Chapter 2
The Laplace Equation as the Prototype of an Elliptic Partial Differential Equation of Second Order

2.1 Harmonic Functions: Representation Formula for the Solution of the Dirichlet Problem on the Ball (Existence Techniques 0)

In this section Ω is a bounded domain in $\mathbb{R}^d$ for which the divergence theorem holds; this means that for any vector field V of class $C^1(\Omega) \cap C^0(\bar{\Omega})$,

$$\int_{\Omega} \operatorname{div} V(x)\mathrm{d}x = \int_{\partial\Omega} V(z) \cdot \nu(z) do(z), \tag{2.1.1}$$

where the dot $\cdot$ denotes the Euclidean product of vectors in $\mathbb{R}^d$, ν is the exterior normal of $\partial\Omega$, and $do(z)$ is the volume element of $\partial\Omega$. Let us recall the definition of the divergence of a vector field $V = (V^1, \ldots, V^d) : \Omega \to \mathbb{R}^d$:

$$\operatorname{div} V(x) := \sum_{i=1}^{d} \frac{\partial V^i}{\partial x^i}(x).$$

In order that (2.1.1) holds, it is, for example, sufficient that $\partial\Omega$ be of class C^1.

Lemma 2.1.1. *Let $u, v \in C^2(\bar{\Omega})$. Then we have Green's 1st formula*

$$\int_{\Omega} v(x)\Delta u(x)\mathrm{d}x + \int_{\Omega} \nabla u(x) \cdot \nabla v(x)\mathrm{d}x = \int_{\partial\Omega} v(z)\frac{\partial u}{\partial \nu}(z) do(z) \tag{2.1.2}$$

(here, ∇u is the gradient of u), and Green's 2nd formula

$$\int_{\Omega} \{v(x)\Delta u(x) - u(x)\Delta v(x)\}\,\mathrm{d}x = \int_{\partial\Omega} \left\{v(z)\frac{\partial u}{\partial \nu}(z) - u(z)\frac{\partial v}{\partial \nu}(z)\right\} do(z). \tag{2.1.3}$$

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Proof. With $V(x) = v(x)\nabla u(x)$, (2.1.2) follows from (2.1.1). Interchanging u and v in (2.1.2) and subtracting the resulting formula from (2.1.2) yield (2.1.3). □

In the sequel we shall employ the following notation:

$$B(x,r) := \left\{y \in \mathbb{R}^d : |x-y| \le r\right\} \qquad \text{(closed ball)}$$

and

$$\mathring{B}(x,r) := \left\{y \in \mathbb{R}^d : |x-y| < r\right\} \qquad \text{(open ball)}$$

for $r > 0$, $x \in \mathbb{R}^d$.

Definition 2.1.1. A function $u \in C^2(\Omega)$ is called harmonic (in Ω) if

$$\Delta u = 0 \quad \text{in } \Omega.$$

In Definition 2.1.1, Ω may be an arbitrary open subset of $\mathbb{R}^d$. We begin with the following simple observation:

Lemma 2.1.2. *The harmonic functions in Ω form a vector space.*

Proof. This follows because Δ is a *linear* differential operator. □

Examples of harmonic functions:

1. In $\mathbb{R}^d$, all constant functions and, more generally, all affine linear functions are harmonic.
2. There also exist harmonic polynomials of higher order, for example,

$$u(x) = \left(x^1\right)^2 - \left(x^2\right)^2$$

   for $x = \left(x^1, \dots, x^d\right) \in \mathbb{R}^d$.
3. Let $h : D \to \mathbb{C}$ be holomorphic for some open $D \subset \mathbb{C}$; that means that h is differentiable in D and satisfies

$$\frac{\partial}{\partial \bar{z}} h = 0 \text{ with } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.1.4}$$

   where $z = x + iy$ (with $i := \sqrt{-1}$ being the imaginary unit) is the coordinate on $\mathbb{C}$ and $\bar{z} = x - iy$. (Thus, in contrast to our standard notation, we now write (x, y) in place of (x^1, x^2), as this corresponds to the convention usually employed in complex analysis.) If we decompose $h = u + iv$ into its real and imaginary parts, (2.1.4) becomes the system of Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.1.5}$$

When u and v are twice differentiable (which, in fact, automatically follows from (2.1.4) as one of the basic facts of complex analysis (cf., e.g., [1])—see also Corollary 2.2.1 below), this implies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{2.1.6}$$

i.e., the real and imaginary part of a holomorphic function are harmonic. Conversely, given a harmonic function $u : D \to \mathbb{R}$, as shown in complex analysis, one may then solve (2.1.5) for v to obtain a holomorphic function $h = u + iv : D \to \mathbb{C}$.

When, in analogy to (2.1.4), we also use the notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{2.1.7}$$

we obtain the decomposition for the Laplace operator on $\mathbb{C} \cong \mathbb{R}^2$

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}. \end{aligned} \tag{2.1.8}$$

4. For $x, y \in \mathbb{R}^d$ with $x \neq y$ (be careful: we revert to our original notation, i.e., x, y now are vectors again, not scalar components as in the previous example), we put

$$\Gamma(x, y) := \Gamma(|x-y|) := \begin{cases} \frac{1}{2\pi} \log|x-y| & \text{for } d = 2, \\ \frac{1}{d(2-d)\omega_d} |x-y|^{2-d} & \text{for } d > 2, \end{cases} \tag{2.1.9}$$

where ω_d is the volume of the d-dimensional unit ball $B(0,1) \subset \mathbb{R}^d$.

We have

$$\frac{\partial}{\partial x^i}\Gamma(x, y) = \frac{1}{d\omega_d}\left(x^i - y^i\right)|x-y|^{-d},$$

$$\frac{\partial^2}{\partial x^i \partial x^j}\Gamma(x, y) = \frac{1}{d\omega_d}\left\{|x-y|^2 \delta_{ij} - d\left(x^i - y^i\right)\left(x^j - y^j\right)\right\}|x-y|^{-d-2}.$$

Thus, as a function of x, Γ is harmonic in $\mathbb{R}^d \setminus \{y\}$. Since Γ is symmetric in x and y, it is then also harmonic as a function of y in $\mathbb{R}^d \setminus \{x\}$. The reason for the choice of the constants employed in (2.1.9) will become apparent after (2.1.13) below.

Definition 2.1.2. Γ from (2.1.9) is called the fundamental solution of the Laplace equation.

What is the reason for this particular solution Γ of the Laplace equation in $\mathbb{R}^d \setminus \{y\}$? The answer comes from the rotational symmetry of the Laplace operator. The equation

$$\Delta u = 0$$

is invariant under rotations about an arbitrary center y. (If $A \in \mathrm{O}(d)$ (orthogonal group) and $y \in \mathbb{R}^d$, then for a harmonic $u(x)$, $u(A(x-y)+y)$ is likewise harmonic.) Because of this invariance of the operator, one then also searches for invariant solutions, i.e., solutions of the form

$$u(x) = \varphi(r) \quad \text{with } r = |x-y| \,.$$

The Laplace equation then is transformed into the following equation for φ as a function of r, with $'$ denoting a derivative with respect to r,

$$\varphi''(r) + \frac{d-1}{r}\varphi'(r) = 0.$$

Solutions have to satisfy

$$\varphi'(r) = c r^{1-d}$$

with constant c. Fixing this constant plus one further additive constant leads to the fundamental solution $\Gamma(r)$.

Theorem 2.1.1 (Green representation formula). *If $u \in C^2(\bar{\Omega})$, we have for $y \in \Omega$,*

$$u(y) = \int_{\partial\Omega} \left\{ u(x)\frac{\partial \Gamma}{\partial \nu_x}(x,y) - \Gamma(x,y)\frac{\partial u}{\partial \nu}(x) \right\} do(x) + \int_{\Omega} \Gamma(x,y)\Delta u(x)\mathrm{d}x \tag{2.1.10}$$

(here, the symbol $\frac{\partial}{\partial \nu_x}$ indicates that the derivative is to be taken in the direction of the exterior normal with respect to the variable x).

