Chapter 6
Reaction–Diffusion Equations and Systems

6.1 Reaction–Diffusion Equations

In this section, we wish to study the initial boundary value problem for nonlinear parabolic equations of the form

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= F(x,t,u) \quad \text{for} \quad x \in \Omega, t > 0, \\ u(x,t) &= g(x,t) \quad \text{for} \quad x \in \partial\Omega, t > 0, \\ u(x,0) &= f(x) \quad \text{for} \quad x \in \Omega, \end{aligned} \tag{6.1.1}$$

with given (continuous and smooth) functions g, f and a Lipschitz continuous function F (in fact, Lipschitz continuity is only needed w.r.t. to u; for x and t, continuity suffices). The nonlinearity of this equation comes from the u-dependence of F. While we may consider (6.1.1) as a heat equation with a nonlinear term on the right-hand side, i.e., as a generalization of

$$u_t(x,t) - \Delta u(x,t) = 0 \quad \text{for } x \in \Omega, t > 0 \tag{6.1.2}$$

(with the same boundary and initial values), in the case where F does not depend on the spatial variable x, i.e., $F = F(t,u)$, we may alternatively view (6.1.1) as a generalization of the ODE:

$$\begin{aligned} u_t(t) &= F(t,u) \quad \text{for } t > 0, \\ u(0) &= u_0. \end{aligned} \tag{6.1.3}$$

For such equations, we have, for the case of a Lipschitz continuous F, a local existence theorem, the Picard–Lindelöf theorem. This says that for given initial value u_0, we may find some $t_0 > 0$ with the property that a unique solution exists for $0 \le t < t_0$. When F is bounded, solutions exist for all t, as follows from

J. Jost, *Partial Differential Equations*, Graduate Texts in Mathematics 214,
DOI 10.1007/978-1-4614-4809-9_6,

an iterated application of the Picard–Lindelöf theorem. When F is unbounded, however, solutions may become infinite in finite time; a standard example is

$$u_t(t) = u^2(t) \tag{6.1.4}$$

with positive initial value u_0. The solution is

$$u(t) = \left(\frac{1}{u_0} - t\right)^{-1} \tag{6.1.5}$$

which for positive u_0 becomes infinite in finite time, at $t = \frac{1}{u_0}$.

We shall see in this section that this qualitative type of behavior, in particular the local (in time) existence result, carries over to the reaction–diffusion equation (6.1.1). In fact, the local existence can be shown like the Picard–Lindelöf theorem by an application of the Banach fixed-point theorem; here, of course, we need to utilize also the results for the heat equation (6.1.2) established in Sect. 5.3. We shall thus start by establishing the local existence result:

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2, and let*

$$g \in C^0(\partial\Omega \times [0, t_0]), \quad f \in C^0(\bar{\Omega}),$$
$$\text{with } g(x, 0) = f(x) \quad \text{for } x \in \partial\Omega,$$

and let

$$F \in C^0(\bar{\Omega} \times [0, t_0] \times \mathbb{R})$$

be locally bounded, i.e., given $\eta > 0$ and $f \in C^0(\bar{\Omega})$, there exists $M = M(\eta)$ with

$$|F(x, t, v(x))| \leq M \quad \text{for } x \in \bar{\Omega}, t \in [0, t_0], |v(x) - f(x)| \leq \eta, \tag{6.1.6}$$

and locally Lipschitz continuous w.r.t. u, i.e., there exists a constant $L = L(\eta)$ with

$$|F(x, t, u_1(x)) - F(x, t, u_2(x))| \leq L|u_1(x) - u_2(x)|$$
$$\text{for } x \in \bar{\Omega}, t \in [0, t_0], \|u_1 - f\|_{C^0(\bar{\Omega})}, \|u_2 - f\|_{C^0(\bar{\Omega})} < \eta. \tag{6.1.7}$$

(Of course, (6.1.6) follows from (6.1.7), but it is convenient to list it separately.)

Then there exists some $t_1 \leq t_0$ for which the initial boundary value problem

$$\begin{aligned} u_t(x, t) - \Delta u(x, t) &= F(x, t, u) \quad \text{for } x \in \Omega,\ 0 < t \leq t_1, \\ u(x, t) &= g(x, t) \quad \text{for } x \in \partial\Omega,\ 0 < t \leq t_1, \\ u(x, 0) &= f(x) \quad \text{for } x \in \Omega, \end{aligned} \tag{6.1.8}$$

admits a unique solution that is continuous on $\bar{\Omega} \times [0, t_1]$.

Proof. Let $q(x, y, t)$ be the heat kernel of Ω of Corollary 5.3.1. According to (5.3.28), a solution then needs to satisfy

$$\begin{aligned} u(x,t) = &\int_0^t \int_\Omega q(x,y,t-\tau)F(y,\tau,u(y,\tau))\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_\Omega q(x,y,t)f(y)\mathrm{d}y \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau)g(y,\tau)do(y)\mathrm{d}\tau. \end{aligned} \tag{6.1.9}$$

A solution of (6.1.9) then is a fixed point of

$$\begin{aligned} \Phi : v \mapsto &\int_0^t \int_\Omega q(x,y,t-\tau)F(y,\tau,v(y,\tau))\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_\Omega q(x,y,t)f(y)\mathrm{d}y \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau)g(y,\tau)do(y)\mathrm{d}\tau \end{aligned} \tag{6.1.10}$$

which maps $C^0(\bar{\Omega} \times [0, t_0])$ to itself. We consider the set

$$A := \{v \in C^0(\bar{\Omega} \times [0, t_1]) : \sup_{x\in\bar{\Omega}, 0\le t\le t_1} |v(x,t) - f(x)| < \eta\}. \tag{6.1.11}$$

