# Chapter 12
# Strong Solutions

## 12.1 The Regularity Theory for Strong Solutions

We start with an elementary observation: Let $v \in C_0^3(\Omega)$. Then

$$\begin{aligned}\|D^2 v\|^2_{L^2(\Omega)} &= \int_\Omega \sum_{i,j=1}^d v_{x^i x^j} v_{x^i x^j} = -\int_\Omega \sum_{i,j=1}^d v_{x^i x^j x^i} v_{x^j} \\ &= \int_\Omega \sum_{i=1} v_{x^i x^i} \sum_{j=1}^d v_{x^j x^j} = \|\Delta v\|^2_{L^2(\Omega)} . \end{aligned} \tag{12.1.1}$$

Thus, the $L^2$-norm of $\Delta v$ controls the $L^2$-norms of all second derivatives of $v$. Therefore, if $v$ is a solution of the differential equation

$$\Delta v = f,$$

the $L^2$-norm of $f$ controls the $L^2$-norm of the second derivatives of $v$. This is a result in the spirit of elliptic regularity theory as encountered in Sect. 11.2 (cf. Theorem 11.2.1). In the preceding computation, however, we have assumed that, firstly, $v$ is thrice continuously differentiable and, secondly, that it has compact support. The aim of elliptic regularity theory, however, is to deduce such regularity results, and also, one typically encounters nonvanishing boundary terms on $\partial\Omega$. Thus, our assumptions are inappropriate, and we need to get rid of them. This is the content of this section.

We shall first discuss an elementary special case of the Calderon–Zygmund inequality. Let $f \in L^2(\Omega)$, $\Omega$ open and bounded in $\mathbb{R}^d$. We define the Newton potential of $f$ as

$$w(x) := \int_\Omega \Gamma(x, y) f(y) \mathrm{d}y \tag{12.1.2}$$

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using the fundamental solution constructed in Sect. 2.1,

$$\Gamma(x,y) = \begin{cases} \frac{1}{2\pi}\log|x-y| & \text{for } d = 2, \\ \frac{1}{d(2-d)\omega_d}|x-y|^{2-d} & \text{for } d > 2. \end{cases}$$

**Theorem 12.1.1.** *Let $f \in L^2(\Omega)$ and let $w$ be the Newton potential of $f$. Then $w \in W^{2,2}(\Omega)$, $\Delta w = f$ almost everywhere in $\Omega$, and*

$$\left\| D^2 w \right\|_{L^2(\mathbb{R}^d)} = \| f \|_{L^2(\Omega)} \tag{12.1.3}$$

*($w$ is called a strong solution of $\Delta w = f$, because this equation holds almost everywhere).*

*Proof.* We first assume $f \in C_0^\infty(\Omega)$. Then $w \in C^\infty(\mathbb{R}^d)$. Let $\Omega \subset\subset \Omega_0$, $\Omega_0$ bounded with a smooth boundary. We first wish to show that for $x \in \Omega$,

$$\begin{aligned} \frac{\partial^2}{\partial x^i \partial x^j} w(x) = &\int_{\Omega_0} \frac{\partial^2}{\partial x^i \partial x^j}\Gamma(x,y)(f(y)-f(x))\mathrm{d}y \\ &+ f(x)\int_{\partial\Omega_0} \frac{\partial}{\partial x^i}\Gamma(x,y)\nu^j\, do(y), \end{aligned} \tag{12.1.4}$$

where $\nu = (\nu^1, \ldots, \nu^d)$ is the exterior normal and $do(y)$ yields the induced measure on $\partial\Omega_0$. This is an easy consequence of the fact that

$$\begin{aligned} \left| \frac{\partial^2}{\partial x^i \partial x^j}\Gamma(x,y)(f(y)-f(x)) \right| &\le \text{const} \frac{1}{|x-y|^d} |f(y)-f(x)| \\ &\le \text{const} \frac{1}{|x-y|^{d-1}} \|f\|_{C^1}. \end{aligned}$$

In other words, the singularity under the integral sign is integrable. (Namely, one simply considers

$$v_\varepsilon(x) = \int \frac{\partial}{\partial x^i}\Gamma(x,y)\eta_\varepsilon(x-y)f(y)\mathrm{d}y,$$

with $\eta_\varepsilon(y) = 0$ for $|y| \le \varepsilon$, $\eta_\varepsilon(y) = 1$ for $|y| \ge 2\varepsilon$ and $|D\eta_\varepsilon| \le \frac{2}{\varepsilon}$, and shows that as $\varepsilon \to 0$, $D_j v_\varepsilon$ converges to the right-hand side of (12.1.4).)

*Remark.* Equation (12.1.4) continues to hold for a Hölder continuous $f$, cf. Sect. 13.1 below, since in that case, one can estimate the integrand by

$$\text{const} \frac{1}{|x-y|^{d-\alpha}} \|f\|_{C^\alpha}$$

$(0 < \alpha < 1)$.

