Chapter 3
The Maximum Principle

Throughout this chapter, Ω is a bounded domain in $\mathbb{R}^d$. All functions u are assumed to be of class $C^2(\Omega)$.

3.1 The Maximum Principle of E. Hopf

We wish to study linear elliptic differential operators of the form

$$Lu(x) = \sum_{i,j=1}^{d} a^{ij}(x)u_{x^i x^j}(x) + \sum_{i=1}^{d} b^i(x)u_{x^i}(x) + c(x)u(x),$$

where we impose the following conditions on the coefficients:

1. Symmetry: $a^{ij}(x) = a^{ji}(x)$ for all i, j and $x \in \Omega$ (this is no serious restriction).
2. Ellipticity: There exists a constant $\lambda > 0$ with

   $$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi^i\xi^j \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d$$

   (this is the key condition).

   In particular, the matrix $(a^{ij}(x))_{i,j=1,\ldots,d}$ is positive definite for all x, and the smallest eigenvalue is greater than or equal to λ.
3. Boundedness of the coefficients: There exists a constant K with

   $$\left|a^{ij}(x)\right|, \left|b^i(x)\right|, |c(x)| \le K \quad \text{for all } i, j \text{ and } x \in \Omega.$$

Obviously, the Laplace operator satisfies all three conditions. The aim of this chapter is to prove maximum principles for solutions of $Lu = 0$. It turns out that for

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that purpose, we need to impose an additional condition on the sign of $c(x)$, since otherwise no maximum principle can hold, as the following simple example demonstrates: The Dirichlet problem

$$u''(x) + u(x) = 0 \quad \text{on } (0, \pi),$$
$$u(0) = 0 = u(\pi),$$

has the solutions

$$u(x) = \alpha \sin x$$

for arbitrary α, and depending on the sign of α, these solutions assume a strict interior maximum or minimum at $x = \pi/2$. The Dirichlet problem

$$u''(x) - u(x) = 0,$$
$$u(0) = 0 = u(\pi),$$

however, has 0 as its only solution.

As a start, let us present a proof of the weak maximum principle for subharmonic functions (Lemma 2.2.1) that does not depend on the mean value formulae:

Lemma 3.1.1. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $\Delta u \geq 0$ in Ω. Then*

$$\sup_{\Omega} u = \max_{\partial\Omega} u. \tag{3.1.1}$$

(Since u is continuous and Ω is bounded, and the closure $\bar{\Omega}$ thus is compact, the supremum of u on Ω coincides with the maximum of u on $\bar{\Omega}$.)

Proof. We first consider the case where we even have

$$\Delta u > 0 \quad \text{in } \Omega.$$

Then u cannot assume an interior maximum at some $x_0 \in \Omega$, since at such a maximum, we would have

$$u_{x^i x^i}(x_0) \leq 0 \quad \text{for } i = 1, \ldots, d,$$

and thus also

$$\Delta u(x_0) \leq 0.$$

We now come to the general case $\Delta u \geq 0$ and consider the auxiliary function

$$v(x) = \mathrm{e}^{x^1},$$

which satisfies

$$\Delta v = v > 0.$$

For each $\varepsilon > 0$, then

$$\Delta(u + \varepsilon v) > 0 \quad \text{in } \Omega,$$

and from the case studied in the beginning, we deduce

$$\sup_{\Omega}(u + \varepsilon v) = \max_{\partial\Omega}(u + \varepsilon v).$$

Then

$$\sup_{\Omega} u + \varepsilon \inf_{\Omega} v \leq \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} v,$$

and since this holds for every $\varepsilon > 0$, we obtain (3.1.1). □

Theorem 3.1.1. *Assume* $c(x) \equiv 0$*, and let u satisfy in* Ω

$$Lu \geq 0,$$

i.e.,

$$\sum_{i,j=1}^{d} a^{ij}(x) u_{x^i x^j} + \sum_{i=1}^{d} b^i(x) u_{x^i} \geq 0. \tag{3.1.2}$$

Then also

$$\sup_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x). \tag{3.1.3}$$

In the case $Lu \leq 0$*, a corresponding result holds for the infimum.*

Proof. As in the proof of Lemma 3.1.1, we first consider the case

$$Lu > 0.$$

Since at an interior maximum x_0 of u, we must have

$$u_{x^i}(x_0) = 0 \quad \text{for } i = 1, \ldots, d,$$

and

$$(u_{x^i x^j}(x_0))_{i,j=1,\ldots,d} \quad \text{negative semidefinite},$$

and thus by the ellipticity condition also

$$Lu(x_0) = \sum_{i,j=1}^{d} a^{ij}(x_0) u_{x^i x^j}(x_0) \leq 0,$$

such an interior maximum cannot occur.

Returning to the general case $Lu \geq 0$, we now consider the auxiliary function

$$v(x) = \mathrm{e}^{\alpha x^1}$$

for $\alpha > 0$. Then

$$Lv(x) = \left(\alpha^2 a^{11}(x) + \alpha b^1(x)\right) v(x).$$

Since Ω and the coefficients b^i are bounded and the coefficients satisfy $a^{ii}(x) \geq \lambda$, we have for sufficiently large α,

$$Lv > 0,$$

and applying what we have proved already to $u + \varepsilon v$

$$(L(u + \varepsilon v) > 0),$$

the claim follows as in the proof of Lemma 3.1.1. The case $Lu \leq 0$ can be reduced to the previous one by considering $-u$. □

Corollary 3.1.1. *Let L be as in Theorem 3.1.1, and let $f \in C^0(\Omega)$, $\varphi \in C^0(\partial\Omega)$ be given. Then the Dirichlet problem*

$$\begin{aligned} Lu(x) &= f(x) \quad \text{for } x \in \Omega, \\ u(x) &= \varphi(x) \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{3.1.4}$$

admits at most one solution.

