Chapter 5
Existence Techniques II: Parabolic Methods. The Heat Equation

5.1 The Heat Equation: Definition and Maximum Principles

Let $\Omega \in \mathbb{R}^d$ be open, $(0, T) \subset \mathbb{R} \cup \{\infty\}$,

$$\Omega_T := \Omega \times (0, T),$$
$$\partial^* \Omega_T := \left(\bar{\Omega} \times \{0\}\right) \cup \left(\partial\Omega \times \overline{(0, T)}\right). \quad \text{(See Fig. 5.1.)}$$

We call $\partial^* \Omega_T$ the reduced boundary of Ω_T.

For each fixed $t \in (0, T)$ let $u(x, t) \in C^2(\Omega)$, and for each fixed $x \in \Omega$ let $u(x, t) \in C^1((0, T))$. Moreover, let $f \in C^0(\partial^* \Omega_T)$, $u \in C^0(\bar{\Omega}_T)$. We say that u solves the heat equation with boundary values f if

$$\begin{aligned} u_t(x, t) &= \Delta_x u(x, t) \quad &&\text{for } (x, t) \in \Omega_T, \\ u(x, t) &= f(x, t) &&\text{for } (x, t) \in \partial^* \Omega_T. \end{aligned} \tag{5.1.1}$$

Written out with a less compressed notation, the differential equation is

$$\frac{\partial}{\partial t} u(x, t) = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(x, t).$$

Equation (5.1.1) is a linear, parabolic partial differential equation of second order. The reason that here, in contrast to the Dirichlet problem for harmonic functions, we are prescribing boundary values only at the reduced boundary is that for a solution of a parabolic equation, the values of u on $\Omega \times \{T\}$ are already determined by its values on $\partial^* \Omega_T$, as we shall see in the sequel.

The heat equation describes the evolution of temperature in heat-conducting media and is likewise important in many other diffusion processes. For example, if we have a body in $\mathbb{R}^3$ with given temperature distribution at time t_0 and if we keep

J. Jost, *Partial Differential Equations*, Graduate Texts in Mathematics 214,
DOI 10.1007/978-1-4614-4809-9_5,

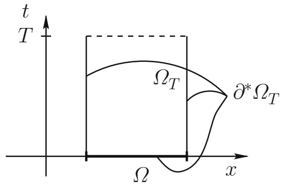

Fig. 5.1

the temperature on its surface constant, this determines its temperature distribution uniquely at all times $t > t_0$. This is a heuristic reason for prescribing the boundary values in (5.1.1) only at the reduced boundary.

Replacing t by $-t$ in (5.1.1) does not transform the heat equation into itself. Thus, there is a distinction between "past" and "future." This is likewise heuristically plausible.

In order to gain some understanding of the heat equation, let us try to find solutions with separated variables, i.e., of the form

$$u(x,t) = v(x)w(t). \tag{5.1.2}$$

Inserting this ansatz into (5.1.1), we obtain

$$\frac{w_t(t)}{w(t)} = \frac{\Delta v(x)}{v(x)}. \tag{5.1.3}$$

Since the left-hand side of (5.1.3) is a function of t only, while the right-hand side is a function of x, each of them has to be constant. Thus

$$\Delta v(x) = -\lambda v(x), \tag{5.1.4}$$

$$w_t(t) = -\lambda w(t), \tag{5.1.5}$$

for some constant λ. We consider the case where we assume homogeneous boundary conditions on $\partial\Omega \times [0,\infty)$, i.e.,

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega,$$

or equivalently,

$$v(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{5.1.6}$$

From (5.1.4) we then get through multiplication by v and integration by parts

$$\int_\Omega |Dv(x)|^2 \mathrm{d}x = -\int_\Omega v(x)\Delta v(x)\mathrm{d}x = \lambda \int_\Omega v(x)^2 \mathrm{d}x.$$

Consequently,

$$\lambda \geq 0$$

(and this is the reason for introducing the minus sign in (5.1.4) and (5.1.5)).

A solution v of (5.1.4) and (5.1.6) that is not identically 0 is called an eigenfunction of the Laplace operator, and λ an eigenvalue. We shall see in Sect. 11.5 that the eigenvalues constitute a discrete sequence $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n \to \infty$ for $n \to \infty$. Thus, a nontrivial solution of (5.1.4) and (5.1.6) exists precisely if $\lambda = \lambda_n$, for some $n \in \mathbb{N}$. The solution of (5.1.5) then is simply given by

$$w(t) = w(0)\mathrm{e}^{-\lambda t}.$$

So, if we denote an eigenfunction for the eigenvalue λ_n by v_n, we obtain the solution

$$u(x,t) = v_n(x)w(0)\mathrm{e}^{-\lambda_n t}$$

of the heat equation (5.1.1), with the homogeneous boundary condition

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega$$

and the initial condition

$$u(x,0) = v_n(x)w(0).$$

This seems to be a rather special solution. Nevertheless, in a certain sense, this is the prototype of a solution. Namely, because (5.1.1) is a linear equation, any linear combination of solutions is a solution itself, and so we may take sums of such solutions for different eigenvalues λ_n. In fact, as we shall demonstrate in Sect. 11.5, any L^2-function on Ω, and thus in particular any continuous function f on $\bar{\Omega}$, assuming Ω to be bounded, which vanishes on $\partial\Omega$, can be expanded as

$$f(x) = \sum_{n\in\mathbb{N}} \alpha_n v_n(x), \tag{5.1.7}$$

where the $v_n(x)$ are the eigenfunctions of Δ, normalized via

$$\int_\Omega v_n(x)^2 \mathrm{d}x = 1$$

and mutually orthogonal:

$$\int_\Omega v_n(x)v_m(x)\mathrm{d}x = 0 \quad \text{for } n \neq m.$$

Then α_n can be computed as

$$\alpha_n = \int_\Omega v_n(x) f(x) \mathrm{d}x.$$

We then have an expansion for the solution of

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) \quad && \text{for } x \in \Omega, t \geq 0, \\ u(x,t) &= 0 && \text{for } x \in \partial\Omega, t \geq 0, \\ u(x,0) &= f(x) && \left(= \sum_n \alpha_n v_n(x)\right), \quad \text{for } x \in \Omega, \end{aligned} \tag{5.1.8}$$

namely,

$$u(x,t) = \sum_{n \in \mathbb{N}} \alpha_n \mathrm{e}^{-\lambda_n t} v_n(x). \tag{5.1.9}$$

Since all the λ_n are nonnegative, we see from this representation that all the "modes" $\alpha_n v_n(x)$ of the initial values f are decaying in time for a solution of the heat equation. In this sense, the heat equation regularizes or smoothes out its initial values. In particular, since thus all factors $\mathrm{e}^{-\lambda_n t}$ are less than or equal to 1 for $t \geq 0$, the series (5.1.9) converges in $L^2(\Omega)$, because (5.1.7) does.

If instead of the heat equation we considered the backward heat equation

$$u_t = -\Delta u,$$

then the analogous expansion would be $u(x,t) = \sum_n \alpha_n \mathrm{e}^{\lambda_n t} v_n(x)$, and so the modes would grow, and differences would be exponentially enlarged, and in fact, in general, the series will no longer converge for positive t. This expresses the distinction between "past" and "future" built into the heat equation and alluded to above.

If we write

$$q(x,y,t) := \sum_{n \in \mathbb{N}} \mathrm{e}^{-\lambda_n t} v_n(x) v_n(y), \tag{5.1.10}$$

and if we can use the results of Sect. 11.5 to show the convergence of this series, we may represent the solution $u(x,t)$ of (5.1.8) as

$$\begin{aligned} u(x,t) &= \sum_{n \in \mathbb{N}} \mathrm{e}^{-\lambda_n t} v_n(x) \int_\Omega v_n(y) f(y) \mathrm{d}y \quad \text{by (5.1.9)} \\ &= \int_\Omega q(x,y,t) f(y) \mathrm{d}y. \end{aligned} \tag{5.1.11}$$

Instead of demonstrating the convergence of the series (5.1.10) and that $u(x,t)$ given by (5.1.9) is smooth for $t > 0$ and permits differentiation under the sum, in this chapter, we shall pursue a different strategy to construct the "heat kernel" $q(x,y,t)$ in Sect. 5.3.

For $x, y \in \mathbb{R}^n$, $t, t_0 \in \mathbb{R}$, $t \neq t_0$, we define the heat kernel at (y, t_0) as

$$\Lambda(x,y,t,t_0) := \frac{1}{(4\pi\,|t-t_0|)^{\frac{d}{2}}}\,\mathrm{e}^{\frac{|x-y|^2}{4(t_0-t)}}.$$

We then have

$$\begin{aligned}
\Lambda_t(x,y,t,t_0) &= -\frac{d}{2(t-t_0)}\Lambda(x,y,t,t_0) + \frac{|x-y|^2}{4(t_0-t)^2}\Lambda(x,y,t,t_0),\\
\Lambda_{x_i}(x,y,t,t_0) &= \frac{x^i-y^i}{2(t_0-t)}\Lambda(x,y,t,t_0),\\
\Lambda_{x_i x_i}(x,y,t,t_0) &= \frac{(x^i-y^i)^2}{4(t_0-t)^2}\Lambda(x,y,t,t_0) + \frac{1}{2(t_0-t)}\Lambda(x,y,t,t_0),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\Delta_x \Lambda(x,y,t,t_0) &= \frac{|x-y|^2}{4(t_0-t)^2}\Lambda(x,y,t,t_0) + \frac{d}{2(t_0-t)}\Lambda(x,y,t,t_0)\\
&= \Lambda_t(x,y,t,t_0).
\end{aligned}$$

The heat kernel thus is a solution of (5.1.1). The heat kernel Λ is similarly important for the heat equation as the fundamental solution Γ is for the Laplace equation.

We first wish to derive a representation formula for solutions of the (homogeneous and inhomogeneous) heat equation that will permit us to compute the values of u at time T from the values of u and its normal derivative on $\partial^*\Omega_T$. For that purpose, we shall first assume that u solves the equation

$$u_t(x,t) = \Delta u(x,t) + \varphi(x,t) \quad \text{in } \Omega_T$$

for some bounded integrable function $\varphi(x,t)$ and that $\Omega \subset \mathbb{R}^d$ is bounded and such that the divergence theorem holds. Let v satisfy $v_t = -\Delta v$ on Ω_T. Then

$$\begin{aligned}
\int_{\Omega_T} v\varphi\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T} v(u_t - \Delta u)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_\Omega\left(\int_0^T v(x,t)u_t(x,t)\,\mathrm{d}t\right)\mathrm{d}x - \int_0^T\left(\int_\Omega v\Delta u\,\mathrm{d}x\right)\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= \int_\Omega \left[v(x,T)u(x,T) - v(x,0)u(x,0) - \int_0^T v_t(x,t)u(x,t)\mathrm{d}t \right] \mathrm{d}x \\
&\quad - \int_0^T \left(\int_\Omega u \Delta v \mathrm{d}x \right) \mathrm{d}t - \int_0^T \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) do\, \mathrm{d}t \\
&= \int_{\Omega\times\{T\}} vu \,\mathrm{d}x - \int_{\Omega\times\{0\}} vu\,\mathrm{d}x - \int_0^T \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) do\, \mathrm{d}t.
\end{aligned} \tag{5.1.12}$$

For $v(x,t) := \Lambda(x,y,T+\varepsilon,t)$ with $T > 0$ and $y \in \Omega^d$ fixed we then have, because of $v_t = -\Delta v$,

$$\begin{aligned}
\int_{\Omega\times\{T\}} \Lambda u \,\mathrm{d}x &= \int_{\Omega_T} \Lambda \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega\times\{0\}} \Lambda u\,\mathrm{d}x \\
&\quad + \int_0^T \left(\int_{\partial\Omega} \left(\Lambda \frac{\partial u}{\partial \nu} - u\frac{\partial \Lambda}{\partial \nu} \right) do \right) \mathrm{d}t.
\end{aligned} \tag{5.1.13}$$

For $\varepsilon \to 0$, the term on the left-hand side becomes

$$\lim_{\varepsilon\to 0} \int_\Omega \Lambda(x,y,T+\varepsilon,T)u(x,T)\mathrm{d}x = u(y,T).$$