Here, we choose $t_1 > 0$ so small that

$$t_1 M \le \frac{\eta}{2} \tag{6.1.12}$$

and

$$t_1 L < 1. \tag{6.1.13}$$

For $v \in A$

$$\begin{aligned} |\Phi(v)(x,t) - f(x)| \le &\left| \int_0^t \int_\Omega q(x,y,t-\tau)F(y,\tau,v(y,\tau))\mathrm{d}y\,\mathrm{d}\tau \right| \\ &+ \left| \int_\Omega q(x,y,t)f(y)\mathrm{d}y \right. \\ &\left. - \int_0^t \int_{\partial\Omega} \frac{\partial q}{\partial \nu_y}(x,y,t-\tau)g(y,\tau)do(y)\mathrm{d}\tau - f(x) \right| \\ \le &\, tM + c_{f,g}(t), \end{aligned} \tag{6.1.14}$$

where we have used (5.3.40) and $c_{f,g}(t)$ controls the difference of the solution $u_0(x,t)$ at time t of the heat equation with initial values f and boundary values g from its initial values, i.e., $\sup_{x\in\bar\Omega}|u_0(x,t)-f(x)|$. That latter quantity can be made arbitrarily small, for example, smaller than $\frac{\eta}{2}$ by choosing t sufficiently small, by continuity of the solution of the heat equation (see Theorem 5.3.3). Together with (6.1.12), we then have, by choosing t_1 sufficiently small,

$$|\Phi(v)(x,t)-f(x)|<\eta, \tag{6.1.15}$$

that is, $\Phi(v)\in A$. Thus, Φ maps the set A to itself.

We shall now show that Φ is a contraction on A: for $v,w\in A$, using (5.3.40) again, and our Lipschitz condition (6.1.7),

$$\begin{aligned}
&\sup_{x\in\bar\Omega,0\le t\le t_1}|\Phi(v)(x,t)-\Phi(w)(x,t)|\\
&=\sup_{x\in\bar\Omega,0\le t\le t_1}\Big|\int_0^t\int_\Omega q(x,y,t-\tau)(F(y,\tau,v(y,\tau))\\
&\quad -F(y,\tau,w(y,\tau)))\mathrm{d}y\ \mathrm{d}\tau\Big|\\
&\le t_1L\sup_{x\in\bar\Omega,0\le t\le t_1}|v(x,t)-w(x,t)|,
\end{aligned}\tag{6.1.16}$$

with $t_1L<1$ by (6.1.13). Thus, Φ is a contraction on A, and the Banach fixed-point theorem (see Theorem A.1 of the appendix) yields the existence of a unique fixed point in A that then is a solution of our problem (6.1.8). We still need to exclude that there exists a solution outside A, but this is simple as the next lemma shows. □

Lemma 6.1.1. *Let $u_1(x,t),u_2(x,t)\in C^0(\bar\Omega\times[0,T])$ be solutions of (6.1.8) with $u_i(x,t)=g(x,t)$ for $x\in\partial\Omega, 0\le t\le T$, $|u_i(x,0)-f(x)|\le\frac{\eta}{2}$ for $x\in\bar\Omega$, $i=1,2$. Then there exists a constant $K=K(\eta)$ with*

$$\sup_{x\in\bar\Omega}|u_1(x,t)-u_2(x,t)|\le \mathrm{e}^{Kt}\sup_{x\in\bar\Omega}|u_1(x,0)-u_2(x,0)|\quad \text{for } 0\le t\le T. \tag{6.1.17}$$

Proof. By the representation formula (5.3.28),

$$\begin{aligned}
u_1(x,t)-u_2(x,t)&=\int_\Omega q(x,y,t)(u_1(y,0)-u_2(y,0))\mathrm{d}y\\
&\quad+\int_0^t\int_\Omega q(x,y,t-\tau)(F(y,\tau,u_1(y,\tau))\\
&\quad-F(y,\tau,u_2(y,\tau)))\mathrm{d}y\,\mathrm{d}\tau
\end{aligned}\tag{6.1.18}$$

Then, as long as $\sup_x|u_i(x,t)-f(x)|\le\eta$, we have the bound from (6.1.7):

$$|F(x,t,u_1(x,t))-F(x,t,u_2(x,t))|\le L|u_1(x,t)-u_2(x,t)|. \tag{6.1.19}$$

Using (5.3.37) and (6.1.19) in (6.1.18), we obtain

$$\sup_{x\in\bar{\Omega}} |u_1(x,t) - u_2(x,t)| \le \sup_{x\in\bar{\Omega}} |u_1(x,0) - u_2(x,0)| + L\int_0^t \sup_{x\in\bar{\Omega}} |u_1(x,\tau) - u_2(x,\tau)|\mathrm{d}\tau \tag{6.1.20}$$

which implies the claim by the following general calculus inequality. □

Lemma 6.1.2. *Let the integrable function $\phi : [0,T] \to \mathbb{R}^+$ satisfy*

$$\phi(t) \le \phi(0) + c\int_0^t \phi(\tau)\mathrm{d}\tau \tag{6.1.21}$$

for all $0 \le t \le T$ and some constant c. Then for $0 \le t \le T$

$$\phi(t) \le \mathrm{e}^{ct}\phi(0). \tag{6.1.22}$$

Proof. From (6.1.21)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-ct}\int_0^t \phi(\tau)\mathrm{d}\tau\right) \le \mathrm{e}^{-ct}\phi(0);$$

hence

$$\mathrm{e}^{-ct}\int_0^t \phi(\tau)\mathrm{d}\tau \le \frac{1-\mathrm{e}^{-ct}}{c}\phi(0),$$

from which, with (6.1.21), the desired inequality (6.1.22) follows. □

We have the following important consequence of Theorem 6.1.1, a global existence theorem:

Corollary 6.1.1. *Under the assumptions of Theorem 6.1.1, suppose that the solution $u(x,t)$ of (6.1.8) satisfies the a priori bound*

$$\sup_{x\in\bar{\Omega},0\le\tau\le t} |u(x,\tau)| \le K \tag{6.1.23}$$

for all times t for which it exists, with some fixed constant K. Then the solution $u(x,t)$ exists for all times $0 \le t < \infty$.