Proof. The difference $v(x) = u_1(x) - u_2(x)$ of two solutions satisfies

$$\begin{aligned} Lv(x) &= 0 \quad \text{in } \Omega, \\ v(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and by Theorem 3.1.1 it then has to vanish identically on Ω. □

Theorem 3.1.1 supposes $c(x) \equiv 0$. This assumption can be weakened as follows:

Corollary 3.1.2. *Suppose $c(x) \leq 0$ in Ω. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$Lu \geq 0 \quad \text{in } \Omega.$$

With $u^+(x) := \max(u(x), 0)$, we then have

$$\sup_{\Omega} u^+ \leq \max_{\partial\Omega} u^+. \tag{3.1.5}$$

Proof. Let $\Omega^+ := \{x \in \Omega : u(x) > 0\}$. Because of $c \leq 0$, we have in Ω^+,

$$\sum_{i,j=1}^{d} a^{ij}(x)u_{x^i x^j} + \sum_{i=1}^{d} b^i(x)u_{x^i} \geq 0,$$

and hence by Theorem 3.1.1,

$$\sup_{\Omega^+} u \leq \max_{\partial\Omega^+} u. \tag{3.1.6}$$

We have

$$u = 0 \quad \text{on } \partial\Omega^+ \cap \Omega \quad \text{(by continuity of } u\text{)},$$

$$\max_{\partial\Omega^+ \cap \partial\Omega} u \leq \max_{\partial\Omega} u,$$

and hence, since $\partial\Omega^+ = (\partial\Omega^+ \cap \Omega) \cup (\partial\Omega^+ \cap \partial\Omega)$,

$$\max_{\partial\Omega^+} u \leq \max_{\partial\Omega} u^+. \tag{3.1.7}$$

Since also

$$\sup_{\Omega} u^+ = \sup_{\Omega^+} u, \tag{3.1.8}$$

(3.1.5) follows from (3.1.6) and (3.1.7). □

We now come to the strong maximum principle of E. Hopf:

Theorem 3.1.2. *Suppose $c(x) \equiv 0$, and let u satisfy in Ω,*

$$Lu \geq 0. \tag{3.1.9}$$

If u assumes its maximum in the interior of Ω, it has to be constant. More generally, if $c(x) \leq 0$, u has to be constant if it assumes a nonnegative interior maximum.

For the proof, we need the boundary point lemma of E. Hopf:

Lemma 3.1.2. *Suppose $c(x) \leq 0$ and*

$$Lu \geq 0 \quad \text{in } \Omega' \subset \mathbb{R}^d,$$

and let $x_0 \in \partial\Omega'$. Moreover, assume

(i) u is continuous at x_0.
(ii) $u(x_0) \geq 0$ if $c(x) \not\equiv 0$.
(iii) $u(x_0) > u(x)$ for all $x \in \Omega'$.
(iv) There exists a ball $\mathring{B}(y, R) \subset \Omega'$ with $x_0 \in \partial B(y, R)$.

We then have, with $r := |x - y|$,

$$\frac{\partial u}{\partial r}(x_0) > 0,$$

provided that this derivative (in the direction of the exterior normal of Ω'*) exists.*

Proof. We may assume

$$\partial B(y, R) \cap \partial \Omega' = \{x_0\}.$$

For $0 < \rho < R$, on the annular region $\overset{\circ}{B}(y, R) \setminus B(y, \rho)$, we consider the auxiliary function

$$v(x) := \mathrm{e}^{-\gamma |x-y|^2} - \mathrm{e}^{-\gamma R^2}.$$

We have

$$\begin{aligned} Lv(x) = \Bigg\{ & 4\gamma^2 \sum_{i,j=1}^{d} a^{ij}(x) \left(x^i - y^i\right)\left(x^j - y^j\right) \\ & - 2\gamma \sum_{i=1}^{d} a^{ii}(x) + b^i(x)\left(x^i - y^i\right) \Bigg\} \mathrm{e}^{-\gamma |x-y|^2} \\ & + c(x)\left(\mathrm{e}^{-\gamma |x-y|^2} - \mathrm{e}^{-\gamma R^2}\right). \end{aligned}$$

For sufficiently large γ, because of the assumed boundedness of the coefficients of L and the ellipticity condition, we have

$$Lv \geq 0 \quad \text{in } \overset{\circ}{B}(y, R) \setminus B(y, \rho). \tag{3.1.10}$$

By (iii) and (iv),

$$u(x) - u(x_0) < 0 \quad \text{for } x \in \overset{\circ}{B}(y, R).$$

Therefore, we may find $\varepsilon > 0$ with

$$u(x) - u(x_0) + \varepsilon v(x) \leq 0 \quad \text{for } x \in \partial B(y, \rho). \tag{3.1.11}$$

Since $v = 0$ on $\partial B(y, R)$, (3.1.11) continues to hold on $\partial B(y, R)$. On the other hand,

$$L\left(u(x) - u(x_0) + \varepsilon v(x)\right) \geq -c(x)u(x_0) \geq 0 \tag{3.1.12}$$

by (3.1.10) and (ii) and because of $c(x) \leq 0$. Thus, we may apply Corollary 3.1.2 on $\overset{\circ}{B}(y, R) \setminus B(y, \rho)$ and obtain

$$u(x) - u(x_0) + \varepsilon v(x) \leq 0 \quad \text{for } x \in \overset{\circ}{B}(y, R) \setminus B(y, \rho).$$