Furthermore, $\Lambda(x,y,T+\varepsilon,t)$ is uniformly continuous in ε, x, t for $\varepsilon \geq 0$, $x \in \partial\Omega$, and $0 \leq t \leq T$ or for $x \in \Omega$, $t = 0$. Thus (5.1.13) implies, letting $\varepsilon \to 0$,

$$\begin{aligned}
u(y,T) &= \int_{\Omega_T} \Lambda(x,y,T,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Lambda(x,y,T,0)u(x,0)\,\mathrm{d}x \\
&\quad + \int_0^T \left(\int_{\partial\Omega} \left(\Lambda(x,y,T,t)\frac{\partial u(x,t)}{\partial \nu} - u(x,t)\frac{\partial \Lambda(x,y,T,t)}{\partial \nu} \right) do \right) \mathrm{d}t.
\end{aligned} \tag{5.1.14}$$

This formula, however, does not yet solve the initial boundary value problem, since in (5.1.14), in addition to $u(x,t)$ for $x \in \partial\Omega$, $t > 0$, and $u(x,0)$, also the normal derivative $\frac{\partial u}{\partial \nu}(x,t)$ for $x \in \partial\Omega$, $t > 0$, enters. Thus we should try to replace $\Lambda(x,y,T,t)$ by a kernel that vanishes on $\partial\Omega \times (0,\infty)$. This is the task that we shall address in Sect. 5.3. Here, we shall modify the construction in a somewhat different manner. Namely, we do not replace the kernel, but change the domain of integration so that the kernel becomes constant on its boundary. Thus, for $\mu > 0$, we let

$$M(y,T;\mu) := \left\{ (x,s) \in \mathbb{R}^d \times \mathbb{R}, s \leq T : \frac{1}{(4\pi(T-s))^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4(T-s)}} \geq \mu \right\}.$$

For any $y \in \Omega, T > 0$, we may find $\mu_0 > 0$ such that for all $\mu > \mu_0$,

$$M(y, T; \mu) \subset \Omega \times [0, T].$$

We always have

$$(y, T) \in M(y, T; \mu),$$

and in fact, $M(y, T; \mu) \cap \{s = T\}$ consists of the single point (y, T). For t falling below T, $M(y, T; \mu) \cap \{s = t\}$ is a ball in $\mathbb{R}^d$ with center (y, t) whose radius first grows but then starts to shrink again if t is decreased further, until it becomes 0 at a certain value of t.

We then perform the above computation on $M(y, T; \mu)$ $(\mu > \mu_0)$ in place of Ω_T, with

$$v(x, t) := \Lambda(x, y, T + \varepsilon, t) - \mu,$$

and as before, we may perform the limit $\varepsilon \searrow 0$. Then

$$v(x, t) = 0 \quad \text{for } (x, t) \in \partial M(y, T; \mu),$$

so that the corresponding boundary term disappears.

Here, we are interested only in the homogeneous heat equation, and so, we put $\varphi = 0$. We then obtain the representation formula

$$\begin{aligned} u(y, T) &= -\int_{\partial M(y,T;\mu)} u(x, t) \frac{\partial \Lambda}{\partial \nu_x}(x, y, T, t) do(x, t) \\ &= \mu \int_{\partial M(y,T;\mu)} u(x, t) \frac{|x - y|}{2(T - t)} do(x, t), \end{aligned} \tag{5.1.15}$$

since

$$\frac{\partial \Lambda}{\partial \nu_x} = -\frac{|x - y|}{2(T - t)} \Lambda = -\frac{|x - y|}{2(T - t)} \mu \quad \text{on } \partial M(y, T; \mu). \tag{5.1.16}$$

In general, the maximum principles for parabolic equations are qualitatively different from those for elliptic equations. Namely, one often gets stronger conclusions in the parabolic case.

Theorem 5.1.1. *Let u be as in the assumptions of (5.1.1). Let $\Omega \subset \mathbb{R}^d$ be open and bounded and*

$$\Delta u - u_t \geq 0 \quad \text{in } \Omega_T. \tag{5.1.17}$$

We then have

$$\sup_{\bar{\Omega}_T} u = \sup_{\partial^* \Omega_T} u. \tag{5.1.18}$$

(If $T < \infty$, we can take max *in place of* sup.*)*

Proof. Without loss of generality $T < \infty$.

(i) Suppose first

$$\Delta u - u_t > 0 \quad \text{in } \Omega_T. \tag{5.1.19}$$

For $0 < \varepsilon < T$, by continuity of u and compactness of $\bar{\Omega}_{T-\varepsilon}$, there exists $(x_0, t_0) \in \bar{\Omega}_{T-\varepsilon}$ with

$$u(x_0, t_0) = \max_{\bar{\Omega}_{T-\varepsilon}} u. \tag{5.1.20}$$

If we had $(x_0, t_0) \in \Omega_{T-\varepsilon}$, then $\Delta u(x_0, t_0) \leq 0$, $\nabla u(x_0, t_0) = 0$, $u_t(x_0, t_0) = 0$ would lead to a contradiction; hence we must have $(x_0, t_0) \in \partial \Omega_{T-\varepsilon}$. For $t = T - \varepsilon$ and $x \in \Omega$, we would get $\Delta u(x_0, t_0) \leq 0$, $u_t(x_0, t_0) \geq 0$, likewise contradicting (5.1.19). Thus we conclude that

$$\max_{\bar{\Omega}_{T-\varepsilon}} u = \max_{\partial^* \Omega_{T-\varepsilon}} u, \tag{5.1.21}$$

and for $\varepsilon \to 0$, (5.1.21) yields the claim, since u is continuous.

(ii) If we have more generally $\Delta u - u_t \geq 0$, we let $v := u - \varepsilon t$, $\varepsilon > 0$. We have

$$v_t = u_t - \varepsilon \leq \Delta u - \varepsilon = \Delta v - \varepsilon < \Delta v,$$

and thus by (i),

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Omega}_T}(v + \varepsilon t) \leq \max_{\bar{\Omega}_T} v + \varepsilon T = \max_{\partial^* \Omega_T} v + \varepsilon T \leq \max_{\partial^* \Omega_T} u + \varepsilon T,$$

and $\varepsilon \to 0$ yields the claim.

□

Theorem 5.1.1 directly leads to a uniqueness result:

Corollary 5.1.1. *Let u, v be solutions of (5.1.1) with $u = v$ on $\partial^* \Omega_T$, where $\Omega \subset \mathbb{R}^d$ is bounded. Then $u = v$ on $\bar{\Omega}_T$.*

Proof. We apply Theorem 5.1.1 to $u - v$ and $v - u$. □

This uniqueness holds only for bounded Ω, however. If, for example, $\Omega = \mathbb{R}^d$, uniqueness holds only under additional assumptions on the solution u.

Theorem 5.1.2. *Let* $\Omega = \mathbb{R}^d$ *and suppose*

$$\begin{aligned} \Delta u - u_t &\geq 0 && \text{in } \Omega_T, \\ u(x,t) &\leq M\mathrm{e}^{\lambda|x|^2} && \text{in } \Omega_T \text{ for } M, \lambda > 0, \\ u(x,0) &= f(x) && x \in \Omega = \mathbb{R}^d. \end{aligned} \tag{5.1.22}$$

Then

$$\sup_{\bar{\Omega}_T} u \leq \sup_{\mathbb{R}^d} f. \tag{5.1.23}$$

Remark. This maximum principle implies the uniqueness of solutions of the differential equation

$$\begin{aligned} \Delta u &= u_t && \text{on } \Omega_T = \mathbb{R}^d \times (0,T), \\ u(x,0) &= f(x) && \text{for } x \in \mathbb{R}^d, \\ u(x,t) &\leq M\mathrm{e}^{\lambda|x|^2} && \text{for } (x,t) \in \Omega_T. \end{aligned}$$

The condition (5.1.22) is a condition for the growth of u at infinity. If this condition does not hold, there are counterexamples for uniqueness. For example, let us choose

$$u(x,t) := \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}$$

with

$$g(t) := \begin{cases} \mathrm{e}^{\frac{-1}{t^k}} & t > 0, \text{ for some } k > 1, \\ 0 & t = 0, \end{cases}$$

$$v(x,t) := 0 \quad \text{for all } (x,t) \in \mathbb{R} \times (0,\infty).$$

Then u and v are solutions of (5.1.1) with $f(x) = 0$. For further details we refer to the book of John [14].

Proof of Theorem 5.1.2: Since we can divide the interval $(0,T)$ into subintervals of length $\tau < \frac{1}{4\lambda}$, it suffices to prove the claim for $T < \frac{1}{4\lambda}$, because we shall then get

$$\sup_{\mathbb{R}^d \times [0,k\tau]} u \leq \sup_{\mathbb{R}^d \times [0,(k-1)\tau]} u \leq \cdots \leq \sup_{\mathbb{R}^d} f(x).$$

Thus let $T < \frac{1}{4\lambda}$. We may then find $\varepsilon > 0$ with

$$T + \varepsilon < \frac{1}{4\lambda}. \tag{5.1.24}$$

For fixed $y \in \mathbb{R}^d$ and $\delta > 0$, we consider

$$v^\delta(x,t) := u(x,t) - \delta\Lambda(x,y,t,T+\varepsilon), \quad 0 \le t \le T. \tag{5.1.25}$$

It follows that

$$v_t^\delta - \Delta v^\delta = u_t - \Delta u \le 0, \tag{5.1.26}$$

since Λ is a solution of the heat equation. For $\Omega^\rho := B(y,\rho)$, we thus obtain from Theorem 5.1.1

$$v^\delta(y,t) \le \max_{\partial^*\Omega^\rho} v^\delta. \tag{5.1.27}$$

Moreover,

$$v^\delta(x,0) \le u(x,0) \le \sup_{\mathbb{R}^d} f, \tag{5.1.28}$$

and for $|x-y| = \rho$,

$$\begin{aligned} v^\delta(x,t) &\le M\mathrm{e}^{\lambda|x|^2} - \delta\frac{1}{(4\pi(T+\varepsilon-t))^{\frac{d}{2}}}\exp\left(\frac{\rho^2}{4(T+\varepsilon-t)}\right) \\ &\le M\mathrm{e}^{\lambda(|y|+\rho)^2} - \delta\frac{1}{(4\pi(T+\varepsilon))^{\frac{d}{2}}}\exp\left(\frac{\rho^2}{4(T+\varepsilon)}\right). \end{aligned}$$

Because of (5.1.24), for sufficiently large ρ, the second term has a larger exponent than the first, and so the whole expression can be made arbitrarily negative; in particular, we can achieve that it is not larger than $\sup_{\mathbb{R}^d} f$. Consequently,

$$v^\delta \le \sup_{\mathbb{R}^d} f \quad \text{on } \partial^*\Omega^\rho. \tag{5.1.29}$$

Thus, (5.1.27) and (5.1.29) yield

$$\begin{aligned} v^\delta(y,t) &= u(y,t) - \delta\Lambda(y,y,t,T+\varepsilon) = u(y,t) - \delta\frac{1}{(4\pi(T+\varepsilon-t))^{\frac{d}{2}}} \\ &\le \sup_{\mathbb{R}^d} f. \end{aligned}$$

The conclusion follows by letting $\delta \to 0$.

Actually, we can use the representation formula (5.1.12) to obtain a strong maximum principle for the heat equation, in the same manner as the mean value formula could be used to obtain Corollary 2.2.3:

Theorem 5.1.3. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and*

$$\Delta u - u_t = 0 \quad \text{in } \Omega_T,$$

with the regularity properties specified at the beginning of this section. Then if there exists some $(x_0, t_0) \in \Omega \times (0, T]$ with

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u \quad \left(or\ with\ u(x_0, t_0) = \min_{\overline{\Omega_T}} u \right),$$

then u is constant in $\bar{\Omega}_{t_0}$.

Proof. The proof is the same as that of Lemma 2.2.1, using the representation formula (5.1.12). (Note that by applying (5.1.15) to the function $u \equiv 1$, we obtain

$$\mu \int_{\partial M(y,T;\mu)} \frac{|x-y|}{2(T-t)} do(x,t) = 1,$$

and so a general u that solves the heat equation is indeed represented as some average. Also, $M(y,T;\mu_2) \subset M(y,T;\mu_1)$ for $\mu_1 \leq \mu_2$, and as $\mu \to \infty$, the sets $M(y,T;\mu)$ shrink to the point (y,T).) □

Of course, the maximum principle also holds for subsolutions, i.e., if

$$\Delta u - u_t \geq 0 \quad \text{in } \Omega_T.$$

In that case, we get the inequality "≤" in place of "=" in (5.1.15), which is what is required for the proof of the maximum principle. Likewise, the statement with the minimum holds for solutions of

$$\Delta u - u_t \leq 0.$$

Slightly more generally, we even have

Corollary 5.1.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and*

$$\Delta u(x,t) + c(x,t)u(x,t) - u_t(x,t) \geq 0 \quad \text{in } \Omega_T,$$

with some bounded function

$$c(x,t) \leq 0 \quad \text{in } \Omega_T. \tag{5.1.30}$$

Then if there exists some $(x_0, t_0) \in \Omega \times (0, T]$ *with*

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u \geq 0, \tag{5.1.31}$$

then u is constant in $\bar{\Omega}_{t_0}$.