Since

$$\Delta\Gamma(x,y) = 0 \quad \text{for all } x \neq y,$$

for $\Omega_0 = B(x,R)$, $R$ sufficiently large, from (12.1.4), we obtain

$$\Delta w(x) = \frac{1}{d\omega_d R^{d-1}} f(x) \int_{|x-y|=R} \sum_{i=1}^{d} \nu^i(y)\nu^i(y)\, do(y) = f(x). \qquad (12.1.5)$$

Thus, if $f$ has compact support, so does $\Delta w$; let the latter be contained in the interior of $B(0,R)$. Then

$$\begin{aligned} \int_{B(0,R)} \sum_{i,j=1}^{d} \left(\frac{\partial^2}{\partial x^i \partial x^j} w\right)^2 &= -\int_{B(0,R)} \sum_i \frac{\partial}{\partial x^i} w \frac{\partial}{\partial x^i} f \\ &\quad + \int_{\partial B(0,R)} Dw \cdot \frac{\partial}{\partial \nu} Dw\, do(y) \\ &= \int_{B(0,R)} (\Delta w)^2 \\ &\quad + \int_{\partial B(0,R)} Dw \cdot \frac{\partial}{\partial \nu} Dw\, do(y). \end{aligned} \qquad (12.1.6)$$

As $R \to \infty$, $Dw$ behaves like $R^{1-d}$, $D^2 w$ like $R^{-d}$, and therefore, the integral on $\partial B(0,R)$ converges to zero for $R \to \infty$. Because of (12.1.5), (12.1.6) then yields (12.1.3).

In order to treat the general case $f \in L^2(\Omega)$, we argue that by Theorem 10.2.7, for $f \in C_0^\infty(\Omega)$, the $W^{1,2}$-norm of $w$ can be controlled by the $L^2$-norm of $f$.[1] We then approximate $f \in L^2(\Omega)$ by $(f_n) \in C_0^\infty(\Omega)$. Applying (12.1.3) to the differences $(w_n - w_m)$ of the Newton potentials $w_n$ of $f_n$, we see that the latter constitute a Cauchy sequence in $W^{2,2}(\Omega)$. The limit $w$ again satisfies (12.1.3), and since $L^2$-functions are defined almost everywhere, $\Delta w = f$ holds almost everywhere, too. □

The above considerations can also be used to provide a proof of Theorem 11.2.1. We recall that result:

**Theorem 12.1.2.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$, with $f \in L^2(\Omega)$. Then $u \in W^{2,2}(\Omega')$, for every $\Omega' \subset\subset \Omega$, and*

$$\|u\|_{W^{2,2}(\Omega')} \leq \text{const}\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right), \qquad (12.1.7)$$

[1] See the proof of Lemma 10.3.1.

*with a constant depending only on $d$, $\Omega$, and $\Omega'$. Moreover,*

$$\Delta u = f \quad \text{almost everywhere in } \Omega.$$

*Proof.* As before, we first consider the case $u \in C^3(\Omega)$. Let $B(x, R) \subset \Omega$, $\sigma \in (0, 1)$, and let $\eta \in C_0^3(B(x, R))$ be a cutoff function with

$$\begin{aligned} 0 \le \eta(y) \le 1,& \\ \eta(y) = 1 &\quad \text{for } y \in B(x, \sigma R), \\ \eta(y) = 0 &\quad \text{for } y \in \mathbb{R}^d \setminus B\left(x, \frac{1+\sigma}{2} \cdot R\right), \\ |D\eta| &\le \frac{4}{(1-\sigma)R}, \\ \left|D^2\eta\right| &\le \frac{16}{(1-\sigma)^2 R^2}. \end{aligned}$$

We put

$$v := \eta u.$$

Then $v \in C_0^3(B(x, R))$, and (12.1.1) implies

$$\left\| D^2 v \right\|_{L^2(B(x,R))} = \|\Delta v\|_{L^2(B(x,R))}. \tag{12.1.8}$$

Now,

$$\Delta v = \eta \Delta u + 2Du \cdot D\eta + u\Delta\eta,$$

and thus

$$\begin{aligned} \left\| D^2 u \right\|_{L^2(B(x,\sigma R))} &\le \left\| D^2 v \right\|_{L^2(B(x,R))} \\ &\le \text{const}\Big( \|f\|_{L^2(B(x,R))} + \frac{1}{(1-\sigma)R} \|Du\|_{L^2\left(B\left(x, \frac{1+\sigma}{2} \cdot R\right)\right)} \\ &\quad + \frac{1}{(1-\sigma)^2 R^2} \|u\|_{L^2(B(x,R))} \Big). \end{aligned} \tag{12.1.9}$$

Now let $\xi \in C_0^1(B(x, R))$ be a cutoff function with

$$\begin{aligned} 0 \le \xi(y) \le 1,& \\ \xi(y) = 1 &\quad \text{for } y \in B\left(x, \frac{1+\sigma}{2} R\right), \\ |D\xi| &\le \frac{4}{(1-\sigma)R}. \end{aligned}$$

Putting $w = \xi^2 u$ and using that $u$ is a weak solution of $\Delta u = f$, we obtain

$$\int_{B(x,R)} Du \cdot D(\xi^2 u) = -\int_{B(x,R)} f\,\xi^2 u,$$

hence

$$\begin{aligned}\int_{B(x,R)} \xi^2 |Du|^2 &= -2\int_{B(x,R)} \xi u Du \cdot D\xi - \int_{B(x,R)} f\,\xi^2 u \\ &\le \frac{1}{2}\int_{B(x,R)} \xi^2 |Du|^2 + 2\int_{B(x,R)} u^2 |D\xi|^2 \\ &\quad + (1-\sigma)^2 R^2 \int_{B(x,R)} f^2 + \frac{1}{(1-\sigma)^2 R^2}\int_{B(x,R)} u^2.\end{aligned}$$

