Chapter 7
Hyperbolic Equations

7.1 The One-Dimensional Wave Equation and the Transport Equation

The basic prototype of a hyperbolic PDE is the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - \Delta u(x,t) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^d,\ t \in (0,\infty),\ \text{or } t \in \mathbb{R}. \tag{7.1.1}$$

This is a linear second-order PDE, like the Laplace and heat equations. As with the heat equation, we consider t as time and x as a spatial variable. In this introductory section, we consider the case where the spatial variable x is one-dimensional. We then write the wave equation as

$$u_{tt}(x,t) - u_{xx}(x,t) = 0. \tag{7.1.2}$$

We are going to reduce (7.1.2) to two first-order equations, called transport equations. Let $\varphi, \psi \in C^2(\mathbb{R})$. Then

$$u(x,t) = \varphi(x+t) + \psi(x-t) \tag{7.1.3}$$

obviously solves (7.1.2).

This simple fact already leads to the important observation that in contrast to the heat equation, solutions of the wave equation need not be more regular for $t > 0$ than they are at $t = 0$. In particular, they are not necessarily of class C^∞. We shall have more to say about that issue, but right now we first wish to motivate (7.1.3): $\varphi(x+t)$ solves

$$\varphi_t - \varphi_x = 0, \tag{7.1.4}$$

$\psi(x-t)$ solves

$$\psi_t + \psi_x = 0, \tag{7.1.5}$$

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and the wave operator

$$L := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \tag{7.1.6}$$

can be written as

$$L = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right), \tag{7.1.7}$$

i.e., as the product of the two operators occurring in (7.1.4) and (7.1.5). This suggests the transformation of variables

$$\xi = x + t, \quad \eta = x - t. \tag{7.1.8}$$

The wave equation (7.1.2) then becomes

$$u_{\xi\eta}(\xi, \eta) = 0, \tag{7.1.9}$$

and for a solution, u_ξ has to be independent of η, i.e.,

$$u_\xi = \varphi'(\xi) \quad \text{(where “ $'$ ” denotes a derivative as usual),}$$

and consequently,

$$u = \int \varphi'(\xi) + \psi(\eta) = \varphi(\xi) + \psi(\eta). \tag{7.1.10}$$

Thus, (7.1.3) actually is the most general solution of the wave equation (7.1.2).

Remark 7.1.1. Equations (7.1.4) and (7.1.5) are formally quite similar to the Cauchy–Riemann equations (2.1.5). Likewise, the decomposition (7.1.7) is analogous to (2.1.8). In fact, when putting $t = iy$, the two decompositions are the same. From an analytical perspective, however, this similarity is deceptive, as the properties of the corresponding solutions, and hence of solutions of the Laplace and wave equations, resp., are very different, as we shall now further explore.

Since the solution (7.1.3) contains two arbitrary functions, we may prescribe two data at $t = 0$, namely, initial values and initial derivatives, again in contrast to the heat equation, where only initial values could be prescribed. From the initial conditions

$$\begin{aligned} u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x), \end{aligned} \tag{7.1.11}$$

we obtain

$$\begin{aligned} \varphi(x) + \psi(x) &= f(x), \\ \varphi'(x) - \psi'(x) &= g(x), \end{aligned} \tag{7.1.12}$$

and thus

$$\varphi(x) = \frac{f(x)}{2} + \frac{1}{2}\int_0^x g(y)\mathrm{d}y + c,$$
$$\psi(x) = \frac{f(x)}{2} - \frac{1}{2}\int_0^x g(y)\mathrm{d}y - c \tag{7.1.13}$$

with some constant c. Hence we have the following theorem:

Theorem 7.1.1. *The solution of the initial value problem*

$$u_{tt}(x,t) - u_{xx}(x,t) = 0 \quad \text{for } x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$

is given by

$$u(x,t) = \varphi(x+t) + \psi(x-t)$$
$$= \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\mathrm{d}y. \tag{7.1.14}$$

(For u to be of class C^2, we need to require $f \in C^2$, $g \in C^1$.)

The representation formula (7.1.14) leads to a couple of observations:

1. We can explicitly determine the value of the solution g at any time t from its initial data at time 0. In fact, the value of $g(x,t)$ depends not only on the values of χ at the two points $\xi_\pm = x \pm t$ but also on the values of θ inside the interval between ξ_- and ξ_+. Remarkably, the values outside that interval play no role for the value at (x,t).
2. Conversely, the value of χ at some point ξ influences values of the solution g at time t only at the two points $x = \xi \pm t$. Likewise, the values of θ at ξ play a role for the value at time t only in the interval $[\xi - t, \xi + t]$. That means that the effect of the initial data is propagated only inside a wedge with slope 1. In other words, the propagation speed for the effect of initial data is 1, hence in particular finite. This is in stark contrast to the heat equation where the representation formula (5.1.11) tells us that the solution at any time t and at any point x is influenced by the initial values at all places. Therefore, in this sense, the heat equation leads to infinite propagation speed, which clearly is a physical idealization.

   We should point out here that, however, as we shall see in Sect. 7.4, the dependence of the solution of the wave equation for an even number of space dimensions is different from the one in odd dimensions. Thus, the phenomenon just analyzed for the one-dimensional wave equation does not hold in the same manner for even dimensions.

