Chapter 11
Sobolev Spaces and L^2 Regularity Theory

11.1 General Sobolev Spaces. Embedding Theorems of Sobolev, Morrey, and John–Nirenberg

Definition 11.1.1. Let $u : \Omega \to \mathbb{R}$ be integrable, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_d)$,

$$D_{\boldsymbol{\alpha}}\varphi := \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^d}\right)^{\alpha_d} \varphi \quad \text{for } \varphi \in C^{|\boldsymbol{\alpha}|}(\Omega).$$

An integrable function $v : \Omega \to \mathbb{R}$ is called an $\boldsymbol{\alpha}$th weak derivative of u, in symbols $v = D_{\boldsymbol{\alpha}} u$, if

$$\int_\Omega \varphi v \, \mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u D_{\boldsymbol{\alpha}} \varphi \mathrm{d}x \quad \text{for all } \varphi \in C_0^{|\boldsymbol{\alpha}|}(\Omega). \tag{11.1.1}$$

For $k \in \mathbb{N}$, $1 \le p < \infty$, we define the Sobolev space

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D_{\boldsymbol{\alpha}} u \text{ exists and is contained in } L^p(\Omega) \text{ for all } |\boldsymbol{\alpha}| \le k\},$$

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| \le k} \int_\Omega |D_{\boldsymbol{\alpha}} u|^p\right)^{\frac{1}{p}}.$$

The spaces $H^{k,p}(\Omega)$ and $H_0^{k,p}(\Omega)$ are defined to be the closures of $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ and $C_0^\infty(\Omega)$, respectively, with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$. Occasionally, we shall employ the abbreviation $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$.

Concerning notation: The multi-index notation will be used in the present section only. Later on, for $u \in W^{1,p}(\Omega)$, first weak derivatives will be denoted by $D_i u$, $i = 1, \ldots, d$, as in Definition 10.2.1, and we shall denote the vector $(D_1 u, \ldots, D_d u)$

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by Du. Likewise, for $u \in W^{2,p}(\Omega)$, second weak derivatives will be written $D_{ij}u$, $i, j = 1, \ldots, d$, and the matrix of second weak derivatives will be denoted by D^2u.

As in Sect. 10.2, one proves the following lemma:

Lemma 11.1.1. *$W^{k,p}(\Omega) = H^{k,p}(\Omega)$. The space $W^{k,p}(\Omega)$ is complete with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$, i.e., it is a Banach space.*

We now state the Sobolev embedding theorem:

Theorem 11.1.1.

$$H_0^{1,p}(\Omega) \subset \begin{cases} L^{\frac{dp}{d-p}}(\Omega) & \text{for } p < d, \\ C^0(\bar{\Omega}) & \text{for } p > d. \end{cases}$$

Moreover, for $u \in H_0^{1,p}(\Omega)$,

$$\|u\|_{\frac{dp}{d-p}} \leq c\,\|Du\|_p \qquad \text{for } p < d, \tag{11.1.2}$$

$$\sup_{\Omega} |u| \leq c\,|\Omega|^{\frac{1}{d}-\frac{1}{p}} \cdot \|Du\|_p \quad \text{for } p > d, \tag{11.1.3}$$

where the constant c depends on p and d only.

In order to better understand the content of the Sobolev embedding theorem, we first consider the scaling behavior of the expressions involved: Let $f \in H^{1,p}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. We look at the scaling $y = \lambda x$ (with $\lambda > 0$) and

$$f_\lambda(y) := f\left(\frac{y}{\lambda}\right) = f(x).$$

Then, with $y = \lambda x$,

$$\left(\int_{\mathbb{R}^d} |Df_\lambda(y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} = \lambda^{\frac{d-p}{p}} \left(\int_{\mathbb{R}^d} |Df(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

(note that on the left, the derivative is taken with respect to y, and on the right with respect to x; this explains the $-p$ in the exponent) and

$$\left(\int_{\mathbb{R}^d} |f_\lambda(y)|^q \,\mathrm{d}y\right)^{\frac{1}{q}} = \lambda^{\frac{d}{q}} \left(\int_{\mathbb{R}^d} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Thus in the limit $\lambda \to 0$, $\|f_\lambda\|_{L^q}$ is controlled by $\|Df_\lambda\|_{L^p}$ if

$$\lambda^{\frac{d}{q}} \leq \lambda^{\frac{d-p}{p}} \quad \text{for } \lambda < 1$$

holds, i.e.,

$$\frac{d}{q} \geq \frac{d-p}{p},$$

i.e.,

$$q \leq \frac{dp}{d-p} \quad \text{if } p < d.$$

(We have implicitly assumed $\|Df\|_{L^p} > 0$ here, but you will easily convince yourself that this is the essential case of the embedding theorem.) We treat only the limit $\lambda \to 0$ here, since only for $\lambda \leq 1$ (for $f \in H_0^{1,p}(\mathbb{R}^d)$) do we have

$$\operatorname{supp} f_\lambda \subset \operatorname{supp} f,$$

and the Sobolev embedding theorem covers only the case where the functions have their support contained in a fixed bounded set Ω. Looking at the scaling properties for $\lambda \to \infty$, one observes that this assumption on the support is necessary for the theorem. The scaling properties for $p > d$ will be examined after Corollary 11.1.5.

Proof of Theorem 11.1.1: We shall first prove the inequalities (11.1.2) and (11.1.3) for $u \in C_0^1(\Omega)$. We put $u = 0$ on $\mathbb{R}^d \setminus \Omega$ again. As in the proof of Theorem 10.2.2,

$$|u(x)| \leq \int_{-\infty}^{x^i} |D_i u(x^1, \ldots, x^{i-1}, \xi, x^{i+1}, \ldots, x^d)| \, \mathrm{d}\xi \quad \text{with } x = (x^1, \ldots, x^d)$$

for $1 \leq i \leq d$, and hence

$$|u(x)|^d \leq \prod_{i=1}^d \int_{-\infty}^{\infty} |D_i u| \, \mathrm{d}x^i$$

and

$$|u(x)|^{\frac{d}{d-1}} \leq \left(\prod_{i=1}^d \int_{-\infty}^{\infty} |D_i u| \, \mathrm{d}x^i \right)^{\frac{1}{d-1}}.$$

It follows that

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{d}{d-1}} \, \mathrm{d}x^1 \leq \left(\int_{-\infty}^{\infty} |D_1 u| \, \mathrm{d}x^1 \right)^{\frac{1}{d-1}} \left(\prod_{i \neq 1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u| \, \mathrm{d}x^i \mathrm{d}x^1 \right)^{\frac{1}{d-1}},$$

where we have used (A.6) for $p_1 = \cdots = p_{d-1} = d - 1$. Iteratively, we obtain

$$\int_{\Omega} |u(x)|^{\frac{d}{d-1}} \, \mathrm{d}x \leq \left(\prod_{i=1}^d \int_{\Omega} |D_i u| \, \mathrm{d}x \right)^{\frac{1}{d-1}},$$

and hence

$$\|u\|_{\frac{d}{d-1}} \leq \left(\prod_{i=1}^{d} \int_{\Omega} |D_i u| \, \mathrm{d}x \right)^{\frac{1}{d}} \leq \frac{1}{d} \int_{\Omega} \sum_{i=1}^{d} |D_i u| \, \mathrm{d}x,$$

since the geometric mean is not larger than the arithmetic one, and consequently

$$\|u\|_{\frac{d}{d-1}} \leq \frac{1}{d} \|Du\|_1 , \tag{11.1.4}$$

which is (11.1.2) for $p = 1$.

Applying (11.1.4) to $|u|^\gamma$ $(\gamma > 1)$ ($|u|^\gamma$ is not necessarily contained in $C_0^1(\Omega)$, even if u is, but as will be explained at the end of the present proof, by an approximation argument, if shown for $C_0^1(\Omega)$, (11.1.4) continues to hold for $H_0^{1,1}$, and we shall choose γ such that for $u \in H_0^{1,p}(\Omega)$, we have $|u|^\gamma \in H_0^{1,1}(\Omega)$), we obtain

$$\| |u|^\gamma \|_{\frac{d}{d-1}} \leq \frac{\gamma}{d} \int_{\Omega} |u|^{\gamma-1} |Du| \, \mathrm{d}x \leq \frac{\gamma}{d} \left\| |u|^{\gamma-1} \right\|_q \cdot \|Du\|_p \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1 \tag{11.1.5}$$

applying Hölder's inequality (A.4). For $p < d$, $\gamma = \frac{(d-1)p}{d-p}$ satisfies

$$\frac{\gamma d}{d-1} = \frac{(\gamma - 1)p}{p-1},$$

and (11.1.5) yields, taking $q = \frac{p}{p-1}$ into account,

$$\|u\|^{\gamma}_{\frac{\gamma d}{d-1}} \leq \frac{\gamma}{d} \|u\|^{\gamma-1}_{\frac{\gamma d}{d-1}} \cdot \|Du\|_p ,$$

i.e.,

$$\|u\|_{\frac{\gamma d}{d-1}} \leq \frac{\gamma}{d} \|Du\|_p ,$$

which is (11.1.2). In order to establish (11.1.3), we need the following generalization of Lemma 10.2.4:

Lemma 11.1.2. *For* $\mu \in (0, 1]$, $f \in L^1(\Omega)$ *let*

$$(V_\mu f)(x) := \int_{\Omega} |x - y|^{d(\mu-1)} f(y) \mathrm{d}y.$$

Let $1 \leq p \leq q \leq \infty$,

$$0 \leq \delta = \frac{1}{p} - \frac{1}{q} < \mu.$$

Then V_μ *maps* $L^p(\Omega)$ *continuously to* $L^q(\Omega)$*, and for* $f \in L^p(\Omega)$*, we have*

$$\left\| V_\mu f \right\|_q \leq \left(\frac{1-\delta}{\mu-\delta} \right)^{1-\delta} \omega_d^{1-\mu} \, |\Omega|^{\mu-\delta} \, \| f \|_p \, . \tag{11.1.6}$$

Proof. Let

$$\frac{1}{r} := 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta.$$

Then

$$\ell(x-y) := |x-y|^{d(\mu-1)} \in L^r(\Omega),$$

and as in the proof of Lemma 10.2.4, we choose R such that $|\Omega| = |B(x,R)| = \omega_d R^d$, and we estimate as follows:

$$\begin{aligned}
\|\ell\|_r &= \left(\int_\Omega |x-y|^{\frac{d(\mu-1)}{1-\delta}} \, \mathrm{d}y \right)^{1-\delta} \\
&\leq \left(\int_{B(x,R)} |x-y|^{\frac{d(\mu-1)}{1-\delta}} \, \mathrm{d}y \right)^{1-\delta} \\
&= \left(\frac{1-\delta}{\mu-\delta} \right)^{1-\delta} \omega_d^{1-\delta} R^{d(\mu-\delta)} \\
&= \left(\frac{1-\delta}{\mu-\delta} \right)^{1-\delta} \omega_d^{1-\mu} \, |\Omega|^{\mu-\delta} \, .
\end{aligned}$$

We write

$$\ell \, |f| = \ell^{r(1-1/p)} \left(\ell^r \, |f|^p \right)^{\frac{1}{q}} |f|^{p\delta} \, ,$$

and the generalized Hölder inequality (A.6) yields

$$\begin{aligned}
&\left| V_\mu f(x) \right| \\
&\quad \leq \left(\int_\Omega \ell^r(x-y) \, |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{q}} \left(\int_\Omega \ell^r(x-y) \mathrm{d}y \right)^{1-\frac{1}{p}} \left(\int_\Omega |f(y)|^p \, \mathrm{d}y \right)^{\delta} ;
\end{aligned}$$

hence, integrating with respect to x and interchanging the integrations in the first integral, we obtain

$$\left\| V_\mu f \right\|_q \leq \sup_\Omega \left(\int \ell^r(x-y)\mathrm{d}y \right)^{\frac{1}{r}} \|f\|_p \leq \left(\frac{1-\delta}{\mu-\delta} \right)^{1-\delta} \omega_d^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p$$

by the above estimate for $\|\ell\|_r$. □

In order to complete the proof of Theorem 11.1.1, we use (10.2.9), assuming first $u \in C_0^1(\Omega)$ as before, i.e.,

$$u(x) = -\frac{1}{d\omega_d} \int_\Omega \sum_{i=1}^d \frac{(x^i - y^i)}{|x-y|^d} D_i u(y)\mathrm{d}y \tag{11.1.7}$$

for $x \in \Omega$. This implies

$$|u| \leq \frac{1}{d\omega_d} V_{\frac{1}{d}}(|D|). \tag{11.1.8}$$

Inequality (11.1.6) for $q = \infty$, $\mu = 1/d$ then yields (11.1.3), again at this moment for $u \in C_0^1(\Omega)$ only.

If now $u \in H_0^{1,p}(\Omega)$, we approximate u in the $W^{1,p}$-norm by C_0^∞ functions u_n, and apply (11.1.2) and (11.1.3) to the difference $u_n - u_m$. It follows that (u_n) is a Cauchy sequence in $L^{dp/(d-p)}(\Omega)$ (for $p < d$) or $C^0(\bar{\Omega})$ (for $p > d$), respectively. Thus u itself is contained in the same space and satisfies (11.1.2) or (11.1.3), respectively,

Corollary 11.1.1.

$$H_0^{k,p}(\Omega) \subset \begin{cases} L^{\frac{dp}{d-kp}}(\Omega) & \text{for } kp < d, \\ C^m(\Omega) & \text{for } 0 \leq m < k - \frac{d}{p}. \end{cases}.$$

Proof. The first embedding iteratively follows from Theorem 11.1.1, and the second one then from the first and the case $p > d$ in Theorem 11.1.1. □

Corollary 11.1.2. *If $u \in H_0^{k,p}(\Omega)$ for some p and all $k \in \mathbb{N}$, then $u \in C^\infty(\Omega)$.*

The embedding theorems to follow will be used in Chap. 14 only. First we shall present another variant of the Sobolev embedding theorem. For a function $v \in L^1(\Omega)$, we define the mean of v on Ω as

$$\fint_\Omega v(x)\mathrm{d}x := \frac{1}{|\Omega|} \int_\Omega v(x)\mathrm{d}x,$$

$|\Omega|$ denoting the Lebesgue measure of Ω. We then have the following result:

Corollary 11.1.3. *Let* $1 \le p < d$ *and* $u \in H^{1,p}(B(x_0, R))$*. Then*

$$\left(\fint_{B(x_0,R)} |u|^{\frac{dp}{d-p}}\right)^{\frac{d-p}{dp}} \le c_0 \left(R^p \fint_{B(x_0,R)} |Du|^p + \fint_{B(x_0,R)} |u|^p\right)^{\frac{1}{p}}, \tag{11.1.9}$$

where c_0 *depends on* p *and* d *only.*

Proof. Without loss of generality, $x_0 = 0$. Likewise, we may assume $R = 1$, since we may consider the functions $\tilde{u}(x) = u(Rx)$ and check that the expressions in (11.1.9) scale in the right way. Thus, let $u \in H^{1,p}(B(0,1))$. We extend u to the ball $B(0,2)$, by putting

$$u(x) = u\left(\frac{x}{|x|^2}\right) \quad \text{for } |x| > 1.$$

This extension satisfies

$$\|u\|_{H^{1,p}(B(0,2))} \le c_1 \|u\|_{H^{1,p}(B(0,1))}. \tag{11.1.10}$$

Now let $\eta \in C_0^\infty(B(0,2))$ with

$$\eta \ge 0, \quad \eta \equiv 1 \text{ on } B(0,1), \quad |D\eta| \le 2.$$

Then $v = \eta u \in H_0^{1,p}(B(0,2))$, and by (11.1.2),

$$\left(\int_{B(0,2)} |v|^{\frac{dp}{d-p}}\right)^{\frac{d-p}{dp}} \le c_2 \left(\int_{B(0,2)} |Dv|^p\right)^{\frac{1}{p}}. \tag{11.1.11}$$

Since

$$Dv = \eta Du + u D\eta,$$

from the properties of η, we deduce

$$|Dv|^p \le c_3 \left(|Du|^p + |u|^p\right), \tag{11.1.12}$$

and hence with (11.1.10),

$$\int_{B(0,2)} |Dv|^p \le c_4 \left(\int_{B(0,1)} |Du|^p + \int_{B(0,1)} |u|^p\right). \tag{11.1.13}$$

Since on the other hand

$$\int_{B(0,1)} |u|^{\frac{dp}{d-p}} \le \int_{B(0,2)} |v|^{\frac{dp}{d-p}},$$

(11.1.9) follows from (11.1.11) and (11.1.13). □

Later on (in Sect. 14.1), we shall need the following result of John and Inselberg:

Theorem 11.1.2. *Let $B(y_0, R_0)$ be a ball in $\mathbb{R}^d$, $u \in W^{1,1}(B(y_0, R_0))$, and suppose that for all balls $B(y, R) \subset \mathbb{R}^d$,*

$$\int_{B(y,R)\cap B(y_0,R_0)} |Du| \le R^{d-1}. \tag{11.1.14}$$

Then there exist $\alpha > 0$ and $\beta_0 < \infty$ satisfying

$$\int_{B(y_0,R_0)} \mathrm{e}^{\alpha|u-u_0|} \le \beta_0 R_0^d \tag{11.1.15}$$

with

$$u_0 = \frac{1}{\omega_d R_0^d} \int_{B(y_0,R_0)} u \quad (\textit{mean of } u \textit{ on } B(y_0, R_0)).$$