Proof. Our scheme of proof still applies because, since c is nonpositive, at a nonnegative maximum point (x_0, t_0) of u, $c(x_0, t_0)u(x_0, t_0) \leq 0$ which strengthens the inequality used in the proof. □

Again, we obtain a minimum principle when we reverse all signs.
For use in Sect. 6.1 below, we now derive a parabolic version of E.Hopf's boundary point Lemma 3.1.2. Compared with Sect. 3.1, we shall reverse here the scheme of proof, i.e., deduce the boundary point lemma from the strong maximum principle instead of the other way around. This is possible because here we consider less general differential operators than the ones in Sect. 3.1 so that we could deduce our maximum principle from the representation formula. Of course, one can also deduce general Hopf type maximum principles in the parabolic case, in a manner analogous to Sect. 3.1, but we do not pursue that here as it will not yield conceptually or technically new insights.

Lemma 5.1.1. *Suppose the function c is bounded and satisfies* $c(x, t) \leq 0$ *in* Ω_T. *Let u solve the differential inequality*

$$\Delta u(x, t) + c(x, t)u(x, t) - u_t(x, t) \geq 0 \quad \text{in } \Omega_T,$$

and let $(x_0, t_0) \in \partial^* \Omega_T$. *Moreover, assume:*

(i) *u is continuous at* (x_0, t_0).
(ii) $u(x_0, t_0) \geq 0$ *if* $c(x) \not\equiv 0$.
(iii) $u(x_0, t_0) > u(x, t)$ *for all* $(x, t) \in \Omega_T$.
(iv) *There exists a ball* $\mathring{B}((y, t_1), R) \subset \Omega_T$ *with* $(x_0, t_0) \in \partial B((y, t_1), R)$.

We then have, with $r := |(x, t) - (y, t_1)|$,

$$\frac{\partial u}{\partial r}(x_0, t_0) > 0, \tag{5.1.32}$$

provided that this derivative (in the direction of the exterior normal of Ω_T*) exists.*

Proof. With the auxiliary function

$$v(x) := \mathrm{e}^{-\gamma(|x-y|^2+(t-t_1)^2)} - \mathrm{e}^{-\gamma R^2},$$

the proof proceeds as the one of Lemma 3.1.2, employing this time the maximum principle Theorem 5.1.3. □

I do not know of any good recent book that gives a detailed and systematic presentation of parabolic differential equations. Some older, but still useful, references are [9, 23].

5.2 The Fundamental Solution of the Heat Equation. The Heat Equation and the Laplace Equation

We first consider the so-called fundamental solution

$$K(x,y,t) = \Lambda(x,y,t,0) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}, \tag{5.2.1}$$

and we first observe that for all $x \in \mathbb{R}^d$, $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^d} K(x,y,t)\mathrm{d}y &= \frac{1}{(4\pi t)^{\frac{d}{2}}} d\omega_d \int_0^\infty e^{-\frac{r^2}{4t}} r^{d-1}\mathrm{d}r = \frac{1}{\pi^{\frac{d}{2}}} d\omega_d \int_0^\infty e^{-s^2} s^{d-1}\mathrm{d}s \\ &= \frac{1}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-|y|^2}\mathrm{d}y = 1. \end{aligned} \tag{5.2.2}$$

For bounded and continuous $f : \mathbb{R}^d \to \mathbb{R}$, we consider the convolution

$$u(x,t) = \int_{\mathbb{R}^d} K(x,y,t) f(y)\mathrm{d}y = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\mathrm{d}y. \tag{5.2.3}$$

Lemma 5.2.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be bounded and continuous. Then*

$$u(x,t) = \int_{\mathbb{R}^d} K(x,y,t) f(y)\mathrm{d}y$$

is of class C^∞ on $\mathbb{R}^d \times (0,\infty)$, and it solves the heat equation

$$u_t = \Delta u. \tag{5.2.4}$$

Proof. That u is of class C^∞ follows, by differentiating under the integral (which is permitted by standard theorems), from the C^∞ property of $K(x,y,t)$. Consequently, we also obtain

$$\frac{\partial}{\partial t} u(x,t) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} K(x,y,t) f(y)\mathrm{d}y = \int_{\mathbb{R}^d} \Delta_x K(x,y,t) f(y)\mathrm{d}y = \Delta_x u(x,t).$$

□

Lemma 5.2.2. *Under the assumptions of Lemma 5.2.1, we have for every* $x \in \mathbb{R}^d$

$$\lim_{t \to 0} u(x,t) = f(x).$$

Proof.

$$\begin{aligned}
|f(x) - u(x,t)| &= \left| f(x) - \int_{\mathbb{R}^d} K(x,y,t) f(y) \mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^d} K(x,y,t)(f(x) - f(y)) \mathrm{d}y \right| \text{ with (5.2.2)} \\
&= \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_0^\infty \mathrm{e}^{-\frac{r^2}{4t}} r^{d-1} \int_{S^{d-1}} (f(x) - f(x + r\xi))\, do(\xi)\, \mathrm{d}r \right| \\
&= \left| \frac{1}{\pi^{\frac{d}{2}}} \int_0^\infty \mathrm{e}^{-s^2} s^{d-1} \int_{S^{d-1}} \left(f(x) - f(x + 2\sqrt{t} s\xi) \right) do(\xi)\, \mathrm{d}s \right| \\
&= \left| \cdots \int_0^M \cdots + \cdots \int_M^\infty \cdots \right| \\
&\le \sup_{y \in B(x, 2\sqrt{t}M)} |f(x) - f(y)| + 2 \sup_{\mathbb{R}^d} |f| \frac{d\omega_d}{\pi^{\frac{d}{2}}} \int_M^\infty \mathrm{e}^{-s^2} s^{d-1} \mathrm{d}s.
\end{aligned}$$

Given $\varepsilon > 0$, we first choose M so large that the second summand is less than $\varepsilon/2$, and we then choose $t_0 > 0$ so small that for all t with $0 < t < t_0$, the first summand is less than $\varepsilon/2$ as well. This implies the continuity. □

By (5.2.3), we have thus found a solution of the initial value problem

$$\begin{aligned}
u_t(x,t) - \Delta u(x,t) &= 0 \quad \text{for } x \in \mathbb{R}^d, \quad t > 0, \\
u(x,0) &= f(x),
\end{aligned}$$

for the heat equation. By Theorem 5.1.2 this is the only solution that grows at most exponentially.

According to the physical interpretation, $u(x,t)$ is supposed to describe the evolution in time of the temperature for initial values $f(x)$. We should note, however, that in contrast to physically more realistic theories, we here obtain an infinite propagation speed as for any positive time $t > 0$, the temperature $u(x,t)$ at the point x is influenced by the initial values at all arbitrarily faraway points y, although the strength decays exponentially with the distance $|x - y|$.

In the case where f has compact support K, i.e., $f(x) = 0$ for $x \notin K$, the function from (5.2.3) satisfies

$$|u(x,t)| \le \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{\operatorname{dist}(x,K)^2}{4t}} \int_K |f(y)|\, \mathrm{d}y, \tag{5.2.5}$$

which goes to 0 as $t \to \infty$.

Remark. Equation (5.2.5) yields an explicit exponential rate of convergence!

More generally, one is interested in the initial boundary value problem for the inhomogeneous heat equation.

Let $\Omega \subset \mathbb{R}^d$ be a domain, and let $\varphi \in C^0(\Omega \times [0,\infty))$, $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega \times (0,\infty))$ be given. We wish to find a solution of

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = \varphi(x,t) \quad & \text{in } \Omega \times (0,\infty), \\ u(x,0) = f(x) \quad & \text{in } \Omega, \\ u(x,t) = g(x,t) \quad & \text{for } x \in \partial\Omega, \quad t \in (0,\infty). \end{aligned} \tag{5.2.6}$$

In order for this problem to make sense, one should require a compatibility condition between the initial and the boundary values: $f \in C^0(\bar{\Omega})$, $g \in C^0(\partial\Omega \times [0,\infty))$, and

$$f(x) = g(x,0) \quad \text{for } x \in \partial\Omega. \tag{5.2.7}$$

We want to investigate the connection between this problem and the Dirichlet problem for the Laplace equation, and for that purpose, we consider the case where $\varphi \equiv 0$ and $g(x,t) = g(x)$ is independent of t. For the following consideration whose purpose is to serve as motivation, we assume that $u(x,t)$ is differentiable sufficiently many times up to the boundary. (Of course, this is an issue that will need a more careful study later on.) We then compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \frac{1}{2} u_t^2 &= u_t u_{tt} - u_t \Delta u_t - \sum_{i=1}^d u_{x^i t}^2 \\ &= u_t \frac{\partial}{\partial t}(u_t - \Delta u) - \sum_{i=1}^d u_{x^i t}^2 \\ &= -\sum_{i=1}^d u_{x^i t}^2 \leq 0. \end{aligned} \tag{5.2.8}$$

According to Theorem 5.1.1,

$$v(t) := \sup_{x \in \Omega} \left| \frac{\partial u(x,t)}{\partial t} \right|^2$$

then is a nonincreasing function of t.

We now consider

$$E(u(\cdot,t)) = \frac{1}{2} \int_\Omega \sum_{i=1}^d u_{x^i}^2 \, \mathrm{d}x$$

and compute

$$\begin{aligned}\frac{\partial}{\partial t}E(u(\cdot,t)) &= \int_\Omega \sum_{i=1}^d u_{tx^i}u_{x^i}\,\mathrm{d}x\\ &= -\int_\Omega u_t\,\Delta u\,\mathrm{d}x,\ \text{since } u_t(x,t)=\frac{\partial}{\partial t}g(x)=0 \quad \text{for } x\in\partial\Omega\\ &= -\int_\Omega u_t^2\,\mathrm{d}x \le 0. \end{aligned} \tag{5.2.9}$$

With (5.2.8), we then conclude that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}E(u(\cdot,t)) &= -\int_\Omega \frac{\partial}{\partial t}u_t^2\,\mathrm{d}x = -\int_\Omega \Delta u_t^2\,\mathrm{d}x + 2\int_\Omega \sum_{i=1}^d u_{x^i t}^2\,\mathrm{d}x\\ &= -\int_{\partial\Omega}\frac{\partial}{\partial\nu}u_t^2\,do(x) + 2\int_\Omega \sum_{i=1}^d u_{x^i t}^2\,\mathrm{d}x.\end{aligned}$$

Since $u_t^2 \ge 0$ in Ω, $u_t^2 = 0$ on $\partial\Omega$, we have on $\partial\Omega$

$$\frac{\partial}{\partial\nu}u_t^2 \le 0.$$

It follows that

$$\frac{\partial^2}{\partial t^2}E(u(\cdot,t)) \ge 0. \tag{5.2.10}$$

Thus $E(u(\cdot,t))$ is a monotonically nonincreasing and convex function of t. In particular, we obtain

$$\frac{\partial}{\partial t}E(u(\cdot,t)) \le \alpha := \lim_{t\to\infty}\frac{\partial}{\partial t}E(u(\cdot,t)) \le 0. \tag{5.2.11}$$

Since $E(u(\cdot,t)) \ge 0$ for all t, we must have $\alpha = 0$, because otherwise for sufficiently large T,

$$E(u(\cdot,T)) = E(u(\cdot,0)) + \int_0^T \frac{\partial}{\partial t}E(u(\cdot,t))\mathrm{d}t \le E(u(\cdot,0)) + \alpha T < 0.$$

Thus it follows that

$$\lim_{t\to\infty}\int_\Omega u_t^2\,\mathrm{d}x = 0. \tag{5.2.12}$$

In order to get pointwise convergence as well, we have to utilize the maximum principle once more. We extend $u_t^2(x,0)$ from Ω to all of $\mathbb{R}^d$ as a nonnegative, continuous function l with compact support and put

$$v(x,t) := \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} l(y)\mathrm{d}y. \tag{5.2.13}$$

We then have

$$v_t - \Delta v = 0,$$

and since $l \geq 0$, also

$$v \geq 0,$$

and thus in particular

$$v \geq u_t^2 \quad \text{on } \partial\Omega.$$

Thus $w := u_t^2 - v$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} w - \Delta w &\leq 0 \quad \text{in } \Omega, \\ w &\leq 0 \quad \text{on } \partial\Omega, \\ w(x,0) &= 0 \quad \text{for } x \in \Omega, t = 0. \end{aligned} \tag{5.2.14}$$

Theorem 5.1.1 then implies

$$w(x,t) \leq 0,$$

i.e.,

$$u_t^2(x,t) \leq v(x,t) \quad \text{for all } x \in \Omega, t > 0. \tag{5.2.15}$$

Since l has compact support, from Lemma 5.2.2 and (5.2.5),

$$\lim_{t\to\infty} v(x,t) = 0 \quad \text{for all } x \in \Omega,$$

and thus also

$$\lim_{t\to\infty} u_t^2(x,t) = 0 \quad \text{for all } x \in \Omega. \tag{5.2.16}$$

Thus, let our regularity assumptions be valid, and consider a solution of our initial boundary value theorem with boundary values that are constant in time. We conclude that its time derivative goes to 0 as $t \to \infty$. Thus, if we can show that

$u(x,t)$ converges for $t \to \infty$ with respect to x in C^2, the limit function u_∞ needs to satisfy

$$\Delta u_\infty = 0,$$

i.e., be harmonic. If we can even show convergence up to the boundary, then u_∞ satisfies the Dirichlet condition

$$u_\infty(x) = g(x) \quad \text{for } x \in \partial\Omega.$$

From the remark about (5.2.5), we even see that $u_t(x,t)$ converges to 0 exponentially in t.