3. The representation formula (7.1.14) does not require any assumptions on the differentiability of χ and θ, or on ϕ and ψ, in fact, not even their continuity. At first glance, this might look like an oddity or, worse, seem to be a problem, since a function that is not differentiable cannot claim any right for being a solution of a differential equation. It will turn out, however, that it is advantageous to take a more positive look at this issue. In fact, there are many instances of differential equations where we cannot find a differentiable solution. This is particularly relevant when one wishes to understand the formation of singularities where any kind of regular behavior of a solution, like differentiability, breaks down. In many such cases, however, it is still possible to define some notion of generalized solution. Such a generalized solution need not necessarily be differentiable or even continuous. The key point, however, is that it satisfies some relation that a differentiable solution, often called a classical solution in this context, also necessarily would have to satisfy if it existed. That relation, like (7.1.14), is formulated in such a way that it continues to be meaningful for nondifferentiable functions. A function that satisfies such a relation then is called a generalized solution. By this simple device, we have extended our concept of the solution of a differential equation. Since the class of solutions thereby has become larger, it should be correspondingly easier to prove the existence of a solution. This may now appear as a cheap trick, like changing a problem that one wishes to solve, but cannot, to an easier one. This misses the point, however. Often, there is a reason underlying the model, like the formation of a singularity, that prevents the existence of a classical solution, whereas there should still exist some kind of generalized solution . In other cases, a successful strategy for finding a classical solution might consist in first showing the existence of a generalized solution and then proving that such a generalized solution has to be differentiable after all, and therefore a classical solution. This will be a guiding scheme in subsequent chapters where we shall go more deeply into the existence and regularity for elliptic PDEs. Actually, this is what much of the modern theory of PDEs is about.
4. In any case, the representation formula (7.1.14) provides a solution of the wave equation also for non-smooth initial data χ, θ. So, we can solve the wave equation for not necessarily smooth initial data, as we could do for the heat equation by the representation formula (5.1.11). In contrast to the latter, however, which produced a solution that was smooth for $t > 0$ regardless of the initial data, for the wave equation, the solution is not any more regular than the initial values χ. That is, for non-smooth initial data, we also obtain a non-smooth solution only.

7.2 First-Order Hyperbolic Equations

In this section, we generalize the transport equation that we have found in Sect. 7.1.

We start, however, differently, namely, with the system of ordinary differential equations

$$\dot{x}^i(t) = f^i(t, x(t)) \quad \text{for } i = 1, \ldots, d. \tag{7.2.1}$$

We shall often use a vector notation

$$\dot{x}(t) = f(t, x(t)) \tag{7.2.2}$$

with $x = (x^1, \dots, x^d)$ and analogously for f. We shall assume

$$|f(t,x)| \le c_1(1 + |x|) \tag{7.2.3}$$

$$|f(t,x) - f(t,y)| \le c_2|x - y| \quad \text{for all } t \in \mathbb{R}, x, y \in \mathbb{R}^d, \tag{7.2.4}$$

for some constants c_1, c_2, i.e., at most linear growth and Lipschitz continuity of the right-hand side of our equation. Under these conditions, we can use the Picard–Lindelöf theorem to obtain a solution of (7.2.1) for all t for any given initial values $x(0)$.

We now consider the first-order partial differential equation

$$\frac{\partial}{\partial t} h(x,t) + \sum_{i=1}^{d} f^i(t,x) \frac{\partial h(x,t)}{\partial x^i} = 0 \tag{7.2.5}$$

with prescribed initial values $h(x, 0)$. For the special case $d = 1$, $f = 1$, this is the transport equation encountered in Sect. 7.1.

In order to study (7.2.5), we consider the characteristic equation

$$\begin{aligned} Y_t(t,x) &= f(t, Y(t,x)) \\ Y(0,x) &= x. \end{aligned} \tag{7.2.6}$$

This equation is the same as (7.2.1). The method of characteristics reduces a partial differential equation like (7.2.5) to a system of ordinary differential equations of the form (7.2.1).

Equation (7.2.6) can be solved because of (7.2.3) and (7.2.4). For initial values $h(x, 0)$ of class C^1, there then exists a unique solution $h(x,t)$ of (7.2.5) of class C^1 which is characterized by the property that it is constant along characteristics, i.e.,

$$h(Y(t,x),t) = h(x,0) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^d. \tag{7.2.7}$$

To see this, we simply compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} h(Y(t,x),t) &= \frac{\partial}{\partial t} h(Y(t,x),t) + \sum_{i=1}^{d} Y_t^i(t,x) \frac{\partial}{\partial x^i} h(Y(t,x),t) \\ &= \frac{\partial}{\partial t} h(Y(t,x),t) + \sum_{i=1}^{d} f^i(t, Y(t,x)) \frac{\partial}{\partial x^i} h(Y(t,x),t). \end{aligned}$$

Thus, (7.2.5) is satisfied if $h(Y(t,x),t)$ is independent of t, and the initial condition then yields (7.2.7).

When we look at (7.2.5) as a PDE, we see from the method of characteristics that we can solve it for general initial values that are prescribed at some hypersurface that is transversal to the characteristic curves. That means that we can consider some hypersurface $u(\eta), t(\eta)$ for $\eta \in \mathbb{R}^d$ that is nowhere tangential to the characteristic curves $Y(t, x)$ and prescribe that

$$h(x(\eta), t(\eta)) = h_0(\eta) \tag{7.2.8}$$

for some function h_0. We then simply extend h by being constant along the characteristic curve through $x(\eta), t(\eta)$. Of course, since h has to be constant along characteristics, we can prescribe only one value on each characteristic, and this then leads to the condition that the hypersurface along which we prescribe initial values has to be noncharacteristic, i.e., nowhere tangent to a characteristic curve.

We now consider a somewhat different equation that will come up in Sects. 8.2 and 9.1 below, as a so-called continuity equation:

$$\frac{\partial}{\partial t} h(x,t) = \sum_{i=1}^{d} \frac{\partial}{\partial x^i}(-f^i(t,x)h(x,t)). \tag{7.2.9}$$

We rewrite (7.2.9) as

$$\frac{\partial}{\partial t} h(x,t) + \sum_{i=1}^{d} f^i(t,x)\frac{\partial h(x,t)}{\partial x^i} + \sum_{i=1}^{d} \frac{\partial f^i(t,x)}{\partial x^i} h(x,t) = 0, \tag{7.2.10}$$

in order to treat it as an extension of (7.2.5). We let $Z(t, x)$ be the solution of

$$Z_t(t,x) = \frac{\partial f^i(t,x)}{\partial x^i} Z(t,x) \tag{7.2.11}$$

$$Z(0,x) = 1. \tag{7.2.12}$$

The solution of (7.2.10), i.e., of (7.2.9), is determined by

$$h(Y(t,x),t)Z(t,x) = h(x,0) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{7.2.13}$$

that is, by an extension of (7.2.7).

