin and lightweight, and is easy to couple energy into. Typical attenuation in fibers is as low as 0.5 dB/km, although some fibers attenuate over 10 dB/km. In addition to optical fibers, optical waveguides can be fabricated in silicon and other semiconducting materials. Optical resonators are also made and can be integrated on silicon chips.

Application: Detection of Materials with Cavity Resonators Cavity resonators with high-quality factors have a very narrow curve (high response) around resonance. Any change in the dielectric constant inside the cavity affects the resonant frequency through the change in dielectric constant. This may be utilized to measure properties of materials or to measure the presence of materials. One example of this type of measurement is smoke detection or even detection of explosives. Other applications are drying of materials by sensing moisture content (resonant frequency is lower the higher the amount of water in the cavity) or curing of polymers by sensing the amount of solvent in the vicinity of the drying polymer. The basic application is shown in **Figure 17.21**. It consists of a rectangular cavity, with a few holes that allow penetration of the material (gases) to be detected. The shift in resonant frequency is monitored and any shift indicates the presence of a material with a dielectric constant different than air.

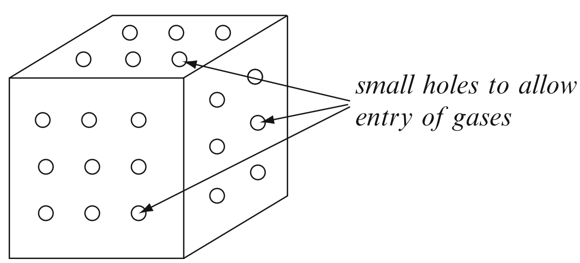


Figure 17.21 A cavity resonator sensor designed for smoke detection

Application: Coupling to Waveguides and Cavities The coupling of electromagnetic energy to waveguides and cavities has not been addressed specifically in the above discussion. If the cavity were ideal, the fields within the cavity would be of infinite amplitude, provided that the necessary modes can be excited. In real cavities, there are always some losses, but these are usually small. The fields are large and the amount of stored energy is also large. However, the small amount of power dissipated has to be compensated for by external sources; otherwise, the cavity would cease to oscillate. This is done by coupling energy into the cavity. A cavity for which the lost energy is exactly balanced is called a critically coupled cavity.

The introduction of energy into the cavity (or a waveguide) can be done in a number of ways. The most obvious of these is to have a source within the cavity that generates the necessary fields. A small loop (**Figure 17.22a**) or a simple probe excitation (**Figure 17.22b**) can be used. A loop generates a magnetic field intensity and this magnetic field intensity excites a mode with magnetic field intensity parallel to that generated by the loop. Different modes can be generated by simply locating the probe or the loop at different locations in the cavity, although, for obvious reasons, these must be close to the outer surfaces of the cavity. Similarly, the cavity can be coupled through a small aperture through which a small amount of energy "leaks" into the cavity (**Figure 17.22c**). In this case, the modes excited are those that have fields parallel to those in the waveguide at the location of the aperture. The three coupling methods in **Figure 17.22** excite different modes. These are shown for a cavity, but identical considerations apply to waveguides.

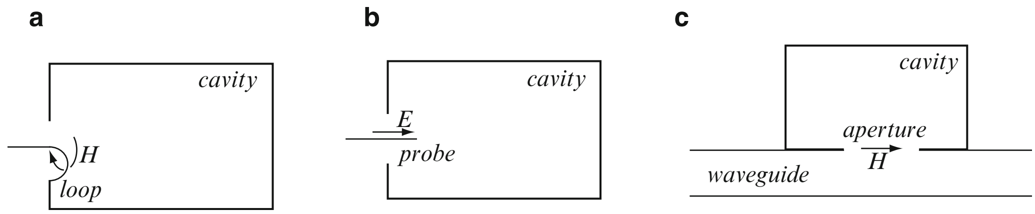


Figure 17.22 (**a**) Coupling to a cavity by a small loop in the cavity. (**b**) Coupling to a cavity by a small probe in the cavity. (**c**) Coupling to a cavity by a small aperture in the wall of the cavity

Application: Frequency Measurement One simple and widely used method for frequency measurement is the tuning of a cavity resonator to resonate at the unknown frequency. Then, by accurate measurements of the cavity dimensions, the frequency may be calculated from **Eq. (17.218)**. In practice, the resonant frequency may be calibrated directly on the cavity. Standard wavemeters are of this type. (They are called wavemeters because often the wavelength is measured rather than the frequency.) Normally, wavemeters are cylindrical cavity resonators as shown in **Figure 17.23**. However, in principle, any cavity resonator may be used. The only requirement is that the modes be separated well and that we either know the mode or the cavity is excited in a known mode.

