Chapter 8
The Heat Equation, Semigroups, and Brownian Motion

8.1 Semigroups

We first want to reinterpret some of our results about the heat equation. For that purpose, we again consider the heat kernel of $\mathbb{R}^d$, which we now denote by $p(x, y, t)$,

$$p(x, y, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}. \tag{8.1.1}$$

For a continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$, by Lemma 5.2.1,

$$u(x, t) = \int_{\mathbb{R}^d} p(x, y, t) f(y) \mathrm{d}y \tag{8.1.2}$$

then solves the heat equation

$$\Delta u(x, t) - u_t(x, t) = 0. \tag{8.1.3}$$

For $t > 0$, and letting C_b^0 denote the class of bounded continuous functions, we define the operator

$$P_t : C_b^0(\mathbb{R}^d) \to C_b^0(\mathbb{R}^d)$$

via

$$(P_t f)(x) = u(x, t), \tag{8.1.4}$$

with u from (8.1.2). By Lemma 5.2.2

$$P_0 f := \lim_{t \to 0} P_t f = f; \tag{8.1.5}$$

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i.e., P_0 is the identity operator. The crucial point is that we have for any $t_1, t_2 \geq 0$,

$$P_{t_1+t_2} = P_{t_2} \circ P_{t_1}. \tag{8.1.6}$$

Written out, this means that for all $f \in C_b^0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{1}{(4\pi\,(t_1+t_2))^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4(t_1+t_2)}} f(y)\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^d} \frac{1}{(4\pi t_2)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-z|^2}{4t_2}} \int_{\mathbb{R}^d} \frac{1}{(4\pi t_1)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|z-y|^2}{4t_1}} f(y)\,\mathrm{d}y\,\mathrm{d}z. \tag{8.1.7}$$

This follows from the formula

$$\frac{1}{(4\pi\,(t_1+t_2))^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4(t_1+t_2)}} = \frac{1}{(4\pi t_2)^{\frac{d}{2}}} \frac{1}{(4\pi t_1)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-z|^2}{4t_2}} \mathrm{e}^{-\frac{|z-y|^2}{4t_1}}\,\mathrm{d}z, \tag{8.1.8}$$

which can be verified by direct computation (cf. also Exercise 5.3).

There exists, however, a deeper and more abstract reason for (8.1.6): $P_{t_1+t_2} f(x)$ is the solution at time $t_1 + t_2$ of the heat equation with initial values f. At time t_1, this solution has the value $P_{t_1} f(x)$. On the other hand, $P_{t_2}(P_{t_1} f)(x)$ is the solution at time t_2 of the heat equation with initial values $P_{t_1} f$. Since by Theorem 5.1.2, the solution of the heat equation is unique within the class of bounded functions and the heat equation is invariant under time translations, it must lead to the same result starting at time 0 with initial values $P_{t_1} f$ and considering the solution at time t_2, or starting at time t_1 with value $P_{t_1} f$ and considering the solution at time $t_1 + t_2$, since the time difference is the same in both cases. This reasoning is also valid for the initial value problem because solutions here are unique as well, by Corollary 5.1.1. We have the following results:

Theorem 8.1.1. *Let $\Omega \subset \mathbb{R}^d$ be bounded and of class C^2, and let $g : \partial\Omega \to \mathbb{R}$ be continuous. For any $f \in C_b^0(\Omega)$, we let*

$$P_{\Omega,g,t} f(x)$$

be the solution of the initial value problem

$$\begin{aligned} \Delta u - u_t &= 0 && \text{in } \Omega \times (0,\infty), \\ u(x,t) &= g(x) && \text{for } x \in \partial\Omega, \\ u(x,0) &= f(x) && \text{for } x \in \Omega. \end{aligned} \tag{8.1.9}$$

We then have

$$P_{\Omega,g,0} f = \lim_{t \searrow 0} P_{\Omega,g,t} f = f \quad \text{for all } f \in C^0(\Omega), \tag{8.1.10}$$

$$P_{\Omega,g,t_1+t_2} = P_{\Omega,g,t_2} \circ P_{\Omega,g,t_1}. \tag{8.1.11}$$

Corollary 8.1.1. *Under the assumptions of Theorem 8.1.1, we have for all* $t_0 \geq 0$ *and for all* $f \in C_b^0(\Omega)$,

$$P_{\Omega,g,t_0} f = \lim_{t \searrow t_0} P_{\Omega,g,t} f.$$

We wish to cover the phenomenon just exhibited by a general definition:

Definition 8.1.1. Let B be a Banach space, and for $t > 0$, let $T_t : B \to B$ be continuous linear operators with:

(i) $T_0 = \mathrm{Id}$
(ii) $T_{t_1+t_2} = T_{t_2} \circ T_{t_1}$ for all $t_1, t_2 \geq 0$
(iii) $\lim_{t \to t_0} T_t v = T_{t_0} v$ for all $t_0 \geq 0$ and all $v \in B$

Then the family $\{T_t\}_{t \geq 0}$ is called a continuous semigroup (of operators).

A different and simpler example of a semigroup is the following: Let B be the Banach space of bounded, uniformly continuous functions on $[0, \infty)$. For $t \geq 0$, we put

$$T_t f(x) := f(x + t). \tag{8.1.12}$$

Then all conditions of Definition 8.1.1 are satisfied. Both semigroups (for the heat semigroup, this follows from the maximum principle) satisfy the following definition:

Definition 8.1.2. A continuous semigroup $\{T_t\}_{t \geq 0}$ of continuous linear operators of a Banach space B with norm $\|\cdot\|$ is called contracting if for all $v \in B$ and all $t \geq 0$,

$$\|T_t v\| \leq \|v\|. \tag{8.1.13}$$

(Here, continuity of the semigroup means continuous dependence of the operators T_t on t.)

8.2 Infinitesimal Generators of Semigroups

If the initial values $f(x) = u(x, 0)$ of a solution u of the heat equation

$$u_t(x, t) - \Delta u(x, t) = 0 \tag{8.2.1}$$

are of class C^2, we expect that

$$\lim_{t \searrow 0} \frac{u(x, t) - u(x, 0)}{t} = u_t(x, 0) = \Delta u(x, 0) = \Delta f(x), \tag{8.2.2}$$

or with the notation

$$u(x,t) = P_t f(u)$$

of the previous section,

$$\lim_{t \searrow 0} \frac{1}{t}(P_t - \mathrm{Id}) f = \Delta f. \tag{8.2.3}$$

We want to discuss this in more abstract terms and verify the following definition:

Definition 8.2.1. Let $\{T_t\}_{t \geq 0}$ be a continuous semigroup on a Banach space B. We put

$$D(A) := \left\{ v \in B : \lim_{t \searrow 0} \frac{1}{t}(T_t - \mathrm{Id})v \text{ exists} \right\} \subset B \tag{8.2.4}$$

and call the linear operator

$$A : D(A) \to B,$$

defined as

$$Av := \lim_{t \searrow 0} \frac{1}{t}(T_t - \mathrm{Id})v, \tag{8.2.5}$$

the infinitesimal generator of the semigroup $\{T_t\}$.

Then $D(A)$ is nonempty, since it contains 0.

Lemma 8.2.1. *For all $v \in D(A)$ and all $t \geq 0$, we have*

$$T_t Av = AT_t v. \tag{8.2.6}$$

Thus A commutes with all the T_t.

Proof. For $v \in D(A)$, we have

$$\begin{aligned}
T_t Av &= T_t \lim_{\tau \searrow 0} \frac{1}{\tau}(T_\tau - \mathrm{Id})v \\
&= \lim_{\tau \searrow 0} \frac{1}{\tau}(T_t T_\tau - T_t)v \text{ (since } T_t \text{ is continuous and linear)} \\
&= \lim_{\tau \searrow 0} \frac{1}{\tau}(T_\tau T_t - T_t)v \text{ (by the semigroup property)} \\
&= \lim_{\tau \searrow 0} \frac{1}{\tau}(T_\tau - \mathrm{Id})T_t v \\
&= AT_t v.
\end{aligned}$$

□

In particular, if $v \in D(A)$, then so is $T_t v$. In that sense, there is no loss of regularity of $T_t v$ when compared with v $(= T_0 v)$.

In the sequel, we shall employ the notation

$$J_\lambda v := \int_0^\infty \lambda e^{-\lambda s} T_s v \, ds \quad \text{for } \lambda > 0 \tag{8.2.7}$$

for a contracting semigroup $\{T_t\}$. The integral here is a Riemann integral for functions with values in some Banach space. The standard definition of the Riemann integral as a limit of step functions easily generalizes to the Banach-space-valued case. The convergence of the improper integral follows from the estimate

$$\begin{aligned}\lim_{K,M\to\infty} \left\| \int_K^M \lambda e^{-\lambda s} T_s v ds \right\| &\le \lim_{K,M\to\infty} \int_K^M \lambda e^{-\lambda s} \|T_s v\| \, ds \\ &\le \lim_{K,M\to\infty} \|v\| \int_K^M \lambda e^{-\lambda s} ds \\ &= 0,\end{aligned}$$

which holds because of the contraction property and the completeness of B.

Since

$$\int_0^\infty \lambda e^{-\lambda s} ds = \int_0^\infty -\frac{d}{ds}\left(e^{-\lambda s}\right) ds = 1, \tag{8.2.8}$$

$J_\lambda v$ is a weighted mean of the semigroup $\{T_t\}$ applied to v. Since

$$\begin{aligned}\|J_\lambda v\| &\le \int_0^\infty \lambda e^{-\lambda s} \|T_s v\| \, ds \\ &\le \|v\| \int_0^\infty \lambda e^{-\lambda s} ds \\ &\quad \text{by the contraction property} \\ &\le \|v\| \end{aligned} \tag{8.2.9}$$

by (8.2.8), $J_\lambda : B \to B$ is a bounded linear operator with norm $\|J_\lambda\| \le 1$.

