Chapter 14
The Moser Iteration Method and the Regularity Theorem of de Giorgi and Nash

14.1 The Moser–Harnack Inequality

In this chapter, as in Chap. 11, we shall consider elliptic differential operators of divergence type. In order to concentrate on the essential aspects and not to burden the proofs with too many technical details, in this chapter we shall omit all lower-order terms and consider only solutions of the homogeneous equation. Thus, we shall investigate (weak) solutions of

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right) = 0,$$

where the coefficients a^{ij} are (measurable and) bounded and satisfy an ellipticity condition. We thus assume that there exist constants $0 < \lambda \le \Lambda < \infty$ with

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \le \Lambda\,|\xi|^2 \tag{14.1.1}$$

for all x in the domain of definition Ω of u and all $\xi \in \mathbb{R}^d$.[1]

[1] As an alternative sometimes adopted in the literature, one could define the constant Λ by the inequality

$$\sup_{i,j,x}\left|a^{ij}(x)\right| \le \Lambda$$

for all $x \in \Omega$. Of course, these two possible definitions of Λ are not equivalent, but the relevant difference is only that, in the estimates below, Λ would have to be replaced by $d\Lambda$ if the alternative definition were adopted. In fact, in subsequent sections, we shall also switch to that alternative convention.

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Definition 14.1.1. A function $u \in W^{1,2}(\Omega)$ is called a weak subsolution of L, and we write this as $Lu \geq 0$, if for all $\varphi \in H_0^{1,2}(\Omega)$, $\varphi \geq 0$ in Ω,

$$\int_\Omega \sum_{i,j} a^{ij}(x) D_i u D_j \varphi \mathrm{d}x \leq 0. \tag{14.1.2}$$

Similarly, it is called a weak supersolution ($Lu \leq 0$), if we have $\geq$ in (13.1.2).

Inequalities like $\varphi \geq 0$ are assumed to hold pointwise almost everywhere, here and in the sequel. Likewise, sup and inf will denote the essential supremum and infimum, respectively. Finally, as always, $⨍$ will denote the average mean integral:

$$⨍_\Omega \varphi \mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega \varphi \mathrm{d}x.$$

In order to familiarize ourselves with the notions of sub- and supersolutions, we shall demonstrate the following useful lemma.

Lemma 14.1.1. *(i) Let u be a subsolution, i.e. $u \in C^2(\Omega)$, $Lu \geq 0$, and let $f \in C^2(\mathbb{R})$ be convex with $f' \geq 0$. Then $f \circ u$ is a subsolution as well.*

(ii) Let u be a supersolution, $f \in C^2(\mathbb{R})$ concave with $f' \geq 0$. Then $f \circ u$ is a supersolution as well.

(iii) Let u be a solution, and $f \in C^2(\mathbb{R})$ convex. Then $f \circ u$ is a subsolution.

Proof.

$$L(f \circ u) = \sum_{i,j} \frac{\partial}{\partial x^j} \left(a^{ij} f'(u) \frac{\partial u}{\partial x^i} \right) = f''(u) \sum_{i,j} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + f'(u) Lu, \tag{14.1.3}$$

which implies all the inequalities claimed. □

We now wish to verify that the assertions of Lemma 14.1.1 continue to hold for *weak* (sub-, super-)solutions. We assume that $f'(u)$ and $f''(u)$ satisfy approximate integrability conditions to make the chain rules for weak derivatives

$$D_i(f \circ u) = f'(u) D_i(u)$$

and

$$D_i(f' \circ u) = f''(u) D_i u \quad \text{for } i = 1, \dots, d$$

valid. (By Lemma 10.2.3 this holds if, for example,

$$\sup_{y \in \mathbb{R}} |f'(y)| + \sup_{y \in \mathbb{R}} |f''(y)| < \infty.)$$

We obtain

$$\int_\Omega \sum_{i,j} a^{ij} D_i(f \circ u) D_j \varphi = \int \sum_{i,j} a^{ij} f'(u) D_i u D_j \varphi$$
$$= \int \sum_{i,j} a^{ij} D_i u D_j (f'(u)\varphi)$$
$$- \int \sum_{i,j} a^{ij} D_i u f''(u) D_j u \varphi.$$

The last integral is nonnegative because of the ellipticity condition, if f is convex, i.e., $f''(u) \geq 0$, and $\varphi \geq 0$, and consequently yields a nonpositive contribution because of the minus sign in front of it, if u is a weak subsolution and $f'(u) \geq 0$. Therefore, under those assumptions,

$$\int_\Omega \sum_{i,j} a^{ij} D_i(f \circ u) D_j \varphi \leq 0,$$

and $f \circ u$ is a weak subsolution.