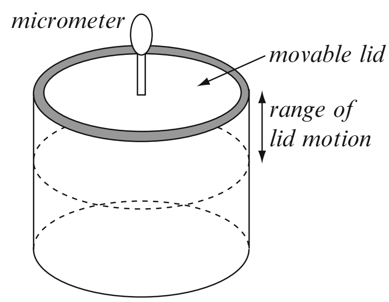


Figure 17.23 A cylindrical cavity wavemeter

17.13 Summary

The theme of this chapter is propagation of waves in waveguides. The starting point is a general description of longitudinal and transverse electric and magnetic field intensities from Maxwell's equations. These, together with the effect of the conducting surfaces, define the properties of the waves. The treatment is separated into ***transverse electromagnetic*** **(TEM)**, ***transverse electric*** **(TE)**, and ***transverse magnetic*** **(TM)** waves.

Assumptions: direction of propagation is z, time-harmonic fields: ($e^{j\omega t}$ time dependency).

General equations—transverse field components in general, lossy media:

$$E_x = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial E_z}{\partial x} - j\omega\mu\frac{\partial H_z}{\partial y}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.18}$$

$$E_y = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial E_z}{\partial y} + j\omega\mu\frac{\partial H_z}{\partial x}\right) \quad \left[\frac{\text{V}}{\text{m}}\right] \tag{17.19}$$

$$H_x = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial H_z}{\partial x} + j\omega\varepsilon\frac{\partial E_z}{\partial y}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.20}$$

$$H_y = \frac{1}{\gamma^2 + k^2}\left(-\gamma\frac{\partial H_z}{\partial y} - j\omega\varepsilon\frac{\partial E_z}{\partial x}\right) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{17.21}$$

Given $E_z, H_z, \gamma = \alpha + j\beta$, and $k = \omega\sqrt{\mu\varepsilon}$, we can obtain E_x, E_y, H_x, H_y. These fields are used to define TEM, TE, and TM waves by setting conditions on E_z and H_z. **Table 17.2** summarizes the properties of TE, TM, and TEM waves. Some of the properties that show the differences between the three types of waves under lossless conditions are shown in the following table.

	TEM waves	TE waves	TM waves
Conditions on E_z, H_z	$E_z = 0, H_z = 0$	$E_z = 0, H_z \neq 0$	$E_z \neq 0, H_z = 0$
Propagation constant	$\gamma_{TEM} = j\omega\sqrt{\mu\varepsilon}$	$\gamma_{TE} = j\omega\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}$	$\gamma_{TM} = \gamma_{TE}$
Equations	See note 1	(17.27) through (17.30)	(17.53) through (17.56)
Phase velocity [m/s]	$v_p = 1/\sqrt{\mu\varepsilon}$	$v_{TE} = 1/\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}$	$v_{TM} = v_{TE}$
Wave impedance [Ω]	$\eta = \sqrt{\mu/\varepsilon}$	$Z_{TE} = \eta/\sqrt{1-(f_c/f)^2}$	$Z_{TM} = \eta\sqrt{1-(f_c/f)^2}$

f_c is the cutoff frequency, below which waves do not propagate.

Notes:

(1) TEM waves are plane waves as discussed in **Chapters** 12 and 13.
(2) The equations for TE and TM waves are obtained by substituting the conditions from the first row of the table into **Eqs.** (17.18) through (17.21).
(3) In TE waves there is no longitudinal electric field, whereas in TM waves there is no longitudinal magnetic field.
(4) Phase velocity is given for lossless media.

Properties in lossy media can be easily obtained by modifying γ as follows ($\gamma = \alpha + j\beta$):

$$\gamma_{TE} = \gamma_{TM} = j\sqrt{(\omega^2\mu\varepsilon - k_c^2) - j\omega\mu\sigma} \tag{17.45}$$

Below cutoff the waves are said to be attenuated. The attenuation constant for evanescent waves (below cutoff, at a frequency $f < f_c$) is

$$\alpha_e = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_c^2}{f^2} - 1} \quad \left[\frac{\text{Np}}{\text{m}}\right] \tag{17.39}$$

The attenuation below cutoff is very high so that we may safely say that waves do not propagate.