If we know already that the Dirichlet problem

$$\begin{aligned} \Delta u_\infty &= 0 \quad \text{in } \Omega, \\ u_\infty &= g \quad \text{on } \partial\Omega \end{aligned} \tag{5.2.17}$$

admits a solution, it is easy to show that any solution $u(x,t)$ of the heat equation with appropriate boundary values converges to u_∞. Namely, we even have the following result:

Theorem 5.2.1. *Let Ω be a bounded domain in $\mathbb{R}^d$, and let $g(x,t)$ be continuous on $\partial\Omega \times (0,\infty)$, and suppose*

$$\lim_{t\to\infty} g(x,t) = g(x) \quad \textit{uniformly in } x \in \partial\Omega. \tag{5.2.18}$$

Let $F(x,t)$ be continuous on $\Omega \times (0,\infty)$, and suppose

$$\lim_{t\to\infty} F(x,t) = F(x) \quad \textit{uniformly in } x \in \Omega. \tag{5.2.19}$$

Let $u(x,t)$ be a solution of

$$\begin{aligned} \Delta u(x,t) - \frac{\partial}{\partial t} u(x,t) &= F(x,t) \quad \textit{for } x \in \Omega, \quad 0 < t < \infty, \\ u(x,t) &= g(x,t) \quad \textit{for } x \in \partial\Omega, \quad 0 < t < \infty. \end{aligned} \tag{5.2.20}$$

Let $v(x)$ be a solution of

$$\begin{aligned} \Delta v(x) &= F(x) \quad \textit{for } x \in \Omega, \\ v(x) &= g(x) \quad \textit{for } x \in \partial\Omega. \end{aligned} \tag{5.2.21}$$

We then have

$$\lim_{t\to\infty} u(x,t) = v(x) \quad \textit{uniformly in } x \in \Omega. \tag{5.2.22}$$

Proof. We consider the difference

$$w(x,t) = u(x,t) - v(x). \tag{5.2.23}$$

Then

$$\begin{aligned} \Delta w(x,t) - \frac{\partial}{\partial t} w(x,t) &= F(x,t) - F(x) \quad \text{in } \Omega \times (0,\infty), \\ w(x,t) &= g(x,t) - g(x) \quad \text{in } \partial\Omega \times (0,\infty), \end{aligned} \tag{5.2.24}$$

and the claim follows from the following lemma: □

Lemma 5.2.3. *Let Ω be a bounded domain in $\mathbb{R}^d$, let $\phi(x,t)$ be continuous on $\Omega \times (0,\infty)$, and suppose*

$$\lim_{t\to\infty} \phi(x,t) = 0 \quad \textit{uniformly in } x \in \Omega. \tag{5.2.25}$$

Let $\gamma(x,t)$ be continuous on $\partial\Omega \times (0,\infty)$, and suppose

$$\lim_{t\to\infty} \gamma(x,t) = 0 \quad \textit{uniformly in } x \in \partial\Omega. \tag{5.2.26}$$

Let $w(x,t)$ be a solution of

$$\begin{aligned} \Delta w(x,t) - \frac{\partial}{\partial t} w(x,t) &= \phi(x,t) \quad \textit{in } \Omega \times (0,\infty), \\ w(x,t) &= \gamma(x,t) \quad \textit{in } \partial\Omega \times (0,\infty). \end{aligned} \tag{5.2.27}$$

Then

$$\lim_{t\to\infty} w(x,t) = 0 \quad \textit{uniformly in } x \in \Omega. \tag{5.2.28}$$

Proof. We choose $R > 0$ such that

$$2x^1 < R \quad \text{for all } x = (x^1, \ldots, x^d) \in \Omega, \tag{5.2.29}$$

and consider

$$k(x) := e^R - e^{x^1}. \tag{5.2.30}$$

Then

$$\Delta k = -e^{x^1}.$$

With $\kappa := \inf_{x\in\Omega} e^{x^1}$, we thus have

$$\Delta k \le -\kappa. \tag{5.2.31}$$

We consider, with constants η, c_0, τ to be determined, and with

$$\kappa_0 := \inf_{x\in\Omega} k(x), \quad \kappa_1 := \sup_{x\in\Omega} k(x),$$

the expression

$$m(x,t) := \eta\frac{k(x)}{\kappa} + \eta\frac{k(x)}{\kappa_0} + c_0\frac{k(x)}{\kappa_0}\mathrm{e}^{-\frac{\kappa}{\kappa_1}(t-\tau)} \tag{5.2.32}$$

in $\Omega \times [\tau, \infty)$.

Then

$$\begin{aligned}&\Delta m(x,t) - \frac{\partial}{\partial t}m(x,t)\\ &\quad < -\eta - \eta\frac{\kappa}{\kappa_0} - c_0\frac{\kappa}{\kappa_0}\mathrm{e}^{-\frac{\kappa}{\kappa_1}(t-\tau)} + c_0\frac{\kappa_1}{\kappa_0}\frac{\kappa}{\kappa_1}\mathrm{e}^{-\frac{\kappa}{\kappa_1}(t-\tau)} < -\eta.\end{aligned} \tag{5.2.33}$$

Furthermore,

$$m(x,\tau) > c_0 \quad \text{for } x \in \Omega, \tag{5.2.34}$$

$$m(x,t) > \eta \quad \text{for } (x,t) \in \partial\Omega \times [\tau,\infty). \tag{5.2.35}$$

By our assumptions (5.2.25) and (5.2.26), for every η, there exists some $\tau = \tau(\eta)$ with

$$|\phi(x,t)| < \eta \quad \text{for } x \in \Omega, \quad t \geq \tau, \tag{5.2.36}$$

$$|\gamma(x,t)| < \eta \quad \text{for } x \in \partial\Omega, \quad t \geq \tau. \tag{5.2.37}$$

In (5.2.32) we now put

$$\tau = \tau(\eta), \quad c_0 = \sup_{x\in\Omega} |w(x,\tau)|\,.$$

Then

$$m(x,\tau) \pm w(x,\tau) \geq 0 \qquad \text{for } x \in \Omega \text{ by (5.2.34)},$$

$$m(x,t) \pm w(x,t) \geq 0 \qquad \text{for } x \in \partial\Omega,\ t \geq \tau,$$

by (5.2.35), (5.2.37), and (5.2.27);

$$\left(\Delta - \frac{\partial}{\partial t}\right)(m(x,t) \pm w(x,t)) \leq 0 \qquad \text{for } x \in \Omega,\ t \geq \tau,$$

by (5.2.33), (5.2.36), and (5.2.27).

It follows from Theorem 5.1.1 (observe that it is irrelevant that our functions are defined only on $\Omega \times [\tau, \infty)$ instead of $\Omega \times [0, \infty)$, and initial values are given on $\Omega \times \{\tau\}$) that

$$\begin{aligned} |w(x,t)| &\leq m(x,t) \quad \text{for } x \in \Omega, \quad t > \tau, \\ &\leq \eta \left(\frac{\kappa_1}{\kappa} + \frac{\kappa_1}{\kappa_0} \right) + c_0 \frac{\kappa_1}{\kappa_0} \mathrm{e}^{-\frac{\kappa}{\kappa_1}(t-\tau)}, \end{aligned}$$

and this becomes smaller than any given $\varepsilon > 0$ if $\eta > 0$ from (5.2.36) and (5.2.37) is sufficiently small and $t > \tau(\eta)$ is sufficiently large. □

5.3 The Initial Boundary Value Problem for the Heat Equation

In this section, we wish to study the initial boundary value problem for the inhomogeneous heat equation

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \varphi(x,t) \quad \text{for } x \in \Omega, t > 0, \\ u(x,t) &= g(x,t) \quad \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) &= f(x) \quad \text{for } x \in \Omega, \end{aligned} \tag{5.3.1}$$

with given (continuous and smooth) functions φ, g, f. We shall need some preparations.

Lemma 5.3.1. *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^d$. Then for every $\alpha < \frac{d}{2} + 1$, $T > 0$, there exists a constant $c = c(\alpha, d, \Omega)$ such that for all $x_0, x \in \partial\Omega$, $0 < t \leq T$, letting ν denote the exterior normal of $\partial\Omega$, we have*

$$\left| \frac{\partial K}{\partial \nu_x}(x, x_0, t) \right| \leq c t^{-\alpha} |x - x_0|^{-d+2\alpha}.$$

Proof.

$$\frac{\partial}{\partial \nu_x} K(x, x_0, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \frac{\partial}{\partial \nu_x} \mathrm{e}^{-\frac{|x-x_0|^2}{4t}} = -\frac{1}{(4\pi t)^{\frac{d}{2}}} \frac{(x - x_0) \cdot \nu_x}{2t} \mathrm{e}^{-\frac{|x-x_0|^2}{4t}}.$$

As we are assuming that the boundary of Ω is a manifold of class C^2, and since $x, x_0 \in \partial\Omega$, and ν_x is normal to $\partial\Omega$, we have

$$|(x - x_0) \cdot \nu_x| \leq c_1 |x - x_0|^2$$

with a constant c_1 depending on the geometry of $\partial\Omega$. Thus

$$\left|\frac{\partial}{\partial \nu_x} K(x, x_0, t)\right| \leq c_2 t^{-\frac{d}{2}-1} |x - x_0|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4t}} \tag{5.3.2}$$

with some constant c_2. With a parameter $\beta > 0$, we now consider the function

$$\psi(s) := s^{\beta} \mathrm{e}^{-s} \quad \text{for } s > 0. \tag{5.3.3}$$

Inserting $s = \frac{|x-x_0|^2}{4t}$, $\beta = \frac{d}{2} + 1 - \alpha$, we obtain from (5.3.3)

$$\mathrm{e}^{-\frac{|x-x_0|^2}{4t}} \leq c_3 |x - x_0|^{-d-2+2\alpha} \, t^{\frac{d}{2}+1-\alpha}, \tag{5.3.4}$$

with c_3 depending on β, i.e., on d and α. Inserting (5.3.4) into (5.3.2) yields the assertion. □

Lemma 5.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2 with exterior normal ν, and let $\gamma \in C^0(\partial\Omega \times [0, T])$ $(T > 0)$. We put*

$$v(x, t) := -\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x, y, \tau)\gamma(y, t - \tau) do(y) \mathrm{d}\tau. \tag{5.3.5}$$

We then have

$$\begin{aligned} &v \in C^{\infty}(\Omega \times [0, T]), \\ &v(x, 0) = 0 \quad \textit{for all } x \in \Omega, \end{aligned} \tag{5.3.6}$$

and for all $x_0 \in \partial\Omega$, $0 < t \leq T$,

$$\lim_{x \to x_0} v(x, t) = \frac{\gamma(x_0, t)}{2} - \int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x_0, y, \tau)\gamma(y, t - \tau) do(y) \mathrm{d}\tau. \tag{5.3.7}$$

Here, we require that the convergence of x to x_0 takes place in some cone (of angle smaller than $\pi/2$) about the normal to the boundary.