In the same manner, one treats the weak versions of the other assertions of Lemma 14.1.1 to obtain the following result:

Lemma 14.1.2. *Under the corresponding assumptions, the assertions of Lemma 14.1.1 hold for weak (sub-,super-)solutions, provided that the chain rule for weak derivatives is satisfied for $f \in C^2(\mathbb{R})$.*

From Lemma 14.1.2 we derive the following result:

Lemma 14.1.3. *Let $u \in W^{1,2}(\Omega)$ be a weak subsolution of L, and $k \in \mathbb{R}$. Then*

$$v(x) := \max(u(x), k)$$

is a weak subsolution as well.

Proof. We consider the function

$$f : \mathbb{R} \to \mathbb{R},$$
$$f(y) := \max(y, k).$$

Then

$$v = f \circ u.$$

We approximate f by a sequence $(f_n)_{n \in N}$ of convex functions of class C^2 with

$$f_n(y) = f(y) \quad \text{for } y \notin \left(k - \frac{1}{n}, k + \frac{1}{n}\right)$$

and

$$|f_n'(y)| \leq 1 \qquad \text{for all } y.$$

Then, as in the proofs of Lemmas 10.2.2 and 10.2.3, by an approximation argument, $f_n \circ u$ converges to $v = f \circ u$ in $W^{1,2}$. Therefore,

$$\int_\Omega \sum_{i,j} a^{ij} D_i v D_j \varphi = \lim_{n\to\infty} \int_\Omega \sum_{i,j} a^{ij} D_i (f_n \circ u) D_j \varphi$$
$$\leq 0 \quad \text{for } \varphi \in H_0^{1,2}(\Omega), \varphi \geq 0$$

by Lemma 14.1.2. □

Remark. Of course, we also have a result analogous to Lemma 14.1.3 for weak supersolutions. For $k \in \mathbb{R}$, if $u \in W^{1,2}(\Omega)$ is a weak supersolution, then so is

$$\min(u(x), k).$$

We now come to the fundamental estimates of J. Moser:

Theorem 14.1.1. *Let u be a subsolution in the ball $B(x_0, 4R) \subset \mathbb{R}^d$ ($R > 0$), and assume $p > 1$. Then*

$$\sup_{B(x_0,R)} u \leq c_1 \left(\frac{p}{p-1}\right)^{\frac{2}{p}} \left(\fint_{B(x_0,2R)} (\max(u(x),0))^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{14.1.4}$$

with a constant c_1 depending only on d and $\frac{\Lambda}{\lambda}$.

Remark. If u is positive, then obviously $\max(u, 0) = u$ in (14.1.4), and this case will constitute our main application of this result.

Theorem 14.1.2. *Let u be a* positive *supersolution in $B(x_0, 4R) \subset \mathbb{R}^d$. For $0 < p < \frac{d}{d-2}$, and if $d \geq 3$, then*

$$\left(\fint_{B(x_0,2R)} u^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{c_2}{\left(\frac{d}{d-2} - p\right)^2} \inf_{B(x_0,R)} u, \tag{14.1.5}$$

with c_2 again depending on d and $\frac{\Lambda}{\lambda}$ only. If $d = 2$, this estimate holds for any $0 < p < \infty$, with a constant c_2 depending on p and $\frac{\Lambda}{\lambda}$ in place of $c_2/\left(\frac{d}{d-2} - p\right)^2$.

Remark. In order to see the necessity of the condition $p < \frac{d}{d-2}$, we let L be the Laplace operator Δ and

$$u(x) = \min\left(|x|^{2-d}, k\right) \quad \text{for some } k > 0.$$

According to the remark after Lemma 14.1.3, because $|x|^{2-d}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$, this is a weak supersolution on $\mathbb{R}^d$. If we then let k increase, we see that the $L^{\frac{d}{d-2}}$-norm can no longer be controlled by the infimum.

From Theorems 14.1.1 and 14.1.2, we derive Harnack-type inequalities for solutions of $Lu = 0$. These two theorems directly yield the following corollary:

Corollary 14.1.1. *Let u be a positive (weak) solution of $Lu = 0$ in the ball $B(x_0, 4R) \subset \mathbb{R}^d$ ($R > 0$). Then*

$$\sup_{B(x_0,R)} u \le c_3 \inf_{B(x_0,R)} u, \tag{14.1.6}$$

with c_3 depending on d and $\frac{\Lambda}{\lambda}$ only.

For general domains, we have the following result:

Corollary 14.1.2. *Let u be a positive (weak) solution of $Lu = 0$ in a domain Ω of $\mathbb{R}^d$, and let $\Omega_0 \subset\subset \Omega$. Then*

$$\sup_{\Omega_0} u \le c \inf_{\Omega_0} u, \tag{14.1.7}$$

with c depending on d, Ω, Ω_0, and $\frac{\Lambda}{\lambda}$.