Proof. For sufficiently small $\varepsilon > 0$,

$$B(y,\varepsilon) \subset \Omega,$$

since Ω is open. We apply (2.1.3) for $v(x) = \Gamma(x,y)$ and $\Omega \setminus B(y,\varepsilon)$ (in place of Ω). Since Γ is harmonic in $\Omega \setminus \{y\}$, we obtain

$$\int_{\Omega\setminus B(y,\varepsilon)} \Gamma(x,y)\Delta u(x)\mathrm{d}x = \int_{\partial\Omega} \left\{ \Gamma(x,y)\frac{\partial u}{\partial \nu}(x) - u(x)\frac{\partial \Gamma(x,y)}{\partial \nu_x} \right\} do(x) + \int_{\partial B(y,\varepsilon)} \left\{ \Gamma(x,y)\frac{\partial u}{\partial \nu}(x) - u(x)\frac{\partial \Gamma(x,y)}{\partial \nu_x} \right\} do(x). \tag{2.1.11}$$

In the second boundary integral, ν denotes the exterior normal of $\Omega \setminus B(y,\varepsilon)$, hence the interior normal of $B(y,\varepsilon)$.

We now wish to evaluate the limits of the individual integrals in this formula for $\varepsilon \to 0$. Since $u \in C^2(\bar{\Omega})$, Δu is bounded. Since Γ is integrable, the left-hand side of (2.1.11) thus tends to

$$\int_{\Omega} \Gamma(x,y)\Delta u(x)\mathrm{d}x.$$

On $\partial B(y,\varepsilon)$, we have $\Gamma(x,y) = \Gamma(\varepsilon)$. Thus, for $\varepsilon \to 0$,

$$\left| \int_{\partial B(y,\varepsilon)} \Gamma(x,y)\frac{\partial u}{\partial \nu}(x) do(x) \right| \le d\omega_d \varepsilon^{d-1}\Gamma(\varepsilon) \sup_{B(y,\varepsilon)} |\nabla u| \to 0.$$

Furthermore,

$$-\int_{\partial B(y,\varepsilon)} u(x)\frac{\partial \Gamma(x,y)}{\partial \nu_x} do(x) = \frac{\partial}{\partial \varepsilon}\Gamma(\varepsilon)\int_{\partial B(y,\varepsilon)} u(x) do(x)$$

(since ν is the interior normal of $B(y,\varepsilon)$)

$$= \frac{1}{d\omega_d \varepsilon^{d-1}} \int_{\partial B(y,\varepsilon)} u(x) do(x) \to u(y).$$

Altogether, we get (2.1.10). □

Remark. Applying the Green representation formula for a so-called test function $\varphi \in C_0^\infty(\Omega)$,[1] we obtain

$$\varphi(y) = \int_{\Omega} \Gamma(x,y)\Delta\varphi(x)\mathrm{d}x. \tag{2.1.12}$$

This can be written symbolically as

$$\Delta_x \Gamma(x,y) = \delta_y, \tag{2.1.13}$$

where Δ_x is the Laplace operator with respect to x and δ_y is the Dirac delta distribution, meaning that for $\varphi \in C_0^\infty(\Omega)$,

$$\delta_y[\varphi] := \varphi(y).$$

[1] $C_0^\infty(\Omega) := \{f \in C^\infty(\Omega), \operatorname{supp}(f) := \overline{\{x : f(x) \neq 0\}}$ is a compact subset of $\Omega\}$.

In the same manner, $\Delta\Gamma(\,\cdot\,, y)$ is defined as a distribution, i.e.,

$$\Delta\Gamma(\,\cdot\,, y)[\varphi] := \int_\Omega \Gamma(x, y)\Delta\varphi(x)\mathrm{d}x.$$

Equation (2.1.13) explains the terminology "fundamental solution" for Γ, as well as the choice of constant in its definition.

Remark. By definition, a distribution is a linear functional ℓ on C_0^∞ that is continuous in the following sense:
Suppose that $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ satisfies $\varphi_n = 0$ on $\Omega \setminus K$ for all n and some fixed compact $K \subset \Omega$ as well as $\lim_{n\to\infty} D^\alpha\varphi_n(x) = 0$ uniformly in x for all partial derivatives D^α (of arbitrary order). Then

$$\lim_{n\to\infty} \ell[\varphi_n] = 0$$

must hold.

We may draw the following consequence from the Green representation formula: If one knows Δu, then u is completely determined by its values and those of its normal derivative on $\partial\Omega$. In particular, a harmonic function on Ω can be reconstructed from its boundary data. One may then ask conversely whether one can construct a harmonic function for arbitrary given values on $\partial\Omega$ for the function and its normal derivative. Even ignoring the issue that one might have to impose certain regularity conditions like continuity on such data, we shall find that this is not possible in general, but that one can prescribe essentially only one of these two data. In any case, the divergence theorem (2.1.1) for $V(x) = \nabla u(x)$ implies that because of $\Delta = \operatorname{div}\operatorname{grad}$, a harmonic u has to satisfy

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} do(x) = \int_\Omega \Delta u(x)\mathrm{d}x = 0, \tag{2.1.14}$$

so that the normal derivative cannot be prescribed completely arbitrarily.

Definition 2.1.3. A function $G(x, y)$, defined for $x, y \in \bar{\Omega}$, $x \neq y$, is called a *Green function* for Ω if:

1. $G(x, y) = 0$ for $x \in \partial\Omega$.
2. $h(x, y) := G(x, y) - \Gamma(x, y)$ is harmonic in $x \in \Omega$ (thus in particular also at the point $x = y$).

We now assume that a Green function $G(x, y)$ for Ω exists (which indeed is true for all Ω under consideration here) and put $v(x) = h(x, y)$ in (2.1.3) and subtract the result from (2.1.10), obtaining

$$u(y) = \int_{\partial\Omega} u(x)\frac{\partial G(x, y)}{\partial \nu_x} do(x) + \int_\Omega G(x, y)\Delta u(x)\mathrm{d}x. \tag{2.1.15}$$

Equation (2.1.15) in particular implies that a harmonic u is already determined by its boundary values $u_{|\partial\Omega}$.

This construction now raises the converse question: If we are given functions $\varphi : \partial\Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$, can we obtain a solution of the Dirichlet problem for the Poisson equation

$$\begin{aligned} \Delta u(x) &= f(x) \quad \text{for } x \in \Omega, \\ u(x) &= \varphi(x) \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{2.1.16}$$

by the representation formula

$$u(y) = \int_{\partial\Omega} \varphi(x) \frac{\partial G(x, y)}{\partial \nu_x} do(x) + \int_{\Omega} f(x) G(x, y) \mathrm{d}x? \tag{2.1.17}$$

After all, if u is a solution, it does satisfy this formula by (2.1.15).

Essentially, the answer is yes; to make it really work, however, we need to impose some conditions on φ and f. A natural condition should be the requirement that they be continuous. For φ, this condition turns out to be sufficient, provided that the boundary of Ω satisfies some mild regularity requirements. If Ω is a ball, we shall verify this in Theorem 2.1.2 for the case $f = 0$, i.e., the Dirichlet problem for harmonic functions. For f, the situation is slightly more subtle. It turns out that even if f is continuous, the function u defined by (2.1.17) need not be twice differentiable, and so one has to exercise some care in assigning a meaning to the equation $\Delta u = f$. We shall return to this issue in Sects. 12.1 and 13.1 below. In particular, we shall show that if we require a little more about f, namely, that it be Hölder continuous, then the function u given by (2.1.17) is twice continuously differentiable and satisfies

$$\Delta u = f.$$

Analogously, if $H(x, y)$ for $x, y \in \bar{\Omega}$, $x \neq y$ is defined with[2]

$$\frac{\partial}{\partial \nu_x} H(x, y) = \frac{1}{\|\partial\Omega\|} \quad \text{for } x \in \partial\Omega$$

and a harmonic difference $H(x, y) - \Gamma(x, y)$ as before, we obtain

$$\begin{aligned} u(y) = \frac{1}{\|\partial\Omega\|} \int_{\partial\Omega} u(x) do(x) - \int_{\partial\Omega} H(x, y) \frac{\partial u}{\partial \nu}(x) do(x) \\ + \int_{\Omega} H(x, y) \Delta u(x) \mathrm{d}x. \end{aligned} \tag{2.1.18}$$

[2]Here, $\|\partial\Omega\|$ denotes the measure of the boundary $\partial\Omega$ of Ω; it is given as $\int_{\partial\Omega} do(x)$.