We now consider a first-order PDE that is more general than (7.2.9) or (7.2.5) and that exhibits some new phenomena. We cannot present all details here, but wish to provide at least some understanding of the perspicuous phenomena. For a detailed textbook treatment of first-order hyperbolic equations, we refer to [7, 14]. Our PDE is

$$\frac{\partial}{\partial t} h(x,t) + \sum_{i=1}^{d} F^i(t,x,h)\frac{\partial h(x,t)}{\partial x^i}. \tag{7.2.14}$$

Again, we need to make certain structural assumptions on F; for simplicity, we assume here that F is smooth and that it satisfies some growth condition like

$$|F^i(t,x,h)| \le C|h| \tag{7.2.15}$$

for some constant C and all i, t, x.

The crucial aspect here is that the functions F^i may now also depend on the solution h itself. As before, we consider a characteristic equation with appropriate initial condition:

$$Y_t(t,x) = F(t,x,h_0(x)) \tag{7.2.16}$$

$$Y(0,x) = x \tag{7.2.17}$$

for some prescribed function $h_0(x)$. When h then is again constant along such characteristic curves, i.e.,

$$h(Y(t,x),t) = h(Y(0,x),0) \quad \text{for all } t \ge 0, \tag{7.2.18}$$

with initial values

$$h(x,0) = h_0(x), \tag{7.2.19}$$

we obtain a solution of (7.2.14), since then

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} h(Y(t,x),t) &= \frac{\partial}{\partial t} h(Y(t,x),t) + \sum_{i=1}^{n} Y_t^i(t,x) \frac{\partial}{\partial x^i} h(Y(t,x),t) \\ &= \frac{\partial}{\partial t} h(Y(t,x),t) + \sum_{i=1}^{n} F^i(t, Y(t,x), h_0(x)) \frac{\partial}{\partial x^i} h(Y(t,x),t), \end{aligned}$$

and since h is constant on characteristic curves, we have from (7.2.17) and (7.2.19) that

$$h(Y(t,x),t) = h_0(x),$$

which when inserted into the previous equation yields (7.2.14), indeed.

In particular, when F is independent of t, (7.2.16) becomes

$$Y_t(t,x) = F(x,h_0(x)) \tag{7.2.20}$$

whose solution with (7.2.17) is simply

$$Y(t,x) = F(x,h_0(x))t + x, \tag{7.2.21}$$

that is, a straight line with slope $F(x,h_0(x))$.

Now, however, we may have a problem: These straight lines, or more generally, the characteristic curves solving (7.2.16), might intersect for some $t > 0$. When the solution h then has different values along such intersecting curves, we obtain conflicting values at such an intersection point. In other words, at intersections of characteristic curves, the solution is not unambiguously determined. Or, put differently, the system (7.2.14) in general does not possess a smooth solution that exists for all time $t > 0$.

We consider the following example (Burgers' equation):

$$h_t(x,t) + hh_x(x,t) = 0 \tag{7.2.22}$$

with $x \in \mathbb{R}^1$ and initial condition

$$h(x,0) = h_0(x). \tag{7.2.23}$$

The characteristic equation (7.2.21) then becomes

$$Y(t,x) = h_0(x)t + x. \tag{7.2.24}$$

We first consider

$$h_0(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1-x & \text{for } 0 < x < 1, \\ 0 & \text{for } x \ge 1. \end{cases} \tag{7.2.25}$$

Then

$$h(x,t) = \begin{cases} 1 & \text{for } x \le t \\ \frac{1-x}{1-t} & \text{for } t < x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} \tag{7.2.26}$$

is constant along the solutions of (7.2.24). (One checks, for instance, that $h((1-x)t + x, t) = 1 - x$ in the region $t < x < 1$.) The characteristic curves, however, intersect for $t \ge 1$ so that the solution exists only for $t < 1$. One possibility to define a consistent solution also for $t \ge 1$ consists in separating two regions of smoothness by the shock curve $x = \frac{1+t}{2}$, i.e., simply put, for $t \ge 1$,

$$h(x,t) = \begin{cases} 1 & \text{for } x \le \frac{1+t}{2}, \\ 0 & \text{for } x \ge \frac{1+t}{2}. \end{cases} \tag{7.2.27}$$

In fact, the jump of h across the curve $x = \frac{1+t}{2}$ satisfies some consistency condition, the so-called Rankine–Hugoniot condition, which we shall now explain. The idea is the following (considering, for simplicity, only the case of one space dimension, $x \in \mathbb{R}^1$): We consider an equation of the form

$$h_t(x,t) + \Phi(h(x,t))_x = 0. \tag{7.2.28}$$

For instance, in (7.2.22), we may take $\Phi(h) = \frac{h^2}{2}$.

We multiply (7.2.28) by some smooth function $\eta(x,t)$ with compact support in $\Omega \times (0,\infty)$ and integrate to get

$$\int_{t\geq 0}\int_{x\in\mathbb{R}} (h_t(x,t) + \Phi(h(x,t))_x)\eta \, \mathrm{d}x\mathrm{d}t = 0 \tag{7.2.29}$$

and integrate by parts to obtain

$$\int_{t\geq 0}\int_{x\in\mathbb{R}} (h\eta_t + \Phi(h)\eta_x)\mathrm{d}x\mathrm{d}t = 0. \tag{7.2.30}$$

When h is a solution of (7.2.28), this relation then has to hold for all η with compact support, and conversely, when this holds for all such η, and h is differentiable, we may integrate by parts to obtain (7.2.29). When (7.2.29) holds for all smooth functions η with compact support, one may conclude that (7.2.28) holds (this is sometimes called the fundamental lemma of the calculus of variations). Thus, whenever h is differentiable, the relation (7.2.30) for all η is equivalent to the differential equation (7.2.28). The advantage of (7.2.30), however, is that this relation is meaningful even when h is not, or is not known to be, differentiable. It merely has to be integrable, together with $\Phi(h)$. This leads to the following:

Definition 7.2.1. When the identity (7.2.30) holds for all compactly supported smooth η, for some integrable h for which $\Phi(h)$ is also integrable, then h is called a weak solution of (7.2.28).