In particular,

$$\int_{B(y_0,R_0)} \mathrm{e}^{\alpha u} \int_{B(y_0,R_0)} \mathrm{e}^{-\alpha u} = \int_{B(y_0,R_0)} \mathrm{e}^{\alpha(u-u_0)} \int_{B(y_0,R_0)} \mathrm{e}^{-\alpha(u-u_0)} \le \beta_0^2 R_0^{2d}. \tag{11.1.16}$$

More generally, for a measurable set $B \subset \mathbb{R}^d$, and $u \in L^1(B)$, we denote the mean by

$$u_B := \frac{1}{|B|} \int_B u(y)\mathrm{d}y, \tag{11.1.17}$$

$|B|$ being the Lebesgue measure of B. In order to prepare the proof of Theorem 11.1.2, we start with a lemma:

Lemma 11.1.3. *Let $\Omega \subset \mathbb{R}^d$ be convex, $B \subset \Omega$ measurable with $|B| > 0$, $u \in W^{1,1}(\Omega)$. Then we have for almost all $x \in \Omega$,*

$$|u(x) - u_B| \le \frac{(\mathrm{diam}\,\Omega)^d}{d\,|B|} \int_\Omega |x - z|^{1-d}\, |Du(z)|\, \mathrm{d}z. \tag{11.1.18}$$

Proof. As before, it suffices to prove the inequality for $u \in C^1(\Omega)$. Since Ω is convex, if x and y are contained in Ω, so is the straight line joining them, and we have

$$u(x) - u(y) = -\int_0^{|x-y|} \frac{\partial}{\partial r} u\left(x + r\frac{y-x}{|y-x|}\right) \mathrm{d}r,$$

and thus

$$u(x) - u_B = \frac{1}{|B|}\int_B (u(x) - u(y))\mathrm{d}y$$
$$= -\frac{1}{|B|}\int_B \int_0^{|x-y|} \frac{\partial}{\partial r} u\left(x + r\frac{y-x}{|y-x|}\right)\mathrm{d}r\,\mathrm{d}y.$$

This implies

$$|u(x) - u_B| \leq \frac{1}{|B|}\frac{(\operatorname{diam}\Omega)^d}{d}\left|\int_{\substack{|\omega|=1\\ x+r\omega\in\Omega}} \int_0^{|x-y|} \frac{\partial}{\partial r} u(x + r\omega)\mathrm{d}r\,\mathrm{d}\omega\right|, \quad (11.1.19)$$

if instead of over B, we integrate over the ball $B(x, \operatorname{diam}\Omega)) \cap \Omega$, write $\mathrm{d}y = \varrho^{d-1}\mathrm{d}\omega\,\mathrm{d}\varrho$ in polar coordinates, and integrate with respect to ϱ. Thus, as in the proofs of Theorems 2.2.1 and 10.2.2,

$$|u(x) - u_B| \leq \frac{1}{|B|}\frac{(\operatorname{diam}\Omega)^d}{d}\left|\int_0^{|x-y|}\int_{\partial B(x,r)\cap\Omega} \frac{1}{r^{d-1}}\frac{\partial u}{\partial \nu}(z)\mathrm{d}\sigma(z)\mathrm{d}r\right|$$
$$= \frac{1}{|B|}\frac{(\operatorname{diam}\Omega)^d}{d}\left|\int_\Omega \frac{1}{|x-z|^{d-1}}\sum_{i=1}^d \frac{\partial}{\partial z^i}u(z)\frac{x^i - z^i}{|x-z|}\mathrm{d}z\right|$$
$$\leq \frac{(\operatorname{diam}\Omega)^d}{d\,|B|}\int_\Omega \frac{1}{|x-z|^{d-1}}\,|Du(z)|\,\mathrm{d}z. \qquad \square$$

We shall also need the following variant of Lemma 11.1.2:

Lemma 11.1.4. *Let $f \in L^1(\Omega)$, and suppose that for all balls $B(x_0, R) \subset \mathbb{R}^d$,*

$$\int_{\Omega\cap B(x_0,R)} |f| \leq KR^{d(1-\frac{1}{p})} \quad (11.1.20)$$

with some fixed K. Moreover, let $p > 1$, $1/p < \mu$. Then

$$\left|(V_\mu f)(x)\right| \leq \frac{p-1}{\mu p - 1}(\operatorname{diam}\Omega)^{d(\mu-\frac{1}{p})}K \quad (11.1.21)$$
$$\left((V_\mu f)(x) = \int_\Omega |x-y|^{d(\mu-1)} f(y)\mathrm{d}y\right).$$

Proof. We put $f = 0$ in the exterior of Ω. With $r = |x - y|$, then

$$\begin{aligned}
\left|V_\mu f(x)\right| &\le \int_\Omega r^{d(\mu-1)} |f(y)| \, \mathrm{d}y \\
&= \int_0^{\operatorname{diam}\Omega} r^{d(\mu-1)} \int_{\partial B(x,r)} |f(z)| \, \mathrm{d}z \mathrm{d}r \\
&= \int_0^{\operatorname{diam}\Omega} r^{d(\mu-1)} \left(\frac{\partial}{\partial r} \int_{B(x,r)} |f(y)| \, \mathrm{d}y \right) \mathrm{d}r \\
&= (\operatorname{diam}\Omega)^{d(\mu-1)} \int_{B(x,\operatorname{diam}\Omega)} |f(y)| \, \mathrm{d}y \\
&\quad + d(1-\mu) \int_0^{\operatorname{diam}\Omega} r^{d(\mu-1)-1} \int_{B(x,r)} |f(y)| \, \mathrm{d}y \mathrm{d}r \\
&\le K(\operatorname{diam}\Omega)^{d(\mu-1)+d(1-1/p)} \\
&\quad + Kd(1-\mu) \int_0^{\operatorname{diam}\Omega} r^{d(\mu-1)-1+d(1-1/p)} \mathrm{d}r \text{ by (11.1.20)} \\
&= K \frac{1-\frac{1}{p}}{\mu - \frac{1}{p}} (\operatorname{diam}\Omega)^{d(\mu-1/p)}.
\end{aligned}$$

□

Proof of Theorem 11.1.2: Because of (11.1.14), $f = |Du|$ satisfies the inequality (11.1.20) with $K = 1$ and $p = d$. Thus, by Lemma 11.1.4, for $\mu > 1/d$,

$$V_\mu(f)(x) = \int_{B(y_0,R_0)} |x-y|^{d(\mu-1)} |f(y)| \, \mathrm{d}y \le \frac{d-1}{\mu d - 1} (2R_0)^{\mu d - 1}. \quad (11.1.22)$$

In particular, for $s \ge 1$ and $\mu = \frac{1}{d} + \frac{1}{ds}$,

$$V_{\frac{1}{d}+\frac{1}{ds}}(f) \le (d-1)s(2R_0)^{\frac{1}{s}}. \quad (11.1.23)$$

By Lemma 11.1.2, we also have, for $s \ge 1$, $\mu = 1/ds$, $p = q = 1$,

$$\begin{aligned}
\int_{B(y_0,R_0)} V_{\frac{1}{ds}}(f) &\le ds\omega_d^{1-1/ds} |B(y_0,R_0)|^{\frac{1}{ds}} \|f\|_{L^1(B(y_0,R_0))} \\
&\le ds\omega_d R_0^{\frac{1}{s}} R_0^{d-1}
\end{aligned} \quad (11.1.24)$$

by (11.1.20), which, as noted, holds for $K = 1$ and $p = d$. Now

$$|x-y|^{1-d} = |x-y|^{d(\frac{1}{ds}-1)\frac{1}{s}} |x-y|^{d\left(\frac{1}{ds}+\frac{1}{d}-1\right)\left(1-\frac{1}{s}\right)}, \quad (11.1.25)$$

and from Hölder's inequality then

$$V_{\frac{1}{d}}(f) = \int \left(|x-y|^{d\left(\frac{1}{ds}-1\right)\frac{1}{s}}\,|f(y)|^{\frac{1}{s}}\right)\left(|x-y|^{d\left(\frac{1}{ds}+\frac{1}{d}-1\right)\left(1-\frac{1}{s}\right)}\,|f(y)|^{1-\frac{1}{s}}\right)\mathrm{d}y$$
$$\leq V_{\frac{1}{ds}}(f)^{\frac{1}{s}}\,V_{\frac{1}{d}+\frac{1}{ds}}(f)^{1-\frac{1}{s}}. \tag{11.1.26}$$

With (11.1.23) and (11.1.24), this implies

$$\begin{aligned}\int_{B(y_0,R_0)} V_{\frac{1}{d}}(f)^s &\leq ds\omega_d R_0^{d-1+\frac{1}{s}}(d-1)^{s-1}s^{s-1}(2R_0)^{\frac{s-1}{s}}\\ &\leq 2d(d-1)^{s-1}s^s\omega_d R_0^d\\ &= 2\frac{d}{d-1}\omega_d((d-1)s)^s R_0^d.\end{aligned}$$

Thus

$$\begin{aligned}\int_{B(y_0,R_0)}\sum_{n=0}^{\infty}\frac{V_{\frac{1}{d}}(f)^n}{\gamma^n n!} &\leq \frac{2d}{d-1}\omega_d R_0^d\sum_{n=0}^{\infty}\left(\frac{d-1}{\gamma}\right)^n\frac{n^n}{n!}\\ &\leq cR_0^d, \text{ if } \frac{d-1}{\gamma}<\frac{1}{e},\end{aligned}$$

i.e.,

$$\int_{B(y_0,R_0)}\exp\left(\frac{V_{1/d}(f)}{\gamma}\right)\leq cR_0^d. \tag{11.1.27}$$

Now by Lemma 11.1.3

$$|u(x)-u_0|\leq \text{const}\, V_{\frac{1}{d}}(|Du|), \tag{11.1.28}$$

and since we have proved (11.1.27) for $f=|Du|$, (11.1.15) follows.

Before concluding the present section, we would like to derive some further applications of the preceding lemmas, including the following version of the Poincaré inequality:

Corollary 11.1.4. *Let $\Omega\subset\mathbb{R}^d$ be convex, and $u\in W^{1,p}(\Omega)$. We then have for every measurable $B\subset\Omega$ with $|B|>0$,*

$$\left(\int_\Omega |u-u_B|^p\right)^{\frac{1}{p}}\leq\frac{\omega_d^{1-\frac{1}{d}}}{|B|}\,|\Omega|^{\frac{1}{d}}\,(\text{diam}\,\Omega)^d\left(\int_\Omega|Du|^p\right)^{\frac{1}{p}}. \tag{11.1.29}$$

Proof. By Lemma 11.1.3,

$$|u(x)-u_B|\leq\frac{(\text{diam}\,\Omega)^d}{d\,|B|}V_{\frac{1}{d}}(|Du|),$$

and by Lemma 11.1.2, then,

$$\left\| V_{\frac{1}{d}}(|Du|) \right\|_{L^p(\Omega)} \leq d\omega_d^{1-\frac{1}{d}} |\Omega|^{\frac{1}{d}} \|Du\|_{L^p(\Omega)},$$

and these two inequalities imply the claim. □

The next result is due to C.B. Morrey:

Theorem 11.1.3. *Assume $u \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^d$, and that there exist constants $K < \infty$, $0 < \alpha < 1$, such that for all balls $B(x_0, R) \subset \mathbb{R}^d$,*

$$\int_{\Omega \cap B(x_0,R)} |Du| \leq K R^{d-1+\alpha}. \tag{11.1.30}$$

Then we have for every ball $B(z,r) \subset \mathbb{R}^d$,

$$\operatorname*{osc}_{\Omega \cap B(z,r)} u := \sup_{x,y \in B(z,r) \cap \Omega} |u(x) - u(y)| \leq c K r^{\alpha}, \tag{11.1.31}$$

with $c = c(d, \alpha)$.

Proof. We have

$$\operatorname*{osc}_{\Omega \cap B(z,r)} u \leq 2 \sup_{x \in B(z,r) \cap \Omega} \left| u(x) - u_{B(z,r)} \right|$$

$$\leq c_1 \sup_{x \in B(z,r) \cap \Omega} \int_{B(z,r)} |x - y|^{1-d} |Du(y)| \, \mathrm{d}y$$

by Lemma 11.1.3, where c_1 depends on d only, and where we simply put $Du = 0$ on $\mathbb{R}^d \setminus \Omega$.

$$= c_1 \sup_{x \in B(z,r) \cap \Omega} V_{\frac{1}{d}}(|Du|)(x)$$

with the notation of Lemma 11.1.4. With

$$p = \frac{d}{1-\alpha}, \qquad \text{i.e., } \alpha = 1 - \frac{d}{p},$$

and

$$\mu = \frac{1}{d} > \frac{1}{p},$$

$f = |Du|$ then satisfies the assumptions of Lemma 11.1.4, and the preceding estimate together with Lemma 11.1.4 (applied to $B(z,r)$ in place of Ω) then yields

$$\operatorname*{osc}_{\Omega \cap B(z,r)} u \leq c_2 K (\operatorname{diam} B(z,r))^{1-\frac{d}{p}} = c K r^{\alpha}.$$

□

Definition 11.1.2. A function u defined on Ω is called α-Hölder continuous in Ω, for some $0 < \alpha < 1$, if for all $z \in \Omega$,

$$\sup_{x \in \Omega} \frac{|u(x) - u(z)|}{|x - z|^{\alpha}} < \infty. \tag{11.1.32}$$

Notation: $u \in C^{\alpha}(\Omega)$. For $u \in C^{\alpha}(\Omega)$, we put

$$\|u\|_{C^{\alpha}(\Omega)} := \|u\|_{C^{0}(\Omega)} + \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

(For $\alpha = 1$, a function satisfying (11.1.32) is called Lipschitz continuous, and the corresponding space is denoted by $C^{0,1}(\Omega)$.)

If u satisfies the assumptions of Theorem 11.1.3, it thus turns out to be α-Hölder continuous on Ω; this follows by putting $r = \operatorname{dist}(z, \partial\Omega)$ in Theorem 11.1.3. The notion of Hölder continuity will play a crucial role in Chaps. 13 and 14.

Theorem 11.1.3 now implies the following refinement, due to Morrey, of the Sobolev embedding theorem in the case $p > d$:

Corollary 11.1.5. *Let $u \in H_0^{1,p}(\Omega)$ with $p > d$. Then*

$$u \in C^{1-\frac{d}{p}}(\bar{\Omega}).$$

More precisely, for every ball $B(z, r) \subset \mathbb{R}^d$,

$$\operatorname*{osc}_{\Omega \cap B(z,r)} u \le c r^{1-\frac{d}{p}} \|Du\|_{L^p(\Omega)}, \tag{11.1.33}$$

where c depends on d and p only.

Once more, it helps in understanding the content of this embedding theorem if we take a look at the scaling properties of the norms involved: Let $f \in H^{1,p}(\mathbb{R}^d) \cap C^{\alpha}(\mathbb{R}^d)$ with $0 < \alpha < 1$. We again consider the scaling $y = \lambda x$ ($\lambda > 0$) and put

$$f_\lambda(y) = f(x).$$

Then

$$\frac{|f_\lambda(y_1) - f_\lambda(y_2)|}{|y_1 - y_2|^{\alpha}} = \lambda^{-\alpha} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^{\alpha}} \qquad (y_i = \lambda x_i,\ i = 1, 2)$$

and thus (ignoring the lower-order terms like $\|f\|_{C^0}$ in the definition of the norms for simplicity)

$$\|f_\lambda\|_{C^{\alpha}} = \lambda^{-\alpha} \|f\|_{C^{\alpha}},$$

and as has been computed above,

$$\|f_\lambda\|_{H^{1,p}} = \lambda^{\frac{d-p}{p}} \|f\|_{H^{1,p}} .$$

In the limit $\lambda \to 0$, thus $\|f_\lambda\|_{C^\alpha}$ is controlled by $\|Df_\lambda\|_{L^p}$, provided that

$$\lambda^{-\alpha} \leq \lambda^{\frac{d-p}{p}} \quad \text{for } \lambda < 1,$$

i.e.,

$$\alpha \leq 1 - \frac{d}{p} \quad \text{in the case } p > d.$$

Proof of Corollary 11.1.5: By Hölder's inequality

$$\int_{\Omega \cap B(x_0,R)} |Du| \leq |B(x_0,R)|^{1-\frac{1}{p}} \left(\int_{\Omega \cap B(x_0,R)} |Du|^p \right)^{\frac{1}{p}} \tag{11.1.34}$$

$$\leq c_3 \|Du\|_{L^p(\Omega)} R^{d\left(1-\frac{1}{p}\right)} \tag{11.1.35}$$

$$= c_3 \|Du\|_{L^p(\Omega)} R^{d-1+\left(1-\frac{d}{p}\right)}, \tag{11.1.36}$$

where c_3 depends on p and d only. Consequently, the assumptions of Theorem 11.1.3 hold.