Lemma 8.2.2. *For all $v \in B$, we have*

$$\lim_{\lambda\to\infty} J_\lambda v = v. \tag{8.2.10}$$

Proof. By (8.2.8),

$$J_\lambda v - v = \int_0^\infty \lambda e^{-\lambda s} (T_s v - v) ds.$$

For $\delta > 0$, let

$$I_\lambda^1 := \left\| \int_0^\delta \lambda e^{-\lambda s}(T_s v - v) ds \right\|, \quad I_\lambda^2 := \left\| \int_\delta^\infty \lambda e^{-\lambda s}(T_s v - v) ds \right\|.$$

Now let $\varepsilon > 0$ be given. Since $T_s v$ is continuous in s, there exists $\delta > 0$ such that

$$\|T_s v - v\| < \frac{\varepsilon}{2} \quad \text{for } 0 \le s \le \delta$$

and thus also

$$I_\lambda^1 \le \frac{\varepsilon}{2} \int_0^\delta \lambda e^{-\lambda s} ds < \frac{\varepsilon}{2}$$

by (8.2.8). For each $\delta > 0$, there also exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \ge \lambda_0$,

$$\begin{aligned} I_\lambda^2 &\le \int_\delta^\infty \lambda e^{-\lambda s} (\|T_s v\| + \|v\|) ds \\ &\le 2\|v\| \int_\delta^\infty \lambda e^{-\lambda s} ds \text{ (by the contraction property)} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

This easily implies (8.2.10). □

Theorem 8.2.1. *Let $\{T_t\}_{t\ge 0}$ be a contracting semigroup with infinitesimal generator A. Then $D(A)$ is dense in B.*

Proof. We shall show that for all $\lambda > 0$ and all $v \in B$,

$$J_\lambda v \in D(A). \tag{8.2.11}$$

Since by Lemma 8.2.2,

$$\{J_\lambda v : \lambda > 0, v \in B\}$$

is dense in B, this will imply the assertion. We have

$$\begin{aligned} \frac{1}{t}(T_t - \mathrm{Id}) J_\lambda v &= \frac{1}{t} \int_0^\infty \lambda e^{-\lambda s} T_{t+s} v \, ds - \frac{1}{t} \int_0^\infty \lambda e^{-\lambda s} T_s v \, ds \\ &\qquad \text{since } T_t \text{ is continuous and linear} \\ &= \frac{1}{t} \int_t^\infty \lambda e^{\lambda t} e^{-\lambda\sigma} T_\sigma v \, d\sigma - \frac{1}{t} \int_0^\infty \lambda e^{-\lambda s} T_s v \, ds \\ &= \frac{e^{\lambda t} - 1}{t} \int_t^\infty \lambda e^{-\lambda\sigma} T_\sigma v \, d\sigma - \frac{1}{t} \int_0^t \lambda e^{-\lambda s} T_s v \, ds \\ &= \frac{e^{\lambda t} - 1}{t} \left(J_\lambda v - \int_0^t \lambda e^{-\lambda\sigma} T_\sigma v \, d\sigma \right) - \frac{1}{t} \int_0^t \lambda e^{-\lambda s} T_s v \, ds. \end{aligned}$$

The last term, the integral being continuous in s, for $t \to 0$ tends to $-\lambda T_0 v = -\lambda v$, while the first term in the last line tends to $\lambda J_\lambda v$. This implies

$$A J_\lambda v = \lambda \left(J_\lambda - \mathrm{Id} \right) v \quad \text{for all } v \in B, \tag{8.2.12}$$

which in turn implies (8.2.11). □

For a contracting semigroup $\{T_t\}_{t \geq 0}$, we now define operators

$$D_t T_t : D(D_t T_t) (\subset B) \to B$$

by

$$D_t T_t v := \lim_{h \to 0} \frac{1}{h} \left(T_{t+h} - T_t \right) v, \tag{8.2.13}$$

where $D(D_t T_t)$ is the subspace of B where this limit exists.

Lemma 8.2.3. *$v \in D(A)$ implies $v \in D(D_t T_t)$, and we have*

$$D_t T_t v = A T_t v = T_t A v \quad \textit{for } t \geq 0. \tag{8.2.14}$$

Proof. The second equation has already been established as shown in Lemma 8.2.1. We thus have for $v \in D(A)$,

$$\lim_{h \searrow 0} \frac{1}{h} \left(T_{t+h} - T_t \right) v = A T_t v = T_t A v. \tag{8.2.15}$$

Equation (8.2.15) means that the right derivative of $T_t v$ with respect to t exists for all $v \in D(A)$ and is continuous in t. By a well-known calculus lemma, this then implies that the left derivative exists as well and coincides with the right one, implying differentiability and (8.2.14). The proof of the calculus lemma goes as follows: Let $f : [0, \infty) \to B$ be continuous, and suppose that for all $t \geq 0$, the right derivative $d^+ f(t) := \lim_{h \searrow 0} \frac{1}{h} (f(t+h) - f(t))$ exists and is continuous. The continuity of $d^+ f$ implies that on every interval $[0, T]$ this limit relation even holds uniformly in t. In order to conclude that f is differentiable with derivative $d^+ f$, one argues that

$$\begin{aligned} &\lim_{h \searrow 0} \left\| \frac{1}{h} \left(f(t) - f(t-h) \right) - d^+ f(t) \right\| \\ &\leq \lim_{h \searrow 0} \left\| \frac{1}{h} (f((t-h)+h) - f(t-h)) - d^+ f(t-h) \right\| \\ &\quad + \lim_{h \searrow 0} \left\| d^+ f(t-h) - d^+ f(t) \right\| = 0.) \end{aligned}$$

□

We can interpret Lemma 8.2.3 as:

Corollary 8.2.1. *For a contracting semigroup $\{T_t\}_{t\geq 0}$ with infinitesimal generator A and $v \in D(A)$, $u(t) := T_t v$ satisfies*

$$u'(t) = Au(t) \text{ with } u(0) = v. \tag{8.2.16}$$

Proof. Since we have seen in the proof of Lemma 8.2.3 that $u(t)$ is differentiable w.r.t. t, the differential equation (8.2.16) is simply a restatement of (8.2.14), and that u satisfies the initial condition $u(0) = v$ is a reformulation of $T_0 = \mathrm{Id}$. □

Theorem 8.2.2. *For $\lambda > 0$, the operator $(\lambda \,\mathrm{Id} - A) : D(A) \to B$ is invertible (A being the infinitesimal generator of a contracting semigroup), and we have*

$$(\lambda \,\mathrm{Id} - A)^{-1} = R(\lambda, A) := \frac{1}{\lambda} J_\lambda, \tag{8.2.17}$$

i.e.,

$$(\lambda \,\mathrm{Id} - A)^{-1} v = R(\lambda, A)v = \int_0^\infty \mathrm{e}^{-\lambda s} T_s v \,\mathrm{d}s. \tag{8.2.18}$$

Proof. In order that $(\lambda \,\mathrm{Id} - A)$ be invertible, we need to show first that $(\lambda \,\mathrm{Id} - A)$ is injective. So, we need to exclude that there exists $v_0 \in D(A)$, $v_0 \neq 0$, with

$$\lambda v_0 = A v_0. \tag{8.2.19}$$

For such a v_0, we would have by (8.2.14)

$$D_t T_t v_0 = T_t A v_0 = \lambda T_t v_0, \tag{8.2.20}$$

and hence

$$T_t v_0 = \mathrm{e}^{\lambda t} v_0. \tag{8.2.21}$$

Since $\lambda > 0$, for $v_0 \neq 0$, this would violate the contraction property

$$\|T_t v_0\| \leq \|v_0\|,$$

however. Therefore, $(\lambda \,\mathrm{Id} - A)$ is invertible for $\lambda > 0$. In order to obtain (8.2.17), we start with (8.2.12), i.e.,

$$A J_\lambda v = \lambda (J_\lambda - \mathrm{Id}) v,$$

and get

$$(\lambda \,\mathrm{Id} - A) J_\lambda v = \lambda v. \tag{8.2.22}$$

Therefore, $(\lambda \operatorname{Id} - A)$ maps the image of J_λ bijectively onto B. Since this image is dense in $D(A)$ by (8.2.11) and since $(\lambda \operatorname{Id} - A)$ is injective, $(\lambda \operatorname{Id} - A)$ then also has to map $D(A)$ bijectively onto B. Thus, $D(A)$ has to coincide with the image of J_λ, and (8.2.22) then implies (8.2.17). □

Lemma 8.2.4 (Resolvent equation). *Under the assumptions of Theorem 8.2.2, we have for* $\lambda, \mu > 0$,

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A). \tag{8.2.23}$$

Proof.

$$\begin{aligned} R(\lambda, A) &= R(\lambda, A)(\mu \operatorname{Id} - A) R(\mu, A) \\ &= R(\lambda, A)((\mu - \lambda) \operatorname{Id} + (\lambda \operatorname{Id} - A)) R(\mu, A) \\ &= (\mu - \lambda) R(\lambda, A) R(\mu, A) + R(\mu, A). \end{aligned}$$

□

We now want to compute the infinitesimal generators of some examples with the help of the preceding formalism. We begin with the translation semigroup as introduced at the end of Sect. 8.1: B here is the Banach space of bounded, uniformly continuous functions on $[0, \infty)$, and $T_t f(x) = f(x + t)$ for $f \in B$, $x, t \geq 0$. We then have

$$(J_\lambda f)(x) = \int_0^\infty \lambda e^{-\lambda s} f(x + s) ds = \int_x^\infty \lambda e^{-\lambda(s - x)} f(s) ds, \tag{8.2.24}$$

and hence

$$\frac{d}{dx}(J_\lambda f)(x) = -\lambda f(x) + \lambda (J_\lambda f)(x). \tag{8.2.25}$$