When now h jumps from the value h_- to h_+ along a (differentiable) curve $x = \gamma(t)$, then from (7.2.30), we can deduce the jump condition

$$\Phi(h_+) - \Phi(h_-) = \dot{\gamma}(h_+ - h_-). \tag{7.2.31}$$

This can be seen as follows. Let the curve γ divide the (x,t) plane into two regions $X_\pm$ such that h has the limit $h_\pm$ when approaching γ from $X_\pm$. Let η be compactly supported, but not necessarily vanish along γ. From (7.2.30), we obtain

$$0 = \int_{X_-} (h\eta_t + \Phi(h)\eta_x)\mathrm{d}x\mathrm{d}t + \int_{X_+} (h\eta_t + \Phi(h)\eta_x)\mathrm{d}x\mathrm{d}t. \tag{7.2.32}$$

Since η is compactly supported, integration by parts yields

$$\begin{aligned}\int_{X_-} (h\eta_t + \Phi(h)\eta_x)\mathrm{d}x\mathrm{d}t &= -\int_{X_-} (h_t + \Phi(h)_x)\eta \, \mathrm{d}x\mathrm{d}t \\ &\quad + \int_\gamma (h_- n^2 + \Phi(h_-)n^1)\eta \, \mathrm{d}\gamma(t) \\ &= \int_\gamma (h_- n^2 + \Phi(h_-)n^1)\eta \, \mathrm{d}\gamma(t) \end{aligned} \tag{7.2.33}$$

where $n = (n^1, n^2)$ is the unit normal vector of the curve γ. Here, the right-hand side of the formula has a $+$-sign because n is chosen to point from X_- to X_+. When the parametrization $x = \gamma(t)$ is such that X_- is to the left of γ, then

$$n = \frac{1}{\sqrt{1+\dot{\gamma}^2}}(1, -\dot{\gamma}). \tag{7.2.34}$$

By the same argument,

$$\int_{X_+} (h\eta_t + \Phi(h)\eta_x)\mathrm{d}x\mathrm{d}t = -\int_\gamma (h_+ n^2 + \Phi(h_+)n^1)\eta \,\mathrm{d}\gamma(t). \tag{7.2.35}$$

Combining the two relations (7.2.33) and (7.2.35), we obtain

$$\int_\gamma ((\Phi(h_+) - \Phi(x_-))n^1 + (h_+ - h_-)n^2)\eta \,\mathrm{d}\gamma(t) = 0. \tag{7.2.36}$$

Since this holds for all η, with (7.2.34), we conclude the Rankine–Hugoniot jumping relation (7.2.31). We note that this condition does not determine or constrain the jump curve, but only the difference of the values of h on the two sides of that curve. We note, however, that the jump condition (7.2.31) does not determine the jump curve γ itself, but only the magnitude of the discontinuity across it.

For (7.2.27), we have $h_- = 1, h_+ = 0, \Phi(h_-) = \frac{h_-^2}{2} = \frac{1}{2}, \Phi(h_+) = 0, \dot{\gamma} = \frac{1}{2}$, so that (7.2.31) holds, indeed.

We now consider the initial values

$$h_0(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x \ge 0. \end{cases} \tag{7.2.37}$$

In this case, we encounter the opposite problem: There is no characteristic curve in the region $0 < x < t$. One possibility to overcome this problem is by putting

$$h(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{x}{t} & \text{for } 0 < x < t, \\ 1 & \text{for } x \ge t. \end{cases} \tag{7.2.38}$$

This is a so-called rarefaction wave. Equation (7.2.38) again yields a weak solution in the sense of (7.2.30). This, however, is not the only possible consistent solution. Therefore, one needs selection criteria for distinguishing particular solutions.

For instance, for (7.2.28), the solution (7.2.21) of the characteristic equation becomes

$$Y(t,x) = \Phi_h(h_0(x))t + x. \tag{7.2.39}$$

Therefore, when a characteristic curve with value h_- from the left of the curve γ hits a characteristic curve with value h_+ from the right, we should have

$$\Phi_h(h_-) > \Phi_h(h_+). \tag{7.2.40}$$

Finally, at the end of this section, let me present a short discussion, without many details, of the general first-order partial differential equation,

$$F(x^1, \ldots, x^d, u, p_1, \ldots, p_d) \tag{7.2.41}$$

for an unknown function $u(x^1, \ldots, x^d)$, with

$$p_i := u_{x^i} \quad \text{for } i = 1, \ldots, d, \tag{7.2.42}$$

with subscripts denoting partial derivatives. Here, we assume F to be twice continuously differentiable.

The characteristic curves of (7.2.41) are defined as the solutions of

$$\dot{x}^i = F_{p_i} \tag{7.2.43}$$

$$\dot{u} = \sum_{i=1}^{d} p_i F_{p_i} \tag{7.2.44}$$

$$\dot{p}_i = F_{x^i} + F_u p_i \tag{7.2.45}$$

for $i = 1, \ldots, d$. The $\dot{}$ refers to the derivative w.r.t., the new independent variable t, i.e., we consider x^i, u, p_i here as functions of t. Since F is twice continuously differentiable, the right-hand sides of these equations are locally Lipschitz, and these characteristic equations can therefore be locally solved by the Picard–Lindelöf theorem.

We then have

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x(t), u(t), p(t)) = \sum_i F_{x^i} \dot{x}^i + \sum_i F_{p_i} \dot{p}_i + F_u \dot{u} = 0, \tag{7.2.46}$$

i.e., $F \equiv$ const along characteristics. Therefore, the natural strategy to solve (7.2.41) is to propagate the initial values along characteristic curves.

As an example, we briefly discuss the Hamilton–Jacobi equation

$$u_t + H(t, q^1, \ldots, q^n, u_{q^1}, \ldots, u_{q^n}) = 0, \tag{7.2.47}$$

again assuming H to be twice continuously differentiable. In order to reduce (7.2.47) to the form (7.2.41), we simply put

$$(x^1, \ldots, x^d) = (q^1, \ldots, q^n, t), \text{ i.e., } \frac{\mathrm{d}x^d}{\mathrm{d}t} = 1; \text{ hence } p_d = u_t.$$

With $F = u_t + H$, $p_j = u_{q^j}$, the characteristic equations of (7.2.47) then are, for $j = 1, \dots, n$,

$$\dot{q}^j = H_{p_j}, \dot{p}_j = -H_{q^j}, \tag{7.2.48}$$

$$\dot{u} = \sum_{j=1}^{n} p_j F_{p_j} + p_d F_{p_d} = \sum_j p_j H_{p_j} - H, \dot{p}_d = -H_t. \tag{7.2.49}$$

In fact, when q^j and p_j are determined from (7.2.48), then (7.2.49) yields u. The equations (7.2.48) are the Hamilton equations of classical mechanics.