Proof. This Harnack inequality on Ω_0 follows by the standard ball chain argument: Since $\bar{\Omega}_0$ is compact, it can be covered by finitely many balls $B_i := B(x_i, R)$ with $B(x_i, R) \subset \Omega$ (we choose, e.g., $R < \frac{1}{4}\operatorname{dist}(\partial\Omega, \Omega_0)$), $i = 1, \ldots, N$. Now let $y_1, y_2 \in \Omega_0$; without loss of generality $y_1 \in B_k$, $y_2 \in B_{k+m}$ for some $m \ge 1$, and the balls are enumerated in such manner that $B_j \cap B_{j+1} \ne \emptyset$ for $j = k, \ldots, k+m-1$. By applying Corollary 14.1.1 to the balls $B_k, B_{k+1}, \ldots$, we obtain

$$\begin{aligned} u(y_1) &\le \sup_{B_k} u(x) \le c_3 \inf_{B_k} u(x) \\ &\le c_3 \sup_{B_{k+1}} u(x) \quad (\text{since } B_k \cap B_{k+1} \ne \emptyset) \\ &\le c_3^2 \inf_{B_{k+1}} u(x) \le \ldots \\ &\le c_3^{m+1} \inf_{B_{k+m}} u(x) \le c_3^{m+1} u(y_2). \end{aligned}$$

Since y_1 and y_2 are arbitrary, and $m \le N$, it follows that

$$\sup_{\Omega_0} u(x) \le c_3^{N+1} \inf_{\Omega_0} u(x). \tag{14.1.8}$$

□

We now start with the preparations for the proofs of Theorems 14.1.1 and 14.1.2. For positive u and a point x_0, we put

$$\phi(p,R) := \left(\fint_{B(x_0,R)} u^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Lemma 14.1.4.

$$\lim_{p\to\infty} \phi(p,R) = \sup_{B(x_0,R)} u =: \phi(\infty,R), \tag{14.1.9}$$

$$\lim_{p\to-\infty} \phi(p,R) = \inf_{B(x_0,R)} u =: \phi(-\infty,R). \tag{14.1.10}$$

Proof. By Hölder's inequality, $\phi(p,R)$ is monotonically increasing with respect to p. Namely, for $p < p'$ and $u \in L^{p'}(\Omega)$,

$$\left(\frac{1}{|\Omega|}\int_\Omega u^p\right)^{\frac{1}{p}} \leq \frac{1}{|\Omega|^{\frac{1}{p}}} \left(\int_\Omega 1\right)^{\frac{p'-p}{pp'}} \left(\int_\Omega (u^p)^{\frac{p'}{p}}\right)^{\frac{1}{p'}} = \left(\frac{1}{|\Omega|}\int_\Omega u^{p'}\right)^{\frac{1}{p'}}.$$

Moreover,

$$\phi(p,R) \leq \left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} (\sup u)^p\right)^{\frac{1}{p}} = \phi(\infty,R). \tag{14.1.11}$$

On the other hand, by the definition of the essential supremum, for any $\varepsilon > 0$, there exists some $\delta > 0$ with

$$\left|\left\{x \in B(x_0,R) : u(x) \geq \sup_{B(x_0,R)} u - \varepsilon\right\}\right| > \delta.$$

Therefore,

$$\phi(p,R) \geq \left(\frac{1}{|B(x_0,R)|}\int_{\substack{u(x)\geq \sup u-\varepsilon \\ x\in B(x_0,R)}} u^p\right)^{\frac{1}{p}} \geq \left(\frac{\delta}{|B(x_0,R)|}\right)^{\frac{1}{p}} (\sup u - \varepsilon),$$

and hence

$$\lim_{p\to\infty} \phi(p,R) \geq \sup u - \varepsilon$$

for any $\varepsilon > 0$, and thus also

$$\lim_{p\to\infty} \phi(p,R) \geq \sup u. \tag{14.1.12}$$

Inequalities (14.1.11) and (14.1.12) imply (14.1.9), and (14.1.10) is derived similarly (or, alternatively, by applying the preceding argument to $\frac{1}{u}$). □

Lemma 14.1.5. *(i) Let u be a positive subsolution in Ω, and for $q > \frac{1}{2}$, assume*

$$v := u^q \in L^2(\Omega).$$

For any $\eta \in H_0^{1,2}(\Omega)$, we then have

$$\int_\Omega \eta^2 |Dv|^2 \le \frac{\Lambda^2}{\lambda^2}\left(\frac{2q}{2q-1}\right)^2 \int_\Omega |D\eta|^2 v^2. \tag{14.1.13}$$

(ii) If u is a supersolution instead, this inequality holds for $q < \frac{1}{2}$.