Thus, we have an estimate for $\|\xi Du\|_{L^2(B(x,R))}$, and also

$$\begin{aligned}\|Du\|_{L^2\left(B\left(x,\frac{1+\sigma}{2}R\right)\right)} &\le \|\xi Du\|_{L^2(B(x,R))} \\ &\le \text{const}\left(\frac{1}{(1-\sigma)R}\|u\|_{L^2(B(x,R))}\right. \\ &\qquad \left. + (1-\sigma)R\,\|f\|_{L^2(B(x,R))}\right).\end{aligned} \tag{12.1.10}$$

Inequalities (12.1.9) and (12.1.10) yield

$$\left\|D^2 u\right\|_{L^2(B(x,\sigma R))} \le \text{const}\left(\|f\|_{L^2(B(x,R))} + \frac{1}{(1-\sigma)^2 R^2}\|u\|_{L^2(B(x,R))}\right). \tag{12.1.11}$$

In (12.1.11) we put $\sigma = \frac{1}{2}$, and we cover $\Omega'$ by a finite number of balls $B(x, R/2)$ with $R \le \operatorname{dist}(\Omega', \partial\Omega)$ and obtain (12.1.7) for $u \in C^3(\Omega)$. For the general case $u \in W^{1,2}(\Omega)$, we consider the mollifications $u_h$ defined in appendix. Thus, let $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$. Then

$$\int_\Omega Du_h \cdot Dv = -\int f_h\, v, \quad \text{for all } v \in H_0^{1,2}(\Omega),$$

and since $u_h \in C^\infty(\Omega)$, also

$$\Delta u_h = f_h.$$

By Lemma A.3,

$$\|u_h - u\|, \quad \|f_h - f\|_{L^2(\Omega)} \to 0.$$

In particular, the $u_h$ and the $f_h$ satisfy the Cauchy property in $L^2(\Omega)$. We apply (12.1.7) for $u_{h_1} - u_{h_2}$ to obtain

$$\|u_{h_1} - u_{h_2}\|_{W^{2,2}(\Omega')} \leq \text{const}\left(\|u_{h_1} - u_{h_2}\|_{L^2(\Omega)} + \|f_{h_1} - f_{h_2}\|_{L^2(\Omega)}\right).$$

Thus, the $u_h$ satisfy the Cauchy property in $W^{2,2}(\Omega')$. Consequently, the limit $u$ is in $W^{2,2}(\Omega')$ and satisfies (12.1.7). □

If now $f \in W^{1,2}(\Omega)$, then, because $u \in W^{2,2}(\Omega')$ for all $\Omega' \subset\subset \Omega$, $D_i u$ is a weak solution of $\Delta D_i u = D_i f$ in $\Omega'$. We then obtain $D_i u \in W^{2,2}(\Omega'')$ for all $\Omega'' \subset\subset \Omega'$, i.e., $u \in W^{3,2}(\Omega'')$. Iteratively, we thus obtain a new proof of Theorem 11.2.2, which we now recall:

**Theorem 12.1.3.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$. Then $u \in W^{k+2,2}(\Omega_0)$ for all $\Omega_0 \subset\subset \Omega$, and*

$$\|u\|_{W^{k+2,2}(\Omega_0)} \leq \text{const}\left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)}\right),$$

*with a constant depending on $k, d, \Omega$, and $\Omega_0$.*

In the same manner, we also obtain a new proof of Corollary 11.2.1:

**Corollary 12.1.1.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = f$, for $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$.*

*Proof.* Theorem 12.1.3 and Corollary 11.1.2. □

## 12.2 A Survey of the $L^p$-Regularity Theory and Applications to Solutions of Semilinear Elliptic Equations

The results of the preceding section are valid not only for the exponent $p = 2$, but in fact for any $1 < p < \infty$. We wish to explain this result in the present section. The basis of this $L^p$-regularity theory is the Calderon–Zygmund inequality, which we shall only quote here without proof:

**Theorem 12.2.1.** *Let $1 < p < \infty$, $f \in L^p(\Omega)$ ($\Omega \subset \mathbb{R}^d$ open and bounded), and let $w$ be the Newton potential (12.1.1) of $f$. Then $w \in W^{2,p}(\Omega)$, $\Delta w = f$ almost everywhere in $\Omega$, and*

$$\left\|D^2 w\right\|_{L^p(\Omega)} \leq c(d,p)\,\|f\|_{L^p(\Omega)}, \tag{12.2.1}$$

*with the constant $c(d,p)$ depending only on the space dimension $d$ and the exponent $p$.*

In contrast to the case $p = 2$, i.e., Theorem 12.1.1 , where $c(d, 2) = 1$ for all $d$ and the proof is elementary, the proof of the general case is relatively involved; we refer the reader to Bers–Schechter [2] or Gilbarg–Trudinger [12].