The following version of Theorem 11.1.3 is called "Morrey's Dirichlet growth theorem" and is frequently used for showing the regularity of minimizers of variational problems:

Corollary 11.1.6. *Let $u \in W^{1,2}(\Omega)$, and suppose there exist constants $K' < \infty$, $0 < \alpha < 1$ such that for all balls $B(x_0,R) \subset \mathbb{R}^d$,*

$$\int_{\Omega \cap B(x_0,R)} |Du|^2 \leq K' R^{d-2+2\alpha}. \tag{11.1.37}$$

Then $u \in C^\alpha(\bar{\Omega})$, and for all balls $B(z,r)$,

$$\operatorname*{osc}_{B(z,r)\cap\Omega} u \leq c(K')^{\frac{1}{2}} r^\alpha, \tag{11.1.38}$$

with c depending only on d and α.

Proof. By Hölder's inequality

$$\int_{\Omega \cap B(x_0,R)} |Du| \le |B(x_0,R)|^{\frac{1}{2}} \left(\int_{\Omega \cap B(x_0,R)} |Du|^2 \right)^{\frac{1}{2}}$$
$$\le c_4 (K')^{\frac{1}{2}} R^{d-1+\alpha}$$

by (11.1.37), with c_4 depending on d only. Thus, the assumptions of Theorem 11.1.3 hold again. □

Finally, later on (in Sect. 14.4), we shall use the following result of Campanato characterizing Hölder continuity in terms of L^p-approximability by means on balls:

Theorem 11.1.4. *Let $p \ge 1$, $d < \lambda \le d + p$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain for which there exists some $\delta > 0$ with*

$$|B(x_0, r) \cap \Omega| \ge \delta r^d \quad \textit{for all } x_0 \in \Omega, r > 0. \tag{11.1.39}$$

Then a function $u \in L^p(\Omega)$ is contained in $C^\alpha(\Omega)$ for $\alpha = \frac{\lambda - d}{p}$ (or in $C^{0,1}(\Omega)$ in the case $\lambda = d + p$), precisely if there exists a constant $K < \infty$ with

$$\int_{B(x_0,r)\cap\Omega} \left|u(x) - u_{B(x_0,r)}\right|^p \mathrm{d}x \le K^p r^\lambda \quad \textit{for all } x_0 \in \Omega, r > 0 \tag{11.1.40}$$

(where for defining $u_{B(x_0,r)}$, we have extended u by 0 on $\mathbb{R}^d \setminus \Omega$).

Proof. Let $u \in C^\alpha(\Omega)$, $x \in \Omega \cap B(x_0, r)$. We then have

$$\left|u(x) - u_{B(x_0,r)}\right| \le (2r)^\alpha \|u\|_{C^\alpha(\Omega)},$$

and hence

$$\int_{B(x_0,r)\cap\Omega} \left|u - u_{B(x_0,r)}\right|^p \le c_5 \|u\|_{C^\alpha(\Omega)} r^{\alpha p + d},$$

whereby (11.1.40) is satisfied.

In order to prove the converse implication, we start with the following estimate for $0 < r < R$:

$$\left|u_{B(x_0,R)} - u_{B(x_0,r)}\right|^p \le 2^{p-1} \left(\left|u(x) - u_{B(x_0,R)}\right|^p + \left|u(x) - u_{B(x_0,r)}\right|^p \right),$$

and thus, integrating with respect to x on $\Omega \cap B(x_0, r)$ and using (11.1.39),

$$\left|u_{B(x_0,R)} - u_{B(x_0,r)}\right|^p$$
$$\le \frac{2^{p-1}}{\delta r^d} \left(\int_{B(x_0,r)\cap\Omega} \left|u - u_{B(x_0,R)}\right|^p + \int_{B(x_0,r)\cap\Omega} \left|u - u_{B(x_0,r)}\right|^p \right).$$

This implies

$$\left|u_{B(x_0,R)} - u_{B(x_0,r)}\right| \le c_6 K \frac{R^{\frac{\lambda}{p}}}{r^{\frac{d}{p}}}. \tag{11.1.41}$$

We put $R_i = \frac{R}{2^i}$ and obtain from (11.1.41)

$$\left|u_{B(x_0,R_i)} - u_{B(x_0,R_{i+1})}\right| \le c_7 K 2^{i\frac{d-\lambda}{p}} R^{\frac{\lambda-d}{p}}. \tag{11.1.42}$$

For $i < j$, this implies

$$\left|u_{B(x_0,R_i)} - u_{B(x_0,R_j)}\right| \le c_8 K R_i^{\frac{\lambda-d}{p}}. \tag{11.1.43}$$

Thus $\left(u_{B(x_0,R_i)}\right)_{i\in\mathbb{N}}$ constitutes a Cauchy sequence. Since (11.1.41) with $r_i = \frac{r}{2^i}$ also implies

$$\left|u_{B(x_0,R_i)} - u_{B(x_0,r_i)}\right| \le c_6 K \left(\frac{R}{r}\right)^{\frac{\lambda}{p}} r_i^{\frac{\lambda-d}{p}} \to 0 \quad \text{for } i \to \infty$$

because of $\lambda > d$, the limit of this Cauchy sequence does not depend on R. Since by Lemma A.4, $u_{B(x,r)}$ converges in L^1 for $r \to 0$ towards $u(x)$, in the limit $j \to \infty$, we obtain from (11.1.43)

$$\left|u_{B(x_0,R)} - u(x_0)\right| \le c_8 K R^{\frac{\lambda-d}{p}}. \tag{11.1.44}$$

Thus, $u_{B(x_0,R)}$ converges not only in L^1 but also uniformly towards u as $R \to 0$. Since for $R > 0$, $u_{B(x,R)}$ is continuous with respect x, then so is u.

It remains to show that u is α-Hölder continuous. For that purpose, let $x, y \in \Omega$, $R := |x - y|$. Then

$$\begin{aligned}|u(x) - u(y)| \le \left|u_{B(x,2R)} - u(x)\right| + \left|u_{B(x,2R)} - u_{B(y,2R)}\right| \\ + \left|u(y) - u_{B(y,2R)}\right|.\end{aligned} \tag{11.1.45}$$

Now

$$\left|u_{B(x,2R)} - u_{B(y,2R)}\right| \le \left|u_{B(x,2R)} - u(z)\right| + \left|u(z) - u_{B(y,2R)}\right|,$$

and integrating with respect to z on $B(x,2R) \cap B(y,2R) \cap \Omega$, we obtain

$$\begin{aligned}&\left|u_{B(x,2R)} - u_{B(y,2R)}\right| \\ &\le \frac{1}{|B(x,2R) \cap B(y,2R) \cap \Omega|}\Bigg(\int_{B(x,2R)\cap\Omega)} \left|u(z) - u_{B(x,2R)}\right| \mathrm{d}z \\ &\quad + \int_{B(y,2R)\cap\Omega} \left|u(z) - u_{B(y,2R)}\right| \mathrm{d}z\Bigg) \\ &\le \frac{c_9}{|B(x,2R) \cap B(y,2R) \cap \Omega|} K R^{\frac{\lambda-d}{p}+d}\end{aligned}$$

by applying Hölder's inequality. Because of $R = |x - y|$,

$$B(x, R) \subset B(y, 2R),$$

and so by (11.1.39),

$$|B(x, 2R) \cap B(y, 2R) \cap \Omega| \geq |B(x, R) \cap \Omega| \geq \delta R^d.$$

We conclude that

$$\left|u_{B(x,2R)} - u_{B(y,2R)}\right| \leq c_{10} K R^{\frac{\lambda-d}{p}}. \tag{11.1.46}$$

Using (11.1.44) and (11.1.46), we obtain

$$|u(x) - u(y)| \leq c_{11} K |x - y|^{\frac{\lambda-d}{p}}, \tag{11.1.47}$$

which is Hölder continuity with exponent $\alpha = \frac{\lambda-d}{p}$. □

Later on (in Sect. 14.4), we shall use the following local version of Campanato's theorem:

Corollary 11.1.7. *If for all* $0 < r \leq R_0$ *and all* $x \in \Omega_0$*, we have*

$$\int_{B(x_0,r)} \left|u - u_{B(x_0,r)}\right|^p \leq \gamma r^{d+p\alpha}$$

with constants γ *and* $0 < \alpha < 1$*, then* u *is locally* α*-Hölder continuous in* Ω_0 *(this means that* u *is* α*-Hölder continuous in any* $\Omega_1 \subset\subset \Omega_0$*).*

References for this section are Gilbarg–Trudinger [12] and Giaquinta [10].

11.2 L^2-Regularity Theory: Interior Regularity of Weak Solutions of the Poisson Equation

For $u : \Omega \to \mathbb{R}$, we define the difference quotient

$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h} \quad (h \neq 0),$$

e_i being the ith unit vector of $\mathbb{R}^d$ $(i \in \{1, \ldots, d\})$.

Lemma 11.2.1. *Assume* $u \in W^{1,2}(\Omega), \Omega' \subset\subset \Omega, |h| < \operatorname{dist}(\Omega', \partial\Omega)$*. Then* $\Delta_i^h u \in L^2(\Omega')$ *and*

$$\left\|\Delta_i^h u\right\|_{L^2(\Omega')} \leq \|D_i u\|_{L^2(\Omega)} \quad (i = 1, \ldots, d). \tag{11.2.1}$$

Proof. By an approximation argument, it again suffices to consider the case $u \in C^1(\Omega) \cap W^{1,2}(\Omega)$. Then

$$\begin{aligned}\Delta_i^h u(x) &= \frac{u(x + he_i) - u(x)}{h} \\ &= \frac{1}{h}\int_0^h D_i u(x^1, \ldots, x^{i-1}, x^i + \xi, x^{i+1}, \ldots, x^d)\mathrm{d}\xi,\end{aligned}$$

and with Hölder's inequality

$$\left|\Delta_i^h u(x)\right|^2 \leq \frac{1}{h}\int_0^h |D_i u(x_1, \ldots, x_i + \xi, \ldots, x_d)|^2 \, \mathrm{d}\xi,$$

and thus

$$\int_{\Omega'} \left|\Delta_i^h u(x)\right|^2 \mathrm{d}x \leq \frac{1}{h}\int_0^h \int_\Omega |D_i u|^2 \, \mathrm{d}x\mathrm{d}\xi = \int_\Omega |D_i u|^2 \, \mathrm{d}x.$$

□

Conversely, we have the following result:

Lemma 11.2.2. *Let $u \in L^2(\Omega)$, and suppose there exists $K < \infty$ with $\Delta_i^h u \in L^2(\Omega')$ and*

$$\left\|\Delta_i^h u\right\|_{L^2(\Omega')} \leq K \tag{11.2.2}$$

for all $h > 0$ and $\Omega' \subset\subset \Omega$ with $h < \mathrm{dist}(\Omega', \partial\Omega)$. Then the weak derivative $D_i u$ exists and satisfies

$$\|D_i u\|_{L^2(\Omega)} \leq K. \tag{11.2.3}$$

Proof. For $\varphi \in C_0^1(\Omega)$ and $0 < h < \mathrm{dist}(\mathrm{supp}\,\varphi, \partial\Omega)$ ($\mathrm{supp}\,\varphi$ is the closure of $\{x \in \Omega : \varphi(x) \neq 0\}$), we have

$$\int_\Omega \Delta_i^h u\,\varphi = -\int_\Omega u\Delta_i^{-h}\varphi \to -\int_\Omega u D_i\varphi,$$

as $h \to 0$. Thus, we also have

$$\left|\int_\Omega u D_i\varphi\right| \leq K\,\|\varphi\|_{L^2(\Omega)}\,.$$

Since $C_0^1(\Omega)$ is dense in $L^2(\Omega)$, we may thus extend

$$\varphi \mapsto -\int_\Omega u D_i \varphi$$

to a bounded linear functional on $L^2(\Omega)$. According to the Riesz representation theorem as quoted in the appendix, there then exists $v \in L^2(\Omega)$ with

$$\int_\Omega \varphi v = -\int_\Omega u D_i \varphi \quad \text{for all } \varphi \in C_0^1(\Omega).$$

Since this is precisely the equation defining $D_i u$, we must have $v = D_i u$. □

Theorem 11.2.1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$ with $f \in L^2(\Omega)$. For any $\Omega' \subset\subset \Omega$, then $u \in W^{2,2}(\Omega')$, and*

$$\|u\|_{W^{2,2}(\Omega')} \leq \text{const}\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right), \tag{11.2.4}$$

where the constant depends only on $\delta := \text{dist}(\Omega', \partial\Omega)$. Furthermore, $\Delta u = f$ almost everywhere in Ω.

The content of Theorem 11.2.1 is twofold: First, there is a regularity result saying that a weak solution of the Poisson equation is of class $W^{2,2}$ in the interior, and second, we have an estimate for the $W^{2,2}$-norm. The proof will yield both results at the same time. If the regularity result happens to be known already, the estimate becomes much easier. That easier demonstration of the estimate nevertheless contains the essential idea of the proof, and so we present it first. To start with, we shall prove a lemma. The proof of that lemma is typical for regularity arguments for weak solutions, and several of the subsequent estimates will turn out to be variants of that proof. We thus recommend that the reader study the following estimate very carefully.

Our starting point is the relation

$$\int_\Omega Du \cdot Dv = -\int_\Omega f v \quad \text{for all } v \in H_0^{1,2}(\Omega). \tag{11.2.5}$$

(Here, Du is the vector $(D_1 u, \ldots, D_d u)$.)

We need some technical preparation: We construct some $\eta \in C_0^1(\Omega)$ with $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $x \in \Omega'$ and $|D\eta| \leq \frac{2}{\delta}$. Such an η can be obtained by mollification, i.e., by convolution with a smooth kernel as described in Lemma A.2 in the appendix, from the following function η_0:

$$\eta_0(x) := \begin{cases} 1 & \text{for } \text{dist}(x, \Omega') \leq \frac{\delta}{8}, \\ 0 & \text{for } \text{dist}(x, \Omega') \geq \frac{7\delta}{8}, \\ \frac{7}{6} - \frac{4}{3\delta}\,\text{dist}(x, \Omega') & \text{for } \frac{\delta}{8} \leq \text{dist}(x, \Omega') \leq \frac{7\delta}{8}. \end{cases}$$

Thus η_0 is a (piecewise) linear function of $\operatorname{dist}(x, \Omega')$ interpolating between Ω', where it takes the value 1, and the complement of Ω, where it is 0. This is also the purpose of the cutoff function η. If one abandons the requirement of continuous differentiability (which is not essential anyway), one may put more simply

$$\eta(x) := \begin{cases} 1 & \text{for } x \in \Omega', \\ 0 & \text{for } \operatorname{dist}(x, \Omega') \geq \delta, \\ 1 - \frac{1}{\delta}\operatorname{dist}(x, \Omega') & \text{for } 0 \leq \operatorname{dist}(x, \Omega') \leq \delta \end{cases}$$

(note that $\operatorname{dist}(\Omega', \partial\Omega) \geq \delta$). It is not difficult to verify that $\eta \in H_0^{1,2}(\Omega)$, which suffices for the sequel. In (11.2.5), we now use the test function

$$v = \eta^2 u$$

with η of the type just presented. This yields

$$\int_\Omega \eta^2 |Du|^2 + 2\int_\Omega \eta Du \cdot uD\eta = -\int_\Omega \eta^2 f u, \tag{11.2.6}$$

and with the so-called Young inequality

$$\pm ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \quad \text{for } a, b \in \mathbb{R}, \varepsilon > 0 \tag{11.2.7}$$

used with $a = \eta\,|Du|$, $b = u\,|D\eta|$, $\varepsilon = \frac{1}{2}$ in the second integral, and with $a = \eta f$, $b = \eta u$, $\varepsilon = \delta^2$ in the integral on the right-hand side, we obtain

$$\int_\Omega \eta^2 |Du|^2 \leq \frac{1}{2}\int_\Omega \eta^2 |Du|^2 + 2\int_\Omega |D\eta|^2 u^2 + \frac{1}{2\delta^2}\int_\Omega \eta^2 u^2 + \frac{\delta^2}{2}\int_\Omega \eta^2 f^2. \tag{11.2.8}$$

We recall that $0 \leq \eta \leq 1$, $\eta = 1$ on Ω' to see that this yields

$$\int_{\Omega'} |Du|^2 \leq \int_\Omega \eta^2 |Du|^2 \leq \left(\frac{16}{\delta^2} + \frac{1}{\delta^2}\right)\int_\Omega u^2 + \delta^2 \int_\Omega f^2.$$

We record this inequality in the following lemma:

Lemma 11.2.3. *Let u be a weak solution of $\Delta u = f$ with $f \in L^2(\Omega)$. We then have for any $\Omega' \subset\subset \Omega$,*

$$\|Du\|^2_{L^2(\Omega')} \leq \frac{17}{\delta^2}\|u\|^2_{L^2(\Omega)} + \delta^2 \|f\|^2_{L^2(\Omega)}, \tag{11.2.9}$$

where $\delta := \operatorname{dist}(\Omega', \partial\Omega)$.