By (8.2.12), the infinitesimal generator satisfies

$$A J_\lambda f(x) = \lambda (J_\lambda f - f)(x), \tag{8.2.26}$$

and consequently

$$A J_\lambda f = \frac{d}{dx} J_\lambda f. \tag{8.2.27}$$

At the end of the proof of Theorem 8.2.2, we have seen that the image of J_λ coincides with $D(A)$, and we thus have

$$Ag = \frac{d}{dx} g \quad \text{for all } g \in D(A). \tag{8.2.28}$$

We now intend to show that $D(A)$ contains precisely those $g \in B$ for which $\frac{d}{dx} g$ belongs to B as well. For such a g, we define $f \in B$ by

$$\frac{d}{dx} g(x) - \lambda g(x) = -\lambda f(x). \tag{8.2.29}$$

By (8.2.25), we then also have

$$\frac{\mathrm{d}}{\mathrm{d}x}(J_\lambda f)(x) - \lambda J_\lambda f(x) = -\lambda f(x). \tag{8.2.30}$$

Thus

$$\varphi(x) := g(x) - J_\lambda f(x)$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x) = \lambda\varphi(x), \tag{8.2.31}$$

whence $\varphi(x) = c\mathrm{e}^{\lambda x}$, and since $\varphi \in B$, necessarily $c = 0$, and so $g = J_\lambda f$.

We thus have verified that the infinitesimal generator A is given by (8.2.28), with the domain of definition $D(A)$ containing precisely those $g \in B$ for which $\frac{\mathrm{d}}{\mathrm{d}x} g \in B$ as well.

We now wish to generalize this example in the following important direction. We consider a system of autonomous ordinary differential equations:

$$\begin{aligned}\frac{\mathrm{d}x^i}{\mathrm{d}t} &= F^i(x),\ i = 1, \ldots, d,\\ x(0) &= x_0.\end{aligned} \tag{8.2.32}$$

We shall often employ vector notation, i.e., write $x = (x^1, \ldots, x^d)$, etc. We assume here that the F^i are continuously differentiable and that for all $x_0 \in \mathbb{R}^d$, the solution $x(t)$ exists for all $t \in \mathbb{R}$. With

$$S_t(x_0) := x(t), \tag{8.2.33}$$

we can then define a contracting semigroup by

$$U_t f(x_0) := f(S_t(x_0)) \tag{8.2.34}$$

in the Banach space of all continuous functions with bounded support in $\mathbb{R}^d$. This semigroup is called the Koopman semigroup. Except for the more restricted Banach space, this clearly generalizes the semigroup T_t from (8.1.12) which corresponds to the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ ($d = 1$). We then have

$$\begin{aligned}J_\lambda f(x) &= \int_0^\infty \lambda\mathrm{e}^{-\lambda s} f(S_s(x))\mathrm{d}s\\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}(-\mathrm{e}^{-\lambda s}) f(S_s(x))\mathrm{d}s\\ &= \int_0^\infty \mathrm{e}^{-\lambda s} \sum_{i=1}^d \frac{\partial f}{\partial x^i} F^i(x)\mathrm{d}s + f.\end{aligned} \tag{8.2.35}$$

Using (8.2.26) again, we then have

$$AJ_\lambda f = \lambda(J_\lambda f - f) = J_\lambda \sum_i f_{x^i} F^i. \tag{8.2.36}$$

Thus, using again that the image of J_λ consists with $D(A)$, we obtain

$$Ag = \sum_i g_{x^i} F^i \text{ for all } g \in D(A). \tag{8.2.37}$$

Thus, by Corollary 8.2.1, $h(t, x) := U_t f(x)$ satisfies the partial differential equation:

$$\frac{\partial h}{\partial t} - \sum_i \frac{\partial h}{\partial x^i} F^i(x) = 0. \tag{8.2.38}$$

We next wish to study a semigroup that is dual to the Koopman semigroup, the Perron–Frobenius semigroup. We first observe that $U_t f$ is defined by (9.1.1) for any $f \in L^\infty(\mathbb{R}^d)$ (although U_t is not a continuous semigroup on L^∞). We then define a semigroup Q_t on $L^1(\mathbb{R}^d)$ by

$$\int Q_t f(x) g(x) \mathrm{d}x = \int f(x) U_t g(x) \mathrm{d}x \quad \text{for all } f \in L^1, g \in L^\infty. \tag{8.2.39}$$

In order to get a more explicit form of Q_t, we consider $g = \chi_A$, the characteristic function of a measurable set A. Then

$$\begin{aligned} \int_A Q_t f(x) \mathrm{d}x &= \int Q_t f(x) \chi_A(x) \mathrm{d}x \\ &= \int f(x) U_t \chi_A(x) \mathrm{d}x \\ &= \int f(x) \chi_A(S_t(x)) \mathrm{d}x \text{ by (9.1.1)} \\ &= \int_A f(x) S_t(x) \mathrm{d}x \\ &= \int_{S_t^{-1}(A)} f(x) \mathrm{d}x. \end{aligned}$$

We thus obtain

$$\int_A Q_t f(x) \mathrm{d}x = \int_{S_t^{-1}(A)} f(x) \mathrm{d}x \quad \text{for all } f \in L^1. \tag{8.2.40}$$

This is the characteristic property of the Perron–Frobenius semigroup.
Since U_t is contracting, i.e., $\|U_t g\|_\infty \le \|g\|_\infty$ for all g, as is clear from (9.1.1), from Hölder's inequality (A.4), we see that Q_t is contracting as well. Letting A

be the infinitesimal generator of U_t as given in (8.2.37) and denoting by A_* the infinitesimal generator of Q_t, we readily obtain from (8.2.39)

$$\int A_* f(x) g(x) \mathrm{d}x = \int f(x) A g(x) \mathrm{d}x \quad \text{for all } f \in D(A_*), g \in D(A). \tag{8.2.41}$$

When g is continuously differentiable with compact support and f is continuously differentiable, we can insert (8.2.37) and integrate by parts to obtain

$$\int A_* f(x) g(x) \mathrm{d}x = -\int \sum_i \frac{\partial f(x) F^i(x)}{\partial x^i} g(x) \mathrm{d}x. \tag{8.2.42}$$

Since we show in the appendix that the compactly supported differentiable functions are dense in L^1, we infer

$$A_* f = -\sum_i \frac{\partial (f F^i)}{\partial x^i} \tag{8.2.43}$$

for continuously differentiable f. By Corollary 8.2.1 again, $h(t,x) := Q_t f(x)$ satisfies the partial differential equation:

$$\frac{\partial h}{\partial t} + \sum_i \frac{\partial (h F^i)}{\partial x^i} = 0. \tag{8.2.44}$$

This equation has been studied already in Sect. 7.2; see (7.2.9). For more details about the Koopman and Perron–Frobenius semigroups, we refer to [25].

We now want to study the other example from Sect. 8.1, the heat semigroup, according to the same pattern. Let B be the Banach space of bounded, uniformly continuous functions on $\mathbb{R}^d$, and

$$P_t f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int \mathrm{e}^{-\frac{|x-y|^2}{4t}} f(y) \mathrm{d}y \quad \text{for } t > 0. \tag{8.2.45}$$

We now have

$$J_\lambda f(x) = \int_{\mathbb{R}^d} \int_0^\infty \frac{\lambda}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\lambda t - \frac{|x-y|^2}{4t}} \mathrm{d}t f(y) \mathrm{d}y. \tag{8.2.46}$$

We compute

$$\begin{aligned}
\Delta J_\lambda f(x) &= \int_{\mathbb{R}^d} \int_0^\infty \frac{\lambda}{(4\pi t)^{\frac{d}{2}}} \Delta_x \mathrm{e}^{-\lambda t - \frac{|x-y|^2}{4t}} \mathrm{d}t f(y) \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \int_0^\infty \lambda \mathrm{e}^{-\lambda t} \frac{\partial}{\partial t} \left(\frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \right) \mathrm{d}t f(y) \mathrm{d}y
\end{aligned}$$

$$= -\lambda f(x) - \int_{\mathbb{R}^d} \int_0^\infty \frac{\partial}{\partial t} \left(\lambda e^{-\lambda t}\right) \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}} dt f(y) dy$$
$$= -\lambda f(x) + \lambda J_\lambda f(x).$$

It follows as before that

$$A J_\lambda f = \Delta J_\lambda f, \tag{8.2.47}$$

and thus

$$A g = \Delta g \quad \text{for all } g \in D(A). \tag{8.2.48}$$

We now want to show that this time, $D(A)$ contains all those $g \in B$ for which Δg is contained in B as well. For such a g, we define $f \in B$ by

$$\Delta g(x) - \lambda g(x) = -\lambda f(x) \tag{8.2.49}$$

and compare this with

$$\Delta J_\lambda f(x) - \lambda J_\lambda f(x) = -\lambda f(x). \tag{8.2.50}$$

Thus $\varphi := g - J_\lambda f$ is bounded and satisfies

$$\Delta \varphi - \lambda \varphi = 0 \quad \text{for } \lambda > 0. \tag{8.2.51}$$

The next lemma will imply $\varphi \equiv 0$, whence $g = J_\lambda f$ as desired.