Proof. Suppose the solution exists for $0 \le t \le T$. Then we apply Theorem 6.1.1 at time T instead of 0, with initial values $u(x,T)$ in place of the original initial values $u(x,0)$ and conclude that the solution continues to exist on some interval $[0, T+t_0)$ for some $t_0 > 0$ that only depends on K. We can therefore iterate the procedure to obtain a solution for all time. □

In order to understand the qualitative behavior of solutions of reaction–diffusion equations

$$u_t(x,t) - \Delta u(x,t) = F(t,u) \quad \text{on } \Omega_T, \tag{6.1.24}$$

it is useful to compare them with solutions of the pure reaction equation

$$v_t(x,t) = F(t,v), \tag{6.1.25}$$

which, when the initial values

$$v(x,0) = v_0 \tag{6.1.26}$$

do not depend on x, likewise is independent of the spatial variable x. It therefore satisfies the homogeneous Neumann boundary condition

$$\frac{\partial v}{\partial \nu} = 0, \tag{6.1.27}$$

where ν, as always, is the exterior normal of the domain Ω. Therefore, comparison is easiest when we also assume that u satisfies such a Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{6.1.28}$$

instead of the Dirichlet condition of (6.1.1). We therefore investigate that situation now, even though in Chap. 5 we have not derived existence theorems for parabolic equations with Neumann boundary conditions. For such results, we refer to [9]. We have the following general comparison result:

Lemma 6.1.3. *Let u, v be of class C^2 w.r.t. $x \in \Omega$, of class C^1 w.r.t. $t \in [0,T]$, and satisfy*

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) - F(x,t,u) &\geq v_t(x,t) - \Delta v(x,t) - F(x,t,v) \\ &\qquad \text{for } x \in \Omega, 0 < t \leq T, \\ \frac{\partial u(x,t)}{\partial \nu} &\geq \frac{\partial v(x,t)}{\partial \nu} \quad \text{for } x \in \partial\Omega, 0 < t \leq T, \\ u(x,0) &\geq v(x,0) \quad \text{for } x \in \Omega, \end{aligned} \tag{6.1.29}$$

with our above assumptions on F which in addition is assumed to be continuously differentiable w.r.t. u with $\frac{\partial F}{\partial u} \leq 0$. Then

$$u(x,t) \geq v(x,t) \quad \text{for } x \in \bar{\Omega}, 0 \leq t \leq T. \tag{6.1.30}$$

Proof. $w(x,t) := u(x,t) - v(x,t)$ satisfies $w(x,0) \geq 0$ in Ω and $\frac{\partial w}{\partial \nu} \geq 0$ on $\partial\Omega \times [0,T]$, as well as

$$w_t(x,t) - \Delta w(x,t) - \frac{\mathrm{d}F(x,t,\eta)}{\mathrm{d}u} w(x,t) \geq 0 \tag{6.1.31}$$

with $\eta := su + (1-s)v$ for some $0 < s < 1$. Lemma 5.1.1 and the parabolic maximum principles mentioned there then imply $w \geq 0$, i.e., (6.1.30). □

For example, a solution of

$$u_t - \Delta u = -u^3 \quad \text{for } x \in \bar{\Omega}, t > 0 \tag{6.1.32}$$

with

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \quad \frac{\partial u(x,t)}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega, t > 0 \tag{6.1.33}$$

can be sandwiched between solutions of

$$v_t(t) = -v^3(t), \quad v(0) = m, \quad \text{and } w_t(t) = -w^3(t), \quad w(0) = M \tag{6.1.34}$$

with $m \leq u_0(x) \leq M$, i.e., we have

$$v(t) \leq u(x,t) \leq w(t) \quad \text{for } x \in \bar{\Omega}, t > 0. \tag{6.1.35}$$

Since v and w as solutions of (6.1.34) tend to 0 for $t \to \infty$, we conclude that $u(x,t)$ (assuming that it exists for all $t \geq 0$) also tends to 0 for $t \to \infty$ uniformly in $x \in \Omega$.

We now come to one of the topics that make reaction–diffusion interesting and useful models for pattern formation, namely, travelling waves.
We consider the reaction–diffusion equation in one-dimensional space

$$u_t = u_{xx} + f(u) \tag{6.1.36}$$

and look for solutions of the form

$$u(x,t) = v(x - ct) = v(s), \text{ with } s := x - ct. \tag{6.1.37}$$

This travelling wave solution moves at constant speed c, assumed to be > 0 w.l.o.g, in the increasing x-direction. In particular, if we move the coordinate system with speed c, i.e., keep $x - ct$ constant, then the solution also stays constant. We do not expect such a solution for every wave speed c but at most for particular values that then need to be determined.

A travelling wave solution $v(s)$ of (6.1.36) satisfies the ODE

$$v''(s) + cv'(s) + f(v) = 0, \text{ with } ' = \frac{\mathrm{d}}{\mathrm{d}s}. \tag{6.1.38}$$

When $f \equiv 0$, then a solution must be of the form $v(s) = c_0 + c_1 \mathrm{e}^{-cs}$ and therefore becomes unbounded for $s \to -\infty$, i.e., for $t \to \infty$. In other words, for the heat equation, there is no nontrivial bounded travelling wave. In contrast to this, depending on the precise nonlinear structure of f, such travelling wave solutions may exist for reaction–diffusion equations. This is one of the reasons why such equations are interesting.

As an example, we consider the Fisher equation in one dimension,

$$u_t = u_{xx} + u(1-u). \tag{6.1.39}$$

This is a model for the growth of populations under limiting constraints: The term $-u^2$ on the r.h.s. limits the population size. Due to such an interpretation, one is primarily interested in nonnegative solutions.

We now apply some standard concepts from dynamical systems[1] to the underlying reaction equation

$$u_t = u(1-u). \tag{6.1.40}$$

The fixed points of this equation are $u = 0$ and $u = 1$. The first one is unstable, the second one stable. The travelling wave equation (6.1.38) then is

$$v''(s) + cv'(s) + v(1-v) = 0. \tag{6.1.41}$$

With $w := v'$, this is converted into the first-order system

$$v' = w, \quad w' = -cw - v(1-v). \tag{6.1.42}$$

The fixed points then are $(0,0)$ and $(1,0)$. The eigenvalues of the linearization at $(0,0)$, i.e., of the linear system

$$v' = \mu, \quad \mu' = -c\mu - v, \tag{6.1.43}$$

are

$$\lambda_\pm = \frac{1}{2}\left(-c \pm \sqrt{c^2-4}\right). \tag{6.1.44}$$

For $c^2 \geq 4$, they are both real and negative, and so the solution of (6.1.43) yields a stable node. For $c^2 < 4$, they are conjugate complex with a negative real part, and we obtain a stable spiral. Since a stable spiral oscillates about 0, in that case, we cannot expect a nonnegative solution, and so, we do not consider this case here. Also, for symmetry reasons, we may restrict ourselves to the case $c > 0$, and since we want to exclude the spiral then to $c \geq 2$.