TE and TM Waves in Parallel Plate Waveguides

Mode, m	TE_m	TM_m
Guide phase velocity [m/s]	$v_g = v_p/\sin\theta_i$ (17.70)*	Same as TE
Cutoff frequency [Hz]	$f_{cm} = m/2d\sqrt{\mu\varepsilon}$ (17.80)	Same as TE
Cutoff wave number [rad/m]	$k_{cm} = m\pi/d, \quad m = 1, 2, 3 \ldots$ (17.81)	Same as TE
Cutoff wavelength [m]	$\lambda_{cm} = 2d/m, \quad m = 1, 2, 3 \ldots$ (17.82)	Same as TE
Guide wavelength [m]	$\lambda_g = \lambda/\sqrt{1 - f_{cm}^2/f^2}$ (17.86)	Same as TE
Guide phase velocity [m/s]	$v_g = v_p/\sqrt{1 - f_{cm}^2/f^2}$ (17.85)	Same as TE
Guide phase constant [rad/m]	$\beta_g = \beta\sqrt{1 - f_{cm}^2/f^2}$ (17.87)	Same as TE
Electric field intensity in the waveguide [V/m]	$\mathbf{E}_1(x,z) = \hat{\mathbf{y}}\, jE_0 \sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g}$ (17.92)	$\mathbf{E}_1(x,z) = \hat{\mathbf{x}}\, E_0\frac{\lambda}{\lambda_g}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} + \hat{\mathbf{z}}\, jE_0\frac{\lambda}{\lambda_{cm}}\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g}$ (17.102)

(continued)

(continued)

Mode, m	TE_m	TM_m
Magnetic field intensity in the waveguide [A/m]	$\mathbf{H}_1(x,z) = -\hat{\mathbf{x}} j\frac{E_0}{\eta}\frac{\lambda}{\lambda_g}\sin\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g} - \hat{\mathbf{z}}\frac{E_0}{\eta}\frac{\lambda}{\lambda_{cm}}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g}$ (17.93)	$\mathbf{H}_1(x,z) = \hat{\mathbf{y}}\frac{E_0}{\eta}\cos\left(\frac{m\pi x}{d}\right)e^{-j2\pi z/\lambda_g}$ (17.103)
Time-averaged power density in the waveguide [W/m²]	$\mathcal{P}_{av} = \frac{E_0^2}{2\eta}\frac{\lambda}{\lambda_g}\sin^2\left(\frac{m\pi x}{d}\right)$ (17.97)	$\mathcal{P}_{av} = \frac{E_0^2}{2\eta}\frac{\lambda}{\lambda_{cm}}\cos^2\left(\frac{m\pi x}{d}\right)$ (17.109)
Guide wave impedance [Ω]	$Z_{TE} = \eta\lambda_g/\lambda = \eta/\sqrt{1-f_{cm}^2/f^2}$ (17.98)	$Z_{TM} = \eta/\lambda_g = \eta\sqrt{1-f_{cm}^2/f^2}$ (17.110)