Proof. The claim is trivial for $q = 0$. We put

$$f(u) = u^{2q} \qquad \text{for } q > 0,$$
$$f(u) = -u^{2q} \qquad \text{for } q < 0.$$

By Lemma 14.1.2, $f(u)$ then is a subsolution in case (i), and a supersolution in case (ii). The subsequent calculations are based on that fact. (In the course of the proof there will also arise integrability conditions implying the needed chain rules. For that purpose, the proof of Lemma 10.2.3 requires a slight generalization, utilizing varying Sobolev exponents, the Hölder inequality, and the Sobolev embedding theorem. We leave this as an exercise for the reader.) As a test function in (14.1.2) (or in the corresponding inequality in case (ii), we then use

$$\varphi = f'(u) \cdot \eta^2. \tag{14.1.14}$$

Then

$$\begin{aligned}\int_\Omega \sum_{ij} a^{ij}(x) D_i u D_j \varphi &= \int_\Omega \sum_{i,j} a^{ij} D_i u D_j u f''(u)\eta^2 + \int_\Omega \sum_{i,j} a^{ij} D_i u f'(u) 2\eta D_j \eta \\ &= \int_\Omega 2|q|(2q-1) \sum_{i,j} a^{ij} D_i u D_j u\, u^{2q-2}\eta^2 + \int_\Omega 4|q| \sum_{i,j} a^{ij} D_i u\, u^{2q-1}\eta D_j \eta.\end{aligned} \tag{14.1.15}$$

In case (i), this is ≤ 0. Applying Young's inequality to the last term, for all $\varepsilon > 0$, we obtain

$$2|q|(2q-1)\lambda \int |Du|^2 u^{2q-2}\eta^2 \le 2|q|\Lambda\varepsilon \int |Du|^2 u^{2q-2}\eta^2 + \frac{2|q|\Lambda}{\varepsilon}\int u^{2q} |D\eta|^2.$$

With

$$\varepsilon = \frac{2q-1}{2}\frac{\lambda}{\Lambda},$$

we thus obtain

$$\int |Du|^2 u^{2q-2}\eta^2 \leq \frac{4}{(2q-1)^2}\frac{\Lambda^2}{\lambda^2}\int u^{2q}|D\eta|^2,$$

i.e.,

$$\int |Dv|^2 \eta^2 \leq \frac{\Lambda^2}{\lambda^2}\left(\frac{2q}{2q-1}\right)^2 \int v^2 |D\eta|^2.$$

In case (ii), (14.1.15) is nonnegative, and since in that case also $2q-1 \leq 0$, one can proceed analogously and put

$$\varepsilon = \frac{1-2q}{2}\frac{\lambda}{\Lambda},$$

to obtain (14.1.13) in that case as well. □

We now begin the proofs of Theorems 14.1.1 and 14.1.2. Since the stated inequalities are invariant under scaling, we may assume, without loss of generality, that

$$R = 1 \quad \text{and} \quad x_0 = 0.$$

We shall employ the abbreviation

$$B_r := B(0, r).$$

Let

$$0 < r' < r \leq 2r', \tag{14.1.16}$$

and let $\eta \in H_0^{1,2}(B_r)$ be a cutoff function satisfying

$$\begin{aligned} \eta &\equiv 1 \quad \text{on } B_{r'}, \\ \eta &\equiv 0 \quad \text{on } \mathbb{R}^d \setminus B_r, \\ |D\eta| &\leq \frac{2}{r-r'}. \end{aligned} \tag{14.1.17}$$

For the proof of Theorem 14.1.1, we may assume without loss of generality that u is positive, since otherwise, by Lemma 14.1.3, we may consider the positive subsolutions

$$v_k(x) = \max(u(x), k)$$

for $k > 0$ (or the approximating subsolutions from the proof of that lemma), perform the subsequent reasoning for positive subsolutions, apply the result to the v_k, and finally let k tend to 0.

We consider once more

$$v = u^q$$

and assume that $v \in L^2(\Omega)$. By the Sobolev embedding theorem (Corollary 11.1.3), for $d \geq 3$, we obtain

$$\left(\fint_{B_{r'}} v^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \leq c_4 \left(r'^2 \fint_{B_{r'}} |Dv|^2 + \fint_{B_{r'}} v^2 \right). \tag{14.1.18}$$

If $d = 2$ instead of $\frac{2d}{d-2}$, we may take an arbitrarily large exponent p and proceed analogously. We leave the necessary modifications for the case $d = 2$ to the reader and henceforth treat only the case $d \geq 3$. With (14.1.13) and (14.1.17), (14.1.18) yields

$$\left(\fint_{B_{r'}} v^{\frac{2d}{d-2}} \right)^{\frac{d-2}{d}} \leq \bar{c} \fint_{B_r} v^2 \tag{14.1.19}$$

with

$$\bar{c} \leq c_5 \left(\left(\frac{r'}{r - r'} \right)^2 \left(\frac{2q}{2q-1} \right)^2 + 1 \right). \tag{14.1.20}$$

Thus, we get $v \in L^{\frac{2d}{d-2}}(\Omega)$. We shall iterate that step and realize that higher and higher powers of u are integrable.