Provided that the derivative exists, it follows that

$$\frac{\partial}{\partial r}\left(u(x) - u(x_0) + \varepsilon v(x)\right) \geq 0 \text{ at } x = x_0,$$

and hence for $x = x_0$,

$$\frac{\partial}{\partial r} u(x) \geq -\varepsilon \frac{\partial v(x)}{\partial r} = \varepsilon \left(2\gamma R \mathrm{e}^{-\gamma R^2}\right) > 0. \qquad \square$$

Proof of Theorem 3.1.2: We assume by contradiction that u is not constant but has a maximum m (≥ 0 in case $c \not\equiv 0$) in Ω. We then have

$$\Omega' := \{x \in \Omega : u(x) < m\} \neq \emptyset$$

and

$$\partial\Omega' \cap \Omega \neq \emptyset.$$

We choose some $y \in \Omega'$ that is closer to $\partial\Omega'$ than to $\partial\Omega$. Let $\mathring{B}(y, R)$ be the largest ball with center y that is contained in Ω'. We then get

$$u(x_0) = m \quad \text{for some } x_0 \in \partial B(y, R)$$

and

$$u(x) < u(x_0) \quad \text{for } x \in \Omega'.$$

By Lemma 3.1.2,

$$Du(x_0) \neq 0,$$

which, however, is not possible at an interior maximum point. This contradiction demonstrates the claim. $\square$

3.2 The Maximum Principle of Alexandrov and Bakelman

In this section, we consider differential operators of the same type as in the previous one, but for technical simplicity, we assume that the coefficients $c(x)$ and $b^i(x)$ vanish. While similar results as those presented here continue to hold for vanishing $b^i(x)$ and nonpositive $c(x)$, here we wish only to present the key ideas in a situation that is as simple as possible.

Theorem 3.2.1. *Suppose that* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies*

$$Lu(x) := \sum_{i,j=1}^{d} a^{ij}(x) u_{x^i x^j} \geq f(x), \tag{3.2.1}$$

where the matrix $(a^{ij}(x))$ *is positive definite and symmetric for each* $x \in \Omega$*. Moreover, let*

$$\int_\Omega \frac{|f(x)|^d}{\det(a^{ij}(x))} \mathrm{d}x < \infty. \tag{3.2.2}$$

We then have

$$\sup_\Omega u \leq \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{d\omega_d^{1/d}} \left(\int_\Omega \frac{|f(x)|^d}{\det(a^{ij}(x))} \mathrm{d}x \right)^{1/d}. \tag{3.2.3}$$

In contrast to those estimates that are based on the Hopf maximum principle (cf., e.g., Theorem 3.3.2 below), here we have only an integral norm of f on the right-hand side, i.e., a norm that is weaker than the supremum norm. In this sense, the maximum principle of Alexandrov and Bakelman is stronger than that of Hopf.

For the proof of Theorem 3.2.1, we shall need some geometric constructions. For $v \in C^0(\Omega)$, we define the upper contact set

$$T^+(v) := \left\{ y \in \Omega : \exists p \in \mathbb{R}^d \quad \forall x \in \Omega : v(x) \leq v(y) + p \cdot (x - y) \right\}. \tag{3.2.4}$$

The dot "$\cdot$" here denotes the Euclidean scalar product of $\mathbb{R}^d$. The p that occurs in this definition in general will depend on y; that is, $p = p(y)$. The set $T^+(v)$ is that subset of Ω in which the graph of v lies below a hyperplane in $\mathbb{R}^{d+1}$ that touches the graph of v at $(y, v(y))$. If v is differentiable at $y \in T^+(v)$, then necessarily $p(y) = Dv(y)$. Finally, v is concave precisely if $T^+(v) = \Omega$.

Lemma 3.2.1. *For* $v \in C^2(\Omega)$*, the Hessian*

$$(v_{x^i x^j})_{i,j=1,\dots,d}$$

is negative semidefinite on $T^+(v)$.

Proof. For $y \in T^+(v)$, we consider the function

$$w(x) := v(x) - v(y) - p(y) \cdot (x - y).$$

Then $w(x) \leq 0$ on Ω, since $y \in T^+(v)$ and $w(y) = 0$. Thus, w has a maximum at y, implying that $(w_{x^i x^j}(y))$ is negative semidefinite. Since $v_{x^i x^j} = w_{x^i x^j}$ for all i, j, the claim follows. □

If v is not differentiable at $y \in T^+(v)$, then $p = p(y)$ need not be unique, but there may exist several p's satisfying the condition in (3.2.4). We assign to $y \in T^+(v)$ the set of all those p's, i.e., consider the set-valued map

$$\tau_v(y) := \left\{p \in \mathbb{R}^d : \forall x \in \Omega : v(x) \le v(y) + p \cdot (x-y)\right\}.$$

For $y \notin T^+(v)$, we put $\tau_v(y) := \emptyset$.