Lemma 8.2.5. *Let $\lambda > 0$. There does not exist a bounded $\varphi \not\equiv 0$ with*

$$\Delta \varphi(x) = \lambda \varphi(x) \quad \textit{for all } x \in \mathbb{R}^d. \tag{8.2.52}$$

Proof. For a solution of (8.2.52), we compute

$$\Delta \varphi^2 = 2\,|\nabla \varphi|^2 + 2\varphi \Delta \varphi \quad \left(\text{with } \nabla \varphi = \left(\frac{\partial}{\partial x^1}\varphi, \dots, \frac{\partial}{\partial x^d}\varphi\right)\right)$$
$$= 2\,|\nabla \varphi|^2 + 2\lambda \varphi^2 \qquad \text{by (8.2.52)}. \tag{8.2.53}$$

Let $x_0 \in \mathbb{R}^d$. We choose C^2-functions η_R for $R \geq 1$ with

$$0 \leq \eta_R(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^d, \tag{8.2.54}$$
$$\eta_R(x) = 0 \quad \text{for } |x - x_0| \geq R + 1, \tag{8.2.55}$$
$$\eta_R(x) = 1 \quad \text{for } |x - x_0| \leq R, \tag{8.2.56}$$
$$|\nabla \eta_R(x)| + |\Delta \eta_R(x)| \leq c_0 \quad \text{with a constant } c_0 \text{ that does not depend on } x \text{ and } R. \tag{8.2.57}$$

We compute

$$\begin{aligned}\Delta\left(\eta_R^2\varphi^2\right) &= \eta_R^2\Delta\varphi^2 + \varphi^2\Delta\eta_R^2 + 8\eta_R\varphi\nabla\eta_R\cdot\nabla\varphi\\ &\geq 2\eta_R^2\,|\nabla\varphi|^2 + 2\lambda\eta_R^2\varphi^2 + \left(\Delta\eta_R^2\right)\varphi^2 - 2\eta_R^2\,|\nabla\varphi|^2 - 8\,|\nabla\eta_R|^2\,\varphi^2\\ &\quad\text{by (8.2.53) and the Schwarz inequality}\\ &= 2\lambda\eta_R^2\varphi^2 + \left(\Delta\eta_R^2 - 8\,|\nabla\eta_R|^2\right)\varphi^2. \end{aligned} \tag{8.2.58}$$

Together with (8.2.54)–(8.2.57), this implies

$$0 = \int_{B(x_0,R+1)} \Delta\left(\eta_R^2\varphi^2\right) \geq 2\lambda\int_{B(x_0,R)}\varphi^2 - c_1\int_{B(x_0,R+1)\setminus B(x_0,R)}\varphi^2, \tag{8.2.59}$$

where the constant c_1 does not depend on R.

By assumption, φ is bounded, so

$$\varphi^2 \leq K. \tag{8.2.60}$$

Thus (8.2.59) implies

$$\int_{B(x_0,R)}\varphi^2 \leq \frac{c_2K}{\lambda}R^{d-1}, \tag{8.2.61}$$

where the constant c_2 again is independent of R. Equation (8.2.53) implies that φ is subharmonic. The mean value inequality (cf. Theorem 2.2.2) thus implies

$$\varphi^2(x_0) \leq \frac{1}{\omega_d R^d}\int_{B(x_0,R)}\varphi^2 \leq \frac{c_2K}{\omega_d\lambda R} \quad (\text{by (8.2.61)}) \to 0 \quad \text{for } R\to\infty. \tag{8.2.62}$$

Thus, $\varphi(x_0) = 0$. Since this holds for all $x_0 \in \mathbb{R}^d$, φ has to vanish identically. □

Lemma 8.2.6. *Let B be a Banach space, $L : B \to B$ a continuous linear operator with $\|L\| \leq 1$. Then for every $t \geq 0$ and each $x \in B$, the series*

$$\exp(tL)x := \sum_{\nu=0}^{\infty}\frac{1}{\nu!}(tL)^\nu x$$

converges and defines a continuous semigroup with infinitesimal generator L.

Proof. Because of $\|L\| \leq 1$, we also have

$$\|L^n\| \leq 1 \quad \text{for all } n\in\mathbb{N}. \tag{8.2.63}$$

Thus

$$\left\|\sum_{\nu=m}^{n}\frac{1}{\nu!}(tL)^\nu x\right\| \leq \sum_{\nu=m}^{n}\frac{1}{\nu!}t^\nu\,\|L^\nu x\| \leq \|x\|\sum_{\nu=m}^{n}\frac{t^\nu}{\nu!}. \tag{8.2.64}$$

By the Cauchy property of the real-valued exponential series, the last expression becomes arbitrarily small for sufficiently large m, n, and thus our Banach-space-valued exponential series satisfies the Cauchy property as well, and therefore it converges, since B is complete. The limit $\exp(tL)$ is bounded, because by (8.2.64)

$$\left\| \sum_{\nu=0}^{n} \frac{1}{\nu!}(tL)^{\nu} x \right\| \leq \mathrm{e}^{t} \, \|x\|$$

and thus also

$$\|\exp(tL)x\| \leq \mathrm{e}^{t} \, \|x\| \, . \tag{8.2.65}$$

As for the real exponential series, we have

$$\sum_{\nu=0}^{\infty} \frac{(t+s)^{\nu}}{\nu!} L^{\nu} x = \left(\sum_{\mu=0}^{\infty} \frac{t^{\mu}}{\mu!} L^{\mu} \right) \left(\sum_{\sigma=0}^{\infty} \frac{s^{\sigma}}{\sigma!} L^{\sigma} \right) x, \tag{8.2.66}$$

i.e.,

$$\exp((t+s)L) = \exp tL \circ \exp sL, \tag{8.2.67}$$

whence the semigroup property. Furthermore,

$$\left\| \frac{1}{h} \left(\exp(hL) - \mathrm{Id}\right) x - Lx \right\| \leq \sum_{\nu=2}^{\infty} \frac{h^{\nu-1}}{\nu!} \, \|L^{\nu} x\| \leq \|x\| \sum_{\nu=2}^{\infty} \frac{h^{\nu-1}}{\nu!}.$$

Since the last expression tends to 0 as $h \to 0$, L is the infinitesimal generator of the semigroup $\{\exp(tL)\}_{t \geq 0}$. □

In the same manner as (8.2.67), one proves (cf. (8.2.66)) the following lemma.

Lemma 8.2.7. *Let $L, M : B \to B$ be continuous linear operators satisfying the assumptions of Lemma 8.2.6, and suppose*

$$LM = ML. \tag{8.2.68}$$

Then

$$\exp(t(M+L)) = \exp(tM) \circ \exp(tL). \tag{8.2.69}$$

We have started our discussion with the semigroup of operators T_t, and we have then introduced the operators J_λ and the infinitesimal generator A. In practice, however, it is rather the other way around. The operator A is what is typically given, and it defines some differential equation, as in Corollary 8.2.1. Solving this differential equation then amounts to constructing the semigroup $\{T_t\}$. The Hille–Yosida theorem shows us how to do this. From A, we first construct the J_λ and then with their help the T_t.

Theorem 8.2.3 (Hille–Yosida). *Let $A : D(A) \to B$ be a linear operator whose domain of definition $D(A)$ is dense in the Banach space B. Suppose that the resolvent $R(n, A) = (n \operatorname{Id} - A)^{-1}$ exists for all $n \in \mathbb{N}$ and that*

$$\left\| \left(\operatorname{Id} - \frac{1}{n} A \right)^{-1} \right\| \le 1 \quad \textit{for all } n \in \mathbb{N}. \tag{8.2.70}$$

Then A generates a unique contracting semigroup.

Proof. As before, we put

$$J_n := \left(\operatorname{Id} - \frac{1}{n} A \right)^{-1} \quad \text{for } n \in \mathbb{N} \text{ (cf. Theorem 8.2.2).}$$

The proof will consist of several steps:

(1) We claim

$$\lim_{n \to \infty} J_n x = x \quad \text{for all } x \in B, \tag{8.2.71}$$

and

$$J_n x \in D(A) \quad \text{for all } x \in B. \tag{8.2.72}$$

Namely, for $x \in D(A)$, we first have

$$A J_n x = J_n A x = J_n (A - n \operatorname{Id}) x + n J_n x = n (J_n - \operatorname{Id}) x, \tag{8.2.73}$$

and since by assumption $\|J_n A x\| \le \|Ax\|$, it follows that

$$J_n x - x = \frac{1}{n} J_n A x \to 0 \quad \text{for } n \to \infty.$$

As $D(A)$ is dense in B and the operators J_n are equicontinuous by our assumptions, (8.2.71) follows. Equation (8.2.73) then also implies (8.2.72).

(2) By Lemma 8.2.6, the semigroup

$$\{\exp(s J_n)\}_{s \ge 0}$$

exists, because of (8.2.70). Putting $s = tn$, we obtain the semigroup

$$\{\exp(t n J_n)\}_{t \ge 0}$$

and likewise the semigroup

$$T_t^{(n)} := \exp(tAJ_n) = \exp(tn(J_n - \mathrm{Id})) \quad (t \geq 0)$$

(cf. (8.2.73)). By Lemma 8.2.7, we then have

$$T_t^{(n)} = \exp(-tn)\exp(tnJ_n). \tag{8.2.74}$$

Since by (8.2.70)

$$\|\exp(tnJ_n)x\| \leq \sum_{\nu=0}^{\infty} \frac{(nt)^\nu}{\nu!} \|J_n^\nu x\| \leq \exp(nt)\|x\|,$$

it follows that

$$\left\| T_t^{(n)} \right\| \leq 1, \tag{8.2.75}$$

and thus, in particular, the operators are equicontinuous in $t \geq 0$ and $n \in \mathbb{N}$.