If now u_1 and u_2 are two harmonic functions with

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \text{ on } \partial\Omega,$$

applying (2.1.18) to the difference $u = u_1 - u_2$ yields

$$u_1(y) - u_2(y) = \frac{1}{\|\partial\Omega\|} \int_{\partial\Omega} (u_1(x) - u_2(x))\, do(x). \tag{2.1.19}$$

Since the right-hand side of (2.1.19) is independent of y, $u_1 - u_2$ must be constant in Ω. In other words, a solution of the Neumann boundary value problem

$$\begin{aligned} \Delta u(x) &= 0 && \text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x) && \text{for } x \in \partial\Omega \end{aligned} \tag{2.1.20}$$

is determined only up to a constant, and, conversely, by (2.1.14), a necessary condition for the existence of a solution is

$$\int_{\partial\Omega} g(x) do(x) = 0. \tag{2.1.21}$$

Boundary conditions tend to make the theory of PDEs difficult. Actually, in many contexts, the Neumann condition is more natural and easier to handle than the Dirichlet condition, even though we mainly study Dirichlet boundary conditions in this book as those occur more frequently. There is in fact another, even easier, boundary condition, which actually is not a boundary condition at all, the so-called periodic boundary condition. This means the following. We consider a domain of the form $\Omega = (0, L_1) \times \cdots \times (0, L_d) \subset \mathbb{R}^d$ and require for $u : \bar{\Omega} \to \mathbb{R}$ that

$$u(x_1, \ldots, x_{i-1}, L_i, x_{i+1}, \ldots, x_d) = u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) \tag{2.1.22}$$

for all $x = (x_1, \ldots, x_d) \in \Omega$, $i = 1, \ldots, d$. This means that u can be periodically extended from Ω to all of $\mathbb{R}^d$. A reader familiar with basic geometric concepts will view such a u as a function on the torus obtained by identifying opposite sides in Ω. More generally, one may then consider solutions of PDEs on compact manifolds.

Anyway, we now turn to the Dirichlet problem on a ball. As a preparation, we compute the Green function G for such a ball $B(0, R)$. For $y \in \mathbb{R}^d$, we put

$$\bar{y} := \begin{cases} \frac{R^2}{|y|^2} y & \text{for } y \neq 0, \\ \infty & \text{for } y = 0. \end{cases}$$

($\bar{y}$ is the point obtained from y by reflection across $\partial B(0, R)$.) We then put

$$G(x, y) := \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|y|}{R}|x-\bar{y}|\right) & \text{for } y \neq 0, \\ \Gamma(|x|) - \Gamma(R) & \text{for } y = 0. \end{cases} \tag{2.1.23}$$

For $x \neq y$, $G(x, y)$ is harmonic in x, since for $y \in \mathring{B}(0, R)$, the point $\bar{y}$ lies in the exterior of $B(0, R)$. The function $G(x, y)$ has only one singularity in $B(0, R)$, namely, at $x = y$, and this singularity is the same as that of $\Gamma(x, y)$. The formula

$$G(x, y) = \Gamma\left(\left(|x|^2 + |y|^2 - 2x \cdot y\right)^{1/2}\right) - \Gamma\left(\left(\frac{|x|^2\,|y|^2}{R^2} + R^2 - 2x \cdot y\right)^{1/2}\right) \tag{2.1.24}$$

then shows that for $x \in \partial B(0, R)$, i.e., $|x| = R$, we have indeed

$$G(x, y) = 0.$$

Therefore, the function $G(x, y)$ defined by (2.1.23) is the Green function of $B(0, R)$.

Equation (2.1.24) also implies the symmetry

$$G(x, y) = G(y, x). \tag{2.1.25}$$

Furthermore, since $\Gamma(|x - y|)$ is monotonic in $|x - y|$, we conclude from (2.1.24) that

$$G(x, y) \leq 0 \quad \text{for } x, y \in B(0, R). \tag{2.1.26}$$

Since for $x \in \partial B(0, R)$,

$$|x|^2 + |y|^2 - 2x \cdot y = \frac{|x|^2\,|y|^2}{R^2} + R^2 - 2x \cdot y,$$

(2.1.24) furthermore implies for $x \in \partial B(0, R)$ that

$$\frac{\partial}{\partial \nu_x} G(x, y) = \frac{\partial}{\partial |x|} G(x, y) = \frac{1}{d\omega_d} \frac{|x|}{|x-y|^d} - \frac{1}{d\omega_d} \frac{|x|}{|x-y|^d} \frac{|y|^2}{R^2}$$
$$= \frac{R^2 - |y|^2}{d\omega_d R} \frac{1}{|x-y|^d}.$$

Inserting this result into (2.1.15), we obtain a representation formula for a harmonic $u \in C^2(B(0, R))$ in terms of its boundary values on $\partial B(0, R)$:

$$u(y) = \frac{R^2 - |y|^2}{d\omega_d R} \int_{\partial B(0,R)} \frac{u(x)}{|x-y|^d} do(x). \tag{2.1.27}$$

The regularity condition here can be weakened; in fact, we have the following theorem:

Theorem 2.1.2. ***(Poisson representation formula; solution of the Dirichlet problem on the ball):*** *Let $\varphi : \partial B(0, R) \to \mathbb{R}$ be continuous. Then u, defined by*

$$u(y) := \begin{cases} \frac{R^2-|y|^2}{d\omega_d R} \int_{\partial B(0,R)} \frac{\varphi(x)}{|x-y|^d} do(x) & \text{for } y \in \mathring{B}(0, R), \\ \varphi(y) & \text{for } y \in \partial B(0, R), \end{cases} \tag{2.1.28}$$

is harmonic in the open ball $\mathring{B}(0, R)$ and continuous in the closed ball $B(0, R)$.

Proof. Since G is harmonic in y, so is the kernel of the Poisson representation formula

$$K(x, y) := \frac{\partial G}{\partial \nu_x}(x, y) = \frac{R^2 - |y|^2}{d\omega_d R} |x - y|^{-d} .$$

Thus u is harmonic as well.

It remains only to show continuity of u on $\partial B(0, R)$. We first insert the harmonic function $u \equiv 1$ in (2.1.27), yielding

$$\int_{\partial B(0,R)} K(x, y) do(x) = 1 \quad \text{for all } y \in \mathring{B}(0, R). \tag{2.1.29}$$

We now consider $y_0 \in \partial B(0, R)$. Since φ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$|\varphi(x) - \varphi(y_0)| < \frac{\varepsilon}{2} \quad \text{for } |x - y_0| < 2\delta. \tag{2.1.30}$$

With

$$\mu := \sup_{y \in \partial B(0,R)} |\varphi(y)| ,$$

by (2.1.28) and (2.1.29) we have for $|y - y_0| < \delta$ that

$$\begin{aligned} \Big| u(y) - u(y_0) \Big| &= \left| \int_{\partial B(0,R)} K(x, y) \left(\varphi(x) - \varphi(y_0)\right) do(x) \right| \\ &\le \int_{|x-y_0|\le 2\delta} K(x, y) \left|\varphi(x) - \varphi(y_0)\right| do(x) \\ &\quad + \int_{|x-y_0|>2\delta} K(x, y) \left|\varphi(x) - \varphi(y_0)\right| do(x) \\ &\le \frac{\varepsilon}{2} + 2\mu \left(R^2 - |y|^2\right) R^{d-2} \delta^{-d} . \end{aligned} \tag{2.1.31}$$