Proof. First of all, Lemma 5.3.1, with $\alpha = \frac{3}{4}$, implies that the integral in (5.3.5) indeed exists. The C^∞-regularity of v with respect to x then follows from the corresponding regularity of the kernel K by the change of variables $\sigma = t - \tau$. Equation (5.3.6) is obvious as well. It remains to verify the jump relation (5.3.7). For that purpose, it obviously suffices to investigate

$$-\int_0^{\tau_0} \int_{\partial\Omega \cap B(x_0, \delta)} \frac{\partial K}{\partial \nu_y}(x, y, \tau)\gamma(y, t - \tau) do(y) \mathrm{d}\tau \tag{5.3.8}$$

for arbitrarily small $\tau_0 > 0$, $\delta > 0$. In particular, we may assume that δ_0 and τ are chosen such that for any given $\varepsilon > 0$, we have for $y \in \partial\Omega$, $|y - x_0| < \delta$, and $0 \leq \tau < \tau_0$,

$$|\gamma(x_0, t) - \gamma(y, t - \tau)| < \varepsilon.$$

Thus, we shall have an error of magnitude controlled by ε if in place of (5.3.8), we evaluate the integral

$$-\int_0^{\tau_0} \int_{\partial\Omega \cap B(x_0,\delta)} \frac{\partial K}{\partial \nu_y}(x, y, \tau)\gamma(x_0, t)do(y)\mathrm{d}\tau. \tag{5.3.9}$$

Extracting the factor $\gamma(x_0, t)$ it remains to show that

$$-\lim_{x \to x_0} \int_0^{\tau_0} \int_{\partial\Omega \cap B(x_0,\delta)} \frac{\partial K}{\partial \nu_y}(x, y, \tau)do(y)\mathrm{d}\tau = \frac{1}{2} + O(\delta). \tag{5.3.10}$$

Also, we observe that since γ is continuous, it suffices to show that (5.3.10) holds uniformly in x_0 if x approaches $\partial\Omega$ in the direction normal to $\partial\Omega$. In other words, letting $\nu(x_0)$ denote the exterior normal vector of $\partial\Omega$ at x_0, we may assume

$$x = x_0 - \mu\nu(x_0).$$

In that case, $\mu^2 = |x - x_0|^2$, and since $\partial\Omega$ is of class C^2, for $y \in \partial\Omega$,

$$|x - y|^2 = |y - x_0|^2 + \mu^2 + O\left(|y - x_0|^2 \, |x - x_0|\right).$$

The term $O\left(|y - x_0|^2 \, |x - x_0|\right)$ here is a higher-order term that does not influence the validity of our subsequent limit processes, and so we shall omit it in the sequel for the sake of simplicity. Likewise, for $y \in \partial\Omega$,

$$(x - y) \cdot \nu_y = (x - x_0) \cdot \nu_y + (x_0 - y) \cdot \nu_y = -\mu + O\left(|x_0 - y|^2\right),$$

and the term $O(|x_0 - y|^2)$ may be neglected again.

Thus we approximate

$$\frac{\partial K}{\partial \nu_y}(x, y, \tau) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \frac{(x - y) \cdot \nu_y}{2\tau} \mathrm{e}^{-\frac{|x-y|^2}{4\tau}}$$

by

$$\frac{1}{(4\pi\tau)^{\frac{d}{2}}} \frac{(-\mu)}{2\tau} \mathrm{e}^{-\frac{|x_0-y|^2}{4\tau}} \mathrm{e}^{-\frac{\mu^2}{4\tau}}.$$

This means that we need to estimate the expression

$$\int_0^{\tau_0} \int_{\partial\Omega\cap B(x_0,\delta)} \frac{1}{2(4\pi)^{\frac{d}{2}}} \frac{\mu}{\tau^{\frac{d}{2}+1}} \mathrm{e}^{-\frac{|x_0-y|^2}{4\tau}} \mathrm{e}^{-\frac{\mu^2}{4\tau}} do(y)\mathrm{d}\tau.$$

We introduce polar coordinates with center x_0 and put $r = |x_0 - y|$. We then obtain, again up to a higher-order error term,

$$\mu \mathrm{Vol}(S^{d-2}) \frac{1}{2(4\pi)^{\frac{d}{2}}} \int_0^{\tau_0} \frac{1}{\tau^{\frac{d}{2}+1}} \mathrm{e}^{-\frac{\mu^2}{4\tau}} \int_0^{\delta} \mathrm{e}^{-\frac{r^2}{4\tau}} r^{d-2} \mathrm{d}r\, \mathrm{d}\tau,$$

where S^{d-2} is the unit sphere in $\mathbb{R}^{d-1}$

$$= \frac{\mu \mathrm{Vol}(S^{d-2})}{4\pi^{\frac{d}{2}}} \int_0^{\tau_0} \frac{1}{\tau^{\frac{3}{2}}} \mathrm{e}^{-\frac{\mu^2}{4\tau}} \int_0^{\frac{\delta}{2\tau^{\frac{1}{2}}}} \mathrm{e}^{-s^2} s^{d-2} \mathrm{d}s\, \mathrm{d}\tau$$

$$= \frac{\mathrm{Vol}(S^{d-2})}{2\pi^{\frac{d}{2}}} \int_{\frac{\mu^2}{4\tau_0}}^{\infty} \frac{1}{\sigma^{\frac{1}{2}}} \mathrm{e}^{-\sigma} \int_0^{\frac{\delta\sigma^{\frac{1}{2}}}{\mu}} \mathrm{e}^{-s^2} s^{d-2} \mathrm{d}s\, \mathrm{d}\sigma.$$

In this integral we may let μ tend to 0 and obtain as limit

$$\frac{\mathrm{Vol}(S^{d-2})}{2\pi^{\frac{d}{2}}} \int_0^{\infty} \frac{1}{\sigma^{\frac{1}{2}}} \mathrm{e}^{-\sigma} \int_0^{\infty} \mathrm{e}^{-s^2} s^{d-2} \mathrm{d}s\, \mathrm{d}\sigma = \frac{1}{2}. \tag{5.3.11}$$

By our preceding considerations, this implies (5.3.10).

Equation (5.3.11) is shown with the help of the gamma function

$$\Gamma(x) = \int_0^{\infty} \mathrm{e}^{-t} t^{x-1} \mathrm{d}t \quad \text{for } x > 0.$$

We have

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for all } x > 0,$$

and because of $\Gamma(1) = 1$, then

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}.$$

Moreover,

$$\int_0^{\infty} s^n \mathrm{e}^{-s^2} \mathrm{d}s = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) \quad \text{for all } n \in \mathbb{N}.$$

In particular,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} \mathrm{e}^{-s^2} \mathrm{d}s = \sqrt{\pi}$$

and

$$\pi^{\frac{d}{2}} = \int_{\mathbb{R}^d} e^{-|x|^2} dx = \mathrm{Vol}(S^{d-1}) \int_0^\infty e^{-r^2} r^{d-1} dr = \frac{1}{2}\mathrm{Vol}(S^{d-1})\Gamma\left(\frac{d}{2}\right);$$

hence

$$\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

With these formulae, the integral (5.3.11) becomes

$$\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{2\pi^{\frac{d}{2}}} \Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\frac{d-1}{2}\right) = \frac{1}{2}.$$

□

In an analogous manner, one proves the following lemma:

Lemma 5.3.3. *Under the assumptions of Lemma 5.3.2, for*

$$w(x,t) := \int_0^t \int_{\partial\Omega} K(x,y,\tau)\gamma(y,t-\tau)\, do(y)\, d\tau \tag{5.3.12}$$

($x \in \Omega$, $0 \le t \le T$), we have

$$w \in C^\infty(\Omega \times [0,T]),$$
$$w(x,0) = 0 \quad \textit{for } x \in \Omega. \tag{5.3.13}$$

The function w extends continuously to $\bar{\Omega} \times [0,T]$, and for $x_0 \in \partial\Omega$ we have

$$\lim_{x \to x_0} \nabla_x w(x,t) \cdot \nu(x_0) = \frac{\gamma(x_0,t)}{2} + \int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_{x_0}}(x_0,y,\tau)\gamma(y,t-\tau)\, do(y)\, d\tau, \tag{5.3.14}$$

with the same cone condition as before.

We now want to try first to find a solution of

$$\Delta u - \frac{\partial}{\partial t} u = 0 \qquad \text{in } \Omega \times (0,\infty),$$
$$u(x,0) = 0 \qquad \text{for } x \in \Omega,$$
$$u(x,t) = g(x,t) \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{5.3.15}$$

by Lemma 5.3.2.

We try

$$u(x,t) = -\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x,y,t-\tau)\gamma(y,\tau)\, do(y)\,\mathrm{d}\tau, \tag{5.3.16}$$

with a function $\gamma(x,t)$ yet to be determined. As a consequence of (5.3.7), (5.3.15), γ has to satisfy, for $x_0 \in \partial\Omega$,

$$g(x_0,t) = \frac{1}{2}\gamma(x_0,t) - \int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x_0,y,t-\tau)\gamma(y,\tau)\, do(y)\,\mathrm{d}\tau,$$

i.e.,

$$\gamma(x_0,t) = 2g(x_0,t) + 2\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x_0,y,t-\tau)\gamma(y,\tau)\, do(y)\,\mathrm{d}\tau. \tag{5.3.17}$$

This is a fixed-point equation for γ, and one may attempt to solve it by iteration; i.e., for $x_0 \in \partial\Omega$,

$$\begin{aligned}
\gamma_0(x_0,t) &= 2g(x_0,t),\\
\gamma_n(x_0,t) &= 2g(x_0,t) + 2\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_y}(x_0,y,t-\tau)\gamma_{n-1}(y,\tau)\, do(y)\mathrm{d}\tau
\end{aligned}$$

for $n \in \mathbb{N}$. Recursively, we obtain

$$\gamma_n(x_0,t) = 2g(x_0,t) + 2\int_0^t \int_{\partial\Omega} \sum_{\nu=1}^{n} S_\nu(x_0,y,t-\tau)g(y,\tau)\, do(y)\mathrm{d}\tau \tag{5.3.18}$$

with

$$\begin{aligned}
S_1(x_0,y,t) &= 2\frac{\partial K}{\partial \nu_y}(x_0,y,t),\\
S_{\nu+1}(x_0,y,t) &= 2\int_0^t \int_{\partial\Omega} S_\nu(x_0,z,t-\tau)\frac{\partial K}{\partial \nu_y}(z,y,\tau)\, do(z)\,\mathrm{d}\tau.
\end{aligned}$$

In order to show that this iteration indeed yields a solution, we have to verify that the series

$$S(x_0,y,t) = \sum_{\nu=1}^{\infty} S_\nu(x_0,y,t)$$

converges.

Choosing once more $\alpha = \frac{3}{4}$ in Lemma 5.3.1, we obtain

$$|S_1(x_0, y, t)| \le c t^{-3/4} |x_0 - y|^{-(d-1)+\frac{1}{2}}.$$

Iteratively, we get

$$|S_n(x_0, y, t)| \le c_n t^{-1+\frac{n}{4}} |x_0 - y|^{-(d-1)+\frac{n}{2}}.$$

We now choose $n = \max(4, 2(d-1))$ so that both exponents are positive. If now

$$|S_m(x_0, y, t)| \le \beta_m t^{\alpha} \quad \text{for some constant } \beta_m \text{ and some } \alpha \ge 0,$$

then

$$|S_{m+1}(x_0, y, t)| \le c\beta_0\beta_m \int_0^t (t-\tau)^{\alpha} \tau^{-3/4}\, \mathrm{d}\tau,$$

where the constant c comes from Lemma 5.3.1 and

$$\beta_0 := \sup_{y \in \partial\Omega} \int_{\partial\Omega} |z - y|^{-(d-1)+\frac{1}{2}}\; do(z).$$

Furthermore,

$$\int_0^t (t-\tau)^{\alpha} \tau^{-3/4} \mathrm{d}\tau = \frac{\Gamma(1+\alpha)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}+\alpha\right)} t^{\alpha+1/4},$$

where on the right-hand side we have the gamma function introduced above.