We put $s = 2q$ and assume

$$|s| \geq \mu > 0,$$

choosing an appropriate value for μ later on. Because of $r \leq 2r'$, then

$$\bar{c} \leq c_6 \left(\frac{r'}{r - r'} \right)^2 \left(\frac{s}{s-1} \right)^2, \tag{14.1.21}$$

with c_6 also depending on μ. Thus, by (14.1.19) and (14.1.21), since $v = u^{\frac{s}{2}}$, we get for $s \geq \mu$,

$$\phi\left(\frac{ds}{d-2}, r' \right) = \left(\fint_{B_{r'}} v^{\frac{2d}{d-2}} \right)^{\frac{d-2}{ds}} \leq c_7 \left(\frac{r'}{r - r'} \right)^{\frac{2}{s}} \left(\frac{s}{s-1} \right)^{\frac{2}{s}} \phi(s, r) \tag{14.1.22}$$

with $c_7 = c_6^{\frac{1}{s}}$. For $s \le -\mu$, analogously,

$$\phi\left(\frac{ds}{d-2}, r'\right) \ge \frac{1}{c_7}\left(\frac{r'}{r-r'}\right)^{-\frac{2}{|s|}} \phi(s,r) \tag{14.1.23}$$

(we may omit the term $\left(\frac{s}{s-1}\right)^{-\frac{2}{|s|}}$ here, since it is greater than or equal to 1).

We now wish to complete the proof of Theorem 14.1.1, and therefore, we return to (14.1.22). The decisive insight obtained so far is that we can control the integral of a higher power of u by that of a lower power of u. We now shall simply iterate this estimate to control even higher integral norms of u and from Lemma 14.1.4 then also the supremum of u. For that purpose, let

$$\begin{aligned} s_n &= \left(\frac{d}{d-2}\right)^n p \quad \text{for } p > 1, \\ r_n &= 1 + 2^{-n}, \\ r_n' &= r_{n+1} > \frac{r_n}{2}. \end{aligned}$$

Then (14.1.22) implies

$$\begin{aligned} \phi\left(s_{n+1}, r_{n+1}\right) &\le c_7 \left(\frac{1+2^{-n-1}}{2^{-n-1}} \cdot \frac{\left(\frac{d}{d-2}\right)^n p}{\left(\frac{d}{d-2}\right)^n p - 1}\right)^{\frac{2}{p\left(\frac{d}{d-2}\right)^n}} \phi(s_n, r_n) \\ &\le c_8^{n\left(\frac{d}{d-2}\right)^{-n}} \phi(s_n, r_n), \end{aligned}$$

and iteratively,

$$\phi(s_{n+1}, r_{n+1}) \le c_8^{\sum_{\nu=1}^{n} \nu\left(\frac{d}{d-2}\right)^{-\nu}} \phi(s_1, r_1) \le c_9 \left(\frac{p}{p-1}\right)^{\frac{2}{p}} \phi(p, 2). \tag{14.1.24}$$

(Since we may assume $u \in L^p(\Omega)$, therefore $\phi(s_n, r_n)$ is finite for all $n \in \mathbb{N}$, and thus any power of u is integrable.) Using Lemma 14.1.4, this yields Theorem 14.1.1.

In order to prove Theorem 14.1.2, we now assume $u > \varepsilon > 0$, in order to ensure that $\phi(\sigma, r)$ is finite for $\sigma < 0$. This does not constitute a serious restriction, because once we have proved Theorem 14.1.2 under that assumption, then for positive u, we may apply the result to $u + \varepsilon$. In the resulting inequality for $u + \varepsilon$, namely

$$\left(⨍_{B(x_0,2R)} (u+\varepsilon)^p\right)^{\frac{1}{p}} \le \frac{c_2}{\left(\frac{d}{d-2} - p\right)^2} \inf_{B(x_0,R)} (u+\varepsilon),$$

we then simply let $\varepsilon \to 0$ to deduce the inequality for u itself.

Carrying out the above iteration analogously for $s \le -\mu$ with $r_n = 2 + 2^{-n}$, we deduce from (14.1.23) that

$$\phi(-\mu, 3) \le c_{10}\phi(-\infty, 2) \le c_{10}\phi(-\infty, 1). \tag{14.1.25}$$

By finitely many iteration steps, we also obtain

$$\phi(p, 2) \le c_{11}\phi(\mu, 3). \tag{14.1.26}$$

(The restriction $p < \frac{d}{d-2}$ in Theorem 14.1.2 arises because according to Lemma 14.1.5, in (14.1.19) we may insert $v = u^q$ only for $q < \frac{1}{2}$. The relation $p = 2q\frac{d}{d-2}$ that is needed to control the L^p-norm of u with (14.1.19), by (14.1.20) also yields the factor $\left(\frac{d}{d-2} - p\right)^{-2}$ in (14.1.5).)