$^*v_p = 1/\sqrt{\mu\varepsilon}$

Rectangular Waveguides TM modes, $H_z = 0$. **TE modes,** $E_z = 0$

Mode, m	$TM_{m,n}$	$TE_{m,n}$
Longitudinal field	$E_z(x,y,z) = E_0\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma z}$ (17.128)	$H_z(x,y,z) = H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$ (17.164)
Cutoff frequency [Hz]	$f_{cmn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ (17.138)	Same as TM
Cutoff wave number [rad/m]	$k_{cmn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ (17.137)	Same as TM
Cutoff wavelength [m]	$\lambda_{cmn} = \frac{1}{\sqrt{(m/2a)^2 + (n/2b)^2}}$ (17.139)	Same as TM
Guide wavelength [m]	$\lambda_g = \frac{\lambda}{\sqrt{1-f_{cmn}^2/f^2}}$ (17.143)	Same as TM
Guide phase velocity [m/s]	$v_g = \frac{v_p}{\sqrt{1-f_{cmn}^2/f^2}}$ (17.142)	Same as TM
Guide phase constant [rad/m]	$\beta_g = \beta\sqrt{1-\frac{f_{cmn}^2}{f^2}}$ (17.141)	Same as TM
Transverse fields	See **Eqs.** (17.147) through (17.150)	See **Eqs.** (17.165) through (17.168)
Time-averaged power in the waveguide [W]	$P = \frac{\omega\varepsilon\beta_g E_0^2 ab}{8k_{cmn}^2}$ (17.153)	$P = \frac{\omega\mu\beta_g H_0^2 ab}{8k_{cmn}^2}$ (17.176)
Guide wave impedance [Ω]	$Z_{TM} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\frac{f_{cmn}^2}{f^2}} = \eta\frac{\lambda}{\lambda_g}$ (17.145)	$Z_{TE} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1-f_{cmn}^2/f^2}} = \eta\frac{\lambda_g}{\lambda}$ (17.170)
Valid modes	All modes with $m \neq 0$, $n \neq 0$	All modes except $m = 0$, $n = 0$

Note: Range (bandwidth) for each mode is the range between its cutoff frequency and the cutoff frequency of the next, higher mode.

Power propagated in the TE_{10} mode

$$P(\mathrm{TE}_{10}) = \frac{\omega\mu\beta_g H_0^2 a^3 b}{4\pi^2} \quad [\mathrm{W}] \tag{17.177}$$

TE_{10} is the most important and most often used mode.

Losses

In waveguides these are due to wall losses and dielectric losses.

Dielectric attenuation constant:

$$\alpha_{dTE} = \alpha_{dTM} \frac{\sigma_d \eta_d}{2\sqrt{1 - f_{cmn}^2/f^2}} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right] \tag{17.178}$$

Wall attenuation constant:

$$\alpha_w = \frac{P_{loss}}{2P_{av}} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right] \tag{17.183}$$

where P_{loss} is power lost in walls.

Attenuation below cutoff:

$$\alpha_{bc} = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{f_{cmn}^2}{f^2} - 1} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right], \quad f < f_{cmn} \tag{17.190}$$

Cavity resonators—treated as shorted waveguides.

Properties: the resonant frequencies (modes) are

$$f_{mnp} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \quad [\mathrm{Hz}] \tag{17.218}$$

TM modes**:** $m,n \neq 0$, p can be zero. Longitudinal (z-directed) field:

$$E_z(x,y,z) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.205}$$

The TM transverse components are obtained from the longitudinal field by adding the conditions imposed by the conducting surfaces that short the guide **(Section 17.9.1)**. These are listed in **Eqs. (17.213)** through **(17.216)**.

TE modes**:** *m or n* can be zero (but not both), $p \neq 0$

Longitudinal field:

$$H_z(x,y,z) = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right) \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \tag{17.225}$$

The TE transverse components are obtained from the longitudinal field by adding the conditions imposed by the conducting surfaces that short the guide **(Section 17.9.2)**. These are listed in **Eqs. (17.226)** through **(17.229)**.

Energy and Losses **Stored energy** in the cavity

$$W_0 = \int_v \left(\frac{\varepsilon \mathbf{E}\cdot\mathbf{E}^*}{4} - \frac{\varepsilon \mathbf{H}\cdot\mathbf{H}^*}{4}\right) dv \quad [\mathrm{J}] \tag{17.233}$$

Power loss

$$P_{loss} = \frac{R_s}{2}\int_s J_s^2 ds = \frac{R_s}{2}\int_s |H_t|^2 ds \quad [\text{W}] \tag{17.234}$$

where R_s is the surface resistance **[Eq. (17.184)]**.

Quality factor of the cavity is the ratio of stored energy and power loss per cycle:

$$Q = 2\pi \frac{W_0}{P_{loss}T} = \frac{\omega_0 W_0}{P_{loss}} \quad [\text{dimensionless}] \tag{17.235}$$

Problems

TE, TM, and TEM Propagation in Parallel Plate Waveguides

17.1 Application: TM Modes in Parallel Plate Waveguides. A parallel plate waveguide is made of two strips, $a = 20$ mm wide, separated by $d = 1$ mm and air filled. Neglect edge effects. For an incident electric field intensity of magnitude $E_i = 1$ V, calculate at a frequency 20 % above the lowest cutoff frequency:

(a) The guide phase velocity, guide wavelength, and wave impedance for TM modes.
(b) The electric field intensity everywhere along the line for TM modes.
(c) The magnetic field intensity along the line for TM modes.
(d) The instantaneous power density in the waveguide.