The eigenvalues of the linearization at $(1,0)$, i.e., of the linear system

$$v' = \mu, \quad \mu' = -c\mu + v, \tag{6.1.45}$$

are

$$\lambda_\pm = \frac{1}{2}\left(-c \pm \sqrt{c^2+4}\right); \tag{6.1.46}$$

[1] Readers who are not familiar with this can consult [17].

they are real and of different signs, and we obtain a saddle. Thus, the stability properties are reversed when compared to (6.1.40) which, of course, results from the fact that $\frac{ds}{dt} = -c$ is negative.

For $c \geq 2$, one finds a solution with $v \geq 0$ from $(1,0)$ to $(0,0)$, i.e., with $v(-\infty) = 1, v(\infty) = 0$. $v' \leq 0$ for this solution. We recall that the value of a travelling wave solution is constant when $x - ct$ is constant. Thus, in the present case, when time t advances, the values for large negative values of x which are close to 1 are propagated to the whole real line, and for $t \to \infty$, the solution becomes 1 everywhere. In this sense, the behavior of the ODE (6.1.40) where a trajectory goes from the unstable fixed point 0 to the stable fixed point 1 is translated into a travelling wave that spreads a nucleus taking the value 1 for $x = -\infty$ to the entire space.

The question for which initial conditions a solution of (6.1.39) evolves to such a travelling wave, and what the value of c then is has been widely studied in the literature since the seminal work of Kolmogorov and his coworkers [22]. For example, they showed when $u(x,0) = 1$ for $x \leq x_1$, $0 \leq u(x,0) \leq 1$ for $x_1 \leq x \leq x_2$, $u(x,0) = 0$ for $x \geq x_2$, then the solution $u(x,t)$ evolves towards a travelling wave with speed $c = 2$. In general, the wave speed c depends on the asymptotic behavior of $u(x,0)$ for $x \to \pm\infty$.

6.2 Reaction–Diffusion Systems

In this section, we extend the considerations of the previous section to systems of coupled reaction–diffusion equations. More precisely, we wish to study the initial boundary value problems for nonlinear parabolic systems of the form

$$u_t^\alpha(x,t) - d_\alpha \Delta u^\alpha(x,t) = F^\alpha(x,t,u) \quad \text{for } x \in \Omega, t > 0, \alpha = 1,\dots,n, \quad (6.2.1)$$

for suitable initial and boundary conditions. Here, $u = (u^1,\dots,u^n)$ consists of n components, the d_α are nonnegative constants, and the functions $F^\alpha(x,t,u)$ are assumed to be continuous w.r.t. x,t and Lipschitz continuous w.r.t. u, as in the preceding section. Again, the u-dependence here is the important one.

We note that in (6.2.1), the different components u^α are only coupled through the nonlinear terms $F(x,t,u)$ while the left-hand side of (6.2.1) for each α only involves u^α, but no other component u^β for $\beta \neq \alpha$. Here, we allow some of the diffusion constants d_α to vanish. The corresponding equation for $u^\alpha(x,t)$ then becomes an ordinary differential equation with the spatial coordinate x assuming the role of a parameter. If we ignore the coupling with other components u^β with positive diffusion constants d_β, then such a $u^\alpha(x,t)$ evolves independently for each position x. In particular, in the absence of diffusion, it is no longer meaningful to impose a Dirichlet boundary condition. When d_α is positive, however, diffusion between the different spatial positions takes place. We have already explained in Sect. 5.1 why the diffusion constants should not be negative.

We first observe that, when we assume that the d_α are positive, the proofs of Theorem 6.1.1 and Corollary 6.1.1 extend to the present case when we make corresponding assumptions on the initial and boundary values. The reason is that the proof of Theorem 6.1.1 only needs norm estimates coming from Lipschitz bounds, but no further detailed knowledge on the structure of the right-hand side. Thus

Corollary 6.2.1. *Let the diffusion constants d_α all be positive. Under the assumptions of Theorem 6.1.1 for the right-hand side components F^α, and with the same type of boundary conditions for the components u^α, suppose that the solution $u(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ of (6.2.1) satisfies the a priori bound*

$$\sup_{x\in\bar{\Omega},0\le\tau\le t} |u(x,\tau)| \le K \tag{6.2.2}$$

for all times t for which it exists, with some fixed constant K. Then the solution $u(x,t)$ exists for all times $0 \le t < \infty$.

In the sequel, we shall assume that the reaction term F depends on u only, but not explicitly on x or t. That is, we shall consider the system

$$u^\alpha_t(x,t) - d_\alpha \Delta u^\alpha(x,t) = F^\alpha(u(x,t)) \quad \text{for } x \in \Omega, t > 0, \alpha = 1,\ldots,n, \tag{6.2.3}$$

with further assumptions on F to be specified in a moment.
For the following considerations, it will be simplest to assume homogeneous Neumann boundary conditions

$$\frac{\partial u^\alpha(x,t)}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0,\ \alpha = 1,\ldots,n. \tag{6.2.4}$$

Again, we assume that the solution $u(x,t)$ stays bounded and consequently exists for all time. We want to compare $u(x,t)$ with its spatial average $\bar{u}$ defined by

$$\bar{u}^\alpha(t) := \frac{1}{\|\Omega\|}\int_\Omega u^\alpha(x,t)\mathrm{d}x, \tag{6.2.5}$$

where $\|\Omega\|$ is the Lebesgue measure of Ω.
We also assume that the right-hand side F is differentiable w.r.t. u, and

$$\sup_{x,t} \|\frac{\mathrm{d}F(u)}{\mathrm{d}u}\| \le L. \tag{6.2.6}$$

Finally, let

$$d_0 := \min_{\alpha=1,\ldots,n} d_\alpha > 0 \tag{6.2.7}$$

and $\lambda_1 > 0$ be the smallest Neumann eigenvalue of Δ on Ω, according to Theorem 11.5.2 below. We then have

Theorem 6.2.1. *Assume that $u(x,t)$ is a bounded solution of (6.2.1) with homogeneous Neumann boundary conditions (6.2.4). Assume that*

$$\delta := d_0\lambda_1 - L > 0. \tag{6.2.8}$$

Then

$$\int_\Omega \sum_{i=1}^{d} |u_{x^i}(x,t)|^2 \mathrm{d}x \le c_1 \mathrm{e}^{-2\delta t} \tag{6.2.9}$$

for a constant c_1, and

$$\int_\Omega |u(x,t) - \bar{u}(t)|^2 \mathrm{d}x \le c_2 \mathrm{e}^{-2\delta t} \tag{6.2.10}$$

for a constant c_2.