Let $u = \varphi(t, q^1, \dots, q^n, \lambda_1, \dots, \lambda_n)$ be a solution of (7.2.47) depending on parameters $\lambda_1, \dots, \lambda_n$, then, since also $u + \text{const}$ is a solution (as H in (7.2.47) does not depend explicitly on u),

$$u = \varphi + \lambda \tag{7.2.50}$$

is called a complete integral if

$$\det(\varphi_{q^j \lambda_k})_{j,k=1,\dots,n}. \tag{7.2.51}$$

With parameters $\mu^1, \dots, \mu^n$,

$$\varphi_{\lambda_j} = \mu^j, \quad \varphi_{q^j} = p_j \tag{7.2.52}$$

then yields a $(2n)$-parameter family of solutions of (7.2.48). This is Jacobi's theorem.

For more details about first-order partial differential equations, we refer to [5, 14], and for the Hamilton–Jacobi equation, we suggest [5, 21].

7.3 The Wave Equation

We return to the wave equation (7.1.1). In order to understand the specific features of this equation better, we shall compare and contrast the wave equation with the Laplace and the heat equations. As in Sect. 5.1, we consider some open $\Omega \subset \mathbb{R}^d$ and try to solve the wave equation on

$$\Omega_T = \Omega \times (0, T) \quad (T > 0)$$

by separating variables, i.e., writing the solution u of

$$\begin{aligned} u_{tt}(x,t) &= \Delta_x u(x,t) \quad &&\text{on } \Omega_T, \\ u(x,t) &= 0 &&\text{for } x \in \partial\Omega, \end{aligned} \tag{7.3.1}$$

as

$$u(x,t) = v(x)w(t) \tag{7.3.2}$$

as in (5.1.2). This yields, as in Sect. 5.1,

$$\frac{w_{tt}(t)}{w(t)} = \frac{\Delta v(x)}{v(x)}, \tag{7.3.3}$$

and since the left-hand side is a function of t, and the right-hand side one of x, each of them is constant, and we obtain

$$\Delta v(x) = -\lambda v(x), \tag{7.3.4}$$

$$w_{tt}(t) = -\lambda w(t), \tag{7.3.5}$$

for some constant $\lambda \geq 0$ (maximum principle).

As in Sect. 5.1, v is thus an eigenfunction of the Laplace operator on Ω with Dirichlet boundary conditions to be studied in more detail in Sect. 11.5 below. From (7.3.5), since $\lambda \geq 0$, w is then of the form

$$w(t) = \alpha \cos \sqrt{\lambda}\, t + \beta \sin \sqrt{\lambda}\, t. \tag{7.3.6}$$

As in Sect. 11.5, referring to the expansions demonstrated in Sect. 11.5, we let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ denote the sequence of Dirichlet eigenvalues of Δ on Ω, and $v_1, v_2, \ldots$ the corresponding orthonormal eigenfunctions, and we represent a solution of our wave equation (7.3.1) as

$$u(x,t) = \sum_{n\in\mathbb{N}} \left(\alpha_n \cos \sqrt{\lambda_n}\, t + \beta_n \sin \sqrt{\lambda_n}\, t\right) v_n(x). \tag{7.3.7}$$

In particular, for $t = 0$, we have

$$u(x,0) = \sum_{n\in\mathbb{N}} \alpha_n v_n(x), \tag{7.3.8}$$

and so the coefficients α_n are determined by the initial values $u(x,0)$. Likewise,

$$u_t(x,0) = \sum_{n\in\mathbb{N}} \beta_n \sqrt{\lambda_n}\, v_n(x) \tag{7.3.9}$$

and so the coefficients β_n are determined by the initial derivatives $u_t(x,0)$ (the convergence of the series in (7.3.9) is addressed in Theorem 11.5.1 below). So, in contrast to the heat equation, for the wave equation, we may supplement the Dirichlet data on $\partial\Omega$ by two additional data at $t = 0$, namely, initial values and initial time derivatives.

From the representation formula (7.3.7), we also see, again in contrast to the heat equation, that solutions of the wave equation do not decay exponentially in time but rather that the modes oscillate like trigonometric functions. In fact, there is a conservation principle here; namely, the so-called energy

$$E(t) := \frac{1}{2}\int_{\Omega}\left\{u_t(x,t)^2 + \sum_{i=1}^{d} u_{x^i}(x,t)^2\right\} \mathrm{d}x \tag{7.3.10}$$

is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega}\left\{\left(\sum_n\left(-\alpha_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\,t + \beta_n\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,t\right)v_n(x)\right)^2\right. \\ &\quad \left. + \sum_{i=1}^{d}\left(\sum_n\left(\alpha_n\cos\sqrt{\lambda_n}\,t + \beta_n\sin\sqrt{\lambda_n}\,t\right)\frac{\partial}{\partial x_i}v_n(x)\right)^2\right\}\mathrm{d}x \\ &= \frac{1}{2}\sum_n \lambda_n(\alpha_n^2 + \beta_n^2), \end{aligned} \tag{7.3.11}$$

since

$$\int_{\Omega} v_n(x)v_m(x)\mathrm{d}x = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{i=1}^{d}\int_{\Omega}\frac{\partial}{\partial x_i}v_n(x)\frac{\partial}{\partial x_i}v_m(x) = \begin{cases} \lambda_n & \text{for } n = m, \\ 0 & \text{otherwise} \end{cases}$$

(see Theorem 11.5.1). Equation (7.3.11) implies that E does not depend on t, and we conclude that the energy for a solution u of (7.3.1), represented by (7.3.7), is conserved in time.