For estimating the second integral, note that because of $|y - y_0| < \delta$, for $|x - y_0| > 2\delta$ also $|x - y| \geq \delta$. Having chosen ε, we have fixed δ. Then, for showing continuity, we may assume that y is sufficiently close to y_0. Thus, since $|y_0| = R$, for sufficiently small $|y - y_0|$, then also the second term on the right-hand side of (2.1.31) becomes smaller than $\varepsilon/2$, and we see that u is continuous at y_0. □

Corollary 2.1.1. *For $\varphi \in C^0(\partial B(0, R))$, there exists a unique solution $u \in C^2(\overset{\circ}{B}(0, R)) \cap C^0(B(0, R))$ of the Dirichlet problem*

$$\begin{aligned} \Delta u(x) &= 0 && \text{for } x \in \overset{\circ}{B}(0, R), \\ u(x) &= \varphi(x) && \text{for } x \in \partial B(0, R). \end{aligned}$$

Proof. Theorem 2.1.2 shows the existence. Uniqueness follows from (2.1.15); however, in (2.1.15) we have assumed $u \in C^2(B(0, R))$, while more generally, here we consider continuous boundary values. This difficulty is easily overcome: Since u is harmonic in $\overset{\circ}{B}(0, R)$, it is of class C^2 in $\overset{\circ}{B}(0, R)$, for example, by Corollary 2.1.2 below. Consequently, for $|y| < r < R$, applying (2.1.27) with r in place of R, we get

$$u(y) = \frac{r^2 - |y|^2}{d\omega_d r} \int_{\partial B(0,r)} \frac{u(x)}{|x - y|^d} do(x),$$

and since u is continuous in $B(0, R)$, we may let r tend to R in order to get the representation formula in its full generality. □

Corollary 2.1.2. *Any harmonic function $u : \Omega \to \mathbb{R}$ is real analytic in Ω.*

Proof. Let $z \in \Omega$ and choose R such that $B(z, R) \subset \Omega$. Then by (2.1.27), for $y \in \overset{\circ}{B}(z, R)$,

$$u(y) = \frac{R^2 - |y - z|^2}{d\omega_d R} \int_{\partial B(z,R)} \frac{u(x)}{|x - y|^d} do(x),$$

which is a real analytic function of $y \in \overset{\circ}{B}(z, R)$. □

2.2 Mean Value Properties of Harmonic Functions. Subharmonic Functions. The Maximum Principle

Theorem 2.2.1 (Mean value formulae). *A continuous or, more generally, a measurable and locally integrable $u : \Omega \to \mathbb{R}$ is harmonic if and only if for any ball $B(x_0, r) \subset \Omega$,*

$$u(x_0) = S(u, x_0, r) := \frac{1}{d\omega_d r^{d-1}} \int_{\partial B(x_0,r)} u(x) do(x) \quad \textit{(spherical mean)}, \tag{2.2.1}$$

or equivalently, if for any such ball,

$$u(x_0) = K(u, x_0, r) := \frac{1}{\omega_d r^d} \int_{B(x_0,r)} u(x)\mathrm{d}x \qquad \textit{(ball mean).} \tag{2.2.2}$$

Proof. "⇒":
Let u be harmonic. (By definition, u then is twice differentiable, hence continuous, but see Corollary 2.2.1 below on this point.) Then (2.2.1) follows from Poisson's formula (2.1.27) (since we have written (2.1.27) only for the ball $B(0, R)$, take the harmonic function $v(x) := u(x + x_0)$ and apply the formula at the point $x = 0$). Alternatively, we may prove (2.2.1) from the following observation:

Let $u \in C^2(\mathring{B}(y, r))$, $0 < \varrho < r$. Then by (2.1.1)

$$\begin{aligned}
\int_{B(y,\varrho)} \Delta u(x)\mathrm{d}x &= \int_{\partial B(y,\varrho)} \frac{\partial u}{\partial \nu}(x) do(x) \\
&= \int_{\partial B(0,1)} \frac{\partial u}{\partial \varrho}(y + \varrho\omega)\varrho^{d-1}\mathrm{d}\omega \\
&\quad \text{in polar coordinates } \omega = \frac{x-y}{\varrho} \\
&= \varrho^{d-1} \frac{\partial}{\partial \varrho} \int_{\partial B(0,1)} u(y + \varrho\omega)\mathrm{d}\omega \\
&= \varrho^{d-1} \frac{\partial}{\partial \varrho} \left(\varrho^{1-d} \int_{\partial B(y,\varrho)} u(x) do(x) \right) \\
&= d\omega_d \varrho^{d-1} \frac{\partial}{\partial \varrho} S(u, y, \varrho).
\end{aligned} \tag{2.2.3}$$

If u is harmonic, this yields $\frac{\partial}{\partial \varrho} S(u, y, \varrho) = 0$, and so $S(u, y, \varrho)$ is constant in ρ. Because of

$$u(y) = \lim_{\varrho \to 0} S(u, y, \varrho), \tag{2.2.4}$$

for a continuous u this implies the spherical mean value property. Because of

$$K(u, x_0, r) = \frac{d}{r^d} \int_0^r S(u, x_0, \varrho)\varrho^{d-1}\mathrm{d}\varrho, \tag{2.2.5}$$

we also get (2.2.2) if (2.2.1) holds for all radii ϱ with $B(x_0, \varrho) \subset \Omega$.
"⇐":
We point out that in the argument to follow, we do not need the continuity of u; it suffices that u be measurable and locally integrable.

We have just seen that the spherical mean value property implies the ball mean value property. The converse also holds:

If $K(u, x_0, r)$ is constant as a function of r, i.e., by (2.2.5)

$$0 = \frac{\partial}{\partial r} K(u, x_0, r) = \frac{d}{r} S(u, x_0, r) - \frac{d}{r} K(u, x_0, r),$$

then $S(u, x_0, r)$ is likewise constant in r, and by (2.2.4) it thus always has to equal $u(x_0)$.

Suppose now (2.2.1) for $B(x_0, r) \subset \Omega$. We want to show first that u then has to be smooth. For this purpose, we use the following general construction:
Put

$$\varrho(t) := \begin{cases} c_d \exp\left(\frac{1}{t^2-1}\right) & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant c_d is chosen such that

$$\int_{\mathbb{R}^d} \varrho(|x|)\mathrm{d}x = 1.$$

The reader should note that $\varrho(|x|)$ is infinitely differentiable with respect to x. For $f \in L^1(\Omega)$, $B(y, r) \subset \Omega$, we consider the so-called mollification

$$f_r(y) := \frac{1}{r^d} \int_\Omega \varrho\left(\frac{|y-x|}{r}\right) f(x)\mathrm{d}x. \tag{2.2.6}$$

Then f_r is infinitely differentiable with respect to y.

If now (2.2.1) holds, we have

$$\begin{aligned} u_r(y) &= \frac{1}{r^d} \int_0^r \int_{\partial B(y,s)} \varrho\left(\frac{s}{r}\right) u(x) do(x)\mathrm{d}s \\ &= \frac{1}{r^d} \int_0^r \varrho\left(\frac{s}{r}\right) d\omega_d s^{d-1} S(u, y, s)\mathrm{d}s \\ &= u(y) \int_0^1 \varrho(\sigma) d\omega_d \sigma^{d-1}\mathrm{d}\sigma \\ &= u(y) \int_{B(0,1)} \varrho\left(|x|\right) \mathrm{d}x \\ &= u(y). \end{aligned}$$

Thus a function satisfying the mean value property also satisfies

$$u_r(x) = u(x), \quad \text{provided that } B(x, r) \subset \Omega.$$

Thus, with u_r also u is infinitely differentiable. We may thus again consider (2.2.3), i.e.,

$$\int_{B(y,\varrho)} \Delta u(x)\mathrm{d}x = d\omega_d \varrho^{d-1} \frac{\partial}{\partial \varrho} S(u, y, \varrho). \tag{2.2.7}$$

If (2.2.7) holds, then $S(u, x_0, \varrho)$ is constant in ϱ, and therefore, the right-hand side of (2.2.7) vanishes for all y and ϱ with $B(y, \varrho) \subset \Omega$. Thus, also

$$\Delta u(y) = 0$$

for all $y \in \Omega$, and u is harmonic. □

With this observation, we easily obtain the following corollary:

Corollary 2.2.1 (Weyl's lemma). *Let $u : \Omega \to \mathbb{R}$ be measurable and locally integrable in Ω. Suppose that for all $\varphi \in C_0^\infty(\Omega)$,*

$$\int_\Omega u(x)\Delta\varphi(x)\mathrm{d}x = 0.$$

Then u is harmonic and, in particular, smooth.