Thus, under the conditions of the theorem, spatial oscillations decay exponentially, and the solution asymptotically behaves like its spatial average. In the next Sect. 6.3, we shall investigate situations where this does not happen.

Proof. We put, similarly to Sect. 5.2,

$$E(u(\cdot,t)) = \frac{1}{2}\int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{n} \frac{1}{d_\alpha}(u^\alpha_{x^i})^2 \mathrm{d}x$$

and compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(u(\cdot,t)) &= \int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{n} \frac{1}{d_\alpha} u^\alpha_{tx^i} u^\alpha_{x^i}\mathrm{d}x \\
&= \int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{n} \frac{1}{d_\alpha} u^\alpha_{x^i} \frac{\partial(d_\alpha \Delta u^\alpha + F^\alpha(u))}{\partial x^i}\mathrm{d}x \\
&= -\int_\Omega \sum_\alpha (\Delta u^\alpha)^2 \mathrm{d}x + \int_\Omega \sum_{i=1}^{d}\sum_{\alpha=1}^{n} \frac{1}{d_\alpha} u^\alpha_{x^i} \sum_\beta \frac{\partial F^\alpha}{\partial u^\beta} u^\beta_{x^i}, \\
\text{since } \frac{\partial u(x,t)}{\partial \nu} &= 0 \quad \text{for } x \in \partial\Omega \\
&\le -\lambda_1 \int_\Omega \sum_{i=1}^{d}\sum_\alpha (u^\alpha_{x^i})^2 \mathrm{d}x + L\int_\Omega \sum_{i=1}^{d}\sum_\alpha \frac{1}{d_\alpha}(u^\alpha_{x^i})^2 \mathrm{d}x \\
&\le -2\delta E(u(\cdot,t)),
\end{aligned} \tag{6.2.11}$$

using Corollary 11.5.1 below and (6.2.8). This differential inequality by integration readily implies (6.2.9).

By Corollary 11.5.1 again, we have

$$\lambda_1 \int_\Omega |u(x,t) - \bar{u}(t)|^2 \mathrm{d}x \le \int_\Omega \sum_{i=1}^d u_{x^i}(x,t)^2 \mathrm{d}x, \tag{6.2.12}$$

and so (6.2.9) implies (6.2.10). □

We now consider the case where all the diffusion constants d_α in (6.2.3) are equal. After rescaling, we may then assume that all $d_\alpha = 1$ so that we are looking at the system

$$u^\alpha_t(x,t) - \Delta u^\alpha(x,t) = F^\alpha(u(x,t)) \quad \text{for } x \in \Omega, t > 0. \tag{6.2.13}$$

We then have

Theorem 6.2.2. *Assume that $u(x,t)$ is a bounded solution of (6.2.13) with homogeneous Neumann boundary conditions (6.2.4). Assume that*

$$\delta = \lambda_1 - L > 0. \tag{6.2.14}$$

Then

$$\sup_{x\in\Omega} |u(x,t) - \bar{u}(t)| \le c_3 \mathrm{e}^{-2\delta t} \tag{6.2.15}$$

for a constant c_3.

Proof (incomplete). We shall leave out the summation over the index α in our notation, i.e., write u_t^2 or $u_t u_t$ in place of $\sum_{\alpha=1}^n u_t^\alpha u_t^\alpha$ and so on.

As in Sect. 5.2, we compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\frac{1}{2}u_t^2 &= u_t u_{tt} - u_t \Delta u_t - \sum_{i=1}^d u^2_{x^i t} \\ &= u_t \frac{\partial}{\partial t}(u_t - \Delta u) - \sum_{i=1}^d u^2_{x^i t} \\ &\le L u_t^2 - \sum_{i=1}^d (u_{x^i t})^2. \end{aligned} \tag{6.2.16}$$

Therefore, by Corollary 11.5.1,

$$\frac{\partial}{\partial t}\int_\Omega u_t^2 = \int_\Omega \left(\frac{\partial}{\partial t}u_t^2 - \Delta u_t^2\right) \le 2(L - \lambda_1)\int_\Omega u_t^2 \le 0 \tag{6.2.17}$$

by (6.2.14). By parabolic regularity theory (a reference is [9]), we then get control over higher norms of u; this is analogous to elliptic regularity theory but not carried out in detail in this book. Actually, most of what we need can be derived from elliptic regularity theory, except for the following bound which follows from (6.2.17).

$$v(t) := \sup_{x\in\Omega} \left|\frac{\partial u(x,t)}{\partial t}\right|^2$$

is a nonincreasing function of t. In particular, $\frac{\partial u(x,t)}{\partial t}$ remains uniformly bounded in t. Writing our equation for u^α as[1]

$$\Delta u^\alpha(x,t) = \frac{1}{d_\alpha}(u^\alpha_t(x,t) - F^\alpha(x,t,u)), \tag{6.2.18}$$

we may then apply Theorem 13.1.2(a) below to obtain $C^{1,\sigma}$ bounds on $u(x,t)$ as a function of x that are independent of t, for some $0 < \sigma < 1$. Then, first using the Sobolev embedding Theorem 11.1.1 for some $p > d$, and then these pointwise, time-independent bounds on $u(x,t)$ and $\frac{\partial u(x,t)}{\partial x^i}$,

$$\begin{aligned}\sup_{x\in\Omega} |u(x,t) - \bar u(t)| &\le \int_\Omega |u(x,t) - \bar u(t)|^p \mathrm{d}x \\ &\quad + \int_\Omega \sum_i \left|\frac{\partial}{\partial x^i}(u(x,t) - \bar u(t))\right|^p \mathrm{d}x \\ &\le c\int_\Omega |u(x,t) - \bar u(t)|^2 \mathrm{d}x + c\int_\Omega \sum_i \left|\frac{\partial u(x,t)}{\partial x^i}\right|^2 \mathrm{d}x,\end{aligned}$$

for some constant c. From (6.2.9) and (6.2.10), we then obtain (6.2.15). □

A reference for reaction–diffusion equations and systems that we have used in this chapter is [29].