(3) For all $m, n \in \mathbb{N}$, we have

$$J_m J_n = J_n J_m. \tag{8.2.76}$$

Since by (8.2.74), J_n commutes with $T_t^{(n)}$; then also J_m commutes with $T_t^{(n)}$ for all $n, m \in \mathbb{N}$, $t \geq 0$. By Lemmas 8.2.3 and 8.2.6, we have for $x \in B$,

$$D_t T_t^{(n)} x = AJ_n T_t^{(n)} x = T_t^{(n)} AJ_n x; \tag{8.2.77}$$

hence

$$\begin{aligned}
\left\| T_t^{(n)} x - T_t^{(m)} x \right\| &= \left\| \int_0^t D_s \left(T_{t-s}^{(m)} T_s^{(n)} x \right) \mathrm{d}s \right\| \\
&= \left\| \int_0^t T_{t-s}^{(m)} T_s^{(n)} (AJ_n - AJ_m)\, x \,\mathrm{d}s \right\| \\
&\leq t\, \|(AJ_n - AJ_m)x\|
\end{aligned} \tag{8.2.78}$$

with (8.2.75). For $x \in D(A)$, we have by (8.2.73)

$$(AJ_n - AJ_m)\, x = (J_n - J_m)\, Ax. \tag{8.2.79}$$

Equations (8.2.78), (8.2.79), and (8.2.71) imply that for $x \in D(A)$,

$$\left(T_t^{(n)} x \right)_{n \in \mathbb{N}}$$

is a Cauchy sequence and the Cauchy property holds uniformly on $0 \le t \le t_0$, for any t_0. Since the operators $T_t^{(n)}$ are equicontinuous by (8.2.75) and $D(A)$ is dense in B by assumption, then

$$\left(T_t^{(n)} x\right)_{n\in\mathbb{N}}$$

is even a Cauchy sequence for all $x \in B$, again locally uniformly with respect to t. Thus the limit

$$T_t x := \lim_{n\to\infty} T_t^{(n)} x$$

exists locally uniformly in t, and T_t is a continuous linear operator with

$$\|T_t\| \le 1 \tag{8.2.80}$$

(cf. (8.2.75)).

(4) We claim that $(T_t)_{t\ge 0}$ is a semigroup. Namely, since $\{T_t^{(n)}\}_{t\ge 0}$ is a semigroup for all $n \in \mathbb{N}$, using (8.2.75), we get

$$\begin{aligned}\|T_{t+s}x - T_t T_s x\| &\le \left\|T_{t+s}x - T_{t+s}^{(n)}x\right\| + \left\|T_{t+s}^{(n)}x - T_t^{(n)} T_s x\right\| \\ &\quad + \left\|T_t^{(n)} T_s x - T_t T_s x\right\| \\ &\le \left\|T_{t+s}x - T_{t+s}^{(n)}x\right\| + \left\|T_s^{(n)}x - T_s x\right\| \\ &\quad + \left\|\left(T_t^{(n)} - T_t\right) T_s x\right\|,\end{aligned}$$

and this tends to 0 for $n \to \infty$.

(5) By (4) and (8.2.80), $\{T_t\}_{t\ge 0}$ is a contracting semigroup. We now want to show that A is the infinitesimal generator of this semigroup. Letting $\bar{A}$ be the infinitesimal generator, we are thus claiming

$$\bar{A} = A. \tag{8.2.81}$$

Let $x \in D(A)$. From (8.2.71) and (8.2.73), we easily obtain

$$T_t A x = \lim_{n\to\infty} T_t^{(n)} A J_n x, \tag{8.2.82}$$

again locally uniformly with respect to t. Thus, for $x \in D(A)$,

$$\begin{aligned}\lim_{t\searrow 0} \frac{1}{t}(T_t x - x) &= \lim_{t\searrow 0} \frac{1}{t} \lim_{n\to\infty} \left(T_t^{(n)} x - x\right) \\ &= \lim_{t\searrow 0} \frac{1}{t} \lim_{n\to\infty} \int_0^t T_s^{(n)} A J_n x \,\mathrm{d}s \text{ by (8.2.77)} \\ &= \lim_{t\searrow 0} \frac{1}{t} \int_0^t T_s A x \,\mathrm{d}s \\ &= Ax.\end{aligned}$$

Thus, for $x \in D(A)$, we also have $x \in D(\bar{A})$, and $Ax = \bar{A}x$. All that remains is to show that $D(A) = D(\bar{A})$. By the proof of Theorem 8.2.2, $(n\,\mathrm{Id} - \bar{A})$ maps $D(A)$ bijectively onto B. Since $(n\,\mathrm{Id} - A)$ already maps $D(A)$ bijectively onto B, we must have $D(A) = D(\bar{A})$ as desired.

(6) It remains to show the uniqueness of the semigroup $\{T_t\}_{t\geq 0}$ generated by A. Let $\{\bar{T}_t\}_{t\geq 0}$ be another contracting semigroup generated by A. Since A then commutes with $\bar{T}_t$, so do AJ_n and $T_t^{(n)}$. We thus obtain as in (8.2.78) for $x \in D(A)$,

$$\left\| T_t^{(n)}x - \bar{T}_t x \right\| = \left\| \int_0^t D_s\left(\bar{T}_{t-s}T_s^{(n)}x\right)\mathrm{d}s \right\| = \left\| \int_0^t \left(-\bar{T}_{t-s}T_s^{(n)}(A - AJ_n)x\right)\mathrm{d}s \right\|.$$

Then (8.2.71) implies

$$\bar{T}_t x = \lim_{n\to\infty} T_t^{(n)}$$

for all $x \in D(A)$ and then as usual also for all $x \in B$; hence $\bar{T}_t = T_t$. □

We now wish to show that the two examples of the translation and the heat semigroup that we have been considering satisfy the assumptions of the Hille–Yosida theorem. Again, we start with the translation semigroup and continue to employ the previous notation. We had identified

$$A = \frac{\mathrm{d}}{\mathrm{d}x} \tag{8.2.83}$$

as the infinitesimal generator, and we want to show that A satisfies condition (8.2.70). Thus, assume

$$\left(\mathrm{Id} - \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} f = g, \tag{8.2.84}$$

and we have to show that

$$\sup_{x\geq 0} |g(x)| \leq \sup_{x\geq 0} |f(x)|. \tag{8.2.85}$$

Equation (8.2.84) is equivalent to

$$f(x) = g(x) - \frac{1}{n}g'(x). \tag{8.2.86}$$

We first consider the case where g assumes its supremum at some $x_0 \in [0, \infty)$. We then have

$$g'(x_0) \leq 0 \quad (= 0,\ \text{if } x_0 > 0).$$

From this,

$$\sup_x g(x) = g(x_0) \leq g(x_0) - \frac{1}{n} g'(x_0) = f(x_0) \leq \sup_x f(x). \tag{8.2.87}$$

If g does not assume its supremum, we can at least find a sequence $(x_\nu)_{\nu \in \mathbb{N}} \subset [0, \infty)$ with

$$g(x_\nu) \to \sup_x g(x). \tag{8.2.88}$$

We claim that for every $\varepsilon_0 > 0$ there exists $\nu_0 \in \mathbb{N}$ such that for all $\nu \geq \nu_0$,

$$g'(x_\nu) < \varepsilon_0. \tag{8.2.89}$$

Namely, if we had

$$g'(x_\nu) \geq \varepsilon_0 \tag{8.2.90}$$

for some ε_0 and almost all ν, by the uniform continuity of g' that follows from (8.2.86) because $f, g \in B$, there would also exist $\delta > 0$ such that

$$g'(x) \geq \frac{\varepsilon_0}{2} \quad \text{if } |x - x_\nu| \leq \delta$$

for all ν with (8.2.90). Thus we would have

$$g(x_\nu + \delta) = g(x_\nu) + \int_0^\delta g'(x_\nu + t)\mathrm{d}t \geq g(x_\nu) + \frac{\varepsilon_0 \delta}{2}. \tag{8.2.91}$$

On the other hand, by (8.2.88), we may assume

$$g(x_\nu) \geq \sup_x g(x) - \frac{\varepsilon_0 \delta}{4},$$

which in conjunction with (8.2.91) yields the contradiction

$$g(x_\nu + \delta) > \sup g(x).$$

Consequently, (8.2.89) must hold. As in (8.2.87), we now obtain for each $\varepsilon > 0$

$$\begin{aligned} \sup_x g(x) &= \lim_{\nu \to \infty} g(x_\nu) \leq \lim_{\nu \to \infty} \left(g(x_\nu) - \frac{1}{n} g'(x_\nu) \right) + \frac{\varepsilon}{n} \\ &= \lim_{\nu \to \infty} f(x_\nu) + \frac{\varepsilon}{n} \leq \sup_x f(x) + \frac{\varepsilon}{n}. \end{aligned}$$

The case of an infimum is treated analogously, and (8.2.85) follows.

We now want to carry out the corresponding analysis for the heat semigroup, again using the notation already established. In this case, the infinitesimal generator is the Laplace operator:

$$A = \Delta. \tag{8.2.92}$$

We again consider the equation

$$\left(\mathrm{Id} - \frac{1}{n}\Delta\right)^{-1} f = g, \tag{8.2.93}$$

or equivalently,

$$f(x) = g(x) - \frac{1}{n}\Delta g(x), \tag{8.2.94}$$

and we again want to verify (8.2.70), i.e.,

$$\sup_{x\in\mathbb{R}^d} |g(x)| \le \sup_{x\in\mathbb{R}^d} |f(x)|. \tag{8.2.95}$$

Again, we first consider the case where g achieves its supremum at some $x_0 \in \mathbb{R}^d$. Then

$$\Delta g(x_0) \le 0,$$

and consequently,

$$\sup_x g(x) = g(x_0) \le g(x_0) - \frac{1}{n}\Delta g(x_0) = f(x_0) \le \sup_x f(x). \tag{8.2.96}$$

If g does not assume its supremum, we select some $x_0 \in \mathbb{R}^d$, and for every $\eta > 0$, we consider the function

$$g_\eta(x) := g(x) - \eta\,|x - x_0|^2.$$

Since

$$\lim_{|x|\to\infty} g_\eta(x) = -\infty,$$

g_η assumes its supremum at some $x_\eta \in \mathbb{R}^d$. Then

$$\Delta g_\eta(x_\eta) \le 0,$$

i.e.,

$$\Delta g(x_\eta) \le 2d\eta.$$

For $y \in \mathbb{R}^d$, we obtain

$$\begin{aligned} g(y) &\leq g(x_\eta) + \eta\,|y - x_0|^2 \\ &\leq g(x_\eta) - \frac{1}{n}\Delta g(x_\eta) + \eta\left(\frac{2d}{n} + |y - x_0|^2\right) \\ &= f(x_\eta) + \eta\left(\frac{2d}{n} + |y - x_0|^2\right) \\ &\leq \sup_{x \in \mathbb{R}^d} f(x) + \eta\left(\frac{2d}{n} + |y - x_0|^2\right). \end{aligned}$$

Since $\eta > 0$ can be chosen arbitrarily small, we thus get for every $y \in \mathbb{R}^d$

$$g(y) \leq \sup_{x \in \mathbb{R}^d} f(x),$$

i.e., (8.2.95) if we treat the infimum analogously.