Proof. We again consider the mollifications

$$u_r(x) = \frac{1}{r^d} \int_\Omega \varrho\left(\frac{|y-x|}{r}\right) u(y)\mathrm{d}y.$$

For $\varphi \in C_0^\infty$ and $r < \mathrm{dist}(\mathrm{supp}(\varphi), \partial\Omega)$, we obtain

$$\begin{aligned}
\int_\Omega u_r(x)\Delta\varphi(x)\mathrm{d}x &= \int_\Omega \frac{1}{r^d} \int_\Omega \varrho\left(\frac{|y-x|}{r}\right) u(y)\mathrm{d}y\Delta\varphi(x)\mathrm{d}x \\
&= \int_\Omega u(y)\Delta\varphi_r(y)\mathrm{d}y \\
&\quad \text{exchanging the integrals and observing that } (\Delta\varphi)_r = \Delta(\varphi_r), \\
&\quad \text{so that the Laplace operator commutes with the mollification} \\
&= 0,
\end{aligned}$$

since by our assumption for r also $\varphi_r \in C_0^\infty(\Omega)$.

Since u_r is smooth, this also implies

$$\int_\Omega \Delta u_r(x)\varphi(x)\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega_r),$$

with $\Omega_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}$.

Hence,

$$\Delta u_r = 0 \quad \text{in } \Omega_r.$$

Thus, u_r is harmonic in Ω_r.

We consider $R > 0$ and $0 < r \leq \frac{1}{2}R$. Then u_r satisfies the mean value property on any ball with center in Ω_r and radius $\leq \frac{1}{2}R$. Since

$$\begin{aligned}\int_{\Omega_r} |u_r(y)|\,\mathrm{d}y &\leq \int_{\Omega_r} \frac{1}{r^d} \int_{\Omega} \varrho\left(\frac{|x-y|}{r}\right) |u(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{\Omega} |u(x)|\,\mathrm{d}x\end{aligned}$$

obtained by exchanging the integrals and using $\int_{\mathbb{R}^d} \frac{1}{r^d} \varrho\left(\frac{|x-y|}{r}\right) \mathrm{d}y = 1$, the u_r have uniformly bounded norms in $L^1(\Omega)$, if $u \in L^1(\Omega)$. If u is only locally integrable, the preceding reasoning has to be applied locally in Ω, in order to get the local uniform integrability of the u_r. Since this is easily done, we assume for simplicity $u \in L^1(\Omega)$.

Since the u_r satisfy the mean value property on balls of radius $\frac{1}{2}R$, this implies that they are also uniformly bounded (keeping R fixed and letting r tend to 0). Furthermore, because of

$$\begin{aligned}|u_r(x_1) - u_r(x_2)| &\leq \frac{1}{\omega_d}\left(\frac{2}{R}\right)^d \int_{\substack{B(x_1,R/2)\setminus B(x_2,R/2)\\ \cup B(x_2,R/2)\setminus B(x_1,R/2)}} |u_r(x)|\,\mathrm{d}x \\ &\leq \frac{1}{\omega_d}\left(\frac{2}{R}\right)^d \sup |u_r|\, 2\mathrm{Vol}\left(B(x_1, R/2) \setminus B(x_2, R/2)\right),\end{aligned}$$

the u_r are also equicontinuous. Thus, by the Arzela–Ascoli theorem, for $r \to 0$, a subsequence of the u_r converges uniformly towards some continuous function v. We must have $u = v$, because u is (locally) in $L^1(\Omega)$, and so for almost all $x \in \Omega$, $u(x)$ is the limit of $u_r(x)$ for $r \to 0$ (cf. Lemma A.3). Thus, u is continuous, and since all the u_r satisfy the mean value property, so does u. Theorem 2.2.1 now implies the claim. □

Definition 2.2.1. Let $v : \Omega \to [-\infty, \infty)$ be upper semicontinuous, but not identically $-\infty$. Such a v is called *subharmonic* if for every subdomain $\Omega' \subset\subset \Omega$ and every harmonic function $u : \Omega' \to \mathbb{R}$ (we assume $u \in C^0(\bar{\Omega}'))$ with

$$v \leq u \quad \text{on } \partial\Omega',$$

we have

$$v \leq u \quad \text{on } \Omega'.$$

A function $w : \Omega \to (-\infty, \infty]$, lower semicontinuous, $w \not\equiv \infty$, is called *superharmonic* if $-w$ is subharmonic.

Theorem 2.2.2. *A function $v : \Omega \to [-\infty, \infty)$ (upper semicontinuous, $\not\equiv -\infty$) is subharmonic if and only if for every ball $B(x_0, r) \subset \Omega$,*

$$v(x_0) \leq S(v, x_0, r), \tag{2.2.8}$$

or, equivalently, if for every such ball

$$v(x_0) \leq K(v, x_0, r). \tag{2.2.9}$$

Proof. "$\Rightarrow$"
Since v is upper semicontinuous, there exists a monotonically decreasing sequence $(v_n)_{n\in\mathbb{N}}$ of continuous functions with $v = \lim_{n\in\mathbb{N}} v_n$. By Theorem 2.1.2, for every v_n, there exists a harmonic

$$u_n : B(x_0, r) \to \mathbb{R}$$

with

$$u_n|_{\partial B(x_0,r)} = v_n|_{\partial B(x_0,r)} \quad \left(\geq v|_{\partial B(x_0,r)}\right);$$

hence, in particular,

$$S(u_n, x_0, r) = S(v_n, x_0, r).$$

Since v is subharmonic and u_n is harmonic, we obtain

$$v(x_0) \leq u_n(x_0) = S(u_n, x_0, r) = S(v_n, x_0, r).$$

Now $n \to \infty$ yields (2.2.8). The mean value inequality for balls follows from that for spheres (cf. ̃(2.2.5)). For the converse direction, we employ the following lemma:

Lemma 2.2.1. *Suppose v satisfies the mean value inequality (2.2.8) or (2.2.9) for all $B(x_0, r) \subset \Omega$. Then v also satisfies the maximum principle, meaning that if there exists some $x_0 \in \Omega$ with*

$$v(x_0) = \sup_{x\in\Omega} v(x),$$

then v is constant. In particular, if Ω is bounded and $v \in C^0(\bar{\Omega})$, then

$$v(x) \leq \max_{y\in\partial\Omega} v(y) \quad \text{for all } x \in \Omega.$$

Remark. We shall soon see that the assumption of Lemma 2.2.1 is equivalent to v being subharmonic, and therefore, the lemma will hold for subharmonic functions.