The Calderon–Zygmund inequality yields a generalization of Theorem 12.1.2:

**Theorem 12.2.2.** *Let $u \in W^{1,1}(\Omega)$ be a weak solution of $\Delta u = f$, $f \in L^p(\Omega)$, $1 < p < \infty$, i.e.,*

$$\int Du \cdot D\varphi = -\int f\,\varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{12.2.2}$$

*Then $u \in W^{2,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$, and*

$$\|u\|_{W^{2,p}(\Omega')} \leq \text{const}\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right), \tag{12.2.3}$$

*with a constant depending on $p, d, \Omega'$, and $\Omega$. Also,*

$$\Delta u = f \quad \text{almost everywhere in } \Omega. \tag{12.2.4}$$

We do not provide a complete proof of this result either. This time, however, we shall present at least a sketch of the *proof.*
Apart from the fact that (12.1.8) needs to be replaced by the inequality

$$\left\|D^2 v\right\|_{L^p(B(x,R))} \leq \text{ const. } \|\Delta v\|_{L^p(B(x,r))} \tag{12.2.5}$$

coming from the Calderon–Zygmund inequality (Theorem 12.2.1), we may first proceed as in the proof of Theorem 12.1.2 and obtain the estimate

$$\begin{aligned}\left\|D^2 v\right\|_{L^p(B(x,R))} \leq \text{const}\Bigg( & \|f\|_{L^p(B(x,R))} + \frac{1}{(1-\sigma)R}\|Du\|_{L^p(B(x,\frac{1+\sigma}{2}R))} \\ & + \frac{1}{(1-\sigma)^2R^2}\|u\|_{L^p(B(x,r))}\Bigg)\end{aligned} \tag{12.2.6}$$

for $0 < \sigma < 1$, $B(x, R) \subset \Omega$. The second part of the proof, namely, the estimate of $\|Du\|_{L^p}$, however, is much more difficult for $p \neq 2$ than for $p = 2$. One needs an interpolation argument. For details, we refer to Gilbarg–Trudinger [12] or Giaquinta [11]. This ends our sketch of the proof.

The reader may now get the impression that the $L^p$-theory is a technically subtle, but perhaps essentially useless, generalization of the $L^2$-theory. The $L^p$-theory becomes necessary, however, for treating many nonlinear PDEs. We shall now discuss an example of this. We consider the equation

$$\Delta u + \Gamma(u)|Du|^2 = 0 \tag{12.2.7}$$

with a smooth $\Gamma$. We also require that $\Gamma(u)$ be bounded. This holds if we assume that $\Gamma$ itself is bounded, or if we know already that our (weak) solution $u$ is bounded.

Equation (12.2.7) occurs as the Euler–Lagrange equation of the variational problem

$$I(u) := \int_\Omega g(u(x))|Du(x)|^2\,\mathrm{d}x \to \min, \tag{12.2.8}$$

with a smooth $g$ that satisfies the inequalities

$$0 < \lambda \le g(v) \le \Lambda < \infty,\ |g'(v)| \le k < \infty \tag{12.2.9}$$

($g'$ is the derivative of $g$), with constants $\lambda, \Lambda, k$, for all $v$.

In order to derive the Euler–Lagrange equation for (12.2.8), as in Sect. 10.4, for $\varphi \in H_0^{1,2}(\Omega)$, $t \in \mathbb{R}$, we consider

$$I(u + t\varphi) = \int_\Omega g(u + t\varphi)|D(u + t\varphi)|^2\,\mathrm{d}x.$$

In that case,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I(u + t\varphi)_{|t=0} &= \int \left\{2g(u)\sum_i D_i u D_i\varphi + g'(u)|Du|^2\varphi\right\}\,\mathrm{d}x \\ &= \int \left(-2g(u)\Delta u - 2\sum_i D_i g(u) D_i u + g'(u)\,|Du|^2\right)\varphi\,\mathrm{d}x \\ &= \int \left(-2g(u)\Delta u - g'(u)|Du|^2\right)\varphi\,\mathrm{d}x\end{aligned}$$

after integrating by parts and assuming for the moment $u \in C^2$.

The Euler–Lagrange equation stems from requiring that this expression vanishes for all $\varphi \in H_0^{1,2}(\Omega)$, which is the case, for example, if $u$ minimizes $I(u)$ with respect to fixed boundary values. Thus, that equation is

$$\Delta u + \frac{g'(u)}{2g(u)}|Du|^2 = 0. \tag{12.2.10}$$

With $\Gamma(u) := \frac{g'(u)}{2g(u)}$, we have (12.2.7).

In order to apply the $L^p$-theory, we assume that $u$ is a weak solution of (12.2.7) with

$$u \in W^{1,p_1}(\Omega) \quad \text{for some } p_1 > d \tag{12.2.11}$$

(as always, $\Omega \subset \mathbb{R}^d$, and so $d$ is the space dimension).

The assumption (12.2.11) might appear rather arbitrary. It is typical for nonlinear differential equations, however, that some such hypothesis is needed. Although one may show in the present case (see Sect. 14.4 below) that any bounded weak solution $u$ of class $W^{1,2}(\Omega)$ is also contained in $W^{1,p}(\Omega)$ for all $p$, in structurally similar cases, for example, if $u$ is vector-valued instead of scalar-valued [so that in place of a single equation, we have a system of—typically coupled—equations of the type (12.2.7)], there exist examples of solutions of class $W^{1,2}(\Omega)$ that are not contained in any of the spaces $W^{1,p}(\Omega)$ for $p > 2$. We shall display such an example below, see (12.3.4). In other words, for nonlinear equations, one typically needs a certain initial regularity of the solution before the linear theory can be applied.