17.2 Application: TE/TM Waves in Striplines. A parallel plate waveguide is made of two wide strips, separated by a fiberglass sheet $d = 0.5$ mm thick which has a relative permittivity of 3.5. Neglect any effects due to the edges of the strips (i.e., assume the strips are infinitely wide) and calculate:

(a) The lowest TM mode possible.
(b) The lowest TE mode possible.
(c) If a wave at twice the lowest TE cutoff frequency propagates along the waveguide, calculate the wave impedance for the TE and TM modes and compare with the wave impedance for TEM modes.

17.3 Power Relations in Integrated Striplines. In a stripline, the strips are separated a distance 0.02 mm and the strips are 1 mm wide. The material between the strips is air and because the width is much larger than the separation, the edge effects may be neglected. For an incident electric field intensity $E_i = 1$ V/m, calculate:

(a) The total time-averaged power propagated in the lowest TE mode at a frequency 25 % above cutoff.
(b) The total time-averaged power propagated in the lowest TM mode at a frequency 25 % above cutoff.
(c) Compare the results in **(a)** and **(b)** with the power propagated in the TEM mode at the same corresponding frequencies.

17.4 Propagation in Discontinuous Waveguides. Three parallel plate waveguide sections are connected as shown in **Figure 17.24**. The material between the plates is free space. Assume that the three waveguide sections operate in TE modes only. The source on the left supplies power at all frequencies between 1 MHz and 100 GHz. What is the lowest frequency signal received at the receiver? Disregard reflections at the connection between the waveguide sections.

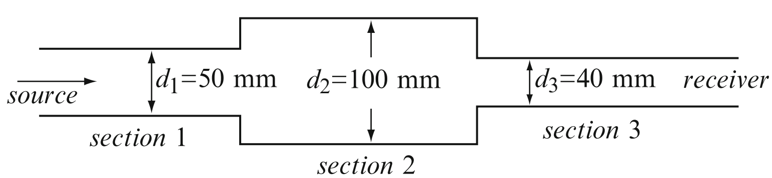


Figure 17.24

17.5 Reflection, Transmission, and SWR in Waveguides. A parallel plate waveguide with dimensions as shown in **Figure 17.25** is very long. A slab of permittivity $\varepsilon_1 = 2.5\varepsilon_0$ [F/m] occupies the right half of the waveguide. Assume TE propagation from left to right, at $f = 2f_c$ [Hz], where f_c is the cutoff frequency of the empty waveguide. Calculate:

(a) The reflection and transmission coefficients at the interface between air and slab.
(b) The standing wave ratio in the waveguide to the left of the interface and to the right of the interface.

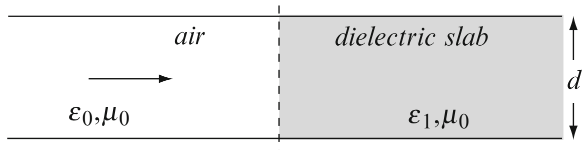


Figure 17.25

17.6 Fields in Shorted Waveguide. A parallel plate waveguide operates in a TM mode and has the following electric and magnetic fields:

$$\mathbf{E} = \hat{\mathbf{x}}\eta(\lambda/\lambda_g)H_0\cos\frac{\pi x}{a}\sin\left(\omega t - (2\pi/\lambda_g)z\right) + \hat{\mathbf{z}}\eta(\lambda/\lambda_c)H_0\sin\frac{\pi x}{a}\cos\left(\omega t - (2\pi/\lambda_g)z\right) \quad [\text{V/m}]$$

$$\mathbf{H} = \hat{\mathbf{y}}H_0\cos\frac{\pi x}{a}\sin\left(\omega t - (2\pi/\lambda_g)z\right) \quad [\text{A/m}]$$

where λ_c [m] is the cutoff wavelength and λ_g [m] is the guide wavelength. The wave propagates in the z direction and the material in the waveguide is free space. Dimensions and coordinates are shown in **Figure 17.26**. Now, a perfect conducting plate is used to short the two parallel plates (dotted line). Calculate the electric and magnetic field intensities to the left of the short. Assume the short is at $z = 0$ for simplicity.