Proof. Assume

$$v(x_0) = \sup_{x\in\Omega} v(x) =: M.$$

Thus,

$$\Omega^M := \{y \in \Omega : v(y) = M\} \neq \emptyset.$$

Let $y \in \Omega^M$, $B(y,r) \subset \Omega$. Since (2.2.8) implies (2.2.9) (cf. (2.2.5)), we may apply (2.2.9) in any case to obtain

$$0 = v(y) - M \leq \frac{1}{\omega_d r^d} \int_{B(y,r)} (v(x) - M)\mathrm{d}x. \tag{2.2.10}$$

Since M is the supremum of v, always $v(x) \leq M$, and we obtain $v(x) = M$ for all $x \in B(y,r)$. Thus Ω^M contains together with y all balls $B(y,r) \subset \Omega$, and it thus has to coincide with Ω, since Ω is assumed to be connected. Thus $u(x) = M$ for all $x \in \Omega$. □

We may now easily conclude the proof of Theorem 2.2.2:
Let u be as in Definition 2.2.1. Then $v-u$ likewise satisfies the mean value inequality, hence the maximum principle, and so

$$v \leq u \quad \text{in } \Omega',$$

if $v \leq u$ on $\partial\Omega'$. □

Corollary 2.2.2. *A function v of class $C^2(\Omega)$ is subharmonic precisely if*

$$\Delta v \geq 0 \quad in \quad \Omega.$$

Proof. "⇐":
Let $B(y,r) \subset \Omega$, $0 < \varrho < r$. Then by (2.2.3)

$$0 \leq \int_{B(y,\varrho)} \Delta v(x)\mathrm{d}x = d\omega_d \varrho^{d-1} \frac{\partial}{\partial\varrho} S(v, y, \varrho).$$

Integrating this inequality yields, for $0 < \varrho < r$,

$$S(v, y, \varrho) \leq S(v, y, r),$$

and since the left-hand side tends to $v(y)$ for $\varrho \to 0$, we obtain

$$v(y) \leq S(v, y, r).$$

By Theorem 2.2.2, v then is subharmonic.
"⇒": Assume $\Delta v(y) < 0$. Since $v \in C^2(\Omega)$, we could then find a ball $B(y,r) \subset \Omega$ with $\Delta v < 0$ on $B(y,r)$. Applying the first part of the proof to $-v$ would yield

$$v(y) > S(v, y, r),$$

and v could not be subharmonic. □

Examples of subharmonic functions:

1. Let $d \geq 2$. We compute

$$\Delta |x|^{\alpha} = (d\alpha + \alpha(\alpha - 2)) |x|^{\alpha-2} .$$

Thus $|x|^{\alpha}$ is subharmonic for $\alpha \geq 2 - d$. (This is not unexpected because $|x|^{2-d}$ is harmonic.)
2. Let $u : \Omega \to \mathbb{R}$ be harmonic and positive, $\beta \geq 1$. Then

$$\begin{aligned} \Delta u^{\beta} &= \sum_{i=1}^{d} \left(\beta u^{\beta-1} u_{x^i x^i} + \beta(\beta-1) u^{\beta-2} u_{x^i} u_{x^i}\right) \\ &= \sum_{i=1}^{d} \beta(\beta-1) u^{\beta-2} u_{x^i} u_{x^i} , \end{aligned}$$

since u is harmonic. Since u is assumed to be positive and $\beta \geq 1$, this implies that u^{β} is subharmonic.
3. Let $u : \Omega \to \mathbb{R}$ again be harmonic and positive. Then

$$\Delta \log u = \sum_{i=1}^{d} \left(\frac{u_{x^i x^i}}{u} - \frac{u_{x^i} u_{x^i}}{u^2} \right) = - \sum_{i=1}^{d} \frac{u_{x^i} u_{x^i}}{u^2},$$

since u is harmonic. Thus, $\log u$ is superharmonic, and $-\log u$ then is subharmonic.
4. The preceding examples can be generalized as follows:
Let $u : \Omega \to \mathbb{R}$ be harmonic, $f : u(\Omega) \to \mathbb{R}$ convex. Then $f \circ u$ is subharmonic. To see this, we first assume $f \in C^2$. Then

$$\begin{aligned} \Delta f(u(x)) &= \sum_{i=1}^{d} \left(f'(u(x)) u_{x^i x^i} + f''(u(x)) u_{x^i} u_{x^i}\right) \\ &= \sum_{i=1}^{d} f''(u(x)) \left(u_{x^i}\right)^2 \quad \text{(since } u \text{ is harmonic)} \\ &\geq 0, \end{aligned}$$

since for a convex C^2-function $f'' \geq 0$. If the convex function f is not of class C^2, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of convex C^2-functions converging to f locally uniformly. By the preceding, $f_n \circ u$ is subharmonic, and hence satisfies the mean value inequality. Since $f_n \circ u$ converges to $f \circ u$ locally uniformly,

$f \circ u$ satisfies the mean value inequality as well and so is subharmonic by Theorem 2.2.2.

We now return to studying harmonic functions. If u is harmonic, u and $-u$ both are subharmonic, and we obtain from Lemma 2.2.1 the following result:

Corollary 2.2.3 (Strong maximum principle). *Let u be harmonic in Ω. If there exists $x_0 \in \Omega$ with*

$$u(x_0) = \sup_{x\in\Omega} u(x) \quad or \quad u(x_0) = \inf_{x\in\Omega} u(x),$$

then u is constant in Ω.

A weaker version of Corollary 2.2.3 is the following:

Corollary 2.2.4 (Weak maximum principle). *Let Ω be bounded and $u \in C^0(\bar{\Omega})$ harmonic. Then for all $x \in \Omega$,*

$$\min_{y\in\partial\Omega} u(y) \leq u(x) \leq \max_{y\in\partial\Omega} u(y).$$

Proof. Otherwise, u would achieve its supremum or infimum in some interior point of Ω. Then u would be constant by Corollary 2.2.3, and the claim would also hold true. □

Corollary 2.2.5 (Uniqueness of solutions of the Poisson equation). *Let $f \in C^0(\Omega)$, Ω bounded, $u_1, u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ solutions of the Poisson equation*

$$\Delta u_i(x) = f(x) \quad for\ x \in \Omega \quad (i = 1, 2).$$

If $u_1(z) \leq u_2(z)$ for all $z \in \partial\Omega$, then also

$$u_1(x) \leq u_2(x) \quad \text{for all } x \in \Omega.$$

In particular, if

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega},$$

then

$$u_1 = u_2.$$

Proof. We apply the maximum principle to the harmonic function $u_1 - u_2$. □

In particular, for $f = 0$, we once again obtain the uniqueness of harmonic functions with given boundary values.

Remark. The reverse implication in Theorem 2.2.1 can also be seen as follows: We observe that the maximum principle needs only the mean value inequalities. Thus, the uniqueness of Corollary 2.2.5 holds for functions that satisfy the mean value formulae. On the other hand, by Theorem 2.1.2, for continuous boundary values there exists a harmonic extension on the ball, and this harmonic extension also satisfies the mean value formulae by the first implication of Theorem 2.2.1. By uniqueness, therefore, any continuous function satisfying the mean value property must be harmonic on every ball in its domain of definition Ω, hence on all of Ω.

As an application of the weak maximum principle we shall show the removability of isolated singularities of harmonic functions:

Corollary 2.2.6. *Let $x_0 \in \Omega \subset \mathbb{R}^d (d \geq 2)$, $u : \Omega \setminus \{x_0\} \to \mathbb{R}$ harmonic and bounded. Then u can be extended as a harmonic function on all of Ω; i.e., there exists a harmonic function*

$$\tilde{u} : \Omega \to \mathbb{R}$$

that coincides with u on $\Omega \setminus \{x_0\}$.

Proof. By a simple transformation, we may assume $x_0 = 0$ and that Ω contains the ball $B(0, 2)$. By Theorem 2.1.2, we may then solve the following Dirichlet problem:

$$\begin{aligned} \Delta\tilde{u} &= 0 \quad \text{in } \mathring{B}(0,1), \\ \tilde{u} &= u \quad \text{on } \partial B(0,1). \end{aligned}$$

We consider the following Green function on $B(0, 1)$ for $y = 0$:

$$G(x) = \begin{cases} \frac{1}{2\pi} \log|x| & \text{for } d = 2, \\ \frac{1}{d(2-d)\omega_d}(|x|^{2-d} - 1) & \text{for } d \geq 3. \end{cases}$$

For $\varepsilon > 0$, we put

$$u_\varepsilon(x) := \tilde{u}(x) - \varepsilon G(x) \quad (0 < |x| \leq 1).$$

First of all,

$$u_\varepsilon(x) = \tilde{u}(x) = u(x) \quad \text{for } |x| = 1. \tag{2.2.11}$$

Since on the one hand, u as a smooth function possesses a bounded derivative along $|x| = 1$, and on the other hand (with $r = |x|$), $\frac{\partial}{\partial r} G(x) > 0$, we obtain, for sufficiently large ε,

$$u_\varepsilon(x) > u(x) \quad \text{for } 0 < |x| < 1.$$

But we also have

$$\lim_{x \to 0} u_\varepsilon(x) = \infty \quad \text{for } \varepsilon > 0.$$

Since u is bounded, consequently, for every $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ with

$$u_\varepsilon(x) > u(x) \quad \text{for } |x| < r(\varepsilon). \tag{2.2.12}$$

From these arguments, we may find a smallest $\varepsilon_0 \geq 0$ with

$$u_{\varepsilon_0}(x) \geq u(x) \quad \text{for } |x| \leq 1.$$

We now wish to show that $\varepsilon_0 = 0$.