It is no longer so easy to verify directly that (8.2.94) is solvable with respect to g for given f. By our previous considerations, however, we already know that Δ generates a contracting semigroup, namely, the heat semigroup, and the solvability of (8.2.94) therefore follows from Theorem 8.2.2. Of course, we could have deduced (8.2.70) in the same way, since it is easy to see that (8.2.70) is also necessary for generating a contracting semigroup. The direct proof given here, however, was simple and instructive enough to be presented.

8.3 Brownian Motion

We consider a particle that moves around in some set S, for simplicity assumed to be a measurable subset of $\mathbb{R}^d$, obeying the following rules: The probability that the particle that is at the point x at time t happens to be in the set $E \subset S$ for $s \geq t$ is denoted by $P(t, x; s, E)$. In particular,

$$\begin{aligned} P(t, x; s, S) &= 1, \\ P(t, x; s, \emptyset) &= 0. \end{aligned}$$

This probability should not depend on the positions of the particles at any times less than t. Thus, the particle has no memory, or, as one also says, the process has the Markov property. This means that for $t < \tau \leq s$, the Chapman–Kolmogorov equation

$$P(t, x; s, E) = \int_S P(\tau, y; s, E)P(t, x; \tau, y)\mathrm{d}y \tag{8.3.1}$$

holds. Here, $P(t,x;\tau,y)$ has to be considered as a probability density, i.e., $P(t,x;\tau,y) \geq 0$ and $\int_S P(t,x;\tau,y)\mathrm{d}y = 1$ for all x,t,τ. We want to assume that the process is homogeneous in time, meaning that $P(t,x;s,E)$ depends only on $(s-t)$. We thus have

$$P(t,x;s,E) = P(0,x;s-t,E) =: P(s-t,x,E),$$

and (8.3.1) becomes

$$P(t+\tau,x,E) := \int_S P(\tau,y,E)P(t,x,y)\mathrm{d}y. \tag{8.3.2}$$

We express this property through the following definition:

Definition 8.3.1. Let $\mathcal{B}$ a σ-additive set of subsets of S with $S \in \mathcal{B}$. For $t > 0$, $x \in S$, and $E \in \mathcal{B}$, let $P(t,x,E)$ be defined satisfying:

(i) $P(t,x,E) \geq 0$, $P(t,x,S) = 1$.
(ii) $P(t,x,E)$ is σ-additive with respect to $E \in \mathcal{B}$ for all t,x.
(iii) $P(t,x,E)$ is $\mathcal{B}$-measurable with respect to x for all t,E.
(iv) $P(t+\tau,x,E) = \int_S P(\tau,y,E)P(t,x,y)\mathrm{d}y$ (Chapman–Kolmogorov equation) for all $t,\tau > 0$, x,E.

Then $P(t,x,E)$ is called a Markov process on $(S,\mathcal{B})$.

Let $L^\infty(S)$ be the space of bounded functions on S. For $f \in L^\infty(S)$, $t > 0$, we put

$$(T_t f)(x) := \int_S P(t,x,y)f(y)\mathrm{d}y. \tag{8.3.3}$$

The Chapman–Kolmogorov equation implies the semigroup property

$$T_{t+s} = T_t \circ T_s \quad \text{for } t,s > 0. \tag{8.3.4}$$

Since, by (i), $P(t,x,y) \geq 0$ and

$$\int_S P(t,x,y)\mathrm{d}y = 1, \tag{8.3.5}$$

it follows that

$$\sup_{x\in S} |T_t f(x)| \leq \sup_{x\in S} |f(x)|, \tag{8.3.6}$$

i.e., the contraction property.

In order that T_t map continuous functions to continuous functions and that $\{T_t\}_{t\geq 0}$ define a continuous semigroup, we need additional assumptions. For simplicity, we consider only the case $S = \mathbb{R}^d$.

Definition 8.3.2. The Markov process $P(t, x, E)$ is called spatially homogeneous if for all translations $i : \mathbb{R}^d \to \mathbb{R}^d$,

$$P(t, i(x), i(E)) = P(t, x, E). \tag{8.3.7}$$

A spatially homogeneous Markov process is called a Brownian motion if for all $\varrho > 0$ and all $x \in \mathbb{R}^d$,

$$\lim_{t \searrow 0} \frac{1}{t} \int_{|x-y|>\varrho} P(t, x, y)\mathrm{d}y = 0. \tag{8.3.8}$$

Theorem 8.3.1. *Let B be the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^d$, equipped with the supremum norm. Let $P(t, x, E)$ be a Brownian motion. We put*

$$(T_t f)(x) := \int_{\mathbb{R}^d} P(t, x, y) f(y)\mathrm{d}y \quad \textit{for } t > 0,$$
$$T_0 f = f.$$

Then $\{T_t\}_{t \geq 0}$ constitutes a contracting semigroup on B.

Proof. As already explained, $P(t, x, E) \geq 0$, $P(t, x, \mathbb{R}^d) = 1$ implies the contraction property:

$$\sup_{x \in \mathbb{R}^d} |(T_t f)(x)| \leq \sup_{x \in \mathbb{R}^d} |f(x)| \quad \text{for all } f \in B, t \geq 0, \tag{8.3.9}$$

and the semigroup property follows from the Chapman–Kolmogorov equation. Let i be a translation of Euclidean space. We put

$$if(x) := f(ix)$$

and obtain

$$\begin{aligned} iT_t f(x) = T_t f(ix) &= \int_{\mathbb{R}^d} P(t, ix, y) f(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} P(t, ix, iy) f(iy)\mathrm{d}y, \\ &\quad \text{since } d(iy) = \mathrm{d}y \text{ for a translation,} \\ &= \int_{\mathbb{R}^d} P(t, x, y) f(iy)\mathrm{d}y, \\ &\quad \text{since the process is spatially homogeneous,} \\ &= T_t i f(x), \end{aligned}$$

i.e.,

$$iT_t = T_t i. \tag{8.3.10}$$

For $x, y \in \mathbb{R}^d$, we may find a translation $i : \mathbb{R}^d \to \mathbb{R}^d$ with

$$ix = y.$$

We then have

$$|(T_t f)(x) - (T_t f)(y)| = |(T_t f)(x) - (iT_t f)(x)| = |T_t(f - if)(x)|.$$

Since f is uniformly continuous, this implies that $T_t f$ is uniformly continuous as well; namely,

$$|T_t(f - if)(x)| = \left|\int P(t, x, z)(f(z) - f(iz))\mathrm{d}z\right| \le \sup_z |f(z) - f(iz)|,$$

and if $|x - y| < \delta$, then also $|z - iz| < \delta$ for all $z \in \mathbb{R}^d$, and δ may be chosen such that this expression becomes smaller than any given $\varepsilon > 0$. Note that this estimate does not depend on t.

It remains to show continuity with respect to t. Let $t \ge s$. For $f \in B$, we consider

$$\begin{aligned}
|T_t f(x) - T_s f(x)| &= |T_\tau g(x) - g(x)| \quad \text{for } \tau := t - s, g := T_s f \\
&= \left|\int_{\mathbb{R}^d} P(\tau, x, y)(g(y) - g(x))\mathrm{d}y\right| \\
&\quad \text{because of } \int_{\mathbb{R}^d} P(t, x, y)\mathrm{d}y = 1 \\
&\le \left|\int_{|x-y|\le\varrho} P(\tau, x, y)(g(y) - g(x))\mathrm{d}y\right| \\
&\quad + \left|\int_{|x-y|>\varrho} P(\tau, x, y)(g(y) - g(x))\mathrm{d}y\right| \\
&\le \left|\int_{|x-y|\le\varrho} P(\tau, x, y)(g(y) - g(x))\mathrm{d}y\right| \\
&\quad + 2\sup_{z\in\mathbb{R}^d} |f(z)| \int_{|x-y|>\varrho} P(\tau, x, y)\mathrm{d}y
\end{aligned}$$

by (8.3.9). Since we have checked already that $g = T_s f$ satisfies the same continuity estimates as f, for given $\varepsilon > 0$, we may choose $\varrho > 0$ so small that the first term on the right-hand side becomes smaller than $\varepsilon/2$. For that value of ϱ

we may then choose τ so small that the second term becomes smaller than $\varepsilon/2$ as well. Note that because of the spatial homogeneity, τ can be chosen independently of x and y. This shows that $\{T_t\}_{t\geq 0}$ is a continuous semigroup, and the proof of Theorem 8.3.1 is complete. □

An example of Brownian motion is given by the heat kernel

$$P(t,x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}. \tag{8.3.11}$$

We shall now see that this already is the typical case of a Brownian motion.