Thus

$$|S_{n+\nu}(x_0, y, t)| \le \beta_n (c\beta_0)^{\nu} t^{\alpha+\nu/4} \prod_{\mu=1}^{\nu} \frac{\Gamma\left(\alpha + \frac{3}{4} + \mu/4\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(\alpha + 1 + \mu/4)}.$$

Since the gamma function grows factorially as a function of its arguments, this implies that

$$\sum_{\nu=1}^{\infty} S_{\nu}(x_0, y, t)$$

converges absolutely and uniformly on $\partial\Omega \times \partial\Omega \times [0, T]$ for every $T > 0$. We thus have the following result:

Theorem 5.3.1. *The initial boundary value problem for the heat equation on a bounded domain $\Omega \subset \mathbb{R}^d$ of class C^2, namely,*

$$\begin{aligned}
\Delta u(x,t) - \frac{\partial}{\partial t} u(x,t) &= 0 && \text{in } \Omega \times (0, \infty),\\
u(x, 0) &= 0 && \text{in } \Omega,\\
u(x,t) &= g(x,t) && \text{for } x \in \partial\Omega, \quad t > 0,
\end{aligned}$$

with given continuous g, admits a unique solution. That solution can be represented as

$$u(x,t) = -\int_0^t \int_{\partial\Omega} \Sigma(x,y,t-\tau)g(y,\tau)\, do(y)\, \mathrm{d}\tau, \tag{5.3.19}$$

where

$$\Sigma(x,y,t) = 2\frac{\partial K}{\partial \nu_y}(x,y,t) + 2\int_0^t \int_{\partial\Omega} \frac{\partial K}{\partial \nu_z}(x,z,t-\tau)\sum_{\nu=1}^{\infty} S_\nu(z,y,\tau)\, do(z)\, \mathrm{d}\tau. \tag{5.3.20}$$

Proof. Since the series $\sum_{\nu=1}^{\infty} S_\nu$ converges,

$$\gamma(x_0,t) = 2g(x_0,t) + 2\int_0^t \int_{\partial\Omega} \sum_{\nu=1}^{\infty} S_\nu(x_0,y,t-\tau)g(y,\tau)\, do(y)\, \mathrm{d}\tau$$

is a solution of (5.3.17). Inserting this into (5.3.16), we obtain (5.3.20). Here, one should note that

$$t^{-3/4}\,|y-x|^{-(d-1)+\frac{1}{2}} \sum_{\nu=1}^{\infty} S_\nu(x_0,y,\tau),$$

and hence also $\Sigma(x,y,t)$ converges absolutely and uniformly on $\partial\Omega \times \partial\Omega \times [0,T]$ for every $T > 0$. Thus, we may differentiate term by term under the integral and show that u solves the heat equation. The boundary values are assumed by construction, and it is clear that u vanishes at $t = 0$. Uniqueness follows from Theorem 5.1.1. □

Definition 5.3.1. Let $\Omega \subset \mathbb{R}^d$ be a domain. A function $q(x,y,t)$ that is defined for $x, y \in \bar{\Omega}$, $t > 0$ is called the heat kernel of Ω if:

(i)
$$\left(\Delta_x - \frac{\partial}{\partial t}\right) q(x,y,t) = 0 \quad \text{for } x, y \in \Omega,\ t > 0, \tag{5.3.21}$$

(ii)
$$q(x,y,t) = 0 \quad \text{for } x \in \partial\Omega, \tag{5.3.22}$$

(iii) and for all continuous $f : \Omega \to \mathbb{R}$

$$\lim_{t\to 0} \int_\Omega q(x,y,t) f(x)\mathrm{d}x = f(y) \quad \text{for all } y \in \Omega. \tag{5.3.23}$$

Corollary 5.3.1. *Any bounded domain $\Omega \subset \mathbb{R}^d$ of class C^2 has a heat kernel, and this heat kernel is of class C^1 on $\bar{\Omega}$ with respect to the spatial variables y. The heat kernel is positive in Ω, for all $t > 0$.*

Proof. For each $y \in \Omega$, by Theorem 5.3.1, we solve the boundary value problem for the heat equation with initial values 0 and

$$g(x,t) = -K(x,y,t).$$

The solution is called $\mu(x,y,t)$, and we put

$$q(x,y,t) := K(x,y,t) + \mu(x,y,t). \tag{5.3.24}$$

Obviously, $q(x,y,t)$ satisfies (i) and (ii), and since

$$\lim_{t\to 0} \mu(x,y,t) = 0,$$

and $K(x,y,t)$ satisfies (iii), then so does $q(x,y,t)$.

Lemma 5.3.3 implies that q can be extended to $\bar{\Omega}$ as a continuously differentiable function of the spatial variables.

That $q(x,y,t) > 0$ for all $x, y \in \Omega, t > 0$ follows from the strong maximum principle (Theorem 5.1.3). Namely,

$$\begin{aligned} q(x,y,t) &= 0 \quad \text{for } x \in \partial\Omega, \\ \lim_{t\to 0} q(x,y,t) &= 0 \quad \text{for } x, y \in \Omega, x \neq y, \end{aligned}$$

while (iii) implies

$$q(x,y,t) > 0 \quad \text{if } |x-y| \text{ and } t > 0 \text{ are sufficiently small.}$$

Thus, $q \geq 0$ and $q \not\equiv 0$, and so, by Theorem 5.1.3,

$$q > 0 \quad \text{in } \Omega \times \Omega \times (0,\infty).$$ □

Lemma 5.3.4 (Duhamel principle). *For all functions u, v on $\bar{\Omega} \times [0,T]$ with the appropriate regularity conditions, we have*

$$\begin{aligned} &\int_0^T \int_\Omega \Big\{ v(x,t)\,(\Delta u(x,T-t) + u_t(x,T-t)) \\ &\qquad - u(x,T-t)\,(\Delta v(x,t) - v_t(x,t)) \Big\}\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^T \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial \nu}(y,T-t) v(y,t) - \frac{\partial v}{\partial \nu}(y,t) u(y,T-t) \right\} do(y)\, \mathrm{d}t \\ &\quad + \int_\Omega \{u(x,0)v(x,T) - u(x,T)v(x,0)\}\, \mathrm{d}x. \end{aligned} \tag{5.3.25}$$

Proof. Same as the proof of (5.1.12) □

Corollary 5.3.2. *If the heat kernel $q(z, w, T)$ of Ω is of class C^1 on $\bar{\Omega}$ with respect to the spatial variables, then it is symmetric with respect to z and w, i.e.,*

$$q(z, w, T) = q(w, z, T) \quad \textit{for all } z, w \in \Omega,\ T > 0. \tag{5.3.26}$$

Proof. In (5.3.25), we put $u(x,t) = q(x,z,t)$, $v(x,t) = q(x,w,t)$. The double integrals vanish by properties (i) and (ii) of Definition 5.3.1. Property (iii) of Definition 5.3.1 then yields $v(z,T) = u(w,T)$, which is the asserted symmetry. □

Theorem 5.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2 with heat kernel $q(x, y, t)$ according to Corollary 5.3.1, and let*

$$\varphi \in C^0(\bar{\Omega} \times [0, \infty)), \quad g \in C^0(\partial\Omega \times (0, \infty)), \quad f \in C^0(\Omega).$$

Then the initial boundary value problem

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \varphi(x,t) \quad &&\textit{for } x \in \Omega,\ t > 0,\\ u(x,t) &= g(x,t) &&\textit{for } x \in \partial\Omega,\ t > 0,\\ u(x,0) &= f(x) &&\textit{for } x \in \Omega, \end{aligned} \tag{5.3.27}$$

admits a unique solution that is continuous on $\bar{\Omega} \times [0, \infty) \setminus \partial\Omega \times \{0\}$ and is represented by the formula

$$\begin{aligned} u(x,t) = &\int_0^t \int_\Omega q(x, y, t-\tau)\varphi(y, \tau)\mathrm{d}y\, \mathrm{d}\tau\\ &+ \int_\Omega q(x, y, t) f(y)\mathrm{d}y\\ &- \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x, y, t-\tau) g(y, \tau) do(y)\mathrm{d}\tau. \end{aligned} \tag{5.3.28}$$

Proof. Uniqueness follows from the maximum principle. We split the existence problem into two subproblems.
We solve

$$\begin{aligned} v_t(x,t) - \Delta v(x,t) &= 0 \quad &&\text{for } x \in \Omega, t > 0,\\ v(x,t) &= g(x,t) &&\text{for } x \in \partial\Omega, t > 0,\\ v(x,0) &= f(x) &&\text{for } x \in \Omega, \end{aligned} \tag{5.3.29}$$

i.e., the homogeneous equation with the prescribed initial and boundary conditions, and

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= \varphi(x,t) && \text{for } x \in \Omega, t > 0, \\ w(x,t) &= 0 && \text{for } x \in \partial\Omega, t > 0, \\ w(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned} \tag{5.3.30}$$

i.e., the inhomogeneous equation with vanishing initial and boundary values.

The solution of (5.3.27) is then given by

$$u = v + w.$$

We first address (5.3.29), and we claim that the solution v can be represented as

$$v(x,t) = \int_\Omega q(x,y,t) f(y)\mathrm{d}y - \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau) g(y,\tau) do(y)\mathrm{d}\tau. \tag{5.3.31}$$

The facts that v solves the heat equation and the initial condition $v(x,0) = f(x)$ follow from the corresponding properties of q. Moreover, $q(x,y,t) = K(x,y,t) + \mu(x,y,t)$ with $\mu(x,y,t)$ coming from the proof of Corollary 5.3.1. By Theorem 5.3.1, this μ can be represented as

$$\mu(x,y,t) = \int_0^t \int_{\partial\Omega} \Sigma(x,z,t-\tau) K(z,y,\tau) do(z)\,\mathrm{d}\tau, \tag{5.3.32}$$

and by Lemma 5.3.3, we have for $y \in \partial\Omega$,

$$\frac{\partial \mu}{\partial \nu_y}(x,y,t) = \frac{\Sigma(x,y,t)}{2} + \int_0^t \int_{\partial\Omega} \Sigma(x,z,t-\tau) \frac{\partial K}{\partial \nu_y}(z,y,\tau) do(z)\,\mathrm{d}\tau. \tag{5.3.33}$$

This means that the second integral on the right-hand side of (5.3.31) is precisely of the type (5.3.19), and thus, by the considerations of Theorem 5.3.1, v indeed satisfies the boundary condition $v(x,t) = g(x,t)$ for $x \in \partial\Omega$, because the first integral of (5.3.31) vanishes on the boundary.

We now turn to (5.3.30). For every $\tau > 0$, we let $z(x,t,\tau)$ be the solution of

$$\begin{aligned} z_t(x,t;\tau) - \Delta z(x,t,\tau) &= 0 && \text{for } x \in \Omega, t > \tau, \\ z(x,t;\tau) &= 0 && \text{for } x \in \partial\Omega, t > \tau, \\ z(x,\tau;\tau) &= \varphi(x,\tau) && \text{for } x \in \Omega. \end{aligned} \tag{5.3.34}$$

This is a special case of (5.3.29), which we already know how to solve, except that the initial conditions are not prescribed at $t = 0$, but at $t = \tau$. This case, however, is trivially reduced to the case of initial conditions at $t = 0$ by replacing t by $t - \tau$, i.e., considering $\zeta(x, t; \tau) = z(x, t + \tau; \tau)$. Thus, (5.3.34) can be solved.