We now consider this issue from a somewhat different perspective. Let u be a solution of the wave equation

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{for } x \in \mathbb{R}^d,\ t > 0. \tag{7.3.12}$$

We again have the energy norm of u:

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^d}\left\{u_t(x,t)^2 + \sum_{i=1}^{d} u_{x^i}(x,t)^2\right\} \mathrm{d}x. \tag{7.3.13}$$

We have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\mathbb{R}^d} \left\{ u_t u_{tt} + \sum_{i=1}^{d} u_{x^i} u_{x^i t} \right\} \mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \left\{ u_t (u_{tt} - \Delta u) + \sum_{i=1}^{d} (u_t u_{x^i})_{x^i} \right\} \mathrm{d}x$$
$$= 0 \tag{7.3.14}$$

if $u(x,t) = 0$ for sufficiently large $|x|$ (where that may depend on t, so that this computation may be applied to solutions of (7.3.12) with compactly supported initial values).

In this manner, it is easy to show the following result about the region of dependency of a solution of (7.3.12), partially generalizing the corresponding results of Sect. 7.4 to arbitrary dimensions:

Theorem 7.3.1. *Let u be a solution of (7.3.12) with*

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \tag{7.3.15}$$

and let $K := \operatorname{supp} f \left(:= \overline{\left\{ x \in \mathbb{R}^d : f(x) \neq 0 \right\}} \right)$ *be compact. Then*

$$u(x,t) = 0 \quad \textit{for} \ \operatorname{dist}(x,K) > t. \tag{7.3.16}$$

Proof. We show that $f(y) = 0$ for all $y \in B(x,T)$ implies $u(x,T) \geq 0$, which is equivalent to our assertion. We put

$$\overline{E}(t) := \frac{1}{2} \int_{B(x,T-t)} \left\{ u_t^2 + \sum_{i=1}^{d} u_{y^i}^2 \right\} \mathrm{d}y \tag{7.3.17}$$

and obtain as in (7.3.14) (cf. (2.1.1))

$$\frac{\mathrm{d}\overline{E}}{\mathrm{d}t} = \int_{B(x,T-t)} \left\{ u_t u_{tt} + \sum u_{y^i} u_{y^i t} \right\} \mathrm{d}y$$
$$- \frac{1}{2} \int_{\partial B(x,T-t)} \left\{ u_t^2 + \sum u_{y^i}^2 \right\} do(y)$$
$$= \int_{\partial B(x,T-t)} \left\{ u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} \left(u_t^2 + \sum u_{y^i}^2 \right) \right\} do(y).$$

By the Schwarz inequality, the integrand is nonpositive, and we conclude that

$$\frac{\mathrm{d}\overline{E}}{\mathrm{d}t} \leq 0 \quad \text{for } t > 0.$$

Since by assumption $\overline{E}(0) = 0$ and $\overline{E}$ is nonnegative, necessarily

$$\overline{E}(t) = 0 \quad \text{for all } t \leq T,$$

and hence

$$u(y,t) = 0 \quad \text{for } |x-y| \leq T-t,$$

so that

$$u(x,T) = 0$$

as desired. □

Theorem 7.3.2. *As in Theorem 7.3.1, let u be a solution of the wave equation with initial values*

$$u(x,0) = f(x) \quad \textit{with compact support}$$

and

$$u_t(x,0) = 0.$$

Then

$$v(x,t) := \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} u(x,s)\mathrm{d}s$$

yields a solution of the heat equation

$$v_t(x,t) - \Delta v(x,t) = 0 \quad \textit{for } x \in \mathbb{R}^d, t > 0$$

with initial values

$$v(x,0) = f(x).$$

Proof. That u solves the heat equation is seen by differentiating under the integral

$$\begin{aligned}\frac{\partial}{\partial t} v(x,t) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}}\right) u(x,s)\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \frac{\partial^2}{\partial s^2}\left(\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}}\right) u(x,s)\mathrm{d}s \\ &\qquad \text{(since the kernel solves the heat equation)} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} \frac{\partial^2}{\partial s^2} u(x,s)\mathrm{d}s\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} \Delta_x u(x, s) ds$$

(since*u*solves the wave equation)

$$= \Delta v(x, t),$$

where we omit the detailed justification of interchanging differentiation and integration here. Then $v(x, 0) = u(x, 0) = f(x)$ follows as in Sect. 5.1. □

7.4 The Mean Value Method: Solving the Wave Equation Through the Darboux Equation

Let $v \in C^0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $r > 0$. As in Sect. 2.2, we consider the spatial mean

$$S(v, x, r) = \frac{1}{d\omega_d r^{d-1}} \int_{\partial B(x,r)} v(y) do(y). \tag{7.4.1}$$

For $r > 0$, we put $S(v, x, -r) := S(v, x, r)$, and $S(v, x, r)$ thus is an even function of $r \in \mathbb{R}$. Since $\frac{\partial}{\partial r} S(v, x, r)|_{r=0} = 0$, the extended function remains sufficiently many times differentiable.

Theorem 7.4.1 (Darboux equation). *For $v \in C^2(\mathbb{R}^d)$,*

$$\left(\frac{\partial}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} \right) S(v, x, r) = \Delta_x S(v, x, r). \tag{7.4.2}$$

Proof. We have

$$S(v, x, r) = \frac{1}{d\omega_d} \int_{|\xi|=1} v(x + r\xi)\, do(\xi),$$

and hence

$$\frac{\partial}{\partial r} S(v, x, r) = \frac{1}{d\omega_d} \int_{|\xi|=1} \sum_{i=1}^{d} \frac{\partial v}{\partial x^i} (x + r\xi)\xi^i\, do(\xi)$$

$$= \frac{1}{d\omega_d r^{d-1}} \int_{\partial B(x,r)} \frac{\partial}{\partial \nu} v(y)\, do(y),$$

where ν is the exterior normal of $B(x, r)$

$$= \frac{1}{d\omega_d r^{d-1}} \int_{B(x,r)} \Delta v(z)\, dz$$

by the Gauss integral theorem. (7.4.3)