6.3 The Turing Mechanism

The turing mechanism is a reaction–diffusion system that has been proposed as a model for biological and chemical pattern formation. We discuss it here in order to show how the interaction between reaction and diffusion processes can give

[1] For this step, we no longer need the assumption that the d_α are all equal, and so, we keep them in the next formula, nor the assumption that F does not depend on x and t, and so, we also allow for that in our formula.

rise to structures that neither of the two processes is capable of creating by itself. The Turing mechanism creates instabilities w.r.t. spatial variables for temporally stable states in a system of two coupled reaction–diffusion equations with different diffusion constants. This is in contrast to the situation considered in the previous §, where we have derived conditions under which a solution asymptotically becomes spatially constant (see Theorems 6.2.1 and 6.2.2). In this section, we shall need to draw upon some results about eigenvalues of the Laplace operator that will only be established in Sect. 11.5 below (see in particular Theorem 11.5.2).

The system is of the form

$$\begin{aligned} u_t &= \Delta u + \gamma f(u,v), \\ v_t &= d\Delta v + \gamma g(u,v), \end{aligned} \tag{6.3.1}$$

where the important parameter is the diffusion constant d that will subsequently be taken > 1. Its relation with the properties of the reaction functions f, g will drive the whole process. The parameter $\gamma > 0$ is only introduced for the subsequent analysis, instead of absorbing it into the functions f and g. Here $u, v : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ for some bounded domain $\Omega \subset \mathbb{R}^d$ of class C^∞, and we fix the initial values

$$u(x,0), v(x,0) \quad \text{for } x \in \Omega,$$

and impose Neumann boundary conditions

$$\frac{\partial u}{\partial n}(x,t) = 0 = \frac{\partial v}{\partial n}(x,t) \quad \text{for all } x \in \partial\Omega, t \geq 0.$$

One can also study Dirichlet type boundary condition, for example, $u = u_0, v = v_0$ on $\partial\Omega$ where u_0 and v_0 are a fixed point of the reaction system as introduced below. In fact, the easiest analysis results when we assume periodic boundary conditions.

In order to facilitate the mathematical analysis, we have rescaled the independent as well as the dependent variables compared to the biological or chemical models treated in the literature on pattern formation. We now present some such examples, again in our rescaled version. All parameters a, b, ρ, K, k in those examples are assumed to be positive.

(1) Schnakenberg reaction

$$\begin{aligned} u_t &= \Delta u + \gamma(a - u + u^2 v), \\ v_t &= d\Delta v + \gamma(b - u^2 v). \end{aligned}$$

(2) Gierer–Meinhardt system

$$\begin{aligned} u_t &= \Delta u + \gamma\left(a - bu + \frac{u^2}{v}\right), \\ v_t &= d\Delta v + \gamma(u^2 - v). \end{aligned}$$

(3) Thomas system

$$u_t = \Delta u + \gamma \left(a - u - \frac{\rho uv}{1 + u + Ku^2} \right),$$
$$v_t = d\Delta v + \gamma \left(\alpha(b - v) - \frac{\rho uv}{1 + u + Ku^2} \right).$$

A slightly more general version of (2) is

(2')

$$u_t = \Delta u + \gamma \left(a - u + \frac{u^2}{v(1 + ku^2)} \right),$$
$$v_t = d\Delta v + \gamma(u^2 - v).$$

We turn to the general discussion of the Turing mechanism. We assume that we have a fixed point (u_0, v_0) of the reaction system:

$$f(u_0, v_0) = 0 = g(u_0, v_0).$$

We furthermore assume that this fixed point is linearly stable. This means that for a solution w of the linearized problem

$$w_t = \gamma A w, \quad \text{with } A = \begin{pmatrix} f_u(u_0, v_0) & f_v(u_0, v_0) \\ g_u(u_0, v_0) & g_v(u_0, v_0) \end{pmatrix}, \tag{6.3.2}$$

we have $w \to 0$ for $t \to \infty$. Thus, all eigenvalues λ of A must have

$$\mathrm{Re}(\lambda) < 0,$$

as solutions are linear combinations of terms behaving like $\mathrm{e}^{\lambda t}$.

The eigenvalues of A are the solutions of

$$\lambda^2 - \gamma(f_u + g_v)\lambda + \gamma^2(f_u g_v - f_v g_u) = 0 \tag{6.3.3}$$

(all derivatives of f and g are evaluated at (u_0, v_0)); hence

$$\lambda_{1,2} = \frac{1}{2}\gamma \left((f_u + g_v) \pm \sqrt{(f_u + g_v)^2 - 4(f_u g_v - f_v g_u)} \right). \tag{6.3.4}$$

We have $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) < 0$ if

$$f_u + g_v < 0, \quad f_u g_v - f_v g_u > 0. \tag{6.3.5}$$

The linearization of the full reaction–diffusion system about (u_0, v_0) is

$$w_t = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \Delta w + \gamma A w. \tag{6.3.6}$$

We let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ on Ω with Neumann boundary conditions, and y_k be a corresponding orthonormal basis of eigenfunctions, as established in Theorem 11.5.2 below,

$$\begin{aligned} \Delta y_k + \lambda_k y_k &= 0 \quad \text{in } \Omega, \\ \frac{\partial y_k}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