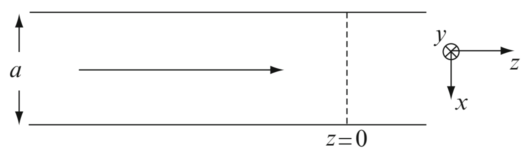


Figure 17.26

17.7 Application: Infrared Detection System. An infrared detection system is made of an optical stripline, used to guide infrared waves, and an infrared detector. The system must operate at a wavelength of 1,200 nm and the detector has an impedance of 50 Ω. The stripline is made of a thin sheet of glass, with relative permittivity $\varepsilon_r = 1.75$, sandwiched between two conducting sheets as shown in **Figure 17.27**. If the stripline must be matched to the detector, calculate:

(a) The lowest possible mode of propagation that may be used.
(b) The thickness d of the glass sheet that will support the mode calculated in **(a)**.

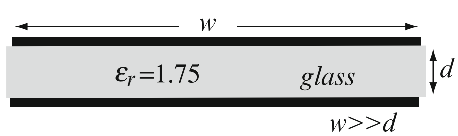


Figure 17.27

17.8 TEM Modes in Parallel Plate Waveguide. A parallel plate waveguide is made with very large (infinite) planar conductors as shown in **Figure 17.28**. At $z = 0$, $\mathbf{E} = \hat{\mathbf{y}}E_0$ [V/m] and the propagation is in the positive z direction. Find **E** and **H** for TEM modes. Assume perfect conductors for the plates and free space between the plates.

17.9 TE Fields in Parallel Plate Waveguide. A parallel plate waveguide is made with very large (infinite) planar conductors as shown in **Figure 17.28**. At $z = 0$, the electric field intensity is directed in the positive y direction and its peak equals E_0. The propagation is in the positive z direction. Find **E** and **H** for TE modes. Assume perfect conductors for the plates and free space between the plates.

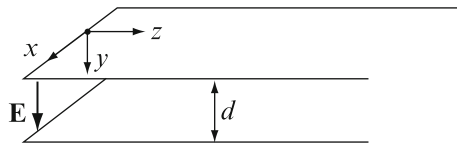


Figure 17.28

17.10 TM Fields in Parallel Plate Waveguide. A parallel plate waveguide is made with very large (infinite) planar conductors as shown in **Figure 17.29**. At $z = 0$, the magnetic field intensity is directed in the positive y direction and its peak equals H_0 [A/m]. The propagation is in the positive z direction. Find **E** and **H** for TM modes. Assume perfect conductors for the plates and free space between the plates.

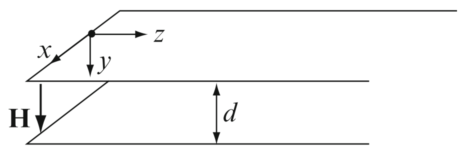


Figure 17.29

TM/TE Modes in Rectangular Waveguides

17.11 Application: Low-Frequency Waveguide–Limitations. An engineer had a bright idea: Why not use rectangular waveguides instead of the coaxial lines used in cable TV? The requirements are as follows: lowest frequency 50 MHz, and the waveguide has a ratio of $a = 2b$.

(a) What must be the dimensions of the waveguide to propagate from 50 MHz and up in the TE_{10} mode?
(b) The normal TV range in the VHF band is up to 150 MHz. Assuming each channel is 6 MHz, how many channels can be propagated in the TE_{10} mode alone?
(c) Is this a bright idea?

17.12 Application: Mode Separation and Bandwidth. The commercial WR 284 rectangular waveguide has internal dimensions of $a = 72.14$ mm and $b = 34.04$ mm. Calculate:

(a) The maximum bandwidth for the TE_{10} mode.
(b) The maximum bandwidth for the TE_{11} mode.

17.13 Application: Modes in Rectangular Waveguide. A WR 112 standard, rectangular waveguide with dimensions $a = 28.50$ mm and $b = 12.62$ mm is used to connect to a radar antenna which operates at a wavelength of 20 mm. Find all propagating modes that can be used at the given wavelength. The waveguide is air filled.