Example 3.2.1. $\Omega = \mathring{B}(0,1)$, $\beta > 0$,

$$v(x) = \beta(1 - |x|).$$

The graph of v thus is a cone with a vertex of height β at 0 and having the unit sphere as its base. We have $T^+(v) = \mathring{B}(0,1)$,

$$\tau_v(y) = \begin{cases} B(0,\beta) & \text{for } y = 0, \\ \left\{-\beta \frac{y}{|y|}\right\} & \text{for } y \neq 0. \end{cases}$$

For the cone with vertex of height β at x_0 and base $\partial B(x_o, R)$,

$$v(x) = \beta\left(1 - \frac{|x - x_0|}{R}\right)$$

and $\Omega = \mathring{B}(x_0, R)$, and analogously,

$$\tau_v\left(\mathring{B}(x_0,R)\right) = \tau_v(x_0) = B(0, \beta/R). \tag{3.2.5}$$

We now consider the image of Ω under τ_v,

$$\tau_v(\Omega) = \bigcup_{y\in\Omega} \tau_v(y) \subset \mathbb{R}^d.$$

We will let $\mathcal{L}_d$ denote d-dimensional Lebesgue measure. Then we have the following lemma:

Lemma 3.2.2. *Let $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

$$\mathcal{L}_d\left(\tau_v(\Omega)\right) \le \int_{T^+(v)} \left|\det\left(v_{x^i x^j}(x)\right)\right| \mathrm{d}x. \tag{3.2.6}$$

Proof. First of all,

$$\tau_v(\Omega) = \tau_v(T^+(v)) = Dv(T^+(v)), \tag{3.2.7}$$

since v is differentiable. By Lemma 3.2.1, the Jacobian matrix of $Dv : \Omega \to \mathbb{R}^d$, namely, $(v_{x^i x^j})$, is negative semidefinite on $T^+(v)$. Thus $Dv - \varepsilon$ Id has maximal rank for $\varepsilon > 0$. From the transformation formula for multiple integrals, we then get

$$\mathcal{L}_d\left((Dv - \varepsilon \,\mathrm{Id})\left(T^+(v)\right)\right) \le \int_{T^+(v)} \left|\det\left(v_{x^i x^j}(x) - \varepsilon\delta_{ij}\right)_{i,j=1,\ldots,d}\right| \mathrm{d}x. \quad (3.2.8)$$

Letting ε tend to 0, the claim follows because of (3.2.7). □

We are now able to *prove Theorem 3.2.1*. We may assume

$$u \le 0 \quad \text{on } \partial\Omega$$

by replacing u by $u - \max_{\partial\Omega} u$ if necessary.

Now let $x_0 \in \Omega$, $u(x_0) > 0$. We consider the function κ_{x_0} on $B(x_0, \delta)$ with $\delta = \mathrm{diam}(\Omega)$ whose graph is the cone with vertex of height $u(x_0)$ at x_0 and base $\partial B(x_0, \delta)$. From the definition of the diameter $\delta = \mathrm{diam}\,\Omega$,

$$\Omega \subset B(x_0, \delta).$$

Since we assume $u \le 0$ on $\partial\Omega$, for each hyperplane that is tangent to this cone there exists some parallel hyperplane that is tangent to the graph of u. (In order to see this, we simply move such a hyperplane parallel to its original position from above towards the graph of u until it first becomes tangent to it. Since the graph of u is at least of height $u(x_0)$, i.e., of the height of the cone, and since $u \le 0$ on $\partial\Omega$ and $\partial\Omega \subset B(x_0, \delta)$, such a first tangency cannot occur at a boundary point of Ω but only at an interior point x_1. Thus, the corresponding hyperplane is contained in $\tau_v(x_1)$.) This means that

$$\tau_{\kappa_{x_0}}(\Omega) \subset \tau_u(\Omega). \quad (3.2.9)$$

By (3.2.5),

$$\tau_{\kappa_{x_0}}(\Omega) = B\left(0, u(x_0)/\delta\right). \quad (3.2.10)$$

Relations (3.2.6), (3.2.9), and (3.2.10) imply

$$\mathcal{L}_d\left(B\left(0, u(x_0)/\delta\right)\right) \le \int_{T^+(u)} \left|\det\left(u_{x^i x^j}(x)\right)\right| \mathrm{d}x,$$

and hence

$$
\begin{aligned}
u(x_0) &\leq \frac{\delta}{\omega_d^{1/d}} \left(\int_{T^+(u)} \left| \det\left(u_{x^i x^j}(x)\right) \right| \mathrm{d}x \right)^{1/d} \\
&= \frac{\delta}{\omega_d^{1/d}} \left(\int_{T^+(u)} (-1)^d \det\left(u_{x^i x^j}(x)\right) \mathrm{d}x \right)^{1/d} \qquad (3.2.11)
\end{aligned}
$$

by Lemma 3.2.1. Without assuming $u \leq 0$ on $\partial\Omega$, we get an additional term $\max_{\partial\Omega} u$ on the right-hand side of (3.2.11). Since the formula holds for all $x_0 \in \Omega$, we have the following result:

Lemma 3.2.3. *For* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$,

$$
\sup_{\Omega} u \leq \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{\omega_d^{1/d}} \left(\int_{T^+(u)} (-1)^d \det\left(u_{x^i x^j}(x)\right) \mathrm{d}x \right)^{1/d}. \qquad (3.2.12)
$$

In order to deduce Theorem 3.2.1 from this result, we need the following elementary lemma:

Lemma 3.2.4. *On* $T^+(u)$,

$$
(-1)^d \det\left(u_{x^i x^j}(x)\right) \leq 1 \det\left(a^{ij}(x)\right) \left(-\frac{1}{d} \sum_{i,j=1}^{d} a^{ij}(x) u_{x^i x^j}(x) \right)^d. \qquad (3.2.13)
$$

Proof. It is well known that for symmetric, positive definite matrices A and B,

$$
\det A \det B \leq \left(\frac{1}{d} \operatorname{trace} AB \right)^d,
$$

which is readily verified by diagonalizing one of the matrices, which is possible if that matrix is symmetric.