Theorem 8.3.2. *Let $P(t,x,E)$ be a Brownian motion that is invariant under all isometries of Euclidean space, i.e.,*

$$P(t,i(x),i(E)) = P(t,x,E) \tag{8.3.12}$$

for all Euclidean isometries i. Then the infinitesimal generator of the contracting semigroup defined by this process is

$$A = c\Delta, \tag{8.3.13}$$

where $c = \text{const} > 0$ and $\Delta =$Laplace operator, and this semigroup then coincides with the heat semigroup up to reparametrization, according to the uniqueness result of Theorem 8.2.3. More precisely, we have

$$P(t,x,y) = \frac{1}{(4\pi ct)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4ct}}. \tag{8.3.14}$$

Proof. (1) Let B again be the Banach space of bounded, uniformly continuous functions on $\mathbb{R}^d$, equipped with the supremum norm. By Theorem 8.3.1, our semigroup operates on B. By Theorem 8.2.1, the domain of definition $D(A)$ of the infinitesimal operator A is dense in B.

(2) We claim that $D(A) \cap C^\infty(\mathbb{R}^d)$ is still dense in B. To verify that, as in Sect. 2.2, we consider mollifications with a smooth kernel, i.e., for $f \in D(A)$,

$$\begin{aligned} f_r(x) &= \frac{1}{r^d}\int_{\mathbb{R}^d} \varrho\left(\frac{|x-y|}{r}\right) f(y)\mathrm{d}y \quad \text{as in (2.2.6)} \\ &= \int_{\mathbb{R}^d} \rho(|z|) f(x-rz)\mathrm{d}z. \end{aligned} \tag{8.3.15}$$

Since we are assuming translation invariance, if the function $f(x)$ is contained in $D(A)$, so is $(i_{rz}f)(x) = f(x-rz)$ for all $r > 0$, $z \in \mathbb{R}^d$ in $D(A)$, and the defining criterion, namely,

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\mathbb{R}^d} P(t,x,y)f(y-rz) - f(x-rz)\right) = 0,$$

holds uniformly in r, z. Approximating the preceding integral by step functions of the form $\sum_\nu c_\nu f(x - rz_\nu)$ (where we have only finitely many summands, since ρ has compact support), we see that since f does, f_r also satisfies $\lim_{t\to 0} \frac{1}{t}\left(\int_{\mathbb{R}^d} P(t,x,y) f_r(y)\,\mathrm{d}y - f_r(x)\right) = 0$, hence is contained in $D(A)$. Since f_r is contained in $C^\infty(\mathbb{R}^d)$ for $r > 0$ and converges to f uniformly as $r \to 0$, the claim follows.

(3) We claim that there exists a function $\varphi \in D(A) \cap C^\infty(\mathbb{R}^d)$ with

$$x^j x^k \frac{\partial^2 \varphi}{\partial x^j \partial x^k}(0) \geq \sum_{j=1}^{d} (x^j)^2 \quad \text{for all } x \in \mathbb{R}^d. \tag{8.3.16}$$

For that purpose, we select $\psi \in B$ with

$$\frac{\partial^2 \psi}{\partial x^j \partial x^k}(0) = 2\delta_{jk} \qquad \left(\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise} \end{cases}\right),$$

and from (2), we find a sequence $(f^{(\nu)})_{\nu\in\mathbb{N}} \subset D(A) \cap C^\infty(\mathbb{R}^d)$, converging uniformly to ψ. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x^j \partial x^k} f_r^{(\nu)}(0) &= \frac{1}{r^d} \int \frac{\partial^2}{\partial x^j \partial x^k} \varrho\left(\frac{|y-x|}{r}\right)\bigg|_{x=0} f^{(\nu)}(y)\,\mathrm{d}y \\
&\to \frac{1}{r^d} \int \frac{\partial^2}{\partial x^j \partial x^k} \varrho\left(\frac{|y-x|}{r}\right)\bigg|_{x=0} \psi(y)\,\mathrm{d}y \quad \text{for } \nu \to \infty \\
&= \frac{1}{r^d} \int \rho\left(\frac{|y-x|}{r}\right) \frac{\partial^2}{\partial x^j \partial x^k} \psi(y)\,\mathrm{d}y \\
&\quad \text{replacing the derivative with respect to } x \text{ by one with} \\
&\quad \text{respect to } y \text{ and integrating by parts} \\
&\to \frac{\partial^2}{\partial x^j \partial x^k} \psi(0) \quad \text{for } r \to 0 \\
&= 2\delta_{jk}.
\end{aligned}$$

We may thus put $\varphi = f_r^{(\nu)}$ for suitable $\nu \in \mathbb{N}$, $r > 0$, in order to achieve (8.3.16). By Euclidean invariance, for every $x_0 \in \mathbb{R}^d$, there then exists a function in $D(A) \cap C^\infty(\mathbb{R}^d)$, again denoted by φ for simplicity, with

$$(x^j - x_0^j)(x^k - x_0^k) \frac{\partial^2 \varphi}{\partial x^j \partial x^k}(x_0) \geq \sum (x^j - x_0^j)^2 \quad \text{for all } x \in \mathbb{R}^d. \tag{8.3.17}$$

(4) For all $x_0 \in \mathbb{R}^d$, $j = 1, \ldots, d$, $r > 0$, $t > 0$,

$$\int_{|x-x_0|\leq r} (x^j - x_0^j) P(t, x_0, x)\mathrm{d}x = 0, \quad x_0 = \left(x_0^1, \ldots, x_0^d\right); \tag{8.3.18}$$

namely, let

$$i : \mathbb{R}^d \to \mathbb{R}^d$$

be the Euclidean isometry defined by

$$\begin{aligned} i(x^j - x_0^j) &= -(x^j - x_0^j), \\ i(x^k - x_0^k) &= x^k - x_0^k \quad \text{for } k \neq j \end{aligned} \tag{8.3.19}$$

(reflection across the hyperplane through x_0 that is orthogonal to the jth coordinate axis). We then have

$$\begin{aligned} \int\limits_{|x-x_0|\le r} (x^j - x_0^j) P(t, x_0, x)\mathrm{d}x &= \int\limits_{|x-x_0|\le r} i(x^j - x_0^j) P(t, ix_0, ix)\mathrm{d}x \\ &= -\int\limits_{|x-x_0|\le r} (x^j - x_0^j) P(t, x_0, x)\mathrm{d}x \end{aligned}$$

because of (8.3.19) and the assumed invariance of P, and this indeed implies (8.3.18).

Similarly, the invariance of P under rotations of $\mathbb{R}^d$ yields

$$\int_{|x-x_0|\le r} (x^j - x_0^j)^2 P(t, x_0, x)\mathrm{d}x = \int_{|x-x_0|\le r} (x^k - x_0^k)^2 P(t, x_0, x)\mathrm{d}x$$
$$\text{for all } x_0 \in \mathbb{R}^d, r > 0, t > 0, j, k = 1, \ldots, d, \tag{8.3.20}$$

and finally as in (8.3.18),

$$\int_{|x_0-x|\le r} (x^j - x_0^j)(x^k - x_0^k) P(t, x_0, x)\mathrm{d}x = 0 \quad \text{for } j \neq k, \tag{8.3.21}$$

if $x_0 \in \mathbb{R}^d, r > 0, t > 0, j, k \in \{1, \ldots, d\}$.

(5) Let $\varphi \in D(A) \cap C^2(\mathbb{R}^d)$. We then obtain the existence of

$$\begin{aligned} A\varphi(x_0) &= \lim_{t\searrow 0} \frac{1}{t} \int_{\mathbb{R}^d} P(t, x_0, x)(\varphi(x) - \varphi(x_0))\mathrm{d}x \\ &= \lim_{t\searrow 0} \frac{1}{t} \int_{|x-x_0|\le\varepsilon} P(t, x_0, x)(\varphi(x) - \varphi(x_0))\mathrm{d}x \text{ by (8.3.8)} \\ &= \lim_{t\searrow 0} \frac{1}{t} \int_{|x-x_0|\le\varepsilon} \sum_{j=1}^{d} (x^j - x_0^j) \frac{\partial\varphi}{\partial x^j}(x_0) P(t, x_0, x)\mathrm{d}x \end{aligned}$$

$$+\lim_{t\searrow 0}\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\frac{1}{2}\sum_{j,k}(x^j-x_0^j)(x^k-x_0^k)$$
$$\times\frac{\partial^2\varphi}{\partial x^j\partial x^k}(x_0+\tau(x-x_0))P(t,x_0,x)\mathrm{d}x$$

by Taylor expansion for some $\tau\in[0,1)$, as $\varphi\in C^2(\mathbb{R}^d)$.

The first term on the right-hand side vanishes by (8.3.18). Thus, the limit for $t\searrow 0$ of the second term exists, and it follows from (8.3.17) and $P(t,x_0,x)\ge 0$ that

$$\limsup_{t\searrow 0}\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\sum(x^j-x_0^j)^2P(t,x_0,x)\mathrm{d}x<\infty. \tag{8.3.22}$$

By (8.3.8), this limit superior does not depend on $\varepsilon>0$, and neither does the corresponding limit inferior.