17.14 Application: Fields and Power in Rectangular Waveguide. A rectangular waveguide is used to transmit power from a generator to a radar antenna. The waveguide is a WR 34 waveguide with internal dimensions of 0.864 cm and 0.432 cm, operating at 23 GHz in the TE_{10} mode. The power delivered is 50 kW:

(a) Calculate the amplitudes of the electric and magnetic field intensities in the waveguide.
(b) Are these amplitudes acceptable in level? Explain.

17.15 Application: Tunnels as Waveguides. The following communication system is proposed for communication in mine tunnels to avoid the need for cables: the tunnel is used as a waveguide 5 m wide and 2 m high.

(a) What is the lowest frequency that may be used?
(b) If it is desired to propagate a single mode, what is the maximum bandwidth that may be used and still guarantee propagation in the lowest mode?

17.16 Discontinuities in Rectangular Waveguide. A very long rectangular waveguide is filled with two materials as shown in **Figure 17.30**. A TM wave propagates in the waveguide. Calculate:

(a) The lowest frequency (cutoff frequency) that will propagate in the waveguide.
(b) The time it takes the wave to propagate between points A and B.

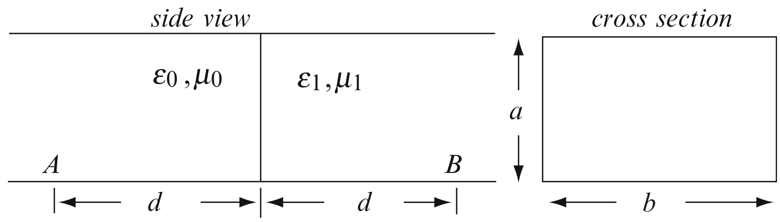


Figure 17.30

17.17 Field Required for Total Power. It is required that a lossless rectangular waveguide carry 100 W of power (time averaged) at 4.5 GHz, in the TE_{10} mode. The waveguide is $a = 47.55$ mm wide and $b = 22.15$ mm high (WR 187 waveguide) and is air filled:

(a) Find the longitudinal and transverse components of the electric field intensity.
(b) Find the longitudinal and transverse components of the magnetic field intensity.

17.18 Power Carried in Lowest Mode. A rectangular waveguide is built such that $b = 0.75a$, with $a = 10$ mm:

(a) Find the lowest cutoff frequency and mode.
(b) Calculate the time-averaged power the wave can propagate at a frequency 25 % above the cutoff in **(a)**, for a given electric field intensity with amplitude $E_0 = 1{,}000$ V/m.
(c) Compare this with another waveguide for which $b = 0.5a$ for the same conditions as in **(b)**. Which waveguide can carry more power for the same field level?

17.19 Maximum Power Handling of a Waveguide. A rectangular waveguide has a width to height ratio $a/b = 2.0$ and the ratio between the operating frequency and the cutoff frequency is $f/f_{c10} = 2.0$ at $f = 10$ GHz. What is the maximum time-averaged power that can be transmitted in the waveguide in the TE_{10} mode without exceeding the breakdown electric field intensity of 3×10^6 V/m in air?

17.20 Application: Optical Waveguide. An optical waveguide is made in the form of a rectangular cross-sectional channel in a silicon substrate, as shown in **Figure 17.31**. The channel is 2 μm wide, 1 μm high, and $\varepsilon = 2\varepsilon_0$ [F/m]:

(a) Explain why this structure can function as a rectangular waveguide and outline the conditions necessary for it to operate. **Hint:** Consider the conditions for total reflection in a dielectric.
(b) Calculate the lowest frequency that can be propagated. Which mode is it and in what range of the spectrum is this propagation possible?
(c) Calculate the peak power that can be propagated at twice the frequency calculated in **(b)** if the peak electric field intensity cannot exceed 1,000 V/mm.

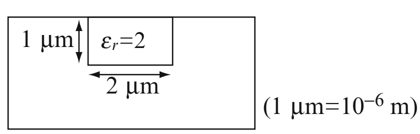


Figure 17.31

17.21 TM Waves in a Waveguide. Write the time domain expressions for the transverse components of the fields for TM propagation in a lossless rectangular waveguide of width a and height b (see **Exercise 17.4**).

17.22 TM Waves in a Waveguide. Find the transverse components for a backward-propagating TM wave in a rectangular waveguide of width a [m] and height b [m]. Assume the backward-propagating wave propagates in the negative z direction and the amplitude of the wave is E_0^- [V/m].