In order to apply the $L^p$-theory to our solution $u$ of (12.2.7), we put

$$f(x) := -\Gamma(u(x))|Du(x)|^2. \tag{12.2.12}$$

Because of (12.2.11) and the boundedness of $\Gamma(u)$, then

$$f \in L^{p_1/2}(\Omega), \tag{12.2.13}$$

and $u$ satisfies

$$\Delta u = f \quad \text{in } \Omega. \tag{12.2.14}$$

By Theorem 12.2.2,

$$u \in W^{2,p_1/2}(\Omega') \quad \text{for any } \Omega' \subset\subset \Omega. \tag{12.2.15}$$

By the Sobolev embedding theorem (Corollaries 11.1.1 and 11.1.3, and Exercise 10.5 of Chap. 11),

$$u \in W^{1,p_2}(\Omega') \quad \text{for any } \Omega' \subset\subset \Omega, \tag{12.2.16}$$

with

$$p_2 = \frac{d\,\frac{p_1}{2}}{d - \frac{p_1}{2}} > p_1 \quad \text{because of } p_1 > d. \tag{12.2.17}$$

Thus,

$$f \in L^{\frac{p_2}{2}}(\Omega') \quad \text{for all } \Omega' \subset\subset \Omega, \tag{12.2.18}$$

and we can apply Theorem 12.2.2 and the Sobolev embedding theorem once more, to obtain

$$u \in W^{2,\frac{p_2}{2}} \cap W^{1,p_3}(\Omega') \quad \text{with } p_3 = \frac{d\,\frac{p_2}{2}}{d - \frac{p_2}{2}} > p_2 \tag{12.2.19}$$

for all $\Omega' \subset\subset \Omega''$. Iterating this procedure, we finally obtain

$$u \in W^{2,q}(\Omega') \quad \text{for all } q. \tag{12.2.20}$$

We now differentiate (12.2.7), in order to obtain an equation for $D_i u$, $i = 1, \ldots, d$:

$$\Delta D_i u + \Gamma'(u) D_i u |Du|^2 + 2\Gamma(u) \sum_j D_j u D_{ij} u = 0. \tag{12.2.21}$$

This time, we put

$$f := -\Gamma'(u) D_i u |Du|^2 - 2\Gamma(u) \sum_j D_j u D_{ij} u. \tag{12.2.22}$$

Then

$$|f| \leq \text{ const } (|Du|^3 + |Du||D^2 u|),$$

and because of (12.2.20) thus

$$f \in L^p(\Omega') \quad \text{for all } p.$$

This means that $v := D_i u$ satisfies

$$\Delta v = f \quad \text{with } f \in L^p(\Omega') \quad \text{for all } p. \tag{12.2.23}$$

By Theorem 12.2.2, we infer

$$v \in W^{2,p}(\Omega') \quad \text{for all } p,$$

i.e.,

$$u \in W^{3,p}(\Omega') \quad \text{for all } p. \tag{12.2.24}$$

We differentiate the equation again, to obtain equations for $D_{ij} u$ $(i, j = 1, \ldots, d)$, apply Theorem 12.2.2, conclude that $u \in W^{4,p}(\Omega')$, etc. Iterating the procedure again (this time with higher-order derivatives instead of higher exponents) and applying the Sobolev embedding theorem (Corollary 11.1.2), we obtain the following result:

**Theorem 12.2.3.** *Let $u \in W^{1,p_1}(\Omega)$, for $p_1 > d$ $(\Omega \subset \mathbb{R}^d)$, be a weak solution of*

$$\Delta u + \Gamma(u)|Du|^2 = 0 \tag{12.2.25}$$

*where $\Gamma$ is smooth and $\Gamma(u)$ is bounded. Then*

$$u \in C^\infty(\Omega).$$

The principle of the preceding iteration process is to use the information about the solution $u$ derived in one step as structural information about the equation satisfied by $u$ in the next step, in order to obtain improved information about $u$. In the example discussed here, we use this information in the right-hand side of the equation, but in Chap. 14 we shall see other instances.

More precisely, for our equation $\Delta u = -\Gamma(u)|Du|^2$, we have used calculus inequalities, like the embedding theorems of Sobolev or Morrey, in order to transfer information from the left-hand side to the right-hand side, and we have used elliptic regularity theory to transfer information in the other direction. In this way, we can work ourselves up to ever higher regularity. Such iteration processes are called bootstrapping; they are typical and essential tools in the study of nonlinear PDEs. Usually, to get the iteration started, one needs to know some initial regularity of the solution, however. In Sect. 14.3, we shall improve Theorem 12.2.3 by showing that we only need to assume the boundedness of $u$ to get its continuity.

