Chapter 1
Introduction: What Are Partial Differential Equations?

As a first answer to the question, What are PDEs, we would like to give a definition:

Definition 1. A PDE is an equation involving derivatives of an unknown function $u\colon \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^d$, $d \geq 2$ (or, more generally, of a differentiable manifold of dimension $d \geq 2$).

Often, one also considers systems of PDEs for vector-valued functions $u\colon \Omega \to \mathbb{R}^N$, or for mappings with values in a differentiable manifold.

The preceding definition, however, is misleading, since in the theory of PDEs one does not study arbitrary equations but concentrates instead on those equations that naturally occur in various applications (physics and other sciences, engineering, economics) or in other mathematical contexts.

Thus, as a second answer to the question posed in the title, we would like to describe some typical examples of PDEs. We shall need a little bit of notation: A partial derivative will be denoted by a subscript,

$$u_{x^i} := \frac{\partial u}{\partial x^i} \quad \text{for } i = 1, \ldots, d.$$

In case $d = 2$, we write x, y in place of x^1, x^2. Otherwise, x is the vector $x = (x^1, \ldots, x^d)$.

Examples. (1) The Laplace equation

$$\Delta u := \sum_{i=1}^{d} u_{x^i x^i} = 0 \quad (\Delta \text{ is called the Laplace operator}),$$

or, more generally, the Poisson equation

$$\Delta u = f \quad \text{for a given function} \quad f : \Omega \to \mathbb{R}.$$

J. Jost, *Partial Differential Equations*, Graduate Texts in Mathematics 214,
DOI 10.1007/978-1-4614-4809-9_1,

For example, the real and imaginary parts u and v of a holomorphic function $u: \Omega \to \mathbb{C}$ ($\Omega \subset \mathbb{C}$ open) satisfy the Laplace equation. This easily follows from the Cauchy–Riemann equations:

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x, \end{aligned} \quad \text{with} \quad z = x + iy$$

implies

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}.$$

The Cauchy–Riemann equations themselves represent a system of PDEs. The Laplace equation also models many equilibrium states in physics, and the Poisson equation is important in electrostatics.

(2) The heat equation: Here, one coordinate t is distinguished as the "time" coordinate, while the remaining coordinates $x^1, \ldots, x^d$ represent spatial variables. We consider

$$u : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \quad \Omega \text{ open in } \mathbb{R}^d, \quad \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\},$$

and pose the equation

$$u_t = \Delta u, \quad \text{where again } \Delta u := \sum_{i=1}^{d} u_{x^i x^i}.$$

The heat equation models heat and other diffusion processes.

(3) The wave equation: With the same notation as in (2), here we have the equation

$$u_{tt} = \Delta u.$$

It models wave and oscillation phenomena.

(4) The Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0$$

(notation as in (2), but with only one spatial coordinate x) models the propagation of waves in shallow waters.

(5) The Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = f,$$

or in higher dimensions

$$\det\, (u_{x^i x^j})_{i,j=1,\ldots,d} = f,$$

with a given function f, is used for finding surfaces (or hypersurfaces) with prescribed curvature.

(6) The minimal surface equation

$$\left(1+u_y^2\right)u_{xx}-2u_xu_yu_{xy}+\left(1+u_x^2\right)u_{yy}=0$$

describes an important class of surfaces in $\mathbb{R}^3$.

(7) The Maxwell equations for the electric field strength $E=(E_1,E_2,E_3)$ and the magnetic field strength $B=(B_1,B_2,B_3)$ as functions of (t,x^1,x^2,x^3):

$$\begin{aligned}
\operatorname{div} B &= 0 && \text{(magnetostatic law)},\\
B_t + \operatorname{curl} E &= 0 && \text{(magnetodynamic law)},\\
\operatorname{div} E &= 4\pi\varrho && \text{(electrostatic law, } \varrho = \text{charge density)},\\
E_t - \operatorname{curl} E &= -4\pi j && \text{(electrodynamic law, } j = \text{current density)},
\end{aligned}$$

where div and curl are the standard differential operators from vector analysis with respect to the variables $(x^1,x^2,x^3)\in\mathbb{R}^3$.