Assume $\varepsilon_0 > 0$. By (2.2.11) and (2.2.12), we could then find z_0, $r(\frac{\varepsilon_0}{2}) < |z_0| < 1$, with

$$u_{\frac{\varepsilon_0}{2}}(z_0) < u(z_0).$$

This would imply

$$\min_{x \in \mathring{B}(0,1) \setminus B(0, r(\frac{\varepsilon_0}{2}))} \left(u_{\frac{\varepsilon_0}{2}}(x) - u(x) \right) < 0,$$

while by (2.2.11), (2.2.12)

$$\min_{y \in \partial B(0,1) \cup \partial B(0, r(\frac{\varepsilon_0}{2}))} \left(u_{\frac{\varepsilon_0}{2}}(y) - u(y) \right) = 0.$$

This contradicts Corollary 2.2.4, because $u_{\frac{\varepsilon_0}{2}} - u$ is harmonic in the annular region considered here. Thus, we must have $\varepsilon_0 = 0$, and we conclude that

$$u \leq u_0 = \tilde{u} \quad \text{in } B(0,1) \setminus \{0\}.$$

In the same way, we obtain the opposite inequality

$$u \geq \tilde{u} \quad \text{in } B(0,1) \setminus \{0\}.$$

Thus, u coincides with $\tilde{u}$ in $B(0,1) \setminus \{0\}$. Since $\tilde{u}$ is harmonic in all of $B(0,1)$, we have found the desired extension. □

From Corollary 2.2.6 we see that not every Dirichlet problem for a harmonic function is solvable. For example, there is no solution of

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \mathring{B}(0,1) \setminus \{0\}, \\ u(x) &= 0 \quad \text{for } |x| = 1, \\ u(0) &= 1. \end{aligned}$$

Namely, by Corollary 2.2.6 any solution u could be extended to a harmonic function on the entire ball $\mathring{B}(0,1)$, but such a harmonic function would have to vanish identically by Corollary 2.2.4, since its boundary values on $\partial B(0,1)$ vanish, and so it could not assume the prescribed value 1 at $x = 0$.

Another consequence of the maximum principle for subharmonic functions is a gradient estimate for solutions of the Poisson equation:

Corollary 2.2.7. *Suppose that in Ω,*

$$\Delta u(x) = f(x)$$

with a bounded function f. Let $x_0 \in \Omega$ and $R := \operatorname{dist}(x_0, \partial\Omega)$. Then

$$|u_{x^i}(x_0)| \le \frac{d}{R} \sup_{\partial B(x_0,R)} |u| + \frac{R}{2} \sup_{B(x_0,R)} |f| \quad \text{for } i = 1, \dots, d. \tag{2.2.13}$$

Proof. We consider the case $i = 1$. For abbreviation, put

$$\mu := \sup_{\partial B(x_0,R)} |u|, \quad M := \sup_{B(x_0,R)} |f|.$$

Without loss of generality, suppose again $x_0 = 0$. The auxiliary function

$$v(x) := \frac{\mu}{R^2} |x|^2 + x^1 \left(R - x^1\right) \left(\frac{d\mu}{R^2} + \frac{M}{2}\right)$$

satisfies, in $B(0, R)$,

$$\begin{aligned} \Delta v(x) &= -M, \\ v\left(0, x^2, \dots, x^d\right) &\ge 0 \quad \text{for all } x^2, \dots, x^d, \\ v(x) &\ge \mu \quad \text{for } |x| = R,\ x^1 \ge 0. \end{aligned}$$

We now consider

$$\bar{u}(x) := \frac{1}{2}\left(u\left(x^1, \dots, x^d\right) - u\left(-x^1, x^2, \dots, x^d\right)\right).$$

In $B(0, R)$, we have

$$\begin{aligned} |\Delta \bar{u}(x)| &\le M, \\ \bar{u}(0, x^2, \dots, x^d) &= 0 \quad \text{for all } x^2, \dots, x^d, \\ |\bar{u}(x)| &\le \mu \quad \text{for all } |x| = R. \end{aligned}$$

We consider the half-ball $B^+ := \{|x| \le R,\ x^1 > 0\}$. The preceding inequalities imply

$$\Delta(v \pm \bar{u}) \le 0 \quad \text{in } \mathring{B}^+,$$
$$v \pm \bar{u} \ge 0 \quad \text{on } \partial B^+.$$

The maximum principle (Lemma 2.2.1) yields

$$|\bar{u}| \le v \quad \text{in } B^+.$$

We conclude that

$$|u_{x^1}(0)| = \lim_{\substack{x^1 \to 0 \\ x^1 > 0}} \left| \frac{\bar{u}(x^1, 0, \ldots, 0)}{x^1} \right| \le \lim_{\substack{x^1 \to 0 \\ x^1 > 0}} \frac{v(x^1, 0, \ldots, 0)}{x^1} = \frac{\mathrm{d}\mu}{R} + \frac{R}{2} M,$$

i.e., (2.2.13). □

Other consequences of the mean value formulae are the following:

Corollary 2.2.8 (Liouville theorem). *Let $u : \mathbb{R}^d \to \mathbb{R}$ be harmonic and bounded. Then u is constant.*

Proof. For $x_1, x_2 \in \mathbb{R}^d$, by (2.2.2) for all $r > 0$,

$$\begin{aligned} u(x_1) - u(x_2) &= \frac{1}{\omega_d r^d} \left(\int_{B(x_1,r)} u(x)\mathrm{d}x - \int_{B(x_2,r)} u(x)\mathrm{d}x \right) \\ &= \frac{1}{\omega_d r^d} \left(\int_{B(x_1,r)\setminus B(x_2,r)} u(x)\mathrm{d}x - \int_{B(x_2,r)\setminus B(x_1,r)} u(x)\mathrm{d}x \right). \end{aligned} \tag{2.2.14}$$

By assumption

$$|u(x)| \le M,$$

and for $r \to \infty$,

$$\frac{1}{\omega_d r^d} \mathrm{Vol}\,(B(x_1, r) \setminus B(x_2, r)) \to 0.$$

This implies that the right-hand side of (2.2.14) converges to 0 for $r \to \infty$. Therefore, we must have

$$u(x_1) = u(x_2).$$

Since x_1 and x_2 are arbitrary, u has to be constant. □

Another *proof* of Corollary 2.2.8 follows from Corollary 2.2.7:

By Corollary 2.2.7, for all $x_0 \in \mathbb{R}^d$, $R > 0$, $i = 1, \ldots, d$,

$$|u_{x^i}(x_0)| \le \frac{d}{R} \sup_{\mathbb{R}^d} |u| .$$

Since u is bounded by assumption, the right-hand side tends to 0 for $R \to \infty$, and it follows that u is constant. This proof also works under the weaker assumption

$$\lim_{R \to \infty} \frac{1}{R} \sup_{B(x_0,R)} |u| = 0.$$

This assumption is sharp, since affine linear functions are harmonic functions on $\mathbb{R}^d$ that are not constant.