Inserting $A = (-u_{x^i x^j})$, $B = (a^{ij})$ (which is possible by Lemma 3.2.1 and the ellipticity assumption), we obtain (3.2.13). □

Inequalities (3.2.12) and (3.2.13) imply

$$
\sup_{\Omega} u \leq \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{d\,\omega_d^{1/d}} \left(\int_{T^+(u)} \frac{\left(-\sum_{i,j=1}^{d} a^{ij}(x) u_{x^i x^j}(x) \right)^d}{\det\left(a^{ij}(x)\right)} \mathrm{d}x \right)^{1/d}. \qquad (3.2.14)
$$

In turn (3.2.14) directly implies Theorem 3.2.1, since by assumption, $-\sum a^{ij} u_{x^i x^j} \leq -f$, and the left-hand side of this inequality is nonnegative on $T^+(u)$ by Lemma 3.2.1. □

We wish to apply Theorem 3.2.1 to some nonlinear equation, namely, the two-dimensional Monge–Ampère equation.

Thus, let Ω be open in $\mathbb{R}^2 = \{(x^1, x^2)\}$, and let $u \in C^2(\Omega)$ satisfy

$$u_{x^1x^1}(x)u_{x^2x^2}(x) - u^2_{x^1x^2}(x) = f(x) \quad \text{in } \Omega, \tag{3.2.15}$$

with given f. In order that (3.2.15) be elliptic:

(i) The Hessian of u must be positive definite, and hence also
(ii) $f(x) > 0$ in Ω.

Condition (3.2) means that u is a convex function. Thus, u cannot assume a maximum in the interior of Ω, but a minimum is possible. In order to control the minimum, we observe that if u is a solution of (3.2.15), then so is $(-u)$. However, Eq. (3.2.15) is no longer elliptic at $(-u)$, since the Hessian of $(-u)$ is negative and not positive, so that Theorem 3.2.1 cannot be applied directly. We observe, however, that Lemma 3.2.3 does not need an ellipticity assumption and obtain the following corollary:

Corollary 3.2.1. *Under the assumptions (3.2) and (3.2), a solution u of the Monge–Ampère equation (3.2.15) satisfies*

$$\inf_{\Omega} u \geq \min_{\partial\Omega} u - \frac{\operatorname{diam}(\Omega)}{\sqrt{\pi}} \left(\int_{\Omega} f(x) \mathrm{d}x \right)^{\frac{1}{2}}.$$

The crucial point here is that the nonlinear Monge–Ampère equation for a solution u can be formally written as a linear differential equation. Namely, with

$$a^{11}(x) = \frac{1}{2}u_{x^2x^2}(x), \qquad a^{12}(x) = a^{21}(x) = \frac{1}{2}u_{x^1x^2}(x),$$
$$a^{22}(x) = \frac{1}{2}u_{x^1x^1}(x)$$

(3.2.15) becomes

$$\sum_{i,j=1}^{2} a^{ij} u_{x^ix^j}(x) = f(x),$$

and is thus of the type considered. Consequently, in order to deduce properties of a solution u, we have only to check whether the required conditions for the coefficients $a^{ij}(x)$ hold under our assumptions about u. It may happen, however, that these conditions are satisfied for some, but not for all, solutions u. For example, under the assumptions (i) and (ii), (3.2.15) was no longer elliptic at the solution $(-u)$.

3.3 Maximum Principles for Nonlinear Differential Equations

We now consider a general differential equation of the form

$$F[u] = F(x, u, Du, D^2u) = 0, \tag{3.3.1}$$

with $F : S := \Omega \times \mathbb{R} \times \mathbb{R}^d \times S(d,\mathbb{R}) \to \mathbb{R}$, where $S(d,\mathbb{R})$ is the space of symmetric, real-valued, $d \times d$ matrices. Elements of S are written as (x, z, p, r); here $p = (p_1, \dots, p_d) \in \mathbb{R}^d$, $r = (r_{ij})_{i,j=1,\dots,d} \in S(d,\mathbb{R})$. We assume that F is differentiable with respect to the r_{ij}.

Definition 3.3.1. The differential equation (3.3.1) is called elliptic at $u \in C^2(\Omega)$ if

$$\left(\frac{\partial F}{\partial r_{ij}}\left(x, u(x), Du(x), D^2u(x)\right)\right)_{i,j=1,\dots,d} \quad \text{is positive definite.} \tag{3.3.2}$$

For example, the Monge–Ampère equation (3.2.15) is elliptic in this sense if the conditions (i) and (ii) at the end of Sect. 3.2 hold.

It is not completely clear what the appropriate generalization of the maximum principle from linear to nonlinear equations is, because in the linear case, we always have to make assumptions on the lower-order terms. One interpretation that suggests a possible generalization is to consider the maximum principle as a statement comparing a solution with a constant that under different conditions was a solution of $Lu \le 0$. Because of the linear structure, this immediately led to a comparison theorem for arbitrary solutions u_1, u_2 of $Lu = 0$. For this reason, in the nonlinear case, we also start with a comparison theorem:

Theorem 3.3.1. *Let $u_0, u_1 \in C^2(\Omega) \cap C^0(\bar{\Omega})$, and suppose*

(i) $F \in C^1(S)$.
(ii) F is elliptic at all functions $tu_1 + (1-t)u_0$, $0 \le t \le 1$.
(iii) For each fixed (x, p, r), F is monotonically decreasing in z.