This implies

$$\frac{\partial^2}{\partial r^2}S(v,x,r) = -\frac{d-1}{d\omega_d r^d}\int_{B(x,r)} \Delta v(z)\mathrm{d}z + \frac{1}{d\omega_d r^{d-1}}\int_{\partial B(x,r)} \Delta v(y)\,do(y)$$
$$= -\frac{d-1}{r}\frac{\partial}{\partial r}S(v,x,r) + \frac{1}{d\omega_d r^{d-1}}\Delta_x\int_{\partial B(x,r)} v(y)\,do(y) \tag{7.4.4}$$

because

$$\Delta_x\int_{\partial B(x,r)} v(y)\,do(y) = \Delta_x\int_{\partial B(x_0,r)} v(x-x_0+y)\,do(y)$$
$$= \int_{\partial B(x_0,r)} \Delta_x v(x-x_0+y)\,do(y)$$
$$= \int_{\partial B(x,r)} \Delta v(y)\,do(y).$$

Equation (7.4.4) is equivalent to (7.4.2). □

Corollary 7.4.1. *Let $u(x,t)$ be a solution of the initial value problem for the wave equation*

$$u_{tt}(x,t) - \Delta(x,t) = 0 \quad \text{for } x\in\mathbb{R}^d,\ t>0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x). \tag{7.4.5}$$

We define the spatial mean

$$M(u,x,r,t) := \frac{1}{d\omega_d r^{d-1}}\int_{\partial B(x,r)} u(y,t)\,do(y).s \tag{7.4.6}$$

We then have

$$\frac{\partial^2}{\partial t^2}M(u,x,r,t) = \left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right)M(u,x,r,t). \tag{7.4.7}$$

Proof. By the first line of (7.4.4),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right)M(u,x,r,t) = \frac{1}{d\omega_d r^{d-1}}\int\limits_{\partial B(x,r)} \Delta_y u(y,t)\,do(y)$$
$$= \frac{1}{d\omega_d r^{d-1}}\int\limits_{\partial B(x,r)} \frac{\partial^2}{\partial t^2}u(y,t)\,do(y),$$

since u solves the wave equation, and this in turn equals

$$\frac{\partial^2}{\partial t^2} M(u, x, r, t).$$

□

For abbreviation, we put

$$w(r, t) := M(u, x, r, t). \tag{7.4.8}$$

Thus w solves the differential equation

$$w_{tt} = w_{rr} + \frac{d-1}{r} w_r \tag{7.4.9}$$

with initial data

$$\begin{aligned} w(r, 0) &= S(f, x, r), \\ w_t(r, 0) &= S(g, x, r).vspace * -3pt \end{aligned} \tag{7.4.10}$$

If the space dimension d equals 3, for a solution w of (7.4.9), $v := rw$ then solves the one-dimensional wave equation

$$v_{tt} = v_{rr} \tag{7.4.11}$$

with initial data

$$\begin{aligned} v(r, 0) &= rS(f, x, r), \\ v_t(r, 0) &= rS(g, x, r). \end{aligned} \tag{7.4.12}$$

By Theorem 7.1.1, this implies

$$\begin{aligned} rM(u, x, r, t) = &\frac{1}{2}\{(r+t)S(f, x, r+t) + (r-t)S(f, x, r-t)\} \\ &+ \frac{1}{2}\int_{r-t}^{r+t} \rho S(g, x, \rho) \mathrm{d}\rho. \end{aligned} \tag{7.4.13}$$

Since $S(f, x, r)$ and $S(g, x, r)$ are even functions of r, we obtain

$$\begin{aligned} M(u, x, r, t) = &\frac{1}{2r}\{(t+r)S(f, x, r+t) - (t-r)S(f, x, t-r)\} \\ &+ \frac{1}{2r}\int_{t-r}^{t+r} \rho S(g, x, \rho) \mathrm{d}\rho. \end{aligned} \tag{7.4.14}$$

We want to let r tend to 0 in this formula. By continuity of u,

$$M(u, x, 0, t) = u(x, t), \tag{7.4.15}$$

and we obtain

$$u(x,t) = tS(g,x,t) + \frac{\partial}{\partial t}(tS(f,x,t)). \tag{7.4.16}$$

By our preceding considerations, every solution of class C^2 of the initial value problem (7.4.5) for the wave equation must be represented in this way, and we thus obtain the following result:

Theorem 7.4.2. *The unique solution of the initial value problem for the wave equation in* 3 *space dimensions,*

$$\begin{aligned} u_{tt}(x,t) - \Delta u(x,t) &= 0 \quad \text{for } x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{7.4.17}$$

for given $f \in C^3(\mathbb{R}^3)$*,* $g \in C^2(\mathbb{R}^3)$*, can be represented as*

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} \left(tg(y) + f(y) + \sum_{i=1}^{3} f_{y^i}(y)(y^i - x^i) \right) do(y). \tag{7.4.18}$$

Proof. First of all, (7.4.16) yields

$$u(x,t) = \frac{1}{4\pi t} \int_{\partial B(x,t)} g(y) do(y) + \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{\partial B(x,t)} f(y) do(y) \right). \tag{7.4.19}$$

In order to carry out the differentiation in the integral, we need to transform the mean value of f back to the unit sphere, i.e.,

$$\frac{1}{4\pi t} \int_{\partial B(x,t)} f(y) do(y) = \frac{t}{4\pi} \int_{|z|=1} f(x+tz) do(z).$$

The Darboux equation implies that u from (7.4.19) solves the wave equation, and the correct initial data result from the relations

$$S(w,x,0) = w(x), \qquad \frac{\partial}{\partial r} S(w,x,r)|_{r=0} = 0$$

satisfied by every continuous w. □

An important observation resulting from (7.4.18) is that for space dimensions 3 (and higher), a solution of the wave equation can be less regular than its initial values. Namely, if $u(x,0) \in C^k$, $u_t(x,0) \in C^{k-1}$, this implies $u(x,t) \in C^{k-1}$, $u_t(x,t) \in C^{k-2}$ for positive t.

Moreover, as in the case $d = 1$, we may determine the regions of influence of the initial data. It is quite remarkable that the value of u at (x, t) depends on the initial data only on the sphere $\partial B(x, t)$, but not on the data in the interior of the ball $B(x, t)$. This is the so-called Huygens principle. This principle, however, holds only in odd dimensions greater than 1, but not in even dimensions. We want to explain this for the case $d = 2$. Obviously, a solution of the wave equation for $d = 2$ can be considered as a solution for $d = 3$ that happens to be independent of the third spatial coordinate x^3.