We then put

$$w(x, t) = \int_0^t z(x, t; \tau)\mathrm{d}\tau. \tag{5.3.35}$$

Then

$$\begin{aligned} w_t(x, t) &= \int_0^t z_t(x, t; \tau)\mathrm{d}\tau + z(x, t; t) = \int_0^t \Delta z(x, t; \tau)\mathrm{d}\tau + \varphi(x, t) \\ &= \Delta w(x, t) + \varphi(x, t) \end{aligned}$$

and

$$\begin{aligned} w(x, t) &= 0 \quad \text{for } x \in \partial\Omega, t > 0, \\ w(x, 0) &= 0 \quad \text{for } x \in \Omega. \end{aligned}$$

Thus, w is a solution of (5.3.30) as required, and the proof is complete, since the representation formula (5.3.28) follows from the one for v and the one for w that, by (5.3.35), comes from integrating the one for z. The latter in turn solves (5.3.34) and so, by what has been proved already, is given by

$$z(x, t; \tau) = \int_\Omega q(x, y, t - \tau)\varphi(y, \tau)\mathrm{d}y.$$

Thus, inserting this into (5.3.35), we obtain

$$w(x, t) = \int_0^t \int_\Omega q(x, y, t - \tau)\varphi(y, \tau)\mathrm{d}y\, \mathrm{d}\tau. \tag{5.3.36}$$

This completes the proof. □

We briefly interrupt our discussion of the solution of the heat equation and record the following simple result on the heat kernel q for subsequent use:

$$\int_\Omega q(x, y, t)\mathrm{d}y \leq 1 \tag{5.3.37}$$

for all $t \geq 0$. To start, we have

$$\lim_{t \to 0} \int_\Omega q(x, y, t)\mathrm{d}y = 1. \tag{5.3.38}$$

This follows from (5.3.23) with $f \equiv 1$ and the proof of Corollary 5.3.1 which enables to replace the integration w.r.t. x in (5.3.23) by the one w.r.t. y in (5.3.38). Next, we observe that

$$\frac{\partial q}{\partial \nu_y}(x, y, t) \leq 0 \tag{5.3.39}$$

because q is nonnegative in Ω and vanishes on the boundary $\partial\Omega$ (see (5.3.22) and Corollary 5.3.1). We then note that the solution of Theorem 5.3.2 for $\varphi \equiv 1, \quad g(x,t) = t$, and $f(x) = 0$ is given by $u(x,t) = t$. In the representation formula (5.3.28), using (5.3.39), this yields

$$\int_0^t \int_\Omega q(x, y, t-\tau)\mathrm{d}y\, \mathrm{d}\tau \leq t, \tag{5.3.40}$$

from which (5.3.37) is derived upon a little reflection.

We now resume the discussion of the solution established in Theorem 5.3.2. We did not claim continuity of our solution at the corner $\partial\Omega \times \{0\}$, and in general, we cannot expect continuity there unless we assume a matching condition between the initial and the boundary values. We do have, however,

Theorem 5.3.3. *The solution of Theorem 5.3.2 is continuous on all of* $\bar{\Omega} \times [0, \infty)$ *when we have the compatibility condition*

$$g(x, 0) = f(x) \quad \textit{for } x \in \partial\Omega. \tag{5.3.41}$$

Proof. While the continuity at the corner $\partial\Omega \times \{0\}$ could also be established from a refinement of our previous considerations, we provide here some independent and simpler reasoning. By the general superposition argument that we have already employed a few times (in particular in the proof of Theorem 5.3.2), it suffices to establish continuity for a solution of

$$\begin{aligned} v_t(x,t) - \Delta v(x,t) &= 0 && \text{for } x \in \Omega,\ t > 0, \\ v(x,t) &= g(x,t) && \text{for } x \in \partial\Omega,\ t > 0, \\ v(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned} \tag{5.3.42}$$

with a continuous g satisfying

$$g(x,0) = 0 \quad \text{for } x \in \partial\Omega, \tag{5.3.43}$$

and for a solution of

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= 0 && \text{for } x \in \Omega,\ t > 0, \\ w(x,t) &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\ w(x,0) &= f(x) && \text{for } x \in \Omega, \end{aligned} \tag{5.3.44}$$

with a continuous f satisfying

$$f(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{5.3.45}$$

(We leave it to the reader to check the case of a solution of the inhomogeneous equation $u_t(x,t) - \Delta u(x,t) = \varphi(x,t)$ with vanishing initial and boundary values.) To deal with the first case, we consider, for $\tau > 0$,

$$\begin{aligned} \tilde{v}_t(x,t) - \Delta\tilde{v}(x,t) &= 0 && \text{for } x \in \Omega,\ t > 0, \\ \tilde{v}(x,t) &= 0 && \text{for } x \in \partial\Omega,\ 0 < t \le \tau, \\ \tilde{v}(x,t) &= g(x,t-\tau) && \text{for } x \in \partial\Omega,\ t > \tau, \\ \tilde{v}(x,0) &= 0 && \text{for } x \in \Omega. \end{aligned} \tag{5.3.46}$$

Since, by (5.3.43), the boundary values are continuous at $t = \tau$, by the boundary continuity result of Theorem 5.3.2, $\tilde{v}(x,\tau)$ is continuous for $x \in \partial\Omega$. Also, by uniqueness, $\tilde{v}(x,t) = 0$ for $0 \le t \le \tau$, because both the boundary and initial values vanish there. Therefore, again by uniqueness, $v(x,t) = \tilde{v}(x,t+\tau)$, and we conclude the continuity of $v(x,0)$ for $x \in \partial\Omega$.

We can now turn to the second case. We consider some bounded C^2 domain $\tilde{\Omega}$ with $\bar{\Omega} \subset \tilde{\Omega}$. We put $f^+(x) := \max(f(x),0)$ for $x \in \Omega$ and $f(x) = 0$ for $x \in \tilde{\Omega}\setminus\Omega$. Then, because of (5.3.45), f^+ is continuous on $\tilde{\Omega}$. We then solve

$$\begin{aligned} \tilde{w}_t(x,t) - \Delta\tilde{w}(x,t) &= 0 && \text{for } x \in \tilde{\Omega},\ t > 0, \\ \tilde{w}(x,t) &= 0 && \text{for } x \in \partial\tilde{\Omega},\ t > 0, \\ \tilde{w}(x,0) &= f^+(x) && \text{for } x \in \tilde{\Omega}. \end{aligned} \tag{5.3.47}$$

By the continuity result of Theorem 5.3.2, $\tilde{w}(x,0)$ is continuous for $x \in \tilde{\Omega}$ and therefore in particular for $x \in \partial\Omega$. Since $f^+(x) = 0$ for $x \in \partial\Omega$, $\tilde{w}(x,t) \to 0$ for $x \in \partial\Omega$ and $t \to 0$. Since the initial values of $\tilde{w}$ are nonnegative, $\tilde{w}(x,t) \ge 0$ for all $x \in \tilde{\Omega}$ and $t \ge 0$ by the maximum principle (Theorem 5.1.1). In particular, $\tilde{w}(x,t) \ge w(x,t)$ for $x \in \partial\Omega$ since $w(x,t) = 0$ there. Since also $\tilde{w}(x,0) = f^+(x) \ge f(x) = w(x,0)$, the maximum principle implies $\tilde{w}(x,t) \ge w(x,t)$ for all $x \in \bar{\Omega}, t \ge 0$. Altogether, $w(x,0) \le 0$ for $x \in \partial\Omega$. Doing the same reasoning with $f^-(x) := \min(f(x),0)$, we conclude that also $w(x,0) \ge 0$ for $x \in \partial\Omega$, i.e., altogether, $w(x,0) = 0$ for $x \in \partial\Omega$. This completes the proof. □

Remark. Theorem 5.3.2 does not claim that u is twice differentiable with respect to x, and in fact, this need not be true for a φ that is merely continuous. However, one may still justify the equation

$$u_t(x,t) - \Delta u(x,t) = \varphi(x,t).$$

We shall return to the analogous issue in the elliptic case in Sects. 12.1 and 13.1. In Sect. 13.1, we shall verify that u is twice continuously differentiable with respect to x if we assume that φ is Hölder continuous.

Here, we shall now concentrate on the case $\varphi = 0$ and address the regularity issue both in the interior of Ω and at its boundary. We recall the representation formula (5.1.14) for a solution of the heat equation on Ω,

$$\begin{aligned} u(x,t) = &\int_{\Omega} K(x,y,t)u(y,0)\,\mathrm{d}y \\ &+ \int_0^t \int_{\partial\Omega} \left(K(x,y,t-\tau)\frac{\partial u(y,\tau)}{\partial \nu} \right. \\ &\left. - \frac{\partial K}{\partial \nu_y}(x,y,t-\tau)u(y,\tau) \right) do(y)\,\mathrm{d}\tau. \end{aligned} \tag{5.3.48}$$

We put $K(x,y,s) = 0$ for $s \le 0$ and may then integrate the second integral from 0 to ∞ instead of from 0 to t. Then $K(x,y,s)$ is of class C^∞ for $x, y \in \mathbb{R}^d$, $s \in \mathbb{R}$, except at $x = y, s = 0$. We thus have the following theorem:

Theorem 5.3.4. *Any solution $u(x,t)$ of the heat equation in a domain Ω is of class C^∞ with respect to $x \in \Omega$, $t > 0$.*

Proof. Since we do not know whether the normal derivative $\frac{\partial u}{\partial \nu}$ exists on $\partial\Omega$ and is continuous there, we cannot apply (5.3.48) directly. Instead, for given $x \in \Omega$, we consider some ball $B(x,r)$ contained in Ω. We then apply (5.3.48) on $\mathring{B}(x,r)$ in place of Ω. Since $\partial B(x,r)$ in Ω is contained in Ω, and u as a solution of the heat equation is of class C^1 there, the normal derivative $\frac{\partial u}{\partial \nu}$ on $\partial B(x,r)$ causes no problem, and the assertion is obtained. □

In particular, the heat kernel $q(x,y,t)$ of a bounded C^2-domain Ω is of class C^∞ with respect to $x, y \in \Omega$, $t > 0$. This also follows directly from (5.3.24), (5.3.32), and (5.3.20) and the regularity properties of $\Sigma(x,y,t)$ established in Theorem 5.3.1. From these solutions it also follows that $\frac{\partial q}{\partial \nu_y}(x,y,t)$ for $y \in \partial\Omega$ is of class C^∞ with respect to $x \in \Omega$, $t > 0$. Thus, one can also use the representation formula (5.3.28) for deriving regularity properties. Putting $q(x,y,s) = 0$ for $s < 0$, we may again extend the second integral in (5.3.28) from 0 to ∞, and we then obtain by integrating by parts, assuming that the boundary values are differentiable with respect to t,

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) = &\int_{\Omega} \frac{\partial}{\partial t} q(x,y,t) f(y)\mathrm{d}y \\ &- \int_0^\infty \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau)\frac{\partial}{\partial \tau} g(y,\tau)\,do(y)\,\mathrm{d}\tau \\ &+ \lim_{\tau\to 0} \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau) g(y,\tau)\,do(y). \end{aligned} \tag{5.3.49}$$

Since $q(x, y, t) = 0$ for $x \in \partial\Omega$, $y \in \Omega$, $t > 0$, also $\frac{\partial q}{\partial \nu_y}(x, y, t - \tau) = 0$ for $x, y \in \partial\Omega, \tau < t$, and

$$\frac{\partial}{\partial t} q(x, y, t) = 0 \quad \text{for } x \in \partial\Omega,\ y \in \Omega,\ t > 0 \tag{5.3.50}$$

(passing to the limit here is again justified by (5.3.32)). Since the second integral in (5.3.49) has boundary values $\frac{\partial}{\partial t} g(x, t)$, we thus have the following result:

Lemma 5.3.5. *Let u be a solution of the heat equation on the bounded C^2-domain Ω with continuous boundary values $g(x, t)$ that are differentiable with respect to t. Then u is also differentiable with respect to t, for $x \in \partial\Omega$, $t > 0$, and we have*

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial}{\partial t} g(x, t) \quad \textit{for } x \in \partial\Omega,\ t > 0. \tag{5.3.51}$$

We are now in position to establish the connection between the heat and Laplace equation rigorously that we had arrived at from heuristic considerations in Sect. 5.2.

Theorem 5.3.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2, and let $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$. Let u be the solution of Theorem 5.3.2 of the initial boundary value problem:*

$$\begin{aligned} \Delta u(x, t) - u_t(x, t) &= 0 && \textit{for } x \in \Omega,\ t > 0, \\ u(x, 0) &= f(x) && \textit{for } x \in \Omega, \\ u(x, t) &= g(x) && \textit{for } x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{5.3.52}$$

Then u converges for $t \to \infty$ uniformly on $\bar{\Omega}$ towards a solution of the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta u(x) &= 0 && \textit{for } x \in \Omega, \\ u(x) &= g(x) && \textit{for } x \in \partial\Omega. \end{aligned} \tag{5.3.53}$$

Proof. We write $u(x, t) = u^1(x, t) + u^2(x, t)$, where u^1 and u^2 both solve the heat equation, and u^1 has the correct initial values, i.e.,

$$u^1(x, 0) = f(x) \quad \text{for } x \in \Omega,$$

while u^2 has the correct boundary values, i.e.,

$$u^2(x, t) = g(x) \quad \text{for } x \in \partial\Omega,\ t > 0,$$

as well as

$$\begin{aligned} u^1(x, t) &= 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \\ u^2(x, 0) &= 0 \quad \text{for } x \in \Omega. \end{aligned}$$

By Lemma 5.2.3, we have

$$\lim_{t\to\infty} u^1(x,t) = 0.$$

Thus, the initial values f are irrelevant, and we may assume without loss of generality that $f \equiv 0$, i.e., $u = u^2$.