So far, we have not used that we are temporarily assuming $u \in W^{2,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$. Now, however, we come to the estimate of the $W^{2,2}$-norm, so we shall need that assumption. Let $u \in W^{2,2}(\Omega') \cap W^{1,2}(\Omega)$ again satisfy

$$\int_\Omega Du \cdot Dv = -\int_\Omega f v \quad \text{for all } v \in H_0^{1,2}(\Omega). \tag{11.2.10}$$

If $\operatorname{supp} v \subset\subset \Omega'$ (i.e., $v \in H_0^{1,2}(\Omega'')$ for some $\Omega'' \subset\subset \Omega'$), we may, assuming $u \in W^{2,2}(\Omega')$, integrate by parts in (11.2.10) to obtain

$$\int_\Omega \left(\sum_{i=1}^d D_i D_i u \right) v = \int_\Omega f v. \tag{11.2.11}$$

This in particular holds for all $v \in C_0^\infty(\Omega')$, and since $C_0^\infty(\Omega')$ is dense in $L^2(\Omega')$, (11.2.11) then also holds for $v \in L^2(\Omega')$, where we have put $v = 0$ in $\Omega \setminus \Omega'$.

We consider the matrix $D^2 u$ of the second weak derivatives of u and obtain

$$\begin{aligned}
\int_{\Omega'} \left|D^2 u\right|^2 &= \int_{\Omega'} \sum_{i,j=1}^d D_i D_j u \cdot D_i D_j u \\
&= \int_{\Omega'} \sum_{i=1}^d D_i D_i u \cdot \sum_{i=1}^d D_j D_j u \\
&\quad + \text{boundary terms that we neglect for the moment (later on, they} \\
&\quad\quad \text{will be converted into interior terms with the help of cutoff functions),} \\
&\quad\quad \text{by an integration by parts that will even require the assumption } u \in W^{3,2}(\Omega') \\
&= \int_{\Omega'} f \sum_{i=1}^d D_j D_j u \\
&\leq \left(\int_{\Omega'} f^2 \right)^{\frac{1}{2}} \left(\int_{\Omega'} \left|D^2 u\right|^2 \right)^{\frac{1}{2}} \quad \text{by Hölder's inequality,}
\end{aligned} \tag{11.2.12}$$

and hence

$$\int_{\Omega'} \left|D^2 u\right|^2 \leq \int_\Omega f^2, \tag{11.2.13}$$

i.e.,

$$\left\| D^2 u \right\|_{L^2(\Omega')}^2 \leq \| f \|_{L^2(\Omega)}^2 . \tag{11.2.14}$$

Taken together, (11.2.9) and (11.2.14) yield

$$\|u\|^2_{W^{2,2}(\Omega')} \le (c_1(\delta)+1)\,\|u\|^2_{L^2(\Omega)} + 2\,\|f\|^2_{L^2(\Omega)}\,. \tag{11.2.15}$$

We now come to the actual Proof of Theorem 11.2.1: Let

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega, \quad \operatorname{dist}(\Omega'', \partial\Omega) \ge \frac{\delta}{4}, \quad \operatorname{dist}(\Omega', \partial\Omega'') \ge \frac{\delta}{4}.$$

We again use

$$\int_\Omega Du\cdot Dv = -\int_\Omega f\cdot v \quad \text{for all } v\in H_0^{1,2}(\Omega). \tag{11.2.16}$$

In the sequel, we consider v with

$$\operatorname{supp} v \subset\subset \Omega''$$

and choose $h>0$ with

$$2h < \operatorname{dist}(\operatorname{supp} v, \partial\Omega'').$$

In (11.2.16), we may then also insert $\Delta_i^h v$ ($i\in\{1,\dots,d\}$) in place of v. We obtain

$$\begin{aligned}\int_{\Omega''} D\Delta_i^h u\cdot Dv &= \int_{\Omega''}\Delta_i^h(Du)\cdot Dv = -\int_{\Omega''} Du\cdot \Delta_i^h Dv\\ &= -\int_{\Omega''} Du\cdot D\left(\Delta_i^h v\right)\\ &= \int_{\Omega''} f\Delta_i^h v \quad \le \|f\|_{L^2(\Omega)}\cdot\|Dv\|_{L^2(\Omega'')}\end{aligned} \tag{11.2.17}$$

by Lemma 11.2.1 and the choice of h. As described above, let $\eta\in C_0^1(\Omega'')$, $0\le \eta\le 1$, $\eta(x)=1$ for $x\in\Omega'$, $|D\eta|\le 8/\delta$. We put

$$v := \eta^2\Delta_i^h u.$$

From (11.2.17), we obtain

$$\begin{aligned}\int_{\Omega''}\left|\eta D\Delta_i^h u\right|^2 &= \int_{\Omega''} D\Delta_i^h u\cdot Dv - 2\int_{\Omega''}\eta D\Delta_i^h u\cdot\Delta_i^h u D\eta\\ &\le \|f\|_{L^2(\Omega)}\left\|D\left(\eta^2\Delta_i^h u\right)\right\|_{L^2(\Omega'')}\\ &\quad + 2\left\|\eta D\Delta_i^h u\right\|_{L^2(\Omega'')}\left\|\Delta_i^h u D\eta\right\|_{L^2(\Omega'')}\,.\end{aligned}$$

With Young's inequality (11.2.7) and employing Lemma 11.2.1 (recall the choice of h), we hence obtain

$$\left\|\eta D\Delta_i^h u\right\|_{L^2(\Omega'')}^2 \leq 2\,\|f\|_{L^2(\Omega)}^2 + \frac{1}{4}\left\|\eta D\Delta_i^h u\right\|_{L^2(\Omega'')}^2 + \frac{1}{4}\left\|\eta D\Delta_i^h u\right\|_{L^2(\Omega'')}^2 + 8\sup|D\eta|^2\,\|D_i u\|_{L^2(\Omega'')}^2 .$$

The essential point in employing Young's inequality here is that the expression $\left\|\eta D\Delta_i^h u\right\|_{L^2(\Omega'')}^2$ occurs on the right-hand side with a smaller coefficient than on the left-hand side, and so the contribution on the right-hand side can be absorbed in the left-hand side. Because of $\eta \equiv 1$ on Ω' and $(a^2+b^2)^{\frac{1}{2}} \leq a+b$ with Lemma 11.2.2, as $h \to \infty$, we obtain

$$\left\|D^2 u\right\|_{L^2(\Omega')} \leq \text{const}\left(\|f\|_{L^2(\Omega)} + \frac{1}{\delta}\|Du\|_{L^2(\Omega'')}\right). \tag{11.2.18}$$

Lemma 11.2.3 (with Ω'' in place of Ω') now implies

$$\|Du\|_{L^2(\Omega'')} \leq c_1\left(\frac{1}{\delta}\|u\|_{L^2(\Omega)} + \delta\,\|f\|_{L^2(\Omega)}\right) \tag{11.2.19}$$

with some constant c_1. Inequality (11.2.4) then follows from (11.2.18) and (11.2.19).

If f happens to be even of class $W^{1,2}(\Omega)$, in (11.2.5) we may insert $D_i v$ in place of v to obtain

$$\int_\Omega D(D_i u)\cdot Dv = -\int_\Omega D_i f\cdot v.$$

Theorem 11.2.1 then implies $D_i u \in W^{2,2}(\Omega')$, i.e., $u \in W^{3,2}(\Omega')$. In this manner, we iteratively obtain the following theorem:

Theorem 11.2.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$, $f \in W^{k,2}(\Omega)$. For any $\Omega' \subset\subset \Omega$ then $u \in W^{k+2,2}(\Omega')$, and*

$$\|u\|_{W^{k+2,2}(\Omega')} \leq \text{const}\left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)}\right),$$

where the constant depends on d, k, and $\text{dist}(\Omega', \partial\Omega)$.

Corollary 11.2.1. *If $u \in W^{1,2}(\Omega)$ is a weak solution of $\Delta u = f$ with $f \in C^\infty(\Omega)$, then also $u \in C^\infty(\Omega)$.*

Proof. From Theorem 11.2.2 and Corollary 11.1.2. □

The regularity theory also easily implies results about removability of isolated singularities. We state and prove the result here for the Laplace equation, leaving it to the reader to identify the necessary or sufficient conditions on the right-hand side f of the Poisson equation for such a result to hold.

Corollary 11.2.2. *Let $u \in (W^{1,2} \cap C^\infty)(\Omega \setminus \{x_0\})$ for some $x_0 \in \Omega \subset \mathbb{R}^d$ for $d > 1$ be a solution of*

$$\Delta u = 0. \tag{11.2.20}$$

Then u extends as a smooth harmonic function to all of Ω.

Proof. We only need to show that u is a weak solution of $\Delta u = 0$ in all of Ω. Corollary 11.2.1 (or in the present special case of harmonic functions even Corollary 2.2.1) then implies that u is smooth in Ω, and hence also solves $\Delta u = 0$ there by continuity of its second derivatives.

In order to show that u is weakly harmonic, we need to verify (10.1.5), i.e.,

$$\int_\Omega \nabla u(x) \cdot \nabla \eta(x) \mathrm{d}x = 0, \tag{11.2.21}$$

for all $\eta \in C_0^\infty(\Omega)$.

Since the result is local, we may assume that Ω is the open unit ball $\mathring{B}(0,1) \subset \mathbb{R}^d$, and $x_0 = 0$.

We now write for $\epsilon > 0$

$$\eta = \eta(\lambda_\epsilon + (1 - \lambda_\epsilon)) \tag{11.2.22}$$

for the cut-off function

$$\lambda_\epsilon(x) \equiv 1 \text{ for } \epsilon \le |x| \le 1$$

$$\lambda_\epsilon(x) = \frac{|x|}{\epsilon} \text{ for } 0 \le |x| \le \epsilon$$

$$\lambda_\epsilon(0) = 0.$$

($\eta\lambda_\epsilon$ is not smooth, but in $W^{1,2}$ if η is, and this suffices for our purposes. Alternatively, we can smooth out λ_ϵ near $|x| = \epsilon$.)
We then have

$$\int_{\mathring{B}(0,1)} \nabla u(x) \cdot \nabla \eta(x) \mathrm{d}x = \int_{\mathring{B}(0,1)} \nabla u(x) \cdot \nabla(\lambda_\epsilon \eta(x)) \mathrm{d}x + \int_{\mathring{B}(0,1)} \nabla u(x) \cdot \nabla((1-\lambda_\epsilon)\eta(x)) \mathrm{d}x. \tag{11.23}$$

The first term on the right hand side is 0 since u is harmonic on $\Omega \setminus \{x_0\}$, that is, on $\mathring{B}(0,1) \setminus \{0\}$. The integrand in the second term vanishes for $|x| \ge \epsilon$. In order to make the left hand side of (11.23) 0, that is, in order to get (11.2.21), we thus need to show that

$$\int_{\mathring{B}(0,\epsilon)} \nabla u(x) \cdot \nabla((1 - \lambda_\epsilon)\eta(x)) \mathrm{d}x \to 0 \tag{11.2.24}$$

as $\epsilon \to 0$. The difficult term is

$$\int_{\overset{\circ}{B}(0,\epsilon)} \eta(x)\nabla u(x) \cdot \nabla((1-\lambda_\epsilon))\mathrm{d}x. \tag{11.2.25}$$

By Hölder's inequality, this term is controlled by

$$\sup|\eta| \left(\int_{\overset{\circ}{B}(0,\epsilon)} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\overset{\circ}{B}(0,\epsilon)} |\nabla \lambda_\epsilon|^2\right)^{\frac{1}{2}} = c\epsilon^{d-2} \sup|\eta| \left(\int_{\overset{\circ}{B}(0,\epsilon)} |\nabla u|^2\right)^{\frac{1}{2}}, \tag{11.2.26}$$

for some constant $c = c(d)$. This goes to 0 for $\epsilon \to 0$ where for $d = 2$ we need to use that $\int_{\overset{\circ}{B}(0,\epsilon)} |\nabla u|^2 \to 0$ for $\epsilon \to 0$ because $u \in W^{1,2}$. Thus, we obtain (11.2.24).

□

Remark. In fact, by choosing the cutoff function $\lambda_\epsilon(x) = \frac{\log \epsilon}{\log |x|}$ for $0 \leq |x| \leq \epsilon$, even in dimension $d = 2$, we do not need to exploit that $\int_{\overset{\circ}{B}(0,\epsilon)} |\nabla u|^2 \to 0$ for $\epsilon \to 0$. Such a logarithmic cutoff function is often useful.

At the end of this section, we wish to record once more a fundamental observation concerning elliptic regularity theory as encountered in the present section for the first time and to be encountered many more times in the subsequent sections. For any u contained in the Sobolev space $W^{2,2}(\Omega)$, we have the trivial estimate

$$\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \leq \text{const}\, \|u\|_{W^{2,2}(\Omega)}$$

(where Δu is to be understood as the sum of the weak pure second derivatives of u). Elliptic regularity theory yields an estimate in the opposite direction; according to Theorem 11.2.1, we have

$$\|u\|_{W^{2,2}(\Omega')} \leq \text{const}(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}) \quad \text{for } \Omega' \subset\subset \Omega.$$

Thus Δu and some lower-order term already control all second derivatives of u. Lemma 11.2.3 shall be interpreted in this sense as well.

The Poincaré inequality states that for every $u \in H_0^{1,2}(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq \text{const}\, \|Du\|_{L^2(\Omega)},$$

while for a harmonic $u \in W^{1,2}(\Omega)$, we have the estimate in the opposite direction,

$$\|Du\|_{L^2(\Omega')} \leq \text{const}\, \|u\|_{L^2(\Omega)}$$

(for $\Omega' \subset\subset \Omega$).

In this sense, in elliptic regularity theory, one has estimates in both directions, one direction resulting from general embedding theorems, and the other one from the elliptic equation. Combining both directions often allows iteration arguments for proving even higher regularity, as we have seen in the present section and as we shall have ample occasion to witness in subsequent sections.

11.3 Boundary Regularity and Regularity Results for Solutions of General Linear Elliptic Equations

With the help of Dirichlet's principle, we have found weak solutions of

$$\Delta u = f \quad \text{in } \Omega$$

with

$$u - g \in H_0^{1,2}(\Omega)$$

for given $f \in L^2(\Omega)$, $g \in H^{1,2}(\Omega)$. In the previous section, we have seen that in the interior of Ω, u is as regular as f allows. It is then natural to ask whether u is regular at $\partial\Omega$ as well, provided that g and $\partial\Omega$ satisfy suitable regularity conditions. A preliminary observation is that a solution of the above Dirichlet problem possesses a global bound that depends only on f and g:

Lemma 11.3.1. *Let u be a weak solution of $\Delta u = f$, $u - g \in H_0^{1,2}(\Omega)$ in the bounded region Ω. Then*

$$\|u\|_{W^{1,2}(\Omega)} \leq c \left(\|g\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} \right), \tag{11.3.1}$$

where the constant c depends only on the Lebesgue measure $|\Omega|$ of Ω and on d.

Proof. We insert the test function $v = u - g$ into the weak differential equation

$$\int_\Omega Du \cdot Dv = -\int_\Omega f v \quad \text{for all } v \in H_0^{1,2}(\Omega)$$

to obtain

$$\begin{aligned}\int_\Omega |Du|^2 &= \int Du \cdot Dg - \int f u + \int fg \\ &\leq \frac{1}{2}\int |Du|^2 + \frac{1}{2}\int |Dg|^2 + \frac{1}{\varepsilon}\int f^2 + \frac{\varepsilon}{2}\int u^2 + \frac{\varepsilon}{2}\int g^2\end{aligned}$$

for any $\varepsilon > 0$, by Young's inequality, and hence

$$\|Du\|_{L^2}^2 \leq \varepsilon \|u\|_{L^2}^2 + \|Dg\|_{L^2}^2 + \frac{2}{\varepsilon} \|f\|_{L^2}^2 + \varepsilon \|g\|_{L^2}^2 ,$$

i.e.,

$$\|Du\|_{L^2} \leq \sqrt{\varepsilon} \|u\|_{L^2} + \|Dg\|_{L^2} + \sqrt{\frac{2}{\varepsilon}} \|f\|_{L^2} + \sqrt{\varepsilon} \|g\|_{L^2} . \tag{11.3.2}$$

Obviously,

$$\|u\|_{L^2} \leq \|u - g\|_{L^2} + \|g\|_{L^2} , \tag{11.3.3}$$

and by the Poincaré inequality

$$\|u - g\|_{L^2} \leq \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} (\|Du\|_{L^2} + \|Dg\|_{L^2}) . \tag{11.3.4}$$

Altogether, it follows that

$$\|Du\|_{L^2} \leq \sqrt{\varepsilon} \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} \|Du\|_{L^2} + \left(1 + \sqrt{\varepsilon} \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} \right) \|Dg\|_{L^2}$$
$$+ 2\sqrt{\varepsilon} \|g\|_{L^2} + \sqrt{\frac{2}{\varepsilon}} \|f\|_{L^2} .$$

We now choose

$$\varepsilon = \frac{1}{4} \left(\frac{\omega_d}{|\Omega|} \right)^{\frac{2}{d}} ,$$

i.e.,

$$\sqrt{\varepsilon} \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} = \frac{1}{2},$$

and obtain

$$\|Du\|_{L^2} \leq 3 \|Dg\|_{L^2} + 2 \left(\frac{\omega_d}{|\Omega|} \right)^{\frac{1}{d}} \|g\|_{L^2} + \sqrt{2} \cdot 4 \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} \|f\|_{L^2} . \tag{11.3.5}$$

Inequalities (11.3.3)–(11.3.5) then also yield an estimate for $\|u\|_{L^2}$, and (11.3.1) follows. □

We also wish to convince ourselves that we can reduce our considerations to the case $u \in H_0^{1,2}(\Omega)$. Namely, we simply consider $\bar{u} := u - g \in H_0^{1,2}(\Omega)$, which satisfies

$$\Delta \bar{u} = \Delta u - \Delta g = f - \Delta g = \bar{f} \tag{11.3.6}$$

in the weak sense. Here, we are assuming $g \in W^{2,2}(\Omega)$, and thus, for $\bar{u} \in H_0^{1,2}(\Omega)$, we obtain the equation

$$\Delta \bar{u} = \bar{f} \tag{11.3.7}$$

with $\bar{f} \in L^2(\Omega)$, again in the weak sense. Since the $W^{2,2}$-norm of u can be estimated by those of $\bar{u}$ and g, it thus suffices to consider vanishing boundary values. We consequently assume that $u \in H_0^{1,2}(\Omega)$ is a weak solution of $\Delta u = f$ in Ω.