The only missing step is

$$\phi(\mu, 3) \le c_{12}\phi(-\mu, 3). \tag{14.1.27}$$

Inequalities (14.1.25)–(14.1.27) imply Theorem 14.1.2. For the proof of (14.1.27), we shall use the theorem of John–Nirenberg (Theorem 11.1.2). For that purpose, we put

$$v = \log u, \quad \varphi = \frac{1}{u}\eta^2$$

with some cutoff function $\eta \in H_0^{1,2}(B_4)$. Then

$$\int_{B_4} \sum_{i,j} a^{ij} D_i\varphi D_j u = -\int_{B_4} \eta^2 \sum a^{ij} D_i v D_j v + \int_{B_4} 2\eta \sum a^{ij} D_i \eta D_j v.$$

Since u is a supersolution, the left-hand side is nonnegative; hence

$$\begin{aligned} \lambda \int_{B_4} \eta^2 |Dv|^2 &\le \int_{B_4} \eta^2 \sum a^{ij} D_i v D_j v \le 2\int_{B_4} \eta \sum a^{ij} D_i \eta D_j v \\ &\le 2\Lambda \left(\int_{B_4} \eta^2 |Dv|^2\right)^{\frac{1}{2}} \left(\int_{B_4} |D\eta|^2\right)^{\frac{1}{2}} \end{aligned}$$

by the Schwarz inequality, and thus

$$\int_{B_4} \eta^2 |Dv|^2 \le 4\left(\frac{\Lambda}{\lambda}\right)^2 \int_{B_4} |D\eta|^2 . \tag{14.1.28}$$

If now $B(y, R) \subset B_{3+\frac{1}{2}}$ is any ball, we choose η satisfying

$$\begin{aligned} \eta &\equiv 1 \quad \text{on } B(y, R), \\ \eta &\equiv 0 \quad \text{outside of } B(y, 2R) \cap B_4, \\ |D\eta| &\le \frac{8}{R}. \end{aligned}$$

With such an η, we obtain from (14.1.28)

$$\fint_{B(y,R)} |Dv|^2 \leq \gamma \frac{1}{R^2} \quad \text{with some constant } \gamma.$$

Thus, by Hölder's inequality,

$$\int_{B(y,R)} |Dv| \leq \omega_d \sqrt{\gamma} R^{d-1}.$$

Now let α be as in Theorem 11.1.2. With $\mu = \frac{\alpha}{\omega_d \sqrt{\gamma}}$, applying that theorem to

$$w = \frac{1}{\omega_d \sqrt{\gamma}} v = \frac{1}{\omega_d \sqrt{\gamma}} \log u,$$

we obtain

$$\int_{B_3} u^{\mu} \int_{B_3} u^{-\mu} \leq \beta^2,$$

and hence

$$\phi(\mu, 3) \leq \beta^{\frac{2}{\mu}} \phi(-\mu, 3),$$

and hence (14.1.27), thus completing the proof.

In order to see what the Harnack inequality for supersolutions can tell us about subsolutions, we now state

Corollary 14.1.3. *Let v be a bounded weak subsolution on $B(x_0, 4R)$. There exists a constant $0 < \delta_0 < 1$, independent of v and R, with*

$$\sup_{B(x_0,R)} v \leq (1 - \delta_0) \sup_{B(x_0,4R)} v + \delta_0 \, v_{B(x_0,R)}. \tag{14.1.29}$$

Proof. We abbreviate

$$v_{+,R} := \sup_{B(x_0,R)} v$$

and have

$$\begin{aligned} v_{+,4R} - v_R &= \fint_{B(x_0,R)} (v_{+,4R} - v) \\ &\leq 2^d \fint_{B(x_0,2R)} |v_{+,4R} - v| \\ &\leq c(v_{+,4R} - v_{+,R}) \end{aligned}$$

by Theorem 14.1.2 (for $p = 1$), since $v_{+,4R} - v$ is a nonnegative supersolution on $B(x_0, 4R)$. Consequently,

$$v_{+,R} \le \frac{c-1}{c} v_{+,4R} + \frac{1}{c} v_{B(x_0,R)}. \qquad \square$$

This corollary tells us that unless v is constant, its supremum on the smaller ball is smaller than the one on the larger ball in a way controlled by the difference between the supremum and the average. Thus, it can be interpreted as a quantitative version of the maximum principle. More generally, a (sub)solution defined on some ball is more tightly controlled on a smaller ball. Such explicit quantitative controls are very important in the regularity theory for solutions of elliptic equations, as we shall see in subsequent sections.