We thus put $x^3 = 0$ in (7.4.19) and integrate on the sphere $\partial B(x, t) = \{y \in \mathbb{R}^3 : (y^1 - x^1)^2 + (y^2 - x^2)^2 + (y^3)^2 = t^2\}$ with surface element

$$do(y) = \frac{t}{|y^3|}\mathrm{d}y^1\mathrm{d}y^2.$$

Since the points (y^1, y^2, y^3) and $(y^1, y^2, -y^3)$ yield the same contributions, we obtain

$$u(x^1, x^2, t) = \frac{1}{2\pi}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}}\mathrm{d}y + \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B(x,t)} \frac{f(y)}{\sqrt{t^2 - |x-y|^2}}\mathrm{d}y\right),$$

where $x = (x^1, x^2)$, $y = (y^1, y^2)$, and the ball $B(x, t)$ now is the two-dimensional one.

The values of u at (x, t) now depend on the values on the whole disk $B(x, t)$ and not only on its boundary $\partial B(x, t)$.

A reference for Sects. 7.3 and 7.4 is John [14].

Summary

In this chapter we have studied the wave equation

$$\frac{\partial^2}{\partial t^2}u(x, t) - \Delta u(x, t) = 0 \quad \text{for } x \in \mathbb{R}^d,\ t > 0$$

with initial data

$$u(x, 0) = f(x),$$
$$\frac{\partial}{\partial t}u(x, 0) = g(x).$$

In contrast to the heat equation, there is no gain of regularity compared to the initial data, and in fact, for $d > 1$, there may even occur a loss of regularity.

As was the case with the Laplace equation, mean value constructions are important for the wave equation, and they permit us to reduce the wave equation for $d > 1$ to the Darboux equation for the mean values, which is hyperbolic as well but involves only one spatial coordinate.

The propagation speed for the wave equation is finite, in contrast to the heat equation. The effect of perturbations sets in sharply, and in odd dimensions greater than 1, it also terminates sharply (Huygens principle).

The energy

$$E(t) = \int_{\mathbb{R}^d} \left(|u_t(x,t)|^2 + |\nabla_x u(x,t)|^2 \right) \mathrm{d}x$$

is constant in time.

By a certain time averaging, a solution of the wave equation yields a solution of the heat equation.

In fact, any solution of the one-dimensional wave equation can be represented as

$$u(x,t) = \varphi(x+t) + \psi(x-t)$$

with arbitrary functions φ, ψ. Since such functions need not be regular, we naturally arrive at a concept of a generalized solution of the wave equation. When φ and ψ are differentiable, they satisfy the transport equations

$$\varphi_t - \varphi_x = 0, \psi_t + \psi_x = 0.$$

We have then considered the more general first-order hyperbolic equation

$$\frac{\partial}{\partial t} h(x,t) + \sum_{i=1}^{d} f^i(t,x) \frac{\partial h(x,t)}{\partial x^i} = 0.$$

This equation is solved by the method of characteristics. That simply means that we let h be constant along characteristic curves, i.e., solutions of the system of ODEs

$$\dot{x}^i(t) = f^i(t, x(t)) \text{ for } i = 1, \ldots, d.$$

The more general hyperbolic equation

$$\frac{\partial}{\partial t} h(x,t) + \sum_{i=1}^{d} f^i(t,x,h) \frac{\partial h(x,t)}{\partial x^i},$$

i.e., where the factors f^i now also may depend on the solution itself, can still be approached by the method of characteristics. Here, however, the problem arises that characteristic curves may intersect, leading to singularities of the solution because of incompatible values along these curves. Conversely, the family of characteristic curves may also leave out some region of space, necessitating some interpolation scheme.

Exercises

7.1. We consider the wave equation in one space dimension,

$$u_{tt} - u_{xx} = 0 \quad \text{for } 0 < x < \pi, t > 0,$$

with initial data

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad u_t(x,0) = \sum_{n=1}^{\infty} \beta_n \sin nx$$

and boundary values

$$u(0,t) = u(\pi,t) = 0 \quad \text{for all } t > 0.$$

Represent the solution as a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin nx$$

and compute the coefficients $\gamma_n(t)$.

7.2. Consider the equation

$$u_t + cu_x = 0$$

for some function $u(x,t)$, $x, t \in \mathbb{R}$, where c is constant. Show that u is constant along any line

$$x - ct = \text{const} = \xi,$$

and thus the general solution of this equation is given as

$$u(x,t) = f(\xi) = f(x - ct)$$

where the initial values are $u(x,0) = f(x)$. Does this differential equation satisfy the Huygens principle?

7.3. We consider the general quasilinear PDE for a function $u(x,y)$ of two variables,

$$au_{xx} + 2bu_{xy} + cu_{yy} = d,$$

where a, b, c, d are allowed to depend on x, y, u, u_x, and u_y. We consider the curve $\gamma(s) = (\varphi(s), \psi(s))$ in the xy-plane, where we wish to prescribe the function u and its first derivatives:

$$u = f(s), \; u_x = g(s), \; u_y = h(s) \quad \text{for } x = \varphi(s), y = \psi(s).$$

Show that for this to be possible, we need the relation

$$f'(s) = g(s)\varphi'(s) + h(s)\psi'(s).$$

For the values of u_{xx}, u_{xy}, u_{yy} along γ, compute the equations

$$\varphi' u_{xx} + \psi' u_{xy} = g',$$
$$\varphi' u_{xy} + \psi' u_{yy} = h'.$$

Conclude that the values of u_{xx}, u_{xy}, and u_{yy} along γ are uniquely determined by the differential equations and the data f, g, h (satisfying the above compatibility conditions), unless

$$a\psi'^2 - 2b\varphi'\psi' + c\varphi'^2 = 0$$

along γ. If this latter equation holds, γ is called a characteristic curve for the solution u of our PDE $au_{xx} + 2bu_{xy} + cu_{yy} = d$. (Since a, b, c, d may depend on u and u_x, u_y, in general it depends not only on the equation, but also on the solution, which curves are characteristic.) How is this existence of characteristic curves related to the classification into elliptic, hyperbolic, and parabolic PDEs discussed in the introduction? What are the characteristic curves of the wave equation $u_{tt} - u_{xx} = 0$?