We now consider a special situation; namely, we assume that in the vicinity of a given point $x_0 \in \partial\Omega$, $\partial\Omega$ contains a piece of a hyperplane; for example, without loss of generality, $x_0 = 0$ and

$$\partial\Omega \cap \mathring{B}(0,R) = \left\{(x^1, \ldots, x^{d-1}, 0)\right\} \cap \mathring{B}(0,R)$$

(here, $\mathring{B}(0,R) = \{x \in \mathbb{R}^d : |x| < R\}$ is the interior of the ball $B(0,R)$) for some $R > 0$. Let

$$B^+(0,R) := \left\{(x^1, \ldots, x^d) \in \mathring{B}(0,R) : x^d > 0\right\} \subset \Omega.$$

If now $\eta \in C_0^1(\mathring{B}(0,R))$, we have

$$\eta^2 u \in H_0^{1,2}(B^+(0,R)),$$

because we are assuming that u vanishes on $\partial\Omega \cap \mathring{B}(0,R)$ in the Sobolev space sense. If now $1 \le i \le d-1$ and $|h| < \operatorname{dist}(\operatorname{supp}\eta, \partial\mathring{B}(0,R))$, we also have

$$\eta^2 \Delta_i^h u \in H_0^{1,2}(B^+(0,R)).$$

Thus, we may proceed as in the proof of Theorem 11.2.1, in order to show that

$$D_{ij} u \in L^2\left(\mathring{B}\left(0, \frac{R}{2}\right)\right) \tag{11.3.8}$$

with a corresponding estimate, provided that i and j are not both equal to d. However, since, from our differential equation, we have

$$D_{dd} u = f - \sum_{j=1}^{d-1} D_{jj} u; \tag{11.3.9}$$

we then also obtain

$$D_{dd}u \in L^2\left(\mathring{B}\left(0, \frac{R}{2}\right)\right),$$

and thus the desired regularity result

$$u \in W^{2,2}\left(\mathring{B}\left(0, \frac{R}{2}\right)\right),$$

as well as the corresponding estimate.

In order to treat the general case, we have to require suitable assumptions for $\partial\Omega$.

Definition 11.3.1. An open and bounded set $\Omega \subset \mathbb{R}^d$ is of class C^k ($k = 0, 1, 2, \ldots, \infty$) if for any $x_0 \in \partial\Omega$ there exist $r > 0$ and a bijective map $\phi : \mathring{B}(x_0, r) \to \phi(\mathring{B}(x_0, r)) \subset \mathbb{R}^d$ ($\mathring{B}(x_0, r) = \{y \in \mathbb{R}^d : |x_0 - y| < r\}$) with the following properties:

(i) $\phi(\Omega \cap \mathring{B}(x_0, r)) \subset \{(x^1, \ldots, x^d) : x^d > 0\}$.
(ii) $\phi(\partial\Omega \cap \mathring{B}(x_0, r)) \subset \{(x^1, \ldots, x^d) : x^d = 0\}$.
(iii) ϕ and ϕ^{-1} are of class C^k.

Remark. This means that $\partial\Omega$ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ of differentiability class C^k.

Definition 11.3.2. Let $\Omega \subset \mathbb{R}^d$ be of class C^k, as defined in Definition 11.3.1. We say that $g : \bar{\Omega} \to \mathbb{R}$ is of class $C^l(\bar{\Omega})$ for $l \le k$ if $g \in C^l(\Omega)$ and if for any $x_0 \in \partial\Omega$ and ϕ as in Definition 11.3.1,

$$g \circ \phi^{-1} : \left\{(x^1, \ldots, x^d) : x^d \ge 0\right\} \to \mathbb{R}$$

is of class C^l.

The crucial idea for boundary regularity is to consider, instead of u, local functions $u \circ \phi^{-1}$ with ϕ as in Definition 11.3.1. As we have argued at the beginning of this section, we may assume that the prescribed boundary values are $g = 0$. Then $u \circ \phi^{-1}$ is defined on some half-ball, and we may therefore carry over the interior regularity theory as just described. However, in general, $u \circ \phi^{-1}$ no longer satisfies the Laplace equation. It turns out, however, that $u \circ \phi^{-1}$ satisfies a more general differential equation that is structurally similar to the Laplace equation and for which one may derive interior regularity in a similar manner.

We have derived a corresponding transformation formula already in Sect. 10.4. Thus $w = u \circ \phi^{-1}$ satisfies a differential equation (10.4.11), i.e.,

$$\frac{1}{\sqrt{g}} \sum_{J=1}^{d} \left(\frac{\partial}{\partial \xi^j}\left(\sqrt{g} \sum_{i=1}^{d} g^{ij} \frac{\partial w}{\partial \xi^i}\right)\right) = 0, \tag{11.3.10}$$

where the positive definite matrix g^{ij} is computed from ϕ and its derivatives [cf. (10.4.7)].

We shall consider an even more general class of elliptic differential equations:

$$\begin{aligned} Lu :=& \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right) + \sum_{j=1}^{d}\frac{\partial}{\partial x^j}\left(b^j(x)u(x)\right) \\ &+ \sum_{i=1}^{d} c^i(x)\frac{\partial}{\partial x^i}u(x) + d(x)u(x) \\ =& f(x). \end{aligned} \tag{11.3.11}$$

We shall need two essential assumptions:

(A1) (Uniform ellipticity) There exist $0 < \lambda \le \Lambda < \infty$ with

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \ge \Lambda\,|\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d.$$

(A2) (Boundedness) There exists some $M < \infty$ with

$$\sup_{x\in\Omega}\left(\|b(x)\|, \|c(x)\|, \|d(x)\|\right) \le M.$$

Here, for instance, $\|b(x)\| = (\sum_j b^j(x)b^j(x))^{1/2}$ is the Euclidean norm of the vector $b(x)$. When one is interested in how the subsequent estimates depend on the dimension d, one should keep in mind that this quantity is bounded from above by $d \sup_i |b^i(x)|$.

A function u is called a weak solution of the Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \quad (f \in L^2(\Omega) \text{ given}), \\ u - g &\in H_0^{1,2}(\Omega), \end{aligned}$$

if for all $v \in H_0^{1,2}(\Omega)$,

$$\begin{aligned} &\int_\Omega \Big\{ \sum_{i,j} a^{ij}(x)D_i u(x)D_j v(x) + \sum_j b^j(x)u(x)D_j v(x) \\ &\quad - \left(\sum_i c^i(x)D_i u(x) + d(x)u(x)\right)v(x)\Big\}\mathrm{d}x = -\int_\Omega f(x)v(x)\mathrm{d}x. \end{aligned} \tag{11.3.12}$$

In order to become a little more familiar with (11.3.12), we shall first try to find out what happens if we insert our test functions that proved successful for the weak Poisson equation, namely, $v = \eta^2 u$ and $v = u - g$. Here η is a cutoff function as described in Sect. 11.2 with respect to $\Omega' \subset\subset \Omega$. With $v = \eta^2 u$, (11.3.12) then becomes

$$\int_\Omega \left\{ \sum \eta^2 a^{ij} D_i u D_j u + 2 \sum \eta a^{ij} u D_i u D_j \eta + \sum \eta^2 b^j u D_j u \right.$$
$$\left. + 2 \sum u^2 b^j \eta D_j \eta - \sum \eta^2 c^i u D_i u - \mathrm{d}\eta^2 u^2 \right\} = -\int f \eta^2 u. \tag{11.3.13}$$

In order to handle the various terms, analogously to (11.2.8), we shall use Young's inequality, this time of the form

$$\sum a^{ij} a_i b_j \le \frac{\varepsilon}{2} \sum a^{ij} a_i a_j + \frac{1}{2\varepsilon} \sum a^{ij} b_i b_j \tag{11.3.14}$$

for $\varepsilon > 0$, $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in \mathbb{R}^d$, and a positive definite matrix $(a^{ij})_{i,j=1,\ldots,d}$. From (A1) and (A2), we thence obtain the following inequalities:

$$2 \sum \eta a^{ij} u D_i u D_j \eta \le \varepsilon \sum \eta^2 a^{ij} D_i u D_j u + \frac{1}{\varepsilon} \sum a^{ij} u^2 D_i \eta D_j \eta$$
$$\sum \eta^2 b^j u D_j u \le \frac{\varepsilon'}{2} \sum \eta^2 D_j u D_j u + \frac{1}{2\varepsilon'} \sum \eta^2 u^2 b^j b^j$$
$$2 \sum u^2 b^j \eta D_j \eta \le \sum u^2 D_j \eta D_j \eta + \sum u^2 \eta^2 b^j b^j$$
$$\sum \eta^2 c^i u D_i u \le \frac{\varepsilon'}{2} \sum \eta^2 D_j u D_j u + \frac{1}{2\varepsilon'} \sum \eta^2 u^2 c^j c^j$$
$$f \eta^2 u^2 \le \frac{1}{2} \eta^2 u^2 + \frac{1}{2} \eta^2 f^2.$$

With the help of these inequalities, (11.3.13) yields

$$\int \eta^2 \sum a^{ij} D_i u D_j u \le \varepsilon \int \eta^2 \sum a^{ij} D_i u D_j u$$
$$+ \varepsilon' \int |Du|^2 \eta^2 + \left(\frac{1}{\varepsilon'} M^2 + M^2 + M + \frac{1}{2} \right) \int \eta^2 u^2$$
$$+ \left(\frac{\Lambda}{\varepsilon} + 1 \right) \int u^2 |D\eta|^2 + \frac{1}{2} \int \eta^2 f^2.$$

We choose $\varepsilon = \frac{1}{2}$ and then $\varepsilon' = \frac{\lambda}{4}$, to obtain, with

$$\int |Du|^2 \eta^2 \leq \frac{1}{\lambda} \int \eta^2 \sum a^{ij} D_i u D_j u \tag{11.3.15}$$

which follows from (A1) again, the desired estimate

$$\int \eta^2 |Du|^2 \leq c_1(\lambda, \Lambda) \int u^2 |D\eta|^2 + c_2(\lambda, M) \int \eta^2 u^2 + c_3(\lambda) \int \eta^2 f^2, \tag{11.3.16}$$

with constants $c_1, c_2, and c_3$ that depend only on the indicated quantities. In fact, as an aside, in the special case where $b = c = d = f = 0$, we simply have

$$\int \eta^2 |Du|^2 \leq 2\frac{\Lambda}{\lambda} \int u^2 |D\eta|^2 .$$

With $\delta = \operatorname{dist}(\Omega', \partial\Omega)$, we can have $\eta = 1$ on Ω' and $|D\eta| \leq \frac{1}{\delta}$ and obtain

$$\int_{\Omega'} |Du|^2 \leq \left(\frac{c_1(\lambda, \Lambda)}{\delta^2} + c_2(\lambda, M) \right) \int_{\Omega} u^2 + c_3(\lambda) \int_{\Omega} f^2. \tag{11.3.17}$$

This is the analogue of Lemma 11.2.3. The global bound of Lemma 11.3.1, however, does not admit a direct generalization. If we insert the test function $u-g$ in (11.3.12), we obtain only (as usual, employing Young's inequality in order to absorb all the terms containing derivatives into the positive definite leading term)

$$\begin{aligned} \int_{\Omega} |Du|^2 &\leq \frac{1}{\lambda} \int \sum a^{ij} D_i u D_j u \\ &\leq c_4(\lambda, \Lambda, M, |\Omega|) \left(\|g\|^2_{W^{1,2}} + \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right). \end{aligned} \tag{11.3.18}$$

Thus, the additional term $\|u\|^2_{L^2(\Omega)}$ appears in the right-hand side. That this is really necessary can already be seen from the differential equation

$$\begin{aligned} u''(t) + \kappa^2 u(t) &= 0 \quad \text{for } 0 < t < \pi, \\ u(0) = u(\pi) &= 0, \end{aligned} \tag{11.3.19}$$

with $\kappa > 0$. Namely, for $\kappa \in \mathbb{N}$, we have the solutions

$$u(t) = b \sin(\kappa t)$$

with $b \in \mathbb{R}$ arbitrary, and these solutions obviously cannot be controlled solely by the right-hand side of the differential equation and the boundary values, because those are all zero. The local interior regularity theory of Sect. 11.2, however, remains fully valid. Namely, we have the following theorem:

Theorem 11.3.1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$; i.e., let (11.3.12) hold. Let the ellipticity assumption (A11.3) hold. Moreover, let all coefficients $a^{ij}(x), \dots, d(x)$ as well as $f(x)$ be of class C^∞. Then also $u \in C^\infty(\Omega)$.*

Remark. Regularity is a local result. Since we assume that all coefficients are C^∞, in particular, on every $\Omega' \subset\subset \Omega$, we have a bound of type (A11.3), with the constant M depending on Ω' here, however.

Let us discuss the Proof of Theorem 11.3.1: We first reduce the proof to the case $b^j, c^i, d \equiv 0$, i.e., to the regularity of weak solutions of

$$Mu := \sum_{i,j} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right) = f(x). \tag{11.3.20}$$

For that purpose, we simply rewrite

$$Lu = f$$

as

$$Mu = -\sum \frac{\partial}{\partial x^j}(b^j(x)u(x)) - \sum c^i(x)\frac{\partial}{\partial x^i}u(x) - d(x)u(x) + f(x). \tag{11.3.21}$$

We then prove the following theorem:

Theorem 11.3.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Mu = f$ with $f \in W^{k,2}(\Omega)$. Assume (A11.3), and that the coefficients $a^{ij}(x)$ of M are of class $C^{k+1}(\Omega)$. Then for every $\Omega' \subset\subset \Omega$,*

$$u \in W^{k+2,2}(\Omega').$$

If

$$\left\|a^{ij}\right\|_{C^{k+1}(\Omega')} \le M_k \quad \text{for all } i, j, \tag{11.3.22}$$

then

$$\|u\|_{W^{k+2,2}(\Omega')} \le c\left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)}\right) \tag{11.3.23}$$

with $c = c(d, \lambda, k, M_k, \operatorname{dist}(\Omega', \partial\Omega))$.

The Sobolev embedding theorem then implies that in case $a^{ij}, f \in C^\infty$, any solution of $Mu = f$ is of class C^∞ as well. The corresponding regularity for solutions of $Lu = f$, as claimed in Theorem 11.3.1, can then be obtained through the following important iteration argument: Since we assume $u \in W^{1,2}(\Omega)$, the right-hand side of (11.3.21) is in $L^2(\Omega)$. According to Theorem 11.3.2, for $k = 0$, then $u \in W^{2,2}(\Omega)$. This in turn implies that the right-hand side of (11.3.21) is in $W^{1,2}(\Omega)$. Thus, we may apply Theorem 11.3.2 for $k = 1$ to obtain $u \in W^{3,2}(\Omega)$. But then, the right-hand side is in $W^{2,2}(\Omega)$; hence $u \in W^{4,2}(\Omega)$, and so on.

In that manner we deduce $u \in W^{m,2}(\Omega)$ for all $m \in \mathbb{N}$, and by the Sobolev embedding theorem, hence that u is in $C^\infty(\Omega)$.