If

$$u_1 \le u_0 \quad \text{on } \partial\Omega$$

and

$$F[u_1] \ge F[u_0] \quad \text{in } \Omega,$$

then either

$$u_1 < u_0 \quad \text{in } \Omega$$

or

$$u_0 \equiv u_1 \quad \text{in } \Omega.$$

Proof. We put

$$
\begin{aligned}
v &:= u_1 - u_0, \\
u_t &:= tu_1 + (1-t)u_0 \quad \text{for } 0 \le t \le 1, \\
a^{ij}(x) &:= \int_0^1 \frac{\partial F}{\partial r_{ij}} \left(x, u_t(x), Du_t(x), D^2u_t(x)\right) \mathrm{d}t, \\
b^i(x) &:= \int_0^1 \frac{\partial F}{\partial p_i} \left(x, u_t(x), Du_t(x), D^2u_t(x)\right) \mathrm{d}t, \\
c(x) &:= \int_0^1 \frac{\partial F}{\partial z} \left(x, u_t(x), Du_t(x), D^2u_t(x)\right) \mathrm{d}t
\end{aligned}
$$

(note that we are integrating a total derivative with respect to t, namely, $\frac{\mathrm{d}}{\mathrm{d}t} F(x, u_t(x), Du_t(x), D^2u_t(x))$, and consequently, we can convert the integral into boundary terms, leading to the correct representation of Lv below; cf. (3.3.3)),

$$
Lv := \sum_{i,j=1}^{d} a^{ij}(x) v_{x^i x^j}(x) + \sum_{i=1}^{d} b^i(x) v_{x^i}(x) + c(x)v(x).
$$

Then

$$
Lv = F[u_1] - F[u_0] \ge 0 \quad \text{in } \Omega. \tag{3.3.3}
$$

The operator L is elliptic because of (ii), and by (iii), $c(x) \le 0$. Thus, we may apply Theorem 3.1.2 for v and obtain the conclusions of the theorem. □

The theorem holds in particular for solutions of $F[u] = 0$. The key point in the proof of Theorem 3.3.1 then is that since the solutions u_0 and u_1 of the *nonlinear* equation $F[u] = 0$ are already given, we may interpret quantities that depend on u_0 and u_1 and their derivatives as coefficients of a linear differential equation for the difference.

We also would like to formulate the following uniqueness result for the Dirichlet problem for $F[u] = f$ with given f:

Corollary 3.3.1. *Under the assumptions of Theorem 3.3.1, suppose $u_0 = u_1$ on $\partial\Omega$ and*

$$
F[u_0] = F[u_1] \quad \text{in } \Omega.
$$

Then $u_0 = u_1$ in Ω.

As an example, we consider the *minimal surface equation*: Let $\Omega \subset \mathbb{R}^2 = \{(x, y)\}$. The minimal surface equation then is the quasilinear equation

$$
\left(1 + u_y^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1 + u_x^2\right) u_{yy} = 0. \tag{3.3.4}
$$

Theorem 3.3.1 implies the following corollary:

Corollary 3.3.2. *Let $u_0, u_1 \in C^2(\Omega)$ be solutions of the minimal surface equation. If the difference $u_0 - u_1$ assumes u maximum or minimum at an interior point of Ω, we have*

$$u_0 - u_1 \equiv \text{const} \quad \text{in } \Omega.$$

We now come to the following maximum principle:

Theorem 3.3.2. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and let $F \in C^2(S)$. Suppose that for some $\lambda > 0$, the ellipticity condition*

$$\lambda\, |\xi|^2 \leq \sum_{i,j=1}^{d} \frac{\partial F}{\partial r_{ij}}(x, z, p, r)\xi^i \xi^j \tag{3.3.5}$$

holds for all $\xi \in \mathbb{R}^d$, $(x, z, p, r) \in S$. Moreover, assume that there exist constants μ_1, μ_2 such that for all (x, z, p),

$$\frac{F(x, z, p, 0)\, \text{sign}(z)}{\lambda} \leq \mu_1\, |p| + \frac{\mu_2}{\lambda}. \tag{3.3.6}$$

If

$$F[u] = 0 \quad \text{in } \Omega,$$

then

$$\sup_{\Omega} |u| \leq \max_{\partial\Omega} |u| + c\frac{\mu_2}{\lambda}, \tag{3.3.7}$$

where the constant c depends on μ_1 and the diameter diam(Ω).

Here, one should think of (3.3.6) as an analogue of the sign condition $c(x) \leq 0$ and the bound for the $b^i(x)$ as well as a bound of the right-hand side f of the equation $Lu = f$.

Proof. We shall follow a similar strategy as in the proof of Theorem 3.3.1 and shall reduce the result to the maximum principle from Sect. 3.1 for linear equations. Here v is an auxiliary function to be determined, and $w := u - v$. We consider the operator

$$Lw := \sum_{i,j=1}^{d} a^{ij}(x) w_{x^i x^j} + \sum_{i=1}^{d} b^i(x) w_{x^i}$$

with

$$a^{ij}(x) := \int_0^1 \frac{\partial F}{\partial r_{ij}}\left(x, u(x), Du(x), tD^2u(x)\right) \mathrm{d}t, \tag{3.3.8}$$

while the coefficients $b^i(x)$ are defined through the following equation:

$$\sum_{i=1}^{d} b^i(x) w_{x^i} = \sum_{i,j=1}^{d} \int_0^1 \Big(\frac{\partial F}{\partial r_{ij}} \left(x, u(x), Du(x), tD^2u(x)\right) - \frac{\partial F}{\partial r_{ij}} \left(x, u(x), Dv(x), tD^2u(x)\right) \Big) \mathrm{d}t \cdot v_{x^i x^j} + F\left(x, u(x), Du(x), 0\right) - F\left(x, u(x), Dv(x), 0\right). \tag{3.3.9}$$