## 12.3 Some Remarks About Semilinear Elliptic Systems; Transformation Rules for Equations and Systems

The results for solutions of semilinear elliptic equations discussed in the previous section, however, no longer hold for *systems* of elliptic equations of the type of (12.2.25). In this section, we wish to briefly discuss such systems, without being able to provide a comprehensive treatment, however. In order to connect with the preceding section, we start with an example. We consider the map already considered in Example (iii) of Sect. 10.2,

$$u : B(0,1)(\subset \mathbb{R}^d) \to \mathbb{R}^d,$$
$$x \mapsto \frac{x}{|x|},$$

which is discontinuous at 0. We have seen there that for $d \geq 3$, $u \in W^{1,2}(B(0,1), \mathbb{R}^d)$ (this means that all components of $u$ are of class $W^{1,2}$). We recall the formula (10.2.2): We let $e_i$ be the $i$th unit vector, i.e., $x = \sum_i x^i e_i$. For $x \neq 0$, we have

$$\frac{\partial}{\partial x^i} \frac{x}{|x|} = \frac{e_i}{|x|} - \frac{x^i x}{|x|^3}; \tag{12.3.1}$$

hence

$$\left| D \frac{x}{|x|} \right|^2 = \frac{d-1}{|x|^2}. \tag{12.3.2}$$

Therefore,

$$\int_{B(0,1)} |Du|^2 < \infty \text{ for } d \geq 3,$$

i.e., $u \in W^{1,2}(B(0,1))$ for $d \geq 3$.

Next, from (12.3.1),

$$\frac{\partial^2}{\partial x^i \partial x^j} \frac{x}{|x|} = -\frac{x^j e_i}{|x|^3} - \frac{x^i e_j}{|x|^3} - \frac{x\delta_{ij}}{|x|^3} + \frac{3x^i x^j x}{|x|^5},$$

with $\delta_{ij} = 1$ for $i = j$ and 0 else. This implies

$$\Delta \frac{x}{|x|} = -\frac{(d-1)x}{|x|^3}, \tag{12.3.3}$$

and from (12.3.2) and (12.3.3)

$$\Delta u + u|Du|^2 = 0. \tag{12.3.4}$$

Written out with indices, this is

$$\Delta u^\alpha + u^\alpha \sum_{i,\beta=1}^{d} |D_i u^\beta|^2 = 0 \quad \text{for } \alpha = 1, \ldots, d.$$

In particular, we see that the equations for the components $u^\alpha$ of $u$ are coupled by the nonlinearity.

Now, we shall show that $u$, even though it is not continuous at $x = 0$, nevertheless is a weak solution of (12.3.4) on the ball $B(0,1)$. We need to verify that, for $\varphi \in H_0^{1,2} \cap L^\infty(B(0,1), \mathbb{R}^d)$,

$$\int_{B(0,1)} \sum_{i=1}^{d} \sum_{\alpha=1}^{d} (D_i u^\alpha D_i \varphi^\alpha - u^\alpha \varphi^\alpha |Du|^2) = 0. \tag{12.3.5}$$

In order to handle the discontinuity at 0, we utilize the Lipschitz cut-off functions introduced in Sect. 10.2,

$$\eta_m := \begin{cases} 1 & \text{if } |x| \leq 2^{-m} \\ \frac{1}{2^{m-1}}\left(\frac{1}{|x|} - 2^{m-1}\right) & \text{if } 2^{-m} \leq |x| \leq 2^{-(m-1)} \\ 0 & \text{if } 2^{-(m-1)} \leq |x| \end{cases}$$

and write $\varphi = (1 - \eta_m)\varphi + \eta_m \varphi$. The first term then vanishes near 0, and since $u$ is smooth away from 0 and satisfies the (12.3.4) there, this term yields 0 in (12.3.5).

When we insert the second term, $\eta_m \varphi$, in (12.3.5), the only contribution that does not obviously go to 0 for $m \to \infty$ is

$$\int \sum_{i=1}^{d} \sum_{\alpha=1}^{d} (D_i u^\alpha (D_i \eta_m) \varphi^\alpha). \tag{12.3.6}$$

However, since

$$\frac{\partial \eta_m}{\partial x^i} = \begin{cases} 2^{1-m} \frac{x^i}{|x|^3} & \text{for } 2^{-m} \le |x| \le 2^{-(m-1)}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\left| \frac{\partial \eta_m}{\partial x^i} \right| \le \frac{2}{|x|},$$

and with Hölder's inequality, we see with (12.3.2) that (12.3.6) does go to 0 for $m \to \infty$.

We conclude that (12.3.5) holds, i.e., $u = \frac{x}{|x|}$ is a weak solution of (12.3.4) on $B(0, 1)$, indeed. Since $u$ is not continuous, we see that solutions of systems of semilinear elliptic equations need not be regular, in contrast to the case for single equations. Our example, originally found in [13], works in dimension $d \ge 3$; for a two-dimensional example, see the exercises.

Semilinear elliptic equations of the type discussed here play an important role in geometry; see [20].