We shall not display all details of the Proof of Theorem 11.3.2 here, since this represents a generalization of the reasoning given in Sect. 11.2 that only needs a more cumbersome notation, but no new ideas. We have already seen how such a generalization works when we inserted the test function $\eta^2 u$ in (11.3.12). The only additional ingredient is certain rules for manipulating difference quotients, like the product rule

$$\begin{aligned}\Delta_l^h(ab)(x) &= \frac{1}{h}\left(a(x+he_l)b(x+he_l) - a(x)b(x)\right)\\ &= a(x+he_l)\Delta_l^h b(x) + \left(\Delta_l^h a(x)\right) b(x).\end{aligned} \tag{11.3.24}$$

For example,

$$\Delta_l^h\left(\sum_{i=1}^d a^{ij}(x)D_i u(x)\right) = \sum_i \left(a^{ij}(x+he_l)\Delta_l^h D_i u(x) + \Delta_l^h a^{ij}(x) D_i u(x)\right). \tag{11.3.25}$$

As before, we use $\Delta_l^{-h} v$ as a test function in place of v, and in the case $\operatorname{supp} v \subset\subset \Omega''$, $2h < \operatorname{dist}(\operatorname{supp} v, \partial\Omega'')$, we obtain

$$\int_{\Omega''} \sum_{i,j} \Delta_l^h\left(a^{ij}(x)D_i u(x)\right) D_j v(x)\mathrm{d}x = \int f(x)\Delta_l^{-h} v(x)\mathrm{d}x. \tag{11.3.26}$$

With (11.3.24) and Lemma 11.2.1, this yields

$$\begin{aligned}&\int_{\Omega''} \sum_{i,j} a^{ij}(x+he_l) D_i \Delta_l^h u(x) D_j v(x)\mathrm{d}x\\ &\qquad \le c_5(d, M_1)\left(\|u\|_{W^{1,2}(\Omega'')} + \|f\|_{L^2(\Omega)}\right)\|Dv\|_{L^2(\Omega'')},\end{aligned} \tag{11.3.27}$$

i.e., an analogue of (11.2.17). Since because of the ellipticity condition (A11.3), we have the estimate

$$\lambda \int_\Omega \left|\eta D\Delta_l^h u(x)\right|^2 \mathrm{d}x \le \int_\Omega \eta^2 \sum_{i,j} a^{ij}(x+he_l)\Delta_l^h D_i u(x) \Delta_l^h D_j u(x)\mathrm{d}x;$$

we can then proceed as in the proofs of Theorems 11.2.1 and 11.2.2. Readers so inclined should face no difficulties in supplying the details.

We now return to the question of boundary regularity and state a theorem:

Theorem 11.3.3. *Let u be a weak solution of $Mu = f$ in Ω with $u - g \in H_0^{1,2}(\Omega)$. As always, suppose (A11.3). Let $f \in W^{k,2}(\Omega)$, $g \in W^{k+2,2}(\Omega)$. Let Ω be of class C^{k+2}, and let the coefficients of M be of class $C^{k+1}(\bar{\Omega})$ (in the sense of Definition 11.3.1). Then*

$$u \in W^{k+2,2}(\Omega),$$

and we have the estimate

$$\|u\|_{W^{k+2,2}(\Omega)} \leq c\left(\|f\|_{W^{k,2}(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)}\right),$$

with c depending on λ, d, and Ω, and on C^{k+1}-bounds for the a^{ij}.

Proof. As explained at the beginning of this section, we may assume that $\partial\Omega$ is locally a hyperplane, by considering the composition $u \circ \phi^{-1}$ in place of u, where ϕ is a diffeomorphism of the type described in Definition 11.3.1. Namely, by (10.4.12), our equation $Mu = f$ gets transformed into an equation

$$\tilde{M}\tilde{u} = \tilde{f}$$

of the same type, with estimates for the coefficients of $\tilde{M}$ following from those for the a^{ij} as well as estimates for the derivatives of ϕ. We have already explained above how to obtain estimates for u in that particular geometric situation. We let this suffice here, instead of offering tedious details without new ideas. □

Remark. As a reference for the regularity theory of weak solutions, we recommend Gilbarg–Trudinger [12].

11.4 Extensions of Sobolev Functions and Natural Boundary Conditions

Most of our preceding results have been formulated for the spaces $H_0^{k,p}(\Omega)$ only, but not for the general Sobolev spaces $W^{k,p}(\Omega) = H^{k,p}(\Omega)$. A technical reason for this is that the mollifications that we have frequently employed use the values of the given function in some full ball about the point under consideration, and this cannot be done at a boundary point if the function is defined only in the domain Ω, perhaps up to its boundary, but not in the exterior of Ω. Thus, it seems natural to extend a given Sobolev function on a domain Ω in $\mathbb{R}^d$ to all of $\mathbb{R}^d$, or at least to some larger domain that contains the closure of Ω in its interior. The problem then is to guarantee that the extended function maintains all the weak differentiability properties of the original function. It turns out that for this to be successfully resolved, we need to impose certain regularity conditions on $\partial\Omega$ as in Definition 11.3.1. In the spirit of that definition, we thus start with the model situation of the domain

$$\mathbb{R}^d_+ := \left\{(x^1, \dots, x^d) \in \mathbb{R}^d, x^d > 0\right\}.$$

If now $u \in C^k(\overline{\mathbb{R}^d_+})$, we define an extension via

$$E_0 u(x) := \begin{cases} u(x) & \text{for } x^d \geq 0, \\ \sum_{j=1}^k a_j u(x^1, \dots, x^{d-1}, -\frac{1}{j}x^d) & \text{for } x^d < 0, \end{cases} \tag{11.4.1}$$

where the a_j are chosen such that

$$\sum_{j=1}^k a_j \left(-\frac{1}{j}\right)^\nu = 1 \quad \text{for } \nu = 0, \dots, k-1. \tag{11.4.2}$$

One readily verifies that the system (11.4.2) is uniquely solvable for the a_j (the determinant of this system is a Vandermonde determinant that is nonzero). One moreover verifies, and this of course is the reason for the choice of the a_j, that the derivatives of $E_0 u$ up to order $k-1$ coincide with the corresponding ones of u on the hyperplane $\left\{x^d = 0\right\}$ and that the derivatives of order k are bounded whenever those of u are. Thus

$$E_0 u \in C^{k-1,1}(\mathbb{R}^d), \tag{11.4.3}$$

where $C^{l,1}(\Omega)$ is defined as the space of l-times continuously differentiable functions on Ω whose lth derivatives are Lipschitz continuous, i.e.,

$$\sup_{x\in\Omega} \frac{|v(x) - v(x_0)|}{|x - x_0|} < \infty$$

for any such derivative v and $x_0 \in \Omega$ (see also Definition 13.1.1 below).

If now Ω is a domain of class C^k in the sense of Definition 11.3.1, and if $u \in C^k(\bar{\Omega})$ (see Definition 11.3.2), we may locally straighten out the boundary with a C^k-diffeomorphism ϕ^{-1}, extend the functions $u \circ \phi^{-1}$ with the above operator E_0, and then take $E_0(u \circ \phi^{-1}) \circ \phi$. This function then defines a local extension of class $C^{k-1,1}$ of u across $\partial\Omega$. In order to obtain a global extension, we simply patch these local extensions together with the help of a partition of unity. This is easy, and the reader may know this construction already, but for completeness, we present the details. We assume that Ω is a bounded domain of class C^k. Thus, $\partial\Omega$ is compact, and so it may be covered by finitely many sets of the type $\Omega \cap \mathring{B}(x_0, r)$ on which a local diffeomorphism with the properties specified in Definition 11.3.1 exists.

We call these sets Ω_ν, $\nu = 1, \ldots, n$, and the corresponding diffeomorphisms ϕ_ν. In addition, we may find an open set $\Omega_0 \subset \Omega$, with $\partial\Omega \cap \bar{\Omega}_0 = \emptyset$, so that

$$\Omega \subset \bigcup_{\nu=0}^{m} \Omega_\nu.$$

We then let φ_ν, $\nu = 0, \ldots, m$, be a partition of unity subordinate to this covering of Ω and put

$$Eu := \varphi_0 u + \sum_{\nu=1}^{m} E_0\left((\varphi_\nu u) \circ \phi_\nu^{-1}\right) \circ \phi_\nu.$$

This then extends u as a $C^{k-1,1}$ function to some open neighborhood Ω' of $\bar{\Omega}$. By taking a $C_0^\infty(\mathbb{R}^d)$ function η with $\eta \equiv 1$ on Ω, $\eta \equiv 0$ in $\mathbb{R}^d \setminus \Omega'$, one may then also extend u to the $C^{k-1,1}(\mathbb{R}^d)$ function ηEu. In fact, this extension lies in $C_0^{k-1,1}(\Omega')$.

This was for C^k-functions, but it may be extended to Sobolev functions by approximation. Again considering the model situation of $\mathbb{R}^d_+$, we observe that $u \in W^{k,p}(\mathbb{R}^d_+)$ can be approximated by the translated mollifications

$$u_h(x + 2he_d) = \frac{1}{h^d} \int_{y^d > 0} u(y) \varrho\left(\frac{x + 2he_d - y}{h}\right) \mathrm{d}y$$

for $h \to 0$ ($h > 0$) (here, e_d is the dth unit vector in $\mathbb{R}^d$). The limit for $h \to 0$ of the extensions $Eu(x + 2he_d)$ then yields the extension $Eu(x)$. One readily verifies that $Eu \in W^{k,p}(\Omega')$ for some domain Ω' containing $\bar{\Omega}$ (for the detailed argument, one needs the extension lemma (Lemma 10.2.2), which obviously holds for all p, not just for $p = 2$) in order to handle the possible discontinuity of the highest-order derivatives along $\partial\Omega$ in the above construction), and that

$$\|Eu\|_{W^{k,p}(\Omega')} \le C \|u\|_{W^{k,p}(\Omega)} \tag{11.4.4}$$

for some constant C depending on Ω (via bounds on the maps ϕ, ϕ^{-1} from Definition 11.3.1) and k. As above, by multiplying by a C_0^∞ function η with $\eta \equiv 1$ on Ω, $\eta \equiv 0$ outside Ω', we may even assume

$$Eu \in H_0^{k,p}(\Omega'). \tag{11.4.5}$$

Equipped with our extension operator E, we may now extend the embedding theorems from the Sobolev spaces $H_0^{k,p}(\Omega)$ to the spaces $W^{k,p}(\Omega)$, if Ω is a C^k-domain. Namely, if $u \in W^{k,p}(\Omega)$, we consider $Eu \in H_0^{k,p}(\Omega')$, which then is contained in $L^{\frac{dp}{d-kp}}(\Omega')$ for $kp < d$, and in $C^m(\Omega')$, respectively, for $0 \le m < k - \frac{d}{p}$, according to Corollary 11.1.1, and thus in $L^{\frac{dp}{d-kp}}(\Omega)$ or $C^m(\Omega)$, by restriction from Ω' to Ω. Since $Eu = u$ on Ω, we have thus proved the following version of the Sobolev embedding theorem:

Theorem 11.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^k. Then*

$$W^{k,p}(\Omega) \subset \begin{cases} L^{\frac{dp}{d-kp}}(\Omega) & \text{for } kp < d, \\ C^m(\bar{\Omega}) & \text{for } 0 \leq m < k - \frac{d}{p}. \end{cases} \tag{11.4.6}$$

In the same manner, we may extend the compactness theorem of Rellich:

Theorem 11.4.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^1. Then any sequence $(u_n)_{n\in\mathbb{N}}$ that is bounded in $W^{1,2}(\Omega)$ contains a subsequence that converges in $L^2(\Omega)$.*

The preceding version of the Sobolev embedding theorem allows us to put our previous existence and regularity results together to obtain a very satisfactory treatment of the Poisson equation in the smooth setting:

Theorem 11.4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^∞, and let $g \in C^\infty(\partial\Omega)$, $f \in C^\infty(\bar{\Omega})$. Then the Dirichlet problem*

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

possesses a (unique) solution u of class $C^\infty(\bar{\Omega})$.

Proof. As explained in the beginning of Sect. 11.3, we may restrict ourselves to the case where $g = 0$, by considering $\bar{u} = u - g$ in place of u, where we have extended g as a C^∞-function to all of $\bar{\Omega}$. (Since $\bar{\Omega}$ is bounded, C^∞-functions on $\bar{\Omega}$ are contained in all Sobolev spaces $W^{k,p}(\bar{\Omega})$.)

In Sect. 10.3, we have seen how Dirichlet's principle produces a weak solution $u \in H_0^{1,2}(\Omega)$ of $\Delta u = f$. We have already observed in Corollary 10.3.1 that such a u is smooth in Ω, but of course this follows also from the more general approach of Sect. 11.2, as stated in Corollary 11.2.1. Regularity up to the boundary, i.e., the result that $u \in C^\infty(\bar{\Omega})$, finally follows from the Sobolev estimates of Theorem 11.3.3 together with the embedding theorem (Theorem 11.4.1). □

Of course, analogous statements can be stated and proved with the concepts and methods developed here in the C^k-case, for any $k \in \mathbb{N}$. In this setting, however, a somewhat more refined result will be obtained below in Theorem 13.3.1.

Likewise, the results extend to more general elliptic operators. Combining Corollary 10.5.2 with Theorems 11.3.3 and 11.4.1, we obtain the following theorem:

Theorem 11.4.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^∞. Let the functions a^{ij} $(i, j = 1, \ldots, d)$ and c be of class C^∞ in Ω and satisfy the assumptions (A)–(D) of Sect. 10.5, and let $f \in C^\infty(\Omega)$, $g \in C^\infty(\partial\Omega)$ be given. Then the Dirichlet problem*

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}u(x)\right) - c(x)u(x) = f(x) \quad in\ \Omega,$$

$$u(x) = g(x) \quad on\ \partial\Omega,$$

admits a (unique) solution of class $C^\infty(\bar{\Omega})$.

It is instructive to compare this result with Theorem 13.3.2 below.

We now address a question that the curious reader may already have wondered about. Namely, what happens if we consider the weak differential equation

$$\int_\Omega Du \cdot Dv + \int_\Omega fv = 0 \quad (f \in L^2(\Omega)) \tag{11.4.7}$$

for all $v \in W^{1,2}(\Omega)$, and not only for those in $H_0^{1,2}(\Omega)$? A solution u again has to be as regular as f and Ω allow, and in fact, the regularity proofs become simpler, since we do not need to restrict our test functions to have vanishing boundary values. In particular we have the following result:

Theorem 11.4.5. *Let (11.4.7) be satisfied for all* $v \in W^{1,2}(\Omega)$*, on some* C^∞*-domain* Ω*, for some function* $f \in C^\infty(\bar{\Omega})$*. Then also*

$$u \in C^\infty(\bar{\Omega}).$$

The *Proof* follows the scheme presented in Sect. 11.3. We obtain differentiability results on the boundary $\partial\Omega$ (note that here we conclude that u is smooth even on the boundary and not only in Ω as in Theorem 11.3.1) by applying the version stated in Theorem 11.4.1 of the Sobolev embedding theorem.

In Sect. 11.5 we shall need regularity results for solutions of

$$\int_\Omega Du \cdot Dv + \mu \int_\Omega u \cdot v = 0 \quad (\mu \in \mathbb{R}), \quad \text{for all } v \in W^{1,2}(\Omega). \tag{11.4.8}$$

We can apply the iteration scheme described in Sect. 11.3 to establish the following corollary:

Corollary 11.4.1. *Let u be a solution of (11.4.8), for all* $v \in W^{1,2}(\Omega)$*. If the domain* Ω *is of class* C^∞*, then* $u \in C^\infty(\bar{\Omega})$*.*

We return to the equation

$$\int_\Omega Du \cdot Dv + \int_\Omega fv = 0$$

on a C^∞-domain Ω, for $f \in C^\infty(\bar{\Omega})$. Since u is smooth up to the boundary by Theorem 11.4.5, we may integrate by parts to obtain

$$-\int_\Omega \Delta u \cdot v + \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v + \int_\Omega fv = 0 \quad \text{for all } v \in W^{1,2}(\Omega). \tag{11.4.9}$$

We know from our discussion of the weak Poisson equation that already if (11.4.7) holds for all $v \in H_0^{1,2}(\Omega)$, then, since u is smooth, necessarily

$$\Delta u = f \quad \text{in } \Omega. \tag{11.4.10}$$

Equation (11.4.9) then implies

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v = 0 \quad \text{for all } v \in W^{1,2}(\Omega).$$

This then implies

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{11.4.11}$$

Thus, u satisfies a homogeneous Neumann boundary condition. Since this boundary condition arises from (11.4.7) when we do not impose any restrictions on v, it then is also called a natural boundary condition.

We add some further easy observations (which have already been made in Sect. 2.1): If u is a solution, so is $u + c$, for any $c \in \mathbb{R}$. Thus, in contrast to the Dirichlet problem, a solution of the Neumann problem is not unique. On the other hand, a solution does not always exist. Namely, we have

$$-\int_{\Omega} \Delta u + \int_{\partial\Omega} \frac{\partial u}{\partial n} = 0,$$

and therefore, using $v \equiv 1$ in (11.4.9), we obtain the condition

$$\int_{\Omega} f = 0 \tag{11.4.12}$$

on f as a necessary condition for the solvability of (11.4.9), hence of (11.4.7). It is not hard to show that this condition is also sufficient, but we do not pursue that point here.

Again, the preceding considerations about the regularity of solutions of the Neumann problem extend to more general elliptic operators, in the same manner as in Sect. 11.3. This is straightforward.

Finally, one may also consider inhomogeneous Neumann boundary conditions; for simplicity, we consider only the Laplace equation, i.e., assume $f = 0$ in the above.

A solution of

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= h \quad \text{on } \partial\Omega, \text{ for some given smooth function } h \text{ on } \partial\Omega, \end{aligned} \tag{11.4.13}$$

can then be obtained by minimizing

$$\frac{1}{2}\int_{\Omega}|Du|^2 - \int_{\partial\Omega} hu \quad \text{in } W^{1,2}(\Omega). \tag{11.4.14}$$

Here, a necessary (and sufficient) condition for solvability is

$$\int_{\partial\Omega} h = 0. \tag{11.4.15}$$

In contrast to the inhomogeneous Dirichlet boundary condition, here the boundary values do not constrain the space in which we seek a minimizer, but rather enter into the functional to be minimized. Again, a weak solution u, i.e., satisfying

$$\int_{\Omega} Du \cdot Dv - \int_{\partial\Omega} hv = 0 \quad \text{for all } v \in W^{1,2}(\Omega), \tag{11.4.16}$$

is determined up to a constant and is smooth up to the boundary, assuming, of course, that $\partial\Omega$ is smooth as before.