(That this is indeed possible follows from the mean value theorem and the assumption $F \in C^2$. It actually suffices to assume that F is twice continuously differentiable with respect to the variables r only.) Then L satisfies the assumptions of Theorem 3.1.1. Now

$$\begin{aligned} Lw &= L(u-v) \\ &= \sum_{i,j=1}^{d} \left(\int_0^1 \frac{\partial F}{\partial r_{ij}} \left(x, u(x), Du(x), tD^2u(x)\right) \mathrm{d}t \right) u_{x^i x^j} + F(x, u(x), Du(x), 0) \\ &\quad - \sum_{i,j=1}^{d} \left(\int_0^1 \frac{\partial F}{\partial r_{ij}} \left(x, u(x), Dv(x), tD^2u(x)\right) \mathrm{d}t \right) v_{x^i x^j} - F(x, u(x), Dv(x), 0) \\ &= F\left(x, u(x), Du(x), D^2u(x)\right) \\ &\quad - \left(\sum_{i,j=1}^{d} \alpha^{ij}(x) v_{x^i x^j} + F\left(x, u(x), Dv(x), 0\right) \right), \end{aligned} \tag{3.3.10}$$

with

$$\alpha^{ij}(x) = \int_0^1 \frac{\partial F}{\partial r_{ij}} \left(x, u(x), Dv(x), tD^2u(x)\right) \mathrm{d}t \tag{3.3.11}$$

(this again comes from the integral of a total derivative with respect to t). Here by assumption

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{d} \alpha^{ij}(x) \xi^i \xi^j \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d. \tag{3.3.12}$$

We now look for an appropriate auxiliary function v with

$$Mv := \sum \alpha^{ij}(x) v_{x^i x^j} + F(x, u(x), Dv(x), 0) \le 0. \tag{3.3.13}$$

We now suppose that for $\delta := \operatorname{diam}(\Omega)$, Ω is contained in the strip $\{0 < x^1 < \delta\}$. We now try

$$v(x) = \max_{\partial\Omega} u^+ + \frac{\mu_2}{\lambda}\left(e^{(\mu_1+1)\delta} - e^{(\mu_1+1)x^1}\right) \tag{3.3.14}$$

$(u^+(x) = \max(0, u(x)))$.

Then

$$\begin{aligned} Mv =& -\frac{\mu_2}{\lambda}(\mu_1+1)^2\alpha^{11}(x)e^{(\mu_1+1)x^1} + F(x, u(x), Dv(x), 0) \\ \le& -\mu_2(\mu_1+1)^2 e^{(\mu_1+1)x^1} + \mu_2\mu_1(\mu_1+1)e^{(\mu_1+1)x^1} + \mu_2 \\ \le& \, 0 \end{aligned}$$

by (3.3.6) and (3.3.12). This establishes (3.3.13). Equation (3.3.10) then implies, even under the assumption $F[u] \ge 0$ in place of $F[u] = 0$,

$$Lw \ge 0.$$

By definition of v, we also have

$$w = u - v \le 0 \quad \text{on } \partial\Omega.$$

Theorem 3.1.1 thus implies

$$u \le v \quad \text{in } \Omega$$

and (3.3.7) follows with $c = e^{(\mu_1+1)\operatorname{diam}(\Omega)} - 1$. More precisely, under the assumption $F[u] \ge 0$, we have proved the inequality

$$\sup_{\Omega} u \le \max_{\partial\Omega} u^+ + c\frac{\mu_2}{\lambda}, \tag{3.3.15}$$

but the inequality in the other direction of course follows analogously, i.e.,

$$\inf_{\Omega} u \ge \min_{\partial\Omega} u^- - c\frac{\mu_2}{\lambda} \tag{3.3.16}$$

$(u^-(x) := \min(0, u(x)))$. □

Theorem 3.3.2 is of interest even in the linear case. Let us look once more at the simple equation

$$\begin{aligned} f''(x) + \kappa f(x) &= 0 \quad \text{for } x \in (0, \pi), \\ f(0) = f(\pi) &= 0, \end{aligned}$$

with constant κ. We may apply Theorem 3.3.2 with $\lambda = 1$, $\mu_1 = 0$,

$$\mu_2 = \begin{cases} \kappa \sup_{(0,\pi)} |f| & \text{for } \kappa > 0, \\ 0 & \text{for } \kappa \leq 0. \end{cases}$$

It follows that

$$\sup_{(0,\pi)} |f| \leq c\kappa \sup_{(0,\pi)} |f|;$$

i.e., if

$$\kappa < \frac{1}{c},$$

we must have $f \equiv 0$. More generally, in place of κ, one may take any function $c(x)$ with $c(x) \leq \kappa$ on $(0, \pi)$ and consider $f''(x) + c(x)f(x) = 0$, without affecting the preceding conclusion. In particular, this allows us to weaken the sign condition $c(x) \leq 0$. The sharpest possible result here is that $f \equiv 0$ if κ is smaller than the smallest eigenvalue λ_1 of $\frac{d^2}{dx^2}$ on $(0, \pi)$, i.e., 1. This analogously generalizes to other linear elliptic equations, for example,

$$\Delta f(x) + \kappa f(x) = 0 \quad \text{in } \Omega,$$
$$f(y) = 0 \quad \text{on } \partial\Omega.$$

Theorem 3.3.2 does imply such a result, but not with the optimal bound λ_1.

A reference for this chapter is Gilbarg–Trudinger [12].