(8) The Navier–Stokes equations for the velocity $v(x,t)$ and the pressure $p(x,t)$ of an incompressible fluid of density ϱ and viscosity η:

$$\varrho v_t^j+\varrho\sum_{i=1}^{3}v^iv_{x^i}^j-\eta\Delta v^j=-p_{x^j}\quad\text{for } j=1,2,3,$$

$$\operatorname{div} v=0$$

$(d=3, v=(v^1,v^2,v^3))$.

(9) The Einstein field equations of the theory of general relativity for the curvature of the metric (g_{ij}) of space-time:

$$R_{ij}-\frac{1}{2}g_{ij}R=\kappa T_{ij}\quad\text{for } i,j=0,1,2,3\quad\text{(the index 0 stands for the time coordinate } t=x^0\text{)}.$$

Here, κ is a constant, T_{ij} is the energy–momentum tensor (considered as given), while

$$R_{ij}:=\sum_{k=0}^{3}\left(\frac{\partial}{\partial x^k}\Gamma_{ij}^k-\frac{\partial}{\partial x^j}\Gamma_{ik}^k+\sum_{l=0}^{3}\left(\Gamma_{lk}^k\Gamma_{ij}^l-\Gamma_{lj}^k\Gamma_{ik}^l\right)\right)$$

(Ricci curvature)

with

$$\Gamma_{ij}^{k} := \frac{1}{2} \sum_{l=0}^{3} g^{kl} \left(\frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right)$$

and

$$(g^{ij}) := (g_{ij})^{-1} \text{ (inverse matrix)}$$

and

$$R := \sum_{i,j=0}^{3} g^{ij} R_{ij} \text{ (scalar curvature).}$$

Thus R and R_{ij} are formed from first and second derivatives of the unknown metric (g_{ij}).

(10) The Schrödinger equation

$$i\hbar u_t = -\frac{\hbar^2}{2m} \Delta u + V(x, u)$$

(m = mass, V = given potential, $u: \Omega \to \mathbb{C}$) from quantum mechanics is formally similar to the heat equation, in particular in the case $V = 0$. The factor i $(= \sqrt{-1})$, however, leads to crucial differences.

(11) The plate equation

$$\Delta \Delta u = 0$$

even contains fourth derivatives of the unknown function.

We have now seen many rather different-looking PDEs, and it may seem hopeless to try to develop a theory that can treat all these diverse equations. This impression is essentially correct, and in order to proceed, we want to look for criteria for classifying PDEs. Here are some possibilities:

(I) Algebraically, i.e., according to the algebraic structure of the equation:

(a) Linear equations, containing the unknown function and its derivatives only linearly. Examples (1), (2), (3), (7), (11), as well as (10) in the case where V is a linear function of u.

An important subclass is that of the linear equations with constant coefficients. The examples just mentioned are of this type; (10), however, only if $V(x,u) = v_0 \cdot u$ with constant v_0. An example of a linear equation with nonconstant coefficients is

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\left(a^{ij}(x)u_{x^j}\right) + \sum_{i=1}^{d} \frac{\partial}{\partial x^i}\left(b^i(x)u\right) + c(x)u = 0$$

with nonconstant functions a^{ij}, b^i, c.

(b) Nonlinear equations.

Important subclasses:

- Quasilinear equations, containing the highest-occurring derivatives of u linearly. This class contains all our examples with the exception of (5).
- Semilinear equations, i.e., quasilinear equations in which the term with the highest-occurring derivatives of u does not depend on u or its lower-order derivatives. Example (6) is a quasilinear equation that is not semilinear.

PDEs that are not quasilinear are called fully nonlinear. Example (5) is a fully nonlinear equation.

Naturally, linear equations are simpler than nonlinear ones. We shall therefore first study some linear equations.