When we impose the Dirichlet boundary conditions $u = u_0, v = v_0$ on $\partial\Omega$ in place of Neumann conditions, we should then use the Dirichlet eigenfunctions established in Theorem 11.5.1. We then look for solutions of (6.3.6) of the form

$$w_k \mathrm{e}^{\lambda t} = \begin{pmatrix} \alpha y_k \\ \beta y_k \end{pmatrix} \mathrm{e}^{\lambda t}$$

with real α, β. Inserting this into (6.3.6) yields

$$\lambda w_k = -\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \lambda_k w_k + \gamma A w_k. \tag{6.3.7}$$

For a nontrivial solution of (6.3.7), λ thus has to be an eigenvalue of

$$\left(\gamma A - \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \lambda_k\right).$$

The eigenvalue equation is

$$\begin{aligned} &\lambda^2 + \lambda(\lambda_k(1+d) - \gamma(f_u + g_v)) \\ &\quad + d\lambda_k{}^2 - \gamma(df_u + g_v)\lambda_k + \gamma^2(f_u g_v - f_v g_u) = 0. \end{aligned} \tag{6.3.8}$$

We denote the solutions by $\lambda(k)_{1,2}$.

Equation (6.3.5) then means that

$$\operatorname{Re} \lambda(0)_{1,2} < 0 \quad (\text{recall } \lambda_0 = 0).$$

We now wish to investigate whether we can have

$$\operatorname{Re} \lambda(k) > 0 \tag{6.3.9}$$

for some higher mode λ_k.

Since by (6.3.5), $\lambda_k > 0, d > 0$, clearly

$$\lambda_k(1+d) - \gamma(f_u + g_v) > 0,$$

we need for (6.3.9) that

$$d\lambda_k{}^2 - \gamma(df_u + g_v)\lambda_k + \gamma^2(f_u g_v - f_v g_u) < 0. \tag{6.3.10}$$

Because of (6.3.5), this can only happen if

$$df_u + g_v > 0.$$

Computing this with the first equation of (6.3.5), we thus need

$$d \neq 1,$$
$$f_u g_v < 0.$$

If we assume

$$f_u > 0, \quad g_v < 0, \tag{6.3.11}$$

then we need

$$d > 1. \tag{6.3.12}$$

This is not enough to get (6.3.10) negative. In order to achieve this for some value of λ_k, we first determine that value μ of λ_k for which the lhs of (6.3.10) is minimized, i.e.,

$$\mu = \frac{\gamma}{2d}(df_u + g_v), \tag{6.3.13}$$

and we then need that the lhs of (6.3.10) becomes negative for $\lambda_k = \mu$. This is equivalent to

$$\frac{(df_u + g_v)^2}{4d} > f_u g_v - f_v g_u. \tag{6.3.14}$$

If (6.3.14) holds, then the lhs of (6.3.10) has two values of λ_k where it vanishes, namely,

$$\begin{aligned}\mu_\pm &= \frac{\gamma}{2d}\left((df_u + g_v) \pm \sqrt{(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u)}\right) \\ &= \frac{\gamma}{2d}\left((df_u + g_v) \pm \sqrt{(df_u - g_v)^2 + 4df_v g_u}\right)\end{aligned} \tag{6.3.15}$$

and it becomes negative for

$$\mu_- < \lambda_k < \mu_+. \tag{6.3.16}$$

We conclude

Lemma 6.3.1. *Suppose (6.3.14) holds. Then* (u_0, v_0) *is spatially unstable w.r.t. the mode* λ_k*, i.e., there exists a solution of (6.3.7) with*

$$Re\ \lambda > 0$$

if λ_k *satisfies (6.3.16), where* $\mu_\pm$ *are given by (6.3.15).*

Equation (6.3.14) is satisfied for

$$d > d_c = -\frac{2f_v g_u - f_u g_v}{f_u^2} + \frac{2}{f_u^2}\sqrt{f_v g_u (f_v g_u - f_u g_v)}\,. \tag{6.3.17}$$

Whether there exists an eigenvalue λ_k of Δ satisfying (6.3.16) depends on the geometry of Ω. In particular, if Ω is small, all nonzero eigenvalues are large (see Corollaries 11.5.2, 11.5.3 for some results in this direction), and so it may happen that for a given Ω, all nonzero eigenvalues are larger than μ_+. In that case, no Turing instability can occur.

We may also view this somewhat differently. Namely, given Ω, we have the smallest nonzero eigenvalue λ_1. Recalling that μ_+ in (6.3.15) depends on the parameter γ, we may choose $\gamma > 0$ so small that

$$\mu_+ < \lambda_1.$$

Then, again, (6.3.16) cannot be solved, and no Turing instability can occur. In other words, for a Turing instability, we need a certain minimal domain size for a given reaction strength or a certain minimal reaction strength for a given domain size.

If the condition (6.3.16) is satisfied for some eigenvalue λ_k, it is also of geometric significance for which value of k this happens. Namely, by Courant's nodal domain theorem (see the remark at the end of Sect. 11.5), the nodal set $\{y_k = 0\}$ of the eigenfunction y_k divides Ω into at most $(k+1)$ regions. On any of these regions, y_k then has a fixed sign, i.e., is either positive or negative on that entire region. Since y_k is the unstable mode, this controls the number of oscillations of the developing instability.

We summarize Turing's result

Theorem 6.3.1. *Suppose that at a solution* (u_0, v_0) *of*

$$f(u_0, v_0) = 0 = g(u_0, v_0), \tag{6.3.18}$$

we have

$$f_u + g_v < 0, \tag{6.3.19}$$

$$f_u g_v - f_v g_u > 0. \tag{6.3.20}$$

Then (u_0, v_0) *is linearly stable for the reaction system*

$$u_t = \gamma f(u, v),$$
$$v_t = \gamma g(u, v).$$

Suppose that $d > 1$ *satisfies*

$$df_u + g_v > 0, \tag{6.3.21}$$

$$(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0. \tag{6.3.22}$$

Then (u_0, v_0) *as a solution of the reaction–diffusion system*

$$u_t = \Delta u + \gamma f(u, v),$$
$$v_t = d\Delta v + \gamma g(u, v)$$

is linearly unstable against spatial oscillations with eigenvalue λ_k *whenever* λ_k *satisfies (6.3.16).*

When we compare (6.3.19) and (6.3.21), we see that necessarily $f_u > 0$, since $d > 1$. Also, f_v and g_u must then have opposite signs for (6.3.19), and let us assume that $g_u > 0$. We then have $f_v < 0$ and $g_v < 0$. We may then interpret u as the density of an activator and v as that of an inhibitor. At (6.3.18), the activator and the inhibitor are in balance. The Turing mechanism tells us that this balance can get destroyed when the inhibitor is diffusing faster than the activator ($d > 1$). Consequently, at some places, the density of the inhibitor can get so low that it no longer inhibits the growth of the activator to keep the latter confined within suitable bounds. This then is the source of the Turing instability. For this mechanism to work, the frequency of the spatial oscillation patterns must be carefully controlled, see (6.3.16).