Corollary 2.2.9 (Harnack inequality). *Let $u : \Omega \to \mathbb{R}$ be harmonic and nonnegative. Then for every subdomain $\Omega' \subset\subset \Omega$ there exists a constant $c = c(d, \Omega, \Omega')$ with*

$$\sup_{\Omega'} u \le c \inf_{\Omega'} u. \tag{2.2.15}$$

Proof. We first consider the special case $\Omega' = \mathring{B}(x_0, r)$, assuming $B(x_0, 4r) \subset \Omega$. Let $y_1, y_2 \in B(x_0, r)$. By (2.2.2),

$$\begin{aligned}
u(y_1) &= \frac{1}{\omega_d r^d} \int_{B(y_1,r)} u(y)\mathrm{d}y \\
&\le \frac{1}{\omega_d r^d} \int_{B(x_0,2r)} u(y)\mathrm{d}y, \\
&\quad \text{since } u \ge 0 \text{ and } B(y_1, r) \subset B(x_0, 2r) \\
&= \frac{3^d}{\omega_d (3r)^d} \int_{B(x_0,2r)} u(y)\mathrm{d}y \\
&\le \frac{3^d}{\omega_d (3r)^d} \int_{B(y_2,3r)} u(y)\mathrm{d}y, \\
&\quad \text{since } u \ge 0 \text{ and } B(x_0, 2r) \subset B(y_2, 3r) \\
&= 3^d u(y_2),
\end{aligned}$$

and in particular,

$$\sup_{B(x_0,r)} u \le 3^d \inf_{B(x_0,r)} u,$$

which is the claim in this special case.

For an arbitrary subdomain $\Omega' \subset\subset \Omega$, we choose $r > 0$ with

$$r < \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega).$$

Since Ω' is bounded and connected, there exists $m \in \mathbb{N}$ such that any two points $y_1, y_2 \in \Omega'$ can be connected in Ω' by a curve that can be covered by at most m balls of radius r with centers in Ω'. Composing the preceding inequalities for all these balls, we get

$$u(y_1) \leq 3^{md} u(y_2).$$

Thus, we have verified the claim for $c = 3^{md}$. □

The Harnack inequality implies the following result:

Corollary 2.2.10 (Harnack convergence theorem). *Let $u_n : \Omega \to \mathbb{R}$ be a monotonically increasing sequence of harmonic functions. If there exists $y \in \Omega$ for which the sequence $(u_n(y))_{n\in\mathbb{N}}$ is bounded, then u_n converges on any subdomain $\Omega' \subset\subset \Omega$ uniformly towards a harmonic function.*

Proof. The monotonicity and boundedness imply that $u_n(y)$ converges for $n \to \infty$. For $\varepsilon > 0$, there thus exists $N \in \mathbb{N}$ such that for $n \geq m \geq N$,

$$0 \leq u_n(y) - u_m(y) < \varepsilon.$$

Then $u_n - u_m$ is a nonnegative harmonic function (by monotonicity), and by Corollary 2.2.9,

$$\sup_{\Omega'}(u_n - u_m) \leq c\varepsilon, \quad (\text{wlog } y \in \Omega'),$$

where c depends on d, Ω, and Ω'. Thus $(u_n)_{n\in\mathbb{N}}$ converges uniformly in all of Ω'. The uniform limit of harmonic functions has to satisfy the mean value formulae as well, and it is hence harmonic itself by Theorem 2.2.1. □

Summary

In this chapter we encountered some basic properties of harmonic functions, i.e., of solutions of the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega,$$

and also of solutions of the Poisson equation

$$\Delta u = f \quad \text{in } \Omega$$

with given f.

We found the unique solution of the Dirichlet problem on the ball (Theorem 2.1.2), and we saw that solutions are smooth (Corollary 2.1.2) and even satisfy explicit estimates (Corollary 2.2.7) and in particular the maximum principle (Corollary 2.2.3, Corollary 2.2.4), which actually already holds for subharmonic functions (Lemma 2.2.1). All these results are typical and characteristic for solutions of elliptic PDEs. The methods presented in this chapter, however, mostly do not readily generalize, since they have used heavily the rotational symmetry of the Laplace operator. In subsequent chapters we thus need to develop different and more general methods in order to show analogues of these results for larger classes of elliptic PDEs.

Exercises

2.1. Determine the Green function of the half-space

$$\{x = (x^1, \dots, x^d) \in \mathbb{R}^d : x^1 > 0\}.$$

2.2. On the unit ball $B(0,1) \subset \mathbb{R}^d$, determine a function $H(x,y)$, defined for $x \neq y$, with

(i) $\frac{\partial}{\partial \nu_x} H(x,y) = 1$ for $x \in \partial B(0,1)$
(ii) $H(x,y) - \Gamma(x,y)$ is a harmonic function of $x \in B(0,1)$. (Here, $\Gamma(x,y)$ is a fundamental solution.)

2.3. Use the result of Exercise 2.2 to study the Neumann problem for the Laplace equation on the unit ball $B(0,1) \subset \mathbb{R}^d$:
Let $g : \partial B(0,1) \to \mathbb{R}$ with $\int_{\partial B(0,1)} g(y)\, do(y) = 0$ be given. We wish to find a solution of

$$\begin{aligned} \Delta u(x) &= 0 && \text{for } x \in \mathring{B}(0,1), \\ \frac{\partial u}{\partial \nu}(x) &= g(x) && \text{for } x \in \partial B(0,1). \end{aligned}$$

2.4. Let $u : B(0,R) \to \mathbb{R}$ be harmonic and nonnegative. Prove the following version of the Harnack inequality:

$$\frac{R^{d-2}(R-|x|)}{(R+|x|)^{d-1}} u(0) \le u(x) \le \frac{R^{d-2}(R+|x|)}{(R-|x|)^{d-1}} u(0)$$

for all $x \in B(0,R)$.

2.5. Let $u : \mathbb{R}^d \to \mathbb{R}$ be harmonic and nonnegative. Show that u is constant (Hint: Use the result of Exercise 2.4.).

2.6. Let u be harmonic with periodic boundary conditions. Use the maximum principle to show that u is constant.

2.7. Let $\Omega \subset \mathbb{R}^3 \setminus \{0\}$, $u : \Omega \to \mathbb{R}$ harmonic. Show that

$$v(x^1, x^2, x^3) := \frac{1}{|x|} u\left(\frac{x^1}{|x|^2}, \frac{x^2}{|x|^2}, \frac{x^3}{|x|^2}\right)$$

is harmonic in the region $\Omega' := \left\{x \in \mathbb{R}^3 : \left(\frac{x^1}{|x|^2}, \frac{x^2}{|x|^2}, \frac{x^3}{|x|^2}\right) \in \Omega\right\}$.

- Is there a deeper reason for this?
- Is there an analogous result for arbitrary dimension d?

2.8. Let Ω be the unbounded region $\{x \in \mathbb{R}^d : |x| > 1\}$. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $\Delta u = 0$ in Ω. Furthermore, assume

$$\lim_{|x| \to \infty} u(x) = 0.$$

Show that

$$\sup_{\Omega} |u| = \max_{\partial\Omega} |u|.$$

2.9. (Schwarz reflection principle):
Let $\Omega^+ \subset \{x^d > 0\}$,

$$\Sigma := \partial\Omega^+ \cap \{x^d = 0\} \neq \emptyset. \tag{2.2.16}$$

Let u be harmonic in Ω^+, continuous on $\Omega^+ \cup \Sigma$, and suppose $u = 0$ on Σ. We put

$$\bar{u}(x^1, \ldots, x^d) := \begin{cases} u(x^1, \ldots, x^d) & \text{for } x^d \geq 0, \\ -u(x^1, \ldots, -x^d) & \text{for } x^d < 0. \end{cases}$$

Show that $\bar{u}$ is harmonic in $\Omega^+ \cup \Sigma \cup \Omega^-$, where $\Omega^- := \{x \in \mathbb{R}^d : (x^1, \ldots, -x^d) \in \Omega^+\}$.

2.10. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain for which the divergence theorem holds. Assume $u \in C^2(\bar{\Omega})$, $u = 0$ on $\partial\Omega$. Show that for every $\varepsilon > 0$,

$$2\int_{\Omega} |\nabla u(x)|^2 \,\mathrm{d}x \leq \varepsilon \int_{\Omega} (\Delta u(x))^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Omega} u^2(x) \,\mathrm{d}x.$$