One easily sees that $q(x, y, t) > 0$ for $x, y \in \Omega$, because $q(x, y, t) = 0$ for all $x \in \partial\Omega$, and by (iii) of Definition 5.3.1, $q(x, y, t) > 0$ for $x, y \in \Omega$ and sufficiently small $t > 0$. Since q solves the heat equation, by the strong maximum principle, q then is indeed positive in the interior of Ω for all $t > 0$ (see Corollary 5.3.1).

Therefore, we always have

$$\frac{\partial q}{\partial \nu_y}(x, y, t) \le 0. \tag{5.3.54}$$

Since $q(x, y, t)$ solves the heat equation with vanishing boundary values, Lemma 5.2.3 also implies

$$\lim_{t\to\infty} q(x, y, t) = 0 \text{ uniformly in } \bar{\Omega} \times \bar{\Omega} \tag{5.3.55}$$

(utilizing the symmetry $q(x, y, t) = q(y, x, t)$ from Corollary 5.3.1). We then have for $t_2 > t_1$,

$$\begin{aligned}
|u(x,t_2) - u(x,t_1)| &= \left| \int_{t_1}^{t_2} \int_{\partial\Omega} \frac{\partial q}{\partial \nu_z}(x,z,t) g(z) do(z) \mathrm{d}t \right| \\
&\le \max_{\partial\Omega} |g| \int_{t_1}^{t_2} \int_{\partial\Omega} \left(-\frac{\partial q}{\partial \nu_z}(x,z,t) \right) do(z) \mathrm{d}t \\
&= -\max |g| \int_{t_1}^{t_2} \int_{\Omega} \Delta_y q(x,y,t) \mathrm{d}y \, \mathrm{d}t \\
&= -\max |g| \int_{t_1}^{t_2} \int_{\Omega} q_t(x,y,t) \mathrm{d}y \, \mathrm{d}t \\
&= -\max |g| \int_{\Omega} \{q(x,y,t_2) - q(x,y,t_1)\} \, \mathrm{d}y \\
&\to 0 \quad \text{for } t_1, t_2 \to \infty \text{ by (5.3.55).}
\end{aligned}$$

Thus $u(x, t)$ converges for $t \to \infty$ uniformly towards some limit function $u(x)$ that then also satisfies the boundary condition

$$u(x) = g(x) \quad \text{for } x \in \partial\Omega.$$

Theorem 5.3.2 also implies

$$u(x) = -\int_0^\infty \int_{\partial\Omega} \frac{\partial q}{\partial \nu_z}(x, z, t)g(z)do(z)\mathrm{d}t.$$

We now consider the derivatives $\frac{\partial}{\partial t}u(x,t) =: v(x,t)$. Then $v(x,t)$ is a solution of the heat equation itself, namely, with boundary values $v(x,t) = 0$ for $x \in \partial\Omega$ by Lemma 5.3.5. By Lemma 5.2.3, v then converges uniformly to 0 on $\bar{\Omega}$ for $t \to \infty$. Therefore, $\Delta u(x,t)$ converges uniformly to 0 in $\bar{\Omega}$ for $t \to \infty$, too. Thus, we must have

$$\Delta u(x) = 0.$$

□

As a consequence of Theorem 5.3.5, we obtain a new proof for the solvability of the Dirichlet problem for the Laplace equation on bounded domains of class C^2, i.e., a special case of Theorem 4.2.2 (together with Lemma 4.4.1):

Corollary 5.3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2, and let $g : \partial\Omega \to \mathbb{R}$ be continuous. Then the Dirichlet problem*

$$\Delta u(x) = 0 \qquad \text{for } x \in \Omega, \tag{5.3.56}$$

$$u(x) = g(x) \quad \text{for } x \in \partial\Omega, \tag{5.3.57}$$

admits a solution that is unique by the maximum principle.

References for this section are Chavel [4] and the sources given there.

5.4 Discrete Methods

Both for the heuristics and for numerical purposes, it can be useful to discretize the heat equation. For that, we shall proceed as in Sect. 4.1 and also keep the notation of that section. In addition to the spatial variables, we also need to discretize the time variable t; the corresponding step size will be denoted by k. It will turn out to be best to choose k different from the spatial grid size h.

The discretization of the heat equation

$$u_t(x,t) = \Delta u(x,t) \tag{5.4.1}$$

is now straightforward:

$$\begin{aligned}
&\frac{1}{k}\left(u^{h,k}(x, t+k) - u^{h,k}(x,t)\right) \\
&\quad = \Delta_h u^{h,k}(x,t) \\
&\quad = \frac{1}{h}^2 \sum_{i=1}^{d} \Big\{ u^{h,k}\left(x^1, \dots, x^{i-1}, x^i + h, x^{i+1}, \dots, x^d, t\right) \\
&\qquad - 2u^{h,k}\left(x^1, \dots, x^d, t\right) + u^{h,k}\left(x^1, \dots, x^i - h, \dots, x^d, t\right) \Big\}.
\end{aligned} \tag{5.4.2}$$

Thus, for discretizing the time derivative, we have selected a forward difference quotient. In order to simplify the notation, we shall mostly write u in place of $u^{h,k}$. Choosing

$$h^2 = 2dk, \tag{5.4.3}$$

the term $u(x,t)$ drops out, and (5.4.2) becomes

$$\begin{aligned} u(x,t+k) = \frac{1}{2d}\sum_{i=1}^{d}\big(&u\big(x^1,\dots,x^i+h,\dots,x^d,t\big) \\ &+u\big(x^1,\dots,x^i-h,\dots,x^d,t\big)\big). \end{aligned} \tag{5.4.4}$$

This means that $u(x,t+k)$ is the arithmetic mean of the values of u at the $2d$ spatial neighbors of (x,t). From this observation, one sees that if the process stabilizes as time grows, one obtains a solution of the discretized Laplace equation asymptotically as in the continuous case.

It is possible to prove convergence results as in Sect. 4.1. Here, however, we shall not carry this out. We wish to remark, however, that the process can become unstable if $h^2 < 2dk$. The reader may try to find some examples. This means that if one wishes h to be small so as to guarantee accuracy of the approximation with respect to the spatial variables, then k has to be extremely small to guarantee stability of the scheme. This makes the scheme impractical for numerical use.

The mean value property of (5.4.4) also suggests the following semidiscrete approximation of the heat equation: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For $\varepsilon > 0$, we put $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Let a continuous function $g : \partial\Omega \to \mathbb{R}$ be given, with a continuous extension to $\bar{\Omega} \setminus \Omega_\varepsilon$, again denoted by g. Finally, let initial values $f : \Omega \to \mathbb{R}$ be given. We put iteratively

$$\begin{aligned} \tilde{u}(x,0) &= f(x) && \text{for } x \in \Omega, \\ \tilde{u}(x,0) &= 0 && \text{for } x \in \mathbb{R}^d \setminus \Omega, \\ u(x,nk) &= \frac{1}{\omega_d \varepsilon^d}\int_{B(x,\varepsilon)} \tilde{u}(y,(n-1)k)\,\mathrm{d}y && \text{for } x \in \Omega, n \in \mathbb{N}, \end{aligned}$$

and

$$\tilde{u}(x,nk) = \begin{cases} u(x,nk) & \text{for } x \in \Omega_\varepsilon, \\ g(x) & \text{for } x \in \mathbb{R}^d \setminus \Omega_\varepsilon, \end{cases} \qquad n \in \mathbb{N}.$$

Thus, in the nth step, the value of the function at $x \in \Omega_\varepsilon$ is obtained as the mean of the values of the preceding step of the ball $B(x,\varepsilon)$. A solution that is time independent then satisfies a mean value property and thus is harmonic in Ω_ε according to the remark after Corollary 2.2.5.

Summary

In this chapter we have investigated the heat equation on a domain $\Omega \in \mathbb{R}^d$:

$$\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = 0 \quad \text{for } x \in \Omega, t > 0.$$

We prescribed initial values

$$u(x,0) = f(x) \quad \text{for } x \in \Omega,$$

and, in the case that Ω has a boundary $\partial\Omega$, also boundary values

$$u(y,t) = g(y,t) \quad \text{for } y \in \partial\Omega, t \geq 0.$$

In particular, we studied the Euclidean fundamental solution

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4t}},$$

and we obtained the solution of the initial value problem on $\mathbb{R}^d$ by convolution:

$$u(x,t) = \int_{\mathbb{R}^d} K(x,y,t) f(y) \mathrm{d}y.$$

If Ω is a bounded domain of class C^2, we established the existence of the heat kernel $q(x,y,t)$, and we solved the initial boundary value problem by the formula

$$u(x,t) = \int_{\Omega} q(x,y,t) f(y) \mathrm{d}y - \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_z}(x,z,t-\tau) g(z,\tau) do(z) \mathrm{d}\tau.$$

In particular, $u(x,t)$ is of class C^∞ for $x \in \Omega$, $t > 0$ because of the corresponding regularity properties of the kernel $q(x,y,t)$. The solutions satisfy a maximum principle saying that a maximum or minimum can be assumed only on $\Omega \times \{0\}$ or on $\partial\Omega \times [0,\infty)$ unless the solution is constant. Consequently, solutions are unique. If the boundary values $g(y)$ do not depend on t, then $u(x,t)$ converges for $t \to \infty$ towards a solution of the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta u(x) &= 0 \quad && \text{in } \Omega, \\ u(x) &= g(x) \quad && \text{for } x \in \partial\Omega. \end{aligned}$$

This yields a new existence proof for that problem, although requiring stronger assumptions for the domain Ω when compared with the existence proof of Chap. 4. The present proof, on the other hand, is more constructive in the sense of giving an explicit prescription for how to reach a harmonic state from some given state f.

Exercises

5.1. Let $\Omega \subset \mathbb{R}^d$ be bounded, $\Omega_T := \Omega \times (0, T)$. Let

$$L := \sum_{i,j=1}^{d} a^{ij}(x,t)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x,t)\frac{\partial}{\partial x^i}$$

be elliptic for all $(x,t) \in \Omega_T$, and suppose

$$u_t \leq Lu,$$

where $u \in C^0(\bar{\Omega}_T)$ is twice continuously differentiable with respect to $x \in \Omega$ and once with respect to $t \in (0, T)$.

Show that

$$\sup_{\Omega_T} u = \sup_{\partial^* \Omega_T} u.$$

5.2. Using the heat kernel $\Lambda(x, y, t, 0) = K(x, y, t)$, derive a representation formula for solutions of the heat equation on Ω_T with a bounded $\Omega \subset \mathbb{R}^d$ and $T < \infty$.

5.3. Show that for K as in Exercise 5.2,

$$K(x, 0, s+t) = \int_{\mathbb{R}^d} K(x, y, t)K(y, 0, s)\mathrm{d}y$$

(a) If $s, t > 0$
(b) If $0 < t < -s$

5.4. Let Σ be the grid consisting of the points (x,t) with $x = nh$, $t = mk$, $n, m \in \mathbb{Z}$, $m \geq 0$, and let v be the solution of the discrete heat equation

$$\frac{v(x, t+k) - v(x,t)}{k} - \frac{v(x+h, t) - 2v(x,t) + v(x-h, t)}{h^2} = 0$$

with $v(x, 0) = f(x) \in C^0(\mathbb{R})$.

Show that for $\frac{k}{h^2} = \frac{1}{2}$,

$$v(nh, mk) = 2^{-m}\sum_{j=0}^{m}\binom{m}{j} f((n - m + 2j)h).$$

Conclude from this that

$$\sup_{\Sigma} |v| \leq \sup_{\mathbb{R}} |f|.$$

5.5. Use the method of Sect. 5.3 to obtain a solution of the Poisson equation on $\Omega \subset \mathbb{R}^d$, a bounded domain of class C^2, continuous boundary values $g : \partial\Omega \to \mathbb{R}$, and continuous right-hand side $\varphi : \Omega \to \mathbb{R}$, i.e., of

$$\begin{aligned} \Delta u(x) &= \varphi(x) \quad \text{for } x \in \Omega, \\ u(x) &= g(x) \quad \text{for } x \in \partial\Omega. \end{aligned}$$

(For the regularity issue, we need to refer to Sect. 13.1.)