Since we assume that Ω is bounded, the eigenvalues λ_k of Δ on Ω are discrete, and so it also depends on the geometry of Ω whether such an eigenvalue in the range determined by (6.3.16) exists. The number k controls the frequency of oscillations of the instability about (u_0, v_0) and thus determines the shape of the resulting spatial pattern.

Thus, in the situation described in Theorem 6.3.1, the equilibrium state (u_0, v_0) is unstable, and in the vicinity of it, perturbations grow at a rate $e^{\mathrm{Re}\lambda}$, where λ solves (6.3.8).

Typically, one assumes, however, that the dynamics is confined within a bounded region in $(\mathbb{R}^+)^2$. This means that appropriate assumptions on f and g for $u = 0$ or $v = 0$, or for u and v large ensure that solutions starting in the positive quadrant can

neither become zero nor unbounded. It is essentially a consequence of the maximum principle that if this holds for the reaction system, then it also holds for the reaction–diffusion system, see the discussion in Sects. 6.1 and 6.2.

Thus, even though (u_0, v_0) is locally unstable, small perturbations grow exponentially; this growth has to terminate eventually, and one expects that the corresponding solution of the reaction–diffusion system settles at a spatially inhomogeneous steady state. This is the idea of the Turing mechanism. This has not yet been demonstrated in full rigour and generality. So far, the existence of spatially heterogeneous solutions has only been shown by singular perturbation analysis near the critical parameter d_c in (6.3.17). Thus, from the global and nonlinear perspective adopted in this book, the topic has not yet received a complete and satisfactory mathematical treatment.

We want to apply Theorem 6.3.1 to the example (1). In that case we have

$$\begin{aligned} u_0 &= a + b, \\ v_0 &= \frac{b}{(a+b)^2} \qquad \text{(of course, } a, b > 0\text{)} \end{aligned}$$

and at (u_0, v_0) then

$$\begin{aligned} f_u &= \frac{b-a}{a+b}\,, \\ f_v &= (a+b)^2, \\ g_u &= -\frac{2b}{a+b}\,, \\ g_v &= -(a+b)^2, \\ f_u g_v - f_v g_u &= (a+b)^2 > 0. \end{aligned}$$

Since we need that f_u and g_v have opposite signs (in order to get $df_u + g_v > 0$ later on), we require

$$b > a.$$

$f_u + g_v < 0$ then implies

$$0 < b - a < (a+b)^3, \tag{6.3.23}$$

while $df_u + g_v > 0$ implies

$$d(b-a) > (a+b)^3. \tag{6.3.24}$$

Finally, $(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0$ requires

$$\left(d(b-a) - (a+b)^3\right)^2 > 4d(a+b)^4. \tag{6.3.25}$$

The parameters a, b, d satisfying (6.3.23)–(6.3.25) constitute the so-called Turing space for the reaction–diffusion system investigated here.
For many case studies of the Turing mechanism in biological pattern formation, we recommend [28].

Summary

In this chapter, we have studied reaction–diffusion equations

$$u_t(x,t) - \Delta u(x,t) = F(x,t,u) \quad \text{for } x \in \Omega, t > 0$$

as well as systems of this structure. They are nonlinear because of the u-dependence of F. Solutions of such equations combine aspects of the linear diffusion equation

$$u_t(x,t) - \Delta u(x,t) = 0$$

and of the nonlinear reaction equation

$$u_t(t) = F(t,u)$$

but can also exhibit genuinely new phenomena like travelling waves.
The Turing mechanism arises in systems of the form

$$\begin{aligned} u_t &= \Delta u + \gamma f(u,v), \\ v_t &= d\Delta v + \gamma g(u,v), \end{aligned}$$

under appropriate conditions, in particular when an inhibitor v diffuses at a faster rate than an enhancer u, i.e., when $d > 1$ and certain conditions on the derivatives f_u, f_v, g_u, g_v are satisfied. A Turing instability means that for such a system, a spatially homogeneous state becomes unstable. Thus, spatially nonconstant patterns will develop. This is obviously a genuinely nonlinear phenomenon.

Exercises

6.1. Consider the nonlinear elliptic equation

$$\begin{aligned} \Delta u(x) + \sigma u(x) - u^3(x) &= 0 \text{ in a domain } \Omega \subset \mathbb{R}^d, \\ u(y) &= 0 \text{ for } y \in \partial\Omega. \end{aligned} \tag{6.3.26}$$

Let λ_1 be the smallest Dirichlet eigenvalue of Ω (cf. Theorem 11.5.1 below). Show that for $\sigma < \lambda_1$, $u \equiv 0$ is the only solution (hint: multiply the equation by u and integrate by parts and use Corollary 11.5.1 below).

6.2. Consider the nonlinear elliptic system

$$d_\alpha \Delta u^\alpha(x) + F^\alpha(x, u) = 0 \quad \text{for } x \in \Omega,\ \alpha = 1, \dots, n, \tag{6.3.27}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u^\alpha(x)}{\partial \nu} = 0 \text{ for } x \in \partial\Omega,\ \alpha = 1, \dots, n. \tag{6.3.28}$$

Assume that

$$\delta = \lambda_1 \min_{\alpha=1,\dots,n} d_\alpha - L > 0 \tag{6.3.29}$$

as in Theorem 6.2.1. Show that $u \equiv \text{const}$.

6.3. Determine the Turing spaces for the Gierer–Meinhardt and Thomas systems.

6.4. Carry out the analysis of the Turing mechanism for periodic boundary conditions.