11.5 Eigenvalues of Elliptic Operators

In this textbook, at several places (see Sects. 5.1, 6.2, 6.3, and 7.1), we have already encountered expansions in terms of eigenfunctions of the Laplace operator. These expansions, however, served as heuristic motivations only, since we did not show the convergence of these expansions. It is the purpose of the present section to carry this out and to study the eigenvalues of the Laplace operator systematically. In fact, our reasoning will also apply to elliptic operators in divergence form,

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right), \tag{11.5.1}$$

for which the coefficients $a^{ij}(x)$ satisfy the assumptions stated in Sect. 11.3 and are smooth in Ω. Nevertheless, since we have already learned in this chapter how to extend the theory of the Laplace operator to such operators, here we shall carry out the analysis only for the Laplace operator. The indicated generalization we shall leave as an easy exercise. We hope that this strategy has the pedagogical advantage of concentrating on the really essential features.

Let Ω be an open and bounded domain in $\mathbb{R}^d$. The eigenvalue problem for the Laplace operator consists in finding nontrivial solutions of

$$\Delta u(x) + \lambda u(x) = 0 \quad \text{in } \Omega, \tag{11.5.2}$$

for some constant λ, the eigenvalue in question. Here one also imposes some boundary conditions on u. In the light of the preceding, it seems natural to require the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{11.5.3}$$

For many applications, however, it is more natural to have the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{11.5.4}$$

instead, where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the exterior normal. Here, in order to make this meaningful, one needs to impose certain restrictions, for example, as in Sect. 2.1, that the divergence theorem is valid for Ω. For simplicity, as in the preceding section, we shall assume that Ω is a C^∞-domain in treating Neumann boundary conditions. In any case, we shall treat the eigenvalue problem for either type of boundary condition.

As with many questions in the theory of PDEs, the situation becomes much clearer when a more abstract approach is developed. Thus, we shall work in some Hilbert space H; for the Dirichlet case, we choose

$$H = H_0^{1,2}(\Omega), \tag{11.5.5}$$

while for the Neumann case, we take

$$H = W^{1,2}(\Omega). \tag{11.5.6}$$

In either case, we shall employ the L^2-product

$$\langle f, g \rangle := \int_\Omega f(x)g(x)\mathrm{d}x$$

for $f, g \in L^2(\Omega)$, and we shall also put

$$\|f\| := \|f\|_{L^2(\Omega)} = \langle f, f \rangle^{\frac{1}{2}}.$$

It is important to realize that we are not working here with the scalar product of our Hilbert space H, but rather with the scalar product of another Hilbert space, namely, $L^2(\Omega)$, into which H is compactly embedded by Rellich's theorem (Theorems 10.2.3 and 11.4.2).

Another useful point in the sequel is the symmetry of the Laplace operator,

$$\langle \Delta\varphi, \psi \rangle = -\langle D\varphi, D\psi \rangle = \langle \varphi, \Delta\psi \rangle \tag{11.5.7}$$

for all $\varphi, \psi \in C_0^\infty(\Omega)$, as well as for $\varphi, \psi \in C^\infty(\Omega)$ with $\frac{\partial\varphi}{\partial n} = 0 = \frac{\partial\psi}{\partial n}$ on $\partial\Omega$. This symmetry will imply that all eigenvalues are real.

We now start our eigenvalue search with

$$\lambda := \inf_{u\in H\setminus\{0\}} \frac{\langle Du, Du\rangle}{\langle u, u\rangle} \quad \left(= \inf_{u\in H\setminus\{0\}} \frac{\|Du\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \right). \tag{11.5.8}$$

We wish to show that (because the expression in (11.5.8) is scaling invariant, in the sense that it is not affected by replacing u by cu for some nonzero constant c) this infimum is realized by some $u \in H$ with

$$\Delta u + \lambda u = 0.$$

We first observe that (because the expression in (11.5.8) is scaling invariant, in the sense that it is not affected by replacing u by cu for some constant c) we may restrict our attention to those u that satisfy

$$\|u\|_{L^2(\Omega)} \,(= \langle u, u\rangle) = 1. \tag{11.5.9}$$

We then let $(u_n)_{n\in\mathbb{N}} \subset H$ be a minimizing sequence with $\langle u_n, u_n\rangle = 1$, and thus

$$\lambda = \lim_{n\to\infty} \langle Du_n, Du_n\rangle. \tag{11.5.10}$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is bounded in H, and by the compactness theorem of Rellich (Theorems 10.2.3 and 11.4.2), a subsequence, again denoted by u_n, converges to some limit u in $L^2(\Omega)$ that then also satisfies $\|u\|_{L^2(\Omega)} = 1$. In fact, since

$$\begin{aligned} &\|D(u_n - u_m)\|^2_{L^2(\Omega)} + \|D(u_n + u_m)\|^2_{L^2(\Omega)} \\ &\quad = 2\,\|Du_n\|^2_{L^2(\Omega)} + 2\,\|Du_m\|^2_{L^2(\Omega)} \quad \text{for all } n, m \in \mathbb{N}, \end{aligned}$$

and

$$\|D(u_n + u_m)\|^2_{L^2(\Omega)} \geq \lambda\,\|u_n + u_m\|^2_{L^2(\Omega)} \quad \text{by definition of } \lambda,$$

we obtain

$$\begin{aligned} \|Du_n - Du_m\|^2_{L^2(\Omega)} &\leq 2\,\|Du_n\|^2_{L^2(\Omega)} + 2\,\|Du_m\|^2_{L^2(\Omega)} \\ &\quad - \lambda\,\|u_n + u_m\|^2_{L^2(\Omega)}\,. \end{aligned} \tag{11.5.11}$$

Since by choice of the sequence $(u_n)_{n\in\mathbb{N}}$, $\|Du_n\|^2_{L^2(\Omega)}$ and $\|Du_m\|^2_{L^2(\Omega)}$ converges to λ, and $\|u_n + u_m\|^2_{L^2(\Omega)}$ converges to 4, since the u_n converge in $L^2(\Omega)$ to an element u of norm 1, the right-hand side of (11.5.11) converges to 0, and so then does the left-hand side. This, together with the L^2-convergence, implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence even in H, and so it also converges to u in H. Thus

$$\frac{\langle Du, Du\rangle}{\langle u, u\rangle} = \lambda. \tag{11.5.12}$$

In the Dirichlet case, the Poincaré inequality (Theorem 10.2.2) implies

$$\lambda > 0.$$

At this point, the assumption enters that Ω as a domain is connected. In the Neumann case, we simply take any nonzero constant c, which now is an element of $H \setminus \{0\}$, to see that

$$0 \le \lambda \le \frac{\langle Dc, Dc\rangle}{\langle c, c\rangle} = 0,$$

i.e.,

$$\lambda = 0.$$

Following standard conventions for the enumeration of eigenvalues, we put

$$\begin{aligned} \lambda &=: \lambda_1 && \text{in the Dirichlet case,} \\ \lambda &=: \lambda_0 (= 0) && \text{in the Neumann case,} \end{aligned}$$

and likewise $u =: u_1$ and $u =: u_0$, respectively.

Let us now assume that we have iteratively determined $((\lambda_0, u_0))$, (λ_1, u_1), $\dots, (\lambda_{m-1}, u_{m-1})$, with

$$\begin{aligned} &(\lambda_0 \le)\, \lambda_1 \le \cdots \le \lambda_{m-1}, \\ &u_i \in L^2(\Omega) \cap C^\infty(\Omega), \end{aligned}$$

$$\begin{aligned} u_i &= 0 \quad \text{on } \partial\Omega \quad \text{in the Dirichlet case, and} \\ \frac{\partial u_i}{\partial n} &= 0 \quad \text{on } \partial\Omega \quad \text{in the Neumann case,} \end{aligned}$$

$$\langle u_i, u_j\rangle = \delta_{ij} \quad \text{for all } i, j \le m-1$$

$$\Delta u_i + \lambda_i u_i = 0 \quad \text{in } \Omega \quad \text{for } i \le m-1. \tag{11.5.13}$$

We define

$$H_m := \{v \in H : \langle v, u_i \rangle = 0 \quad \text{for } i \le m-1\}$$

and

$$\lambda_m := \inf_{u \in H_m \setminus \{0\}} \frac{\langle Du, Du \rangle}{\langle u, u \rangle}. \tag{11.5.14}$$

Since $H_m \subset H_{m-1}$, the infimum over the former space cannot be smaller than the one over the latter, i.e.,

$$\lambda_m \ge \lambda_{m-1}. \tag{11.5.15}$$

Note that H_m is a Hilbert space itself, being the orthogonal complement of a finite-dimensional subspace of the Hilbert space H. Therefore, with the previous reasoning, we may find $u_m \in H_m$ with $\|u_m\|_{L^2(\Omega)} = 1$ and

$$\lambda_m = \frac{\langle Du_m, Du_m \rangle}{\langle u_m, u_m \rangle}. \tag{11.5.16}$$

We now want to verify the smoothness of u_m and Eq. (11.5.13) for $i = m$.

From (11.5.14), (11.5.16), for all $\varphi \in H_m$, $t \in \mathbb{R}$,

$$\frac{\langle D\,(u_m + t\varphi), D(u_m + t\varphi)\rangle}{\langle u_m + t\varphi, u_m + t\varphi \rangle} \ge \lambda_m,$$

where we choose $|t|$ so small that the denominator is bounded away from 0. This expression then is differentiable w.r.t. t near $t = 0$ and has a minimum at 0. Hence the derivative vanishes at $t = 0$, and we get

$$\begin{aligned} 0 &= \frac{\langle Du_m, D\varphi \rangle}{\langle u_m, u_m \rangle} - \frac{\langle Du_m, Du_m \rangle}{\langle u_m, u_m \rangle} \frac{\langle u_m, \varphi \rangle}{\langle u_m, u_m \rangle} \\ &= \langle Du_m, D\varphi \rangle - \lambda_m \langle u_m, \varphi \rangle \quad \text{for all } \varphi \in H_m. \end{aligned}$$

In fact, this relation even holds for all $\varphi \in H$, because for $i \le m-1$,

$$\langle u_m, u_i \rangle = 0$$

and

$$\langle Du_m, Du_i \rangle = \langle Du_i, Du_m \rangle = \lambda_i \langle u_i, u_m \rangle = 0,$$

since $u_m \in H_i$. Thus, u_m satisfies

$$\int_\Omega Du_m \cdot D\varphi - \lambda_m \int_\Omega u_m \varphi = 0 \quad \text{for all } \varphi \in H. \tag{11.5.17}$$

By Theorem 11.3.1 and Corollary 11.4.1, respectively, u_m is smooth, and so we obtain from (11.5.17)

$$\Delta u_m + \lambda_m u_m = 0 \quad \text{in } \Omega.$$

As explained in the preceding section, we also have

$$\frac{\partial u_m}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in the Neumann case. In the Dirichlet case, we have of course

$$u_m = 0 \quad \text{on } \partial\Omega$$

(this holds pointwise if $\partial\Omega$ is smooth, as explained in Sect. 11.4; for a general, not necessarily smooth, $\partial\Omega$, this relation is valid in the sense of Sobolev).

Theorem 11.5.1. *Let $\Omega \subset \mathbb{R}^d$ be connected, open, and bounded. Then the eigenvalue problem*

$$\Delta u + \lambda u = 0, \quad u \in H_0^{1,2}(\Omega)$$

has countably many eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_m \le \cdots$$

with

$$\lim_{m\to\infty} \lambda_m = \infty$$

and pairwise L^2-orthonormal eigenfunctions u_i and $\langle Du_i, Du_i\rangle = \lambda_i$. Any $v \in L^2(\Omega)$ can be expanded in terms of these eigenfunctions,

$$v = \sum_{i=1}^{\infty} \langle v, u_i\rangle u_i \quad \left(\textit{and thus } \langle v, v\rangle = \sum_{i=1}^{\infty} \langle v, u_i\rangle^2\right), \tag{11.5.18}$$

and if $v \in H_0^{1,2}(\Omega)$, we also have

$$\langle Dv, Dv\rangle = \sum_{i=1}^{\infty} \lambda_i \langle v, u_i\rangle^2. \tag{11.5.19}$$

Theorem 11.5.2. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open, and of class C^∞. Then the eigenvalue problem*

$$\Delta u + \lambda u = 0, \quad u \in W^{1,2}(\Omega)$$

has countably many eigenvalues

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_m \leq \cdots$$

with

$$\lim_{n \to \infty} \lambda_m = \infty$$

and pairwise L^2-orthonormal eigenfunctions u_i that satisfy

$$\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Any $v \in L^2(\Omega)$ can be expanded in terms of these eigenfunctions

$$v = \sum_{i=0}^{\infty} \langle v, u_i \rangle u_i \quad \left(\text{and thus } \langle v, v \rangle = \sum_{i=0}^{\infty} \langle v, u_i \rangle^2\right), \tag{11.5.20}$$

and if $v \in W^{1,2}(\Omega)$, also

$$\langle Dv, Dv \rangle = \sum_{i=1}^{\infty} \lambda_i \langle v, u_i \rangle^2. \tag{11.5.21}$$

Remark. Those $v \in L^2(\Omega)$ that are not contained in H can be characterized by the fact that the expression on the right-hand side of (11.5.19) or (11.5.21) diverges.

The Proofs of Theorems 11.5.1 and 11.5.2 are now easy: We first check

$$\lim_{m \to \infty} \lambda_m = \infty.$$

Indeed, otherwise,

$$\|Du_m\| \leq c \quad \text{for all } m \text{ and some constant c.}$$

By Rellich's theorem again, a subsequence of (u_m) would then be a Cauchy sequence in $L^2(\Omega)$. This, however, is not possible, since the u_m are pairwise L^2-orthonormal.

It remains to prove the expansion. For $v \in H$ we put

$$\beta_i := \langle v, u_i \rangle$$

and

$$v_m := \sum_{i \leq m} \beta_i u_i, \quad w_m := v - v_m.$$

Thus, w_m is the orthogonal projection of v onto H_{m+1}, and v_m then is orthogonal to H_{m+1}; hence

$$\langle w_m, u_i \rangle = 0 \quad \text{for } i \le m.$$

Thus also

$$\langle Dw_m, Dw_m \rangle \ge \lambda_{m+1} \langle w_m, w_m \rangle$$

and

$$\langle Dw_m, Du_i \rangle = \lambda_i \langle u_i, w_m \rangle = 0.$$

These orthogonality relations imply

$$\begin{aligned} \langle w_m, w_m \rangle &= \langle v, v \rangle - \langle v_m, v_m \rangle, \\ \langle Dw_m, Dw_m \rangle &= \langle Dv, Dv \rangle - \langle Dv_m, Dv_m \rangle, \end{aligned} \tag{11.5.22}$$

and then

$$\langle w_m, w_m \rangle \le \frac{1}{\lambda_{m+1}} \langle Dv, Dv \rangle,$$

which converges to 0 as the λ_m tend to ∞. Thus, the remainder w_m converges to 0 in L^2, and so

$$v = \lim_{m \to \infty} v_m = \sum_i \langle v, u_i \rangle u_i \quad \text{in } L^2(\Omega).$$

Also,

$$Dv_m = \sum_{i \le m} \beta_i Du_i,$$

and hence

$$\begin{aligned} \langle Dv_m, Dv_m \rangle &= \sum_{i \le m} \beta_i^2 \langle Du_i, Du_i \rangle \quad (\text{since } \langle Du_i, Du_j \rangle = 0 \quad \text{for } i \ne j) \\ &= \sum_{i \le m} \lambda_i \beta_i^2. \end{aligned}$$

Since $\langle Dv_m, Dv_m \rangle \le \langle Dv, Dv \rangle$ by (11.5.22) and the λ_i are nonnegative, this series then converges, and then for $m < n$,

$$\begin{aligned} \langle Dw_m - Dw_n, Dw_m - Dw_n \rangle &= \langle Dv_n - Dv_m, Dv_n - Dv_m \rangle \\ &= \sum_{i=m+1}^{n} \lambda_i \beta_i^2 \to 0 \quad \text{for } m, n \to \infty, \end{aligned}$$

and so $(Dw_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in L^2, and so w_m converges in H, and the limit is the same as the L^2-limit, namely, 0. Therefore, we get (11.5.19) and (11.5.21), namely,

$$\langle Dv, Dv\rangle = \lim_{m\to\infty}\langle Dv_m, Dv_m\rangle = \sum \lambda_i \beta_i^2.$$

The eigenfunctions $(u_m)_{m\in\mathbb{N}}$ thus are an L^2-orthonormal sequence. The closure of the span of the u_m then is a Hilbert space contained in $L^2(\Omega)$ and containing H. Since H (in fact, even $C_0^\infty(\Omega)\cap H$, see the appendix) is dense in $L^2(\Omega)$, this Hilbert space then has to be all of $L^2(\Omega)$. So, the expansions (11.5.18) and (11.5.20) are valid for all $v \in L^2(\Omega)$.
The strict inequality $\lambda_1 < \lambda_2$ in the Dirichlet case will be proved in Theorem 11.5.4 below.