17.23 **TM Waves in a Waveguide**. Find the total wave in a waveguide of width a [m] and height b [m] if a forward-propagating TM wave of amplitude E_0^+ [V/m] and a backward-propagating wave of amplitude E_0^- [V/m] exist.

17.24 **TE Waves in a Waveguide**. Find the transverse components $E_x(x,y,z)$, $H_x(x,y,z)$, $E_y(x,y,z)$, and $H_y(x,y,z)$ for a backward-propagating TE wave in a rectangular waveguide of width a [m] and height b [m]. Assume the wave propagates in the negative z direction and the amplitude of the wave is H_1.

17.25 **TE Waves in a Waveguide**. Find the total transverse TE waves in a waveguide of width a [m] and height b [m] if a forward-propagating wave of amplitude H_0 and a backward-propagating wave of amplitude H_1 exist.

Attenuation and Losses in Rectangular Waveguides

17.26 **Dielectric Losses in Waveguides**. A rectangular waveguide is filled with a lossy dielectric with relative permittivity $\varepsilon_r = 2$ and conductivity $\sigma_d = 10^{-4}$ S/m. Assuming perfectly conducting walls, find:

(a) The attenuation constant in the waveguide at a frequency 1.5 times larger than the lowest cutoff frequency.
(b) The percentage of power loss per meter of the waveguide at $f/f_c = 2$. Assume the power entering a section of the waveguide is P_0 and calculate the power loss as a percentage of this power.

17.27 **Conductor (Wall) Losses**. A rectangular waveguide is made of aluminum, which has conductivity of 3.6×10^7 S/m. The walls of the waveguide are thick and the internal dimensions are $a = 25.4$ mm and $b = 38.1$ mm. Assuming the waveguide is empty (free space), calculate:

(a) The power loss per meter length in the TE_{01} mode at $f/f_c = 2$. Assume the amplitude of the longitudinal magnetic field intensity is 1 A/m.
(b) The attenuation constant due to losses in the walls.

17.28 **Waveguide with Dielectric and Wall Losses**. The rectangular waveguide in **Problem 17.27** is given again, but now the waveguide is filled with a low-loss dielectric with relative permittivity of 2 and conductivity $\sigma_d = 10^{-4}$ S/m. All other parameters including wall conductivity remain the same. Calculate:

(a) The attenuation constant in the waveguide.
(b) The power loss per meter length in the TE_{01} mode at $f/f_c = 2$. Assume the amplitude of the longitudinal magnetic field intensity is 1 A/m.

Cavity Resonators

17.29 **Resonant Frequencies in Rectangular Cavity**. A rectangular cavity resonator is 60 mm long, 30 mm high, and 40 mm wide and is air filled. Calculate:

(a) The TE_{101} resonant frequency.
(b) The next three nondegenerate TE resonant modes. Classify the modes.

17.30 **Resonant Frequencies in Shorted Waveguide**. The WR 284 waveguide is made into a cavity 0.5 m long by shorting the waveguide at two locations. Calculate the first 10 resonant frequencies and classify the modes. The waveguide has dimensions $a = 72.14$ mm and $b = 34.04$ mm.

17.31 **Application: Design of a Cavity for Given Resonant Frequencies**. A cavity resonator is built from a section of a waveguide with dimensions $a = 4.755$ cm and $b = 2.215$ cm by shorting the waveguide with two conducting plates. A rectangular cavity of length d is thus created. The cavity is required to resonate at 8 GHz in the TE_{101} mode:

(a) Find the length of the shorted section necessary.
(b) What is the dominant mode and what is its resonant frequency?

17.32 Application: Parallel Plate Resonator. A parallel plate waveguide is shorted at two locations as shown in **Figure 17.32**. The distance between shorts is d [m]. Calculate the resonant frequencies of the shorted guide, assuming that a TM mode exists in the guide within the shorted section.

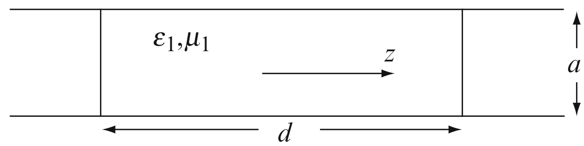


Figure 17.32