(II) According to the order of the highest-occurring derivatives: The Cauchy–Riemann equations and (7) are of first order; (1), (2), (3), (5), (6), (8), (9), (10) are of second order; (4) is of third order; and (11) is of fourth order. Equations of higher order rarely occur, and most important PDEs are second-order PDEs. Consequently, in this textbook we shall almost exclusively study second-order PDEs.

(III) In particular, for second-order equations the following partial classifications turns out to be useful:

Let

$$F\left(x, u, u_{x^i}, u_{x^i x^j}\right) = 0$$

be a second-order PDE. We write the equation in symmetric form, that is, replace $u_{x^i x^j}$ by $\frac{1}{2}(u_{x^i x^j} + u_{x^j x^i})$. We then introduce dummy variables and study the function

$$F\left(x, u, p_i, p_{ij}\right).$$

The equation is called *elliptic* in Ω at $u(x)$ if the matrix

$$F_{p_{ij}}\left(x, u(x), u_{x^i}(x), u_{x^i x^j}(x)\right)_{i,j=1,\ldots,d}$$

is positive definite for all $x \in \Omega$. (If this matrix should happen to be negative definite, the equation becomes elliptic by replacing F by $-F$.) Note that this may depend on the function u. For example, if $f(x) > 0$ in (5), the equation is elliptic for any solution u with $u_{xx} > 0$. (For verifying ellipticity, one should write in place of (5)

$$u_{xx}u_{yy} - u_{xy}u_{yx} - f = 0,$$

which is equivalent to (5) for a twice continuously differentiable u.) Examples (1) and (6) are always elliptic.

The equation is called *hyperbolic* if the above matrix has precisely one negative and $(d-1)$ positive eigenvalues (or conversely, depending on a choice of sign). Example (3) is hyperbolic, and so is (5), if $f(x) < 0$, for a solution u with $u_{xx} > 0$. Finally, an equation that can be written as

$$u_t = F(t, x, u, u_{x^i}, u_{x^i x^j})$$

with elliptic F is called *parabolic*. Note, however, that there is no longer a free sign here, since a negative definite $(F_{p_{ij}})$ is not allowed. Example (2) is parabolic. Obviously, this classification does not cover all possible cases, but it turns out that other types are of minor importance only. Elliptic, hyperbolic, and parabolic equations require rather different theories, with the parabolic case being somewhat intermediate between the elliptic and hyperbolic ones, however.

(IV) According to solvability: We consider a second-order PDE

$$F(x, u, u_{x^i}, u_{x^i x^j}) = 0 \text{ for } u : \Omega \to \mathbb{R},$$

and we wish to impose additional conditions upon the solution u, typically prescribing the values of u or of certain first derivatives of u on the boundary $\partial\Omega$ or part of it.

Ideally, such a boundary value problem satisfies the three conditions of Hadamard for a well-posed problem:

– Existence of a solution u for given boundary values.
– Uniqueness of this solution.
– Stability, meaning continuous dependence on the boundary values.

The third requirement is important, because in applications, the boundary data are obtained through measurements and thus are given only up to certain error margins, and small measurement errors should not change the solution drastically.

The existence requirement can be made more precise in various senses: The strongest one would be to ask that the solution be obtained by an explicit formula in terms of the boundary values. This is possible only in rather special cases, however, and thus one is usually content if one is able to deduce the existence of a solution by some abstract reasoning, for example by deriving a contradiction from the assumption of nonexistence. For such an existence procedure, often nonconstructive techniques are employed, and thus an existence theorem does not necessarily provide a rule for constructing or at least approximating some solution.

Thus, one might refine the existence requirement by demanding a constructive method with which one can compute an approximation that is as accurate as desired. This is particularly important for the numerical approximation of solutions. However, it turns out that it is often easier to treat the two problems separately, i.e., first deducing an abstract existence theorem and then utilizing the insights obtained in doing so for a constructive and numerically stable approximation scheme. Even if the numerical scheme is not rigorously founded, one might be able to use one's knowledge about the existence or nonexistence of a solution for a heuristic estimate of the reliability of numerical results.

Exercise. Find five more examples of important PDEs in the literature.