A moment's reflection also shows that the above procedure produces all the eigenvalues of Δ on H, and that any eigenfunction is a linear combination of the u_i.

An easy consequence of the theorems is the following sharp version of the Poincaré inequality (cf. Theorem 10.2.2):

Corollary 11.5.1. *For $v \in H_0^{1,2}(\Omega)$,*

$$\lambda_1\langle v, v\rangle \le \langle Dv, Dv\rangle, \tag{11.5.23}$$

where λ_1 is the first Dirichlet eigenvalue according to Theorem 11.5.1.
For $v \in H^{1,2}(\Omega)$ with $\frac{\partial v}{\partial \nu}$ on $\partial\Omega$

$$\lambda_1\langle v-\bar{v}, v-\bar{v}\rangle \le \langle Dv, Dv\rangle, \tag{11.5.24}$$

where λ_1 now is the first Neumann eigenvalue according to Theorem 11.5.2, and $\bar{v} := \frac{1}{\|\Omega\|}\int_\Omega v(x)\mathrm{d}x$ is the average of v on Ω ($\|\Omega\|$ is the Lebesgue measure of Ω). Moreover, if such a v with vanishing Neumann boundary values is of class $H^{2,2}(\Omega)$, then also

$$\lambda_1\langle Dv, Dv\rangle \le \langle \Delta v, \Delta v\rangle, \tag{11.5.25}$$

λ_1 again being the first Neumann eigenvalue.

Proof. The inequalities (11.5.23) and (11.5.24) readily follow from (11.5.14), noting that in the second case, $v-\bar{v}$ is orthogonal to the constants, the eigenfunctions for $\lambda_0 = 0$, since

$$\int_\Omega (v(x)-\bar{v})\mathrm{d}x = 0. \tag{11.5.26}$$

As an alternative, and in order to obtain also (11.5.25), we note that $Dv = D(v-\bar{v})$, $\Delta v = \Delta(v-\bar{v})$, and

$$\langle v-\bar{v}, v-\bar{v}\rangle = \sum_{i=1}^\infty \langle v, u_i\rangle^2, \tag{11.5.27}$$

that is, the term for $i = 0$ disappears from the expansion because $v - \bar{v}$ is orthogonal to the constant eigenfunction u_0. Using

$$\langle Dv, Dv \rangle = \sum_{i=1}^{\infty} \lambda_i \langle v, u_i \rangle^2$$

$$\langle \Delta v, \Delta v \rangle = \sum_{i=1}^{\infty} \lambda_i^2 \langle v, u_i \rangle^2$$

and $\lambda_1 \leq \lambda_i$ then yields (11.5.24) and (11.5.25). □

More generally, we can derive Courant's minimax principle for the eigenvalues of Δ:

Theorem 11.5.3. *Under the above assumptions, let P^k be the collection of all k-dimensional linear subspaces of the Hilbert space H. Then the kth eigenvalue of Δ (i.e., λ_k in the Dirichlet case, λ_{k-1} in the Neumann case) is characterized as*

$$\max_{L \in P^{k-1}} \min \left\{ \frac{\langle Du, Du \rangle}{\langle u, u \rangle} : \begin{array}{l} u \neq 0, u \text{ orthogonal to } L, \\ \text{i.e., } \langle u, v \rangle = 0 \quad \text{for all } v \in L \end{array} \right\}, \tag{11.5.28}$$

or dually as

$$\min_{L \in P^k} \max \left\{ \frac{\langle Du, Du \rangle}{\langle u, u \rangle} : u \in L \setminus \{0\} \right\}. \tag{11.5.29}$$

Proof. We have seen that

$$\lambda_m = \min \left\{ \frac{\langle Du, Du \rangle}{\langle u, u \rangle} : u \neq 0, u \text{ orthogonal to the } u_i \text{ with } i \leq m - 1 \right\}. \tag{11.5.30}$$

It is also clear that

$$\lambda_m = \max \left\{ \frac{\langle Du, Du \rangle}{\langle u, u \rangle} : u \neq 0 \text{ linear combination of } u_i \text{ with } i \leq m \right\}, \tag{11.5.31}$$

and in fact, this maximum is realized if u is a multiple of the mth eigenfunction u_m, because $\lambda_i = \frac{\langle Du_i, Du_i \rangle}{\langle u_i, u_i \rangle} \leq \lambda_m$ for $i \leq m$ and the u_i are pairwise orthogonal.

Now let L be another linear subspace of H of the same dimension as the span of the u_i, $i \leq m$. Let L be spanned by vectors v_i, $i \leq m$. We may then find some $v = \sum \alpha_j v_j \in L$ with

$$\langle v, u_i \rangle = \sum_j \alpha_j \langle v_j, u_i \rangle = 0 \quad \text{for } i \leq m - 1. \tag{11.5.32}$$

(This is a system of homogeneous linearly independent equations for the α_j, with one fewer equation than unknowns, and so it can be solved.) Inserting (11.5.32) into the expansion (11.5.19) or (11.5.21), we obtain

$$\frac{\langle Dv, Dv\rangle}{\langle v, v\rangle} = \frac{\sum_{j=m}^{\infty} \lambda_j \langle v, u_j\rangle^2}{\sum_{j=m}^{\infty}\langle v, u_j\rangle^2} \geq \lambda_m.$$

Therefore,

$$\max_{v\in L\setminus\{0\}} \frac{\langle Dv, Dv\rangle}{\langle v, v\rangle} \geq \lambda_m,$$

and (11.5.29) follows. Suitably dualizing the preceding argument, which we leave to the reader, yields (11.5.28). □

While for certain geometrically simple domains, like balls and cubes, one may determine the eigenvalues explicitly; for a general domain, it is a hopeless endeavor to attempt an exact computation of its eigenvalues. One therefore needs approximation schemes, and the minimax principle of Courant suggests one such method, the Rayleigh–Ritz scheme. For that scheme, one selects linearly independent functions $w_1, \ldots, w_k \in H$, which then span a linear subspace L, and seeks the critical values, and in particular the maximum of

$$\frac{\langle Dw, Dw\rangle}{\langle w, w\rangle} \quad \text{for } w \in L.$$

With

$$\begin{aligned} a_{ij} &:= \langle Dw_i, Dw_j\rangle, \quad & A &:= (a_{ij})_{i,j=1,\ldots,k}, \\ b_{ij} &:= \langle w_i, w_j\rangle, & B &:= (b_{ij})_{i,j=1,\ldots,k}, \end{aligned}$$

for

$$w = \sum_{j=1} c_j w_j,$$

then

$$\frac{\langle Dw, Dw\rangle}{\langle w, w\rangle} = \frac{\sum_{i,j=1}^{k} a_{ij} c_i c_j}{\sum_{i,j=1}^{k} b_{ij} c_i c_j},$$

and the critical values are given by the solutions $\mu_1, \ldots, \mu_k$ of

$$\det(A - \mu B) = 0.$$

These values $\mu_1, \dots, \mu_k$ then are taken as approximations of the first k eigenvalues; in particular, if they are ordered such that μ_k is the largest among them, that value is supposed to approximate the kth eigenvalue. One then tries to optimize with respect to the choice of the functions $w_1, \dots, w_k$; i.e., one tries to make μ_k as small as possible, according to (11.5.29), by suitably choosing $w_1, \dots, w_k$.

The characterizations (11.5.28) and (11.5.29) of the eigenvalues have many further useful applications. The basis of those applications is the following simple remark: In (11.5.29), we take the maximum over all $u \in H$ that are contained in some subspace L. If we then enlarge H to some Hilbert space H', then H' contains more such subspaces than H, and so the minimum over all of them cannot increase.

Formally, if we put $P^k(H) := \{k\text{-dimensional linear subspaces of } H\}$, then, if $H \subset H'$, it follows that $P^k(H) \subset P^k(H')$, and so

$$\min_{L \in P^k(H)} \max_{u \in L \setminus \{0\}} \frac{\langle Du, Du \rangle}{\langle u, u \rangle} \geq \min_{L' \in P^k(H')} \max_{u \in L' \setminus \{0\}} \frac{\langle Du, Du \rangle}{\langle u, u \rangle}. \tag{11.5.33}$$

Corollary 11.5.2. *Under the above assumptions, we let* $0 < \lambda_1^D \leq \lambda_2^D \leq \cdots$ *be the Dirichlet eigenvalues, and* $0 = \lambda_0^N < \lambda_1^N \leq \lambda_2^N \leq \cdots$ *be the Neumann eigenvalues. Then*

$$\lambda_{j-1}^N \leq \lambda_j^D \quad \textit{for all } j.$$

Proof. The Hilbert space for the Dirichlet case, namely, $H_0^{1,2}(\Omega)$, is a subspace of that for the Neumann case, namely, $W^{1,2}(\Omega)$, and so (11.5.33) applies. □

The next result states that the eigenvalues decrease if the domain is enlarged:

Corollary 11.5.3. *Let* $\Omega_1 \subset \Omega_2$ *be bounded open subsets of* $\mathbb{R}^d$. *We denote the eigenvalues for the Dirichlet case of the domain* Ω *by* $\lambda_k(\Omega)$. *Then*

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1) \quad \textit{for all } k. \tag{11.5.34}$$

Proof. Any $v \in H_0^{1,2}(\Omega_1)$ can be extended to a function $\tilde{v} \in H_0^{1,2}(\Omega_2)$, simply by putting

$$\tilde{v}(x) = \begin{cases} v(x) & \text{for } x \in \Omega_1, \\ 0 & \text{for } x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

Lemma 10.2.2 tells us that indeed $\tilde{v} \in H_0^{1,2}(\Omega_2)$. Thus, the Hilbert space employed for Ω_1 is contained in that for Ω_2, and the principle (11.5.33) again implies the result for the Dirichlet case. □

Remark. Corollary 11.5.3 is not in general valid for the Neumann case. A first idea to show a result in that case is to extend functions $v \in W^{1,2}(\Omega_1)$ to Ω_2 by the extension operator E constructed in Sect. 11.4. However, this operator does not preserve the norm: In general, $\|Ev\|_{W^{1,2}(\Omega_2)} > \|v\|_{W^{1,2}(\Omega_1)}$, and so this does not represent $W^{1,2}(\Omega_1)$ as a Hilbert subspace of $W^{1,2}(\Omega_2)$. This difficulty makes the Neumann case more involved, and we omit it here.

The next result concerns the first eigenvalue λ_1 of Δ with Dirichlet boundary conditions:

Theorem 11.5.4. *Let λ_1 be the first eigenvalue of Δ on the open and bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Then λ_1 is a simple eigenvalue, meaning that the corresponding eigenspace is one-dimensional. Moreover, an eigenfunction u_1 for λ_1 has no zeros in Ω, and so it is either everywhere positive or negative in Ω.*

Proof. Let

$$\Delta u_1 + \lambda_1 u_1 = 0 \quad \text{in } \Omega.$$

By Corollary 10.2.2, we know that $|u_1| \in W^{1,2}(\Omega)$, and

$$\frac{\langle D|u_1|, D|u_1|\rangle}{\langle |u_1|, |u_1|\rangle} = \frac{\langle Du_1, Du_1\rangle}{\langle u_1, u_1\rangle} = \lambda_1.$$

Therefore, $|u_1|$ also minimizes

$$\frac{\langle Du, Du\rangle}{\langle u, u\rangle},$$

and by the reasoning leading to Theorem 11.5.1, it must also be an eigenfunction with eigenvalue λ_1. Therefore, it is a nonnegative solution of

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega,$$

and by the strong maximum principle (Theorem 3.1.2), it cannot assume a nonpositive interior minimum. Thus, it cannot become 0 in Ω, and so it is positive in Ω. This, however, implies that the original function u_1 cannot become 0 either. Thus, u_1 is of a fixed sign.

This argument applies to all eigenfunctions with eigenvalue λ_1. Since two functions v_1, v_2 neither of which changes sign in Ω cannot satisfy

$$\int_\Omega v_1(x)v_2(x)\mathrm{d}x = 0,$$

i.e., cannot be L^2-orthogonal, the space of eigenfunctions for λ_1 is one-dimensional. □

The classical text on eigenvalue problems is Courant–Hilbert [5].

Remark. More generally, Courant's nodal set theorem holds: Let $\Omega \subset \mathbb{R}^d$ be open and bounded, with Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots$ and corresponding eigenfunctions $u_1, u_2, \ldots$. We call

$$\Gamma^k := \{x \in \Omega : \ u_k(x) = 0\}$$

the nodal set of u_k. The complement $\Omega \setminus \Gamma^k$ then has at most k components.

Summary

In this chapter we have introduced Sobolev spaces as spaces of integrable functions that are not necessarily differentiable in the classical sense, but do possess so-called generalized or weak derivatives that obey the rules for integration by parts. Embedding theorems relate Sobolev spaces to spaces of L^p-functions or of continuous, Hölder continuous, or differentiable functions.

The weak solutions of the Laplace and Poisson equations, obtained in Chap. 10 by Dirichlet's principle, naturally lie in such Sobolev spaces. In this chapter, embedding theorems allow us to show that weak solutions are regular, i.e., differentiable of any order, and hence also solutions in the classical sense.

Based on Rellich's theorem, we have treated the eigenvalue problem for the Laplace operator and shown that any L^2-function admits an expansion in terms of eigenfunctions of the Laplace operator.

Exercises

11.1. Let $u : \Omega \to \mathbb{R}$ be integrable, and let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be multi-indices. Show that if two of the weak derivatives $D_{\boldsymbol{\alpha}+\boldsymbol{\beta}}u$, $D_{\boldsymbol{\alpha}} D_{\boldsymbol{\beta}} u$, $D_{\boldsymbol{\beta}} D_{\boldsymbol{\alpha}} u$ exist, then the third one also exists, and all three of them coincide.

11.2. Let $u, v \in W^{1,1}(\Omega)$ with uv, $uDv + vDu \in L^1(\Omega)$. Then $uv \in W^{1,1}(\Omega)$ as well, and the weak derivative satisfies the product rule

$$D(uv) = uDv + vDu.$$

(For the proof, it is helpful to first consider the case where one of the two functions is of class $C^1(\Omega)$.)

11.3. For $m \geq 2$, $1 \leq q \leq m/2$, $u \in H_0^{2,\frac{m}{q+1}}(\Omega) \cap L^{\frac{m}{q-1}}(\Omega)$ we have $u \in H^{1,\frac{m}{q}}(\Omega)$ and

$$\|Du\|^2_{L^{\frac{m}{q}}(\Omega)} \leq \text{const} \, \|u\|_{L^{\frac{m}{q-1}}(\Omega)} \, \left\|D^2 u\right\|_{L^{\frac{m}{q+1}}(\Omega)} .$$

(Hint: For $p = \frac{m}{q}$,

$$|D_i u|^p = D_i(uD_i u|D_i u|^{p-2}) - uD_i(D_i u|D_i u|^{p-2}).$$

The first term on the right-hand side disappears upon integration over Ω for $u \in C_0^\infty(\Omega)$ (approximation argument!), and for the second one, we utilize the formula

$$D_i(v|v|^{p-2}) = (p-1)(D_i v)|v|^{p-2}.$$

Finally, you need the following version of Hölder's inequality:

$$\|u_1 u_2 u_3\|_{L^1(\Omega)} \leq \|u_1\|_{L^{p_1}(\Omega)} \|u_2\|_{L^{p_2}(\Omega)} \|u_3\|_{L^{p_3}(\Omega)}$$

for $u_i \in L^{p_i}(\Omega)$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ (proof!).)

11.4. Let

$$\Omega_1 := \mathring{B}(0,1) \subset \mathbb{R}^d,$$
$$\Omega_2 := \mathbb{R}^d \setminus \mathring{B}(0,1),$$

i.e., the d-dimensional unit ball and its complement. For which values of k, p, d, α is

$$f(x) := |x|^\alpha$$

in $W^{k,p}(\Omega_1)$ or $W^{k,p}(\Omega_2)$?

11.5. Prove the following version of the Sobolev embedding theorem:
Let $u \in W^{k,p}(\Omega)$, $\Omega' \subset\subset \Omega \subset \mathbb{R}^d$. Then

$$u \in \begin{cases} L^{\frac{dp}{d-kp}}(\Omega') & \text{for } kp < d, \\ C^m(\overline{\Omega'}) & \text{for } 0 \leq m < k - d/p. \end{cases}$$

11.6. State and prove a generalization of Corollary 11.1.5 for $u \in W^{k,p}(\Omega)$ that is analogous to Exercise 11.5.

11.7. Supply the details of the proof of Theorem 11.3.2 (This may sound like a dull exercise after what has been said in the text, but in order to understand the techniques for estimating solutions of PDEs, a certain drill in handling additional lower-order terms and variable coefficients may be needed.)

11.8. Carry out the eigenvalue analysis for the Laplace operator under periodic boundary conditions as defined in Sect. 2.1. In particular, state and prove an analogue of Theorems 11.5.1 and 11.5.2.