A reference for this section is Moser [27].

Krylov and Safonov have shown that solutions of elliptic equations that are not of divergence type satisfy Harnack inequalities as well. In order to describe their results in the simplest case, we again omit all lower-order terms and consider solutions of

$$Mu := \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} u(x) = 0.$$

Here the coefficients $a^{ij}(x)$ again need only be (measurable and) bounded and satisfy the structural condition (14.1.1), i.e.,

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x) \xi_i \xi_j \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d$$

and

$$\sup_{i,j,x} |a^{ij}(x)| \le \Lambda$$

with constants $0 < \lambda < \Lambda < \infty$.

We then have the following theorem:

Theorem 14.1.3. *Let $u \in W^{2,d}(\Omega)$ be positive and satisfy $Mu \ge 0$ almost everywhere in $B(x_0, 4R) \subset \mathbb{R}^d$. For any $p > 0$, we then have*

$$\sup_{B(x_0,R)} u \le c_1 \left(\fint_{B(x_0,2R)} u^p \, dx \right)^{1/p}$$

with a constant c_1 depending on d, $\frac{\Lambda}{\lambda}$, and p.

Theorem 14.1.4. *Let $u \in W^{2,d}(\Omega)$ be positive and satisfy $Mu \leq 0$ almost everywhere in $B(x_0, 4R) \subset \mathbb{R}^d$. Then there exist $p > 0$ and some constant c_2, depending only on d and $\frac{\Lambda}{\lambda}$, such that*

$$\left(\fint_{B(x_0,R)} u^p \, \mathrm{d}x \right)^{1/p} \leq c_2 \inf_{B(x_0,R)} u.$$

As in the case of divergence-type equations (see Sect. 14.2 below), these results imply Harnack inequalities, maximum principles, and the Hölder continuity of solutions $u \in W^{2,d}(\Omega)$ of

$$Mu = 0 \quad \text{almost everywhere } \Omega \subset \mathbb{R}^d.$$

Proofs of the results of Krylov–Safonov can be found in Gilbarg–Trudinger [12].

14.2 Properties of Solutions of Elliptic Equations

In this section we shall apply the Moser–Harnack inequality in order to deduce the Hölder continuity of weak solutions of $Lu = 0$ under the structural condition (14.1.1). That result had originally been proved by E. de Giorgi and J. Nash independently of each other, and with different methods, before J. Moser found the proof presented here, based on the Harnack inequality.

Lemma 14.2.1. *Let $u \in W^{1,2}(\Omega)$ be a weak subsolution of L, i.e.,*

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} \left(a^{ij}(x) \frac{\partial}{\partial x^i} u(x) \right) \geq 0 \textit{ weakly,}$$

with L satisfying the conditions stated in Sect. 14.1. Then u is bounded from above on any $\Omega_0 \subset\subset \Omega$. Thus, if u is a weak solution of $Lu = 0$, it is bounded from above and below on any such Ω_0.

Proof. By Lemma 14.1.3, for any positive k,

$$v(x) := \max(u(x), k)$$

is a positive subsolution (by the way, in place of v, one might also employ the approximating subsolutions $f_n \circ u$ from the proof of Lemma 14.1.3). The local boundedness of v, hence of u, then follows from Theorem 14.1.1, using a ball chain argument as in the proof of Corollary 14.1.2. □

Theorem 14.2.1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of*

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right) = 0, \tag{14.2.1}$$

assuming that the measurable and bounded coefficients $a^{ij}(x)$ satisfy the structural conditions

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j, \quad \left|a^{ij}(x)\right| \le \Lambda \tag{14.2.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^d$, with constants $0 < \lambda < \Lambda < \infty$. Then u is Hölder continuous in Ω. More precisely, for any $\Omega_0 \subset\subset \Omega$, there exist some $\alpha \in (0,1)$ and a constant c with

$$|u(x) - u(y)| \le c\,|x-y|^\alpha \tag{14.2.3}$$

for all $x, y \in \Omega_0$. α depends on d, $\frac{\Lambda}{\lambda}$, and Ω_0, c in addition on $\sup_{\Omega_0} u - \inf_{\Omega_0} u$.

Proof. Let $x \in \Omega$. For $R > 0$ and $B(x,R) \subset \Omega$, we put

$$M(R) := \sup_{B(x,R)} u, \quad m(R) := \inf_{B(x,R)} u.$$

(By Lemma 14.2.1, $-\infty < m(R) \le M(R) < \infty$.) Then

$$\omega(R) := M(R) - m(R)$$

is the oscillation of u in $B(x,R)$, and we plan to prove the inequality

$$\omega(r) \le c_0\left(\frac{r}{R}\right)^\alpha \omega(R) \quad \text{for } 0 < r \le \frac{R}{4} \tag{14.2.4}$$

for some α to be specified. This will then imply

$$u(x) - u(y) \le \sup_{B(x,r)} u - \inf_{B(x,r)} u = \omega(r) \le c_0\frac{\omega(R)}{R^\alpha}\,|x-y|^\alpha. \tag{14.2.5}$$

for all y with $|x-y| = r$. This, in turn, easily implies the claim.