Summary and Perspectives

The maximum principle yields examples of so-called a priori estimates, i.e., estimates that hold for any solution of a given differential equation or class of equations, depending on the given data (boundary values, right-hand side, etc.), without the need to know the solution in advance or without even having to guarantee in advance that a solution exists. Conversely, such a priori estimates often constitute an important tool in many existence proofs. Maximum principles are characteristic for solutions of elliptic (and parabolic) PDEs, and they are not restricted to linear equations. Often, they are even the most important tool for studying certain nonlinear elliptic PDEs.

Exercises

3.1. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be disjoint open sets such that $\bar{\Omega}_1 \cap \bar{\Omega}_2$ contains a smooth hypersurface T, for example,

$$\Omega_1 := \{(x^1, \dots, x^d) : |x| < 1, x^1 > 0\},$$
$$\Omega_2 := \{(x^1, \dots, x^d) : |x| < 1, x^1 < 0\},$$
$$T \;= \{(x^1, \dots, x^d) : |x| < 1, x^1 = 0\}.$$

Let $u \in C^0(\bar{\Omega}_1 \cup \bar{\Omega}_2) \cap C^2(\Omega_1) \cap C^2(\Omega_2)$ be harmonic on Ω_1 and on Ω_2, i.e.,

$$\Delta u(x) = 0, \quad x \in \Omega_1 \cup \Omega_2.$$

Does this imply that u is harmonic on $\Omega_1 \cup \Omega_2 \cup T$?

3.2. Let Ω be open in $\mathbb{R}^2 = \{(x, y)\}$. For a nonconstant solution $u \in C^2(\Omega)$ of the differential equation

$$u_{xy} = 0 \quad \text{in } \Omega,$$

is it possible to assume an interior maximum in Ω?

3.3. Let Ω be open and bounded in $\mathbb{R}^d$. On

$$\Omega \times [0, \infty) \subset \mathbb{R}^{d+1} = \{(x^1, \dots, x^d, t)\},$$

we consider the heat equation

$$u_t = \Delta u, \quad \text{where } \Delta = \sum_{i=1}^{d} \frac{\partial^2}{(\partial x^i)^2}.$$

Show that for bounded solutions $u \in C^2(\Omega \times (0, \infty)) \cap C^0(\bar{\Omega} \times [0, \infty))$,

$$\sup_{\Omega \times [0,\infty)} u \le \sup_{(\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times [0,\infty))} u.$$

3.4. Let $u : \Omega \to \mathbb{R}$ be harmonic, $\Omega' \subset\subset \Omega \subset \mathbb{R}^d$. We then have, for all i, j between 1 and d,

$$\sup_{\Omega'} |u_{x^i x^j}| \le \left(\frac{2d}{\operatorname{dist}(\Omega', \partial\Omega)} \right)^2 \sup_{\Omega} |u|.$$

Prove this inequality. Write down and demonstrate an analogous inequality for derivatives of arbitrary order!

3.5. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\Delta u = u^3, \quad x \in \Omega,$$
$$u \equiv 0, \quad x \in \partial\Omega.$$

Show that $u \equiv 0$ in Ω.

3.6. Prove a version of the maximum principle of Alexandrov and Bakelman for operators

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x)u_{x^i x^j}(x),$$

assuming in place of ellipticity only that $\det(a^{ij}(x))$ is positive in Ω.

3.7. Control the maximum and minimum of the solution u of an elliptic Monge–Ampère equation

$$\det(u_{x^i x^j}(x)) = f(x)$$

in a bounded domain Ω.

3.8. Let $u \in C^2(\Omega)$ be a solution of the Monge–Ampère equation

$$\det(u_{x^i x^j}(x)) = f(x)$$

in the domain Ω with positive f. Suppose there exists $x_0 \in \Omega$ where the Hessian of u is positive definite. Show that the equation then is elliptic at u in all of Ω.

3.9. Let $\mathbb{R}^2 := \{(x^1, x^2)\}$, $\Omega := \mathring{B}(0, R_2) \setminus B(0, R_1)$ with $R_2 > R_1 > 0$. The function $\phi(x^1, x^2) := a + b\log(|x|)$ is harmonic in Ω for all a, b. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be subharmonic, i.e.,

$$\Delta u \geq 0, \quad x \in \Omega.$$

Show that

$$M(r) \leq \frac{M(R_1)\log(\frac{R_2}{r}) + M(R_2)\log(\frac{r}{R_1})}{\log(\frac{R_2}{R_1})}$$

with

$$M(r) := \max_{\partial B(0,r)} u(x)$$

and $R_1 \leq r \leq R_2$.

3.10. Let

$$u_1 := \frac{1}{2} + \frac{1}{2}(x^2 + y^2),$$

$$u_2 := \frac{3}{2} - \frac{1}{2}(x^2 + y^2).$$

Show that u_1 and u_2 solve the Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 1$$

and

$$u_1 = u_2 = 1 \quad \text{on } \partial B(0,1).$$

Is this compatible with the uniqueness result for the Dirichlet problem for nonlinear elliptic PDEs?

3.11. Let $\Omega_T := \Omega \times (0,T)$, and suppose $u \in C^2(\Omega_T) \cap C^0(\bar{\Omega}_T)$ satisfies

$$\begin{aligned} u_t &= \Delta u + u^2 && \text{in } \Omega_T, \\ u(x,t) &> c > 0 && \text{for } (x,t) \in (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T)). \end{aligned}$$

Show that

(a) $u > c$ for all $(x,t) \in \bar{\Omega}_T$.
(b) If in addition $u(x,t) = u(x,0)$ for all $x \in \partial\Omega$ and all t, then $T < \infty$.