In order to see how such semilinear systems naturally arise, we start with the Laplace equation and investigate how it transforms under changes of the independent and the dependent variables. We start with the independent variables; here it suffices to consider the single Laplace equation

$$\Delta u(x) = 0. \tag{12.3.7}$$

We change the independent variables via $\xi = \xi(x)$, and we now compute the Laplacian of $v(\xi(x)) = u(x)$ computed w.r.t. $x$ into a differential equation w.r.t. $\xi$; using $\frac{\partial}{\partial x^i} = \sum_k \frac{\partial \xi^k}{\partial x^i} \frac{\partial}{\partial \xi^k}$, this results in

$$\begin{aligned} \sum_i \frac{\partial^2 v(\xi(x))}{(\partial x^i)^2} &= \sum_i \frac{\partial}{\partial x^i} \left( \sum_k \frac{\partial v}{\partial \xi^k} \frac{\partial \xi^k}{\partial x^i} \right) \\ &= \sum_{i,k,\ell} \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^\ell}{\partial x^i} \frac{\partial^2 v}{\partial \xi^k \partial \xi^\ell} + \sum_{i,k} \frac{\partial^2 \xi^k}{(\partial x^i)}. \end{aligned} \tag{12.3.8}$$

Thus, if we put

$$a^{k\ell} := \sum_i \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^\ell}{\partial x^i}, \tag{12.3.9}$$

$$b^k := \sum_i \frac{\partial^2 \xi^k}{(\partial x^i)^2}, \tag{12.3.10}$$

then this becomes

$$\Delta v(\xi(x))) = \sum_{k,\ell} a^{k\ell} \frac{\partial^2 v(\xi)}{\partial \xi^k \partial \xi^\ell} + \sum_k b^k \frac{\partial v(\xi)}{\partial \xi^k}. \tag{12.3.11}$$

Thus, we have transformed the Laplace equation into another linear equation whose coefficients, in general, are not constant. The coefficients of the leading second-order term depend quadratically on the first derivatives of the coordinate transformation, whereas the additional first-order term depends linearly on the second derivatives of the transformation. Of course, if the coordinate transformation is not singular, then $(a^{k\ell})$ is positive definite, and the new equation

$$\sum_{k,\ell} a^{k\ell} \frac{\partial^2 v(\xi)}{\partial \xi^k \partial \xi^\ell} + \sum_k b^k \frac{\partial v(\xi)}{\partial \xi^k} = 0 \tag{12.3.12}$$

is still elliptic. In particular, the regularity theory for linear elliptic equations as developed in previous chapters applies. Of course, in the present case, we know that a solution has to be smooth as long as the coordinate transformation is smooth, because $u(x)$ is smooth as a solution of the Laplace equation (12.3.7). Of course, we may then also try to revert this procedure and transform an equation of type (12.3.12) into the Laplace equation (12.3.7), but this is not always possible for given $a^{ij}, b^i$ as (12.3.9) and (12.3.10) cannot always be solved for $x = x(\xi)$.

Equation (12.3.12) is not written in divergence form. It is possible, however, to rewrite this equation in divergence form. An easy way to see is described in the exercises.

We now transform the dependent variables. For simplicity of notation, we again start with the scalar equation (12.3.7) and consider the Laplacian of $f \circ u$ for some function $f$. We obtain

$$\begin{aligned}\Delta f \circ u &= \sum_i \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x^i}\right) \\ &= \frac{\partial^2 f}{(\partial u)^2} \sum_i \left(\frac{\partial u}{\partial x^i}\right)^2 + \frac{\partial f}{\partial u} \sum_i \frac{\partial^2 u}{(\partial x^i)^2}.\end{aligned} \tag{12.3.13}$$

The important point here is that we obtain a coefficient $\frac{\partial^2 f(u)}{(\partial u)^2}$ of the linear second-order term that depends on the solution $u$ as well as a nonlinear first-order term $\sum_i \frac{\partial^2 u}{(\partial x^i)^2}$. Thus, the equation $\Delta f \circ u = 0$ now becomes nonlinear. In fact, equations of this type are called semilinear.

When $u$ and $f$ are vectors, $u = (u^1, \ldots, u^n)$, $f = (f^1, \ldots, f^n)$, we obtain the system

$$\Delta f^\mu \circ u = \sum_i \frac{\partial}{\partial x^i}\left(\sum_\alpha \frac{\partial f^\mu}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^i}\right)$$
$$= \sum_{\alpha,\beta}\left(\frac{\partial^2 f^\mu}{\partial u^\alpha \partial u^\beta}\sum_i \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^i}\right) + \sum_\alpha \frac{\partial f^\mu}{\partial u^\alpha}\sum_i \frac{\partial^2 u^\alpha}{(\partial x^i)^2}. \quad (12.3.14)$$

When the transformation $f$ is invertible, i.e., when the Jacobian $\frac{\partial f}{\partial u}$ is invertible, this leads us to semilinear elliptic systems of the form

$$\Delta v^\alpha + \sum_i \sum_{\beta,\gamma} \Gamma^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^i} = 0 \quad (12.3.15)$$

with certain coefficients $\Gamma^\alpha_{\beta\gamma}$. In general, when we transform both the independent and the dependent variables, we arrive at the class of systems of the form

$$\sum_{i,j} a^{ij}\frac{\partial^2 v^\alpha}{\partial x^i \partial x^j} + \sum_i b^i \frac{\partial v^\alpha}{\partial x^i} + \sum_{i,j}\sum_{\beta,\gamma} a^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial v^\beta}{\partial x^i}\frac{\partial v^\gamma}{\partial x^j} = 0. \quad (12.3.16)$$

The important fact is that this class of semilinear elliptic systems is closed under variable transformations. That is, when we perform a transformation of the independent or the dependent variables for a system of the form (12.3.16), we obtain again a system of this type, of course, with different coefficients in general.