We now turn to the proof of (14.2.4):

$$M(R) - u \quad \text{and} \quad u - m(R)$$

are positive solutions of $Lu = 0$ in $B(x,R)$.[1] Thus, by Corollary 14.1.1,

[1] More precisely, these are nonnegative solutions, and as in the proof of Theorem 14.1.2, one adds $\varepsilon > 0$ and lets ε approach to 0.

$$M(R) - m\left(\frac{R}{4}\right) = \sup_{B(x,\frac{R}{4})} (M(R) - u) \leq c_1 \inf_{B(x,\frac{R}{4})} (M(R) - u)$$
$$= c_1 \left(M(R) - M\left(\frac{R}{4}\right)\right),$$

and analogously,

$$M\left(\frac{R}{4}\right) - m(R) = \sup_{B(x,\frac{R}{4})} (u - m(R)) \leq c_1 \inf_{B(x,\frac{R}{4})} (u - m(R))$$
$$= c_1 \left(m\left(\frac{R}{4}\right) - m(R)\right).$$

(By Corollary 14.1.1, c_1 does not depend on R.) Adding these two inequalities yields

$$M\left(\frac{R}{4}\right) - m\left(\frac{R}{4}\right) \leq \frac{c_1 - 1}{c_1 + 1}(M(R) - m(R)). \tag{14.2.6}$$

With $\vartheta := \frac{c_1-1}{c_1+1} < 1$, thus

$$\omega\left(\frac{R}{4}\right) \leq \vartheta\omega(R).$$

Iterating this inequality gives

$$\omega\left(\frac{R}{4^n}\right) \leq \vartheta^n\omega(R) \quad \text{for } n \in \mathbb{N}. \tag{14.2.7}$$

Now let

$$\frac{R}{4^{n+1}} \leq r \leq \frac{R}{4^n}. \tag{14.2.8}$$

We now choose $\alpha > 0$ such that

$$\vartheta \leq \left(\frac{1}{4}\right)^\alpha.$$

Then

$$\begin{aligned}
\omega(r) &\leq \omega\left(\frac{R}{4^n}\right) \quad \text{since } \omega \text{ is obviously monotonically increasing} \\
&\leq \vartheta^n\omega(R) \quad \text{by (14.2.7)} \\
&\leq \left(\frac{1}{4^n}\right)^\alpha \omega(R) \\
&\leq 4^\alpha \left(\frac{r}{R}\right)^\alpha \omega(R) \quad \text{by (14.2.8)},
\end{aligned}$$

whence (14.2.4). □

We now want to prove a strong maximum principle:

Theorem 14.2.2. *Let $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ weakly, the coefficients a^{ij} of L again satisfying*

$$\lambda\,|\xi|^2 \leq \sum_{i,j} a^{ij}(x)\xi_i\xi_j, \quad \left|a^{ij}(x)\right| \leq \Lambda$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^d$. If for some ball $B(y_0, R) \subset\subset \Omega$,

$$\sup_{B(y_0,R)} u = \sup_{\Omega} u, \tag{14.2.9}$$

then u is constant.

Proof. If (14.2.9) holds, we may find some ball $B(x_0, R_0)$ with $B(x_0, 4R_0) \subset \Omega$ and

$$\sup_{B(x_0,R_0)} u = \sup_{\Omega} u. \tag{14.2.10}$$

Without loss of generality $\sup_\Omega u < \infty$ because $\sup_{B(y_0,R)} u < \infty$ by Lemma 14.2.1. For

$$M > \sup_{\Omega} u,$$

$M - u$ then is a positive supersolution, and we may apply Theorem 14.1.2 to it. Passing to the limit, the resulting inequalities then continue to hold for

$$M = \sup_{\Omega} u. \tag{14.2.11}$$

Thus, as in the proof of Corollary 14.1.3, we get from Theorem 14.1.2 for $p = 1$

$$\fint_{B(x_0,2R_0)} (M - u) \leq c \inf_{B(x_0,R_0)} (M - u) = 0$$

by (14.2.10) and (14.2.11). Since by choice of M, we also have $u \leq M$; it follows that

$$u \equiv M \tag{14.2.12}$$

in $B(x_0, 2R_0)$.

Now let $y \in \Omega$. We may find a chain of balls $B(x_i, R_i)$, $i = 0, \dots, m$, with $B(x_i, 4R_i) \subset \Omega$, $B(x_{