(6) Now let $f\in D(A)\cap C^2(\mathbb{R}^d)$. As in (5), we obtain, by Taylor expanding f at x_0,

$$\begin{aligned}
&\frac{1}{t}(T_tf(x_0)-f(x_0))\\
&=\frac{1}{t}\int_{\mathbb{R}^d}(f(x)-f(x_0))P(t,x_0,x)\mathrm{d}x\\
&=\frac{1}{t}\int_{|x-x_0|>\varepsilon}(f(x)-f(x_0))P(t,x_0,x)\mathrm{d}x\\
&\quad+\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\sum_j(x^j-x_0^j)\frac{\partial f}{\partial x^j}(x_0)P(t,x_0,x)\mathrm{d}x\\
&\quad+\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\frac{1}{2}\sum_{j,k}(x^j-x_0^j)(x^k-x_0^k)\frac{\partial^2 f}{\partial x^j\partial x^k}(x_0)P(t,x_0,x)\mathrm{d}x\\
&\quad+\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\sum_{j,k}(x^j-x_0^j)(x^k-x_0^k)\sigma_{ij}(\varepsilon)P(t,x_0,x)\mathrm{d}x
\end{aligned}$$

(where the notation suppresses the x-dependence of the remainder term $\sigma_{ij}(\varepsilon)$, since this converges to 0 for $\varepsilon\to 0$ uniformly in x, since $f\in C^2(\mathbb{R}^d)$)

$$\begin{aligned}
&=\frac{1}{t}\int_{|x-x_0|>\varepsilon}(f(x)-f(x_0))P(t,x_0,x)\mathrm{d}x\\
&\quad+\frac{1}{t}\int_{|x-x_0|\le\varepsilon}\sum_j(x^j-x_0^j)^2\frac{\partial^2 f}{(\partial x^j)^2}(x_0)P(t,x_0,x)\mathrm{d}x
\end{aligned}$$

$$+\frac{1}{t}\int_{|x-x_0|\leq\varepsilon}\sum_{j,k}(x^j-x_0^j)(x^k-x_0^k)\sigma_{ij}(\varepsilon)P(t,x_0,x)\mathrm{d}x$$

$$\text{by (8.3.18) and (8.3.21).} \tag{8.3.23}$$

By (8.3.8), the first term on the right-hand side tends to 0 as $t \to 0$ for every $\varepsilon > 0$. Because of (8.3.22) and $\lim_{\varepsilon\to 0}\sigma_{ij}(\varepsilon) = 0$ (since $f \in C^2$), the last term converges to 0 as $\varepsilon \to 0$ for every $t > 0$. Since we have observed at the end of (5), however, that in the second term on the right-hand side, limits can be performed independently of ε, for all $\varepsilon > 0$, we obtain the existence of

$$\lim_{t\searrow 0}\frac{1}{t}\int_{|x-x_0|\leq\varepsilon}\sum(x^j-x_0^j)^2\frac{\partial^2 f}{(\partial x^j)^2}(x_0)P(t,x_0,x)\mathrm{d}x = Af(x_0), \tag{8.3.24}$$

by performing the limit $t \to 0$ on the right-hand side of (8.3.23).

The argument of (3) shows that for $f \in D(A)$,

$$\frac{\partial^2 f}{(\partial x^j)^2}(x_0)$$

may approximate arbitrary values, and so in particular, we infer the existence of

$$\lim_{t\searrow 0}\frac{1}{t}\int_{|x-x_0|\leq\varepsilon}\sum(x^j-x_0^j)^2P(t,x_0,x)\mathrm{d}x$$

independently of ε. By (8.3.20), for each $j = 1,\ldots,d$,

$$\lim_{t\searrow 0}\frac{1}{t}\int_{|x-x_0|\leq\varepsilon}(x^j-x_0^j)^2P(t,x_0,x)\mathrm{d}x$$

exists and is independent of j and by translation invariance independent of x_0 as well. We thus call this limit c. By (8.3.24), we then have

$$Af(x_0) = c\Delta f(x_0).$$

The rest follows from Theorem 8.2.3. □

Remark. If we assume only spatial homogeneity, i.e., translation invariance, but not invariance under reflections and rotations, the infinitesimal generator still is a second-order differential operator; namely, it is of the form

$$Af(x) = \sum_{j,k=1}^{d}a^{jk}(x)\frac{\partial^2 f}{\partial x^j\partial x^k}(x) + \sum_{j=1}^{d}b^j(x)\frac{\partial f}{\partial x^j}(x)$$

with

$$a^{jk}(x) = \lim_{t \searrow 0} \frac{1}{t} \int_{|y-x| \le \varepsilon} (y^j - x^j)(y^k - x^k) P(t, x, y) \mathrm{d}y,$$

and thus, in particular,

$$a^{jk} = a^{kj}, \quad a^{jj} \ge 0 \quad \text{for all } j, k,$$

and

$$b^j(x) = \lim_{t \searrow 0} \frac{1}{t} \int_{|y-x| \le \varepsilon} (y^j - x^j) P(t, x, y) \mathrm{d}y,$$

where the limits again are independent of $\varepsilon > 0$. The proof can be carried out with the same methods as employed for demonstrating Theorem 8.3.2.

A reference for the present chapter is Yosida [32].

Summary

The heat equation satisfies a Markov property in the sense that the solution $u(x, t)$ at time $t_1 + t_2$ with initial values $u(x, 0) = f(x)$ equals the solution at time t_2 with initial values $u(x, t_1)$. Putting

$$(P_t f)(x) := u(x, t),$$

we thus have

$$(P_{t_1+t_2} f)(x) = P_{t_2}(P_{t_1} f)(x);$$

i.e., P_t satisfies the semigroup property

$$P_{t_1+t_2} = P_{t_2} \circ P_{t_1} \quad \text{for } t_1, t_2 \ge 0.$$

Moreover, $\{P_t\}_{t \ge 0}$ is continuous on the space C^0 in the sense that

$$\lim_{t \searrow t_0} P_t = P_{t_0}$$

for all $t_0 \ge 0$ (in particular, this also holds for $t_0 = 0$, with $P_0 = \mathrm{Id}$).

Moreover, P_t is contracting because of the maximum principle, i.e.,

$$\|P_t f\|_{C^0} \le \|f\|_{C^0} \quad \text{for } t \ge 0, f \in C^0.$$

The infinitesimal generator of the semigroup P_t is the Laplace generator, i.e.,

$$\Delta = \lim_{t \searrow 0} \frac{1}{t}(P_t - \mathrm{Id}).$$

Upon these properties one may found an abstract theory of semigroups in Banach spaces. The Hille–Yosida theorem says that a linear operator $A : D(A) \to B$ whose domain of definition $D(A)$ is dense in the Banach space B and for which $\mathrm{Id} - \frac{1}{n} A$ is invertible for all $n \in \mathbb{N}$, and

$$\left\| (\mathrm{Id} - \frac{1}{n} A)^{-1} \right\| \leq 1$$

generates a unique contracting semigroup of operators

$$T_t : B \to B \quad (t \geq 0).$$

For a stochastic interpretation, one considers the probability density $P(t, x, y)$ that some particle that during the random walk happened to be at the point x at a certain time can be found at y at a time that is larger by the amount t. This constitutes a Markov process inasmuch as this probability density depends only on the time difference, but not on the individual values of the times involved. In particular, $P(t, x, y)$ does not depend on where the particle had been before reaching x (random walk without memory). Such a random walk on the set S satisfies the Chapman–Kolmogorov equation

$$P(t_1 + t_2, x, y) = \int_S P(t_1, x, z) P(t_2, z, y) \mathrm{d}z$$

and thus constitutes a semigroup.

If such a process on $\mathbb{R}^d$ is spatially homogeneous and satisfies

$$\lim_{t \searrow 0} \frac{1}{t} \int_{|x-y|>\rho} P(t, x, y)\, \mathrm{d}y = 0$$

for all $\rho > 0$ and $x \in \mathbb{R}^d$, it is called a Brownian motion. One shows that up to a scaling factor, such a Brownian motion has to be given by the heat semigroup, i.e.,

$$P(t, x, y) = \frac{1}{(4\pi c t)^{d/2}} \mathrm{e}^{-\frac{|x-y|^2}{4ct}}.$$

Exercises

8.1. Let $f \in C^0(\mathbb{R}^d)$ be bounded and $u(x,t)$ a solution of the heat equation:

$$u_t(x,t) = \Delta u(x,t) \quad \text{for } x \in \mathbb{R}^d, t > 0,$$
$$u(x,0) = f(x).$$

Show that the derivatives of u satisfy

$$\left| \frac{\partial}{\partial x^j} u(x,t) \right| \leq \text{const } \sup |f| \cdot t^{-1/2}.$$

(Hint: Use the representation formula (5.2.3) from Sect. 5.2.)

8.2. As in Sect. 8.2, we consider a continuous semigroup

$$\exp(tA) : B \to B \quad (t \geq 0), B \text{ a Banach space.}$$

Let B_1 be another Banach space, and for $t > 0$, suppose

$$\exp(tA) : B_1 \to B$$

is defined, and we have for $0 < t \leq 1$ and for all $\varphi \in B_1$,

$$\| \exp(tA)\varphi\|_B \leq \text{ const } t^{-\alpha} \|\varphi\|_{B_1} \quad \text{for some } \alpha < 1.$$

Finally, let

$$\Phi : B \to B_1$$

be Lipschitz continuous.

Show that for every $f \in B$, there exists $T > 0$ with the property that the evolution equation

$$\frac{\partial v}{\partial t} = Av + \Phi(v(t)) \quad \text{for } t > 0,$$
$$v(0) = f$$

has a unique, continuous solution $v : [0,T] \to B$.
(Hint: Convert the problem into the integral equation

$$v(t) = \exp(tA) f + \int_0^t \exp((t-s)A)\Phi(v(s))\mathrm{d}s,$$

and use the Banach fixed-point theorem (as in the standard proof of the Picard–Lindelöf theorem for ODEs) to obtain a solution of that integral equation.)

8.3. Apply the results of Exercises 8.1 and 8.2 to the initial value problem for the following semilinear parabolic PDE:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + F(t,x,u(x),Du(x)) \quad \text{for } x \in \mathbb{R}^d, t > 0, \\ u(x,0) &= f(x), \end{aligned}$$

for compactly supported $f \in C^0(\mathbb{R}^d)$. We assume that F is smooth with respect to all its arguments.

8.4. Demonstrate the assertion in the remark at the end of Sect. 8.3.