In fact, for the regularity theory of elliptic systems, it is often helpful and important to compose a solution $u = (u^1, \ldots, u^n)$ of a system of the form (12.3.16) with a scalar function $f$ in order to obtain an equation. The fundamental advantage of second-order elliptic equations when compared to systems is that we can apply the maximum principle.

## Summary

A function $u$ from the Sobolev space $W^{2,2}(\Omega)$ is called a strong solution of

$$\Delta u = f$$

if that equation holds for almost all $x$ in $\Omega$.

In this chapter we show that weak solutions of the Poisson equation are strong solutions as well. This makes an alternative approach to regularity theory possible.

More generally, for a weak solution $u \in W^{1,1}(\Omega)$ of

$$\Delta u = f,$$

where $f \in L^p(\Omega)$, one may utilize the Calderon–Zygmund inequality to get the $L^p$-estimate for all $\Omega \subset\subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega')} \leq \text{const}\,(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}).$$

This is valid for all $1 < p < \infty$ (but not for $p = 1$ or $p = \infty$).

This estimate is useful for iteration methods for the regularity of solutions of nonlinear elliptic equations. For example, any solution $u$ of

$$\Delta u + \Gamma(u)|Du|^2 = 0$$

with regular $\Gamma$ is of class $C^\infty(\Omega)$, provided that it satisfies the initial regularity

$$u \in W^{1,p}(\Omega) \quad \text{for some } p > d \text{ (= space dimension).}$$

Such regularity results are no longer true for solutions of semilinear elliptic systems. For instance, the system

$$\Delta u^\alpha + u^\alpha \sum_{i,\beta=1}^{d} |D_i u^\beta|^2 = 0 \quad \text{for } \alpha = 1, \ldots, d$$

admits the singular weak solution $\frac{x}{|x|}$ for $d \geq 3$.

Finally, we have seen that transforming the independent variables in the Laplace equation leads to a linear elliptic equation, whereas a transformation of the dependent variable(s) leads to a semilinear elliptic equation (system).

## Exercises

**12.1.** First, a routine exercise: Extend the reasoning of Sect. 12.2 to elliptic equations of the form

$$\sum_{i,j=1}^{d} D_i(a^{ij} D_j u) + \Gamma(u)|Du|^2 = 0.$$

**12.2.** Transform the Dirichlet integral

$$\int_{\Omega} \sum_i \left( \frac{\partial u}{\partial x^i} \right)^2 \mathrm{d}x^1 \ldots \mathrm{d}x^d$$

via the coordinate transformation $\xi = \xi(x)$ into an integral w.r.t. $\xi$ for $v(\xi) = u(x)$. Write down the Euler–Lagrange equations for the resulting integral. Observe that this equation is in divergence form. Argue that since integral is obtained from the Dirichlet integral by a coordinate transformation, the resulting Euler–Lagrange equation has to be equivalent to the Laplace equation $\Delta u = 0$, the Euler–Lagrange equation of the Dirichlet integral. Therefore, you have found a differential equation in divergence from that must be equivalent to (12.3.12). That means that the latter equation can be rewritten in divergence form. (Of course, this can also be checked directly, but that becomes rather lengthy. Using the transformation formula for the Dirichlet integral as suggested in the present exercise considerably simplifies the required computations, as we only have to transform first, but no second derivatives.)

**12.3.** Using the theorems discussed in Sect. 12.2, derive the following result: Let $u \in W^{1,2}(\Omega)$ be a weak solution of

$$\Delta u = f,$$

with $f \in W^{k,p}(\Omega)$ for some $k \geq 2$ and some $1 < p < \infty$. Then $u \in W^{k+2,p}(\Omega_0)$ for all $\Omega_0 \subset\subset \Omega$, and

$$\|u\|_{W^{k+2,p}(\Omega_0)} \leq \text{ const } (\|u\|_{L^1(\Omega)} + \|f\|_{W^{k,p}(\Omega)}).$$

**12.4.** Consider the equation

$$\Delta u + F(u) = 0$$

with $|F(u)| \leq c|u|^p$ for some $p < \frac{d+2}{d-2}$ if $d > 2$ or $p < \infty$ for $d = 2$.

**12.5.** What assumptions on $F(x, u, Du)$ would you need to show regularity results for (weak, strong) solutions of equations of the form

$$\Delta u(x) + F(x, u(x), Du(x)) = 0?$$

**12.6.** Consider the system for a map $u : B(0, \frac{1}{2})(\subset \mathbb{R}^2) \to \mathbb{R}^2$,

$$\Delta u^1 + \frac{2(u^1 + u^2)}{1 + |u|^2} |Du|^2 = 0$$

$$\Delta u^2 + \frac{2(u^2 - u^1)}{1 + |u|^2} |Du|^2 = 0.$$

Show that

$$u^1(x) = \sin \log \log \frac{1}{|x|}$$

$$u^2(x) = \cos \log \log \frac{1}{|x|}$$

is a bounded weak solution with a singularity at $x = 0$ (cf. Example (iv) in Sect. 10.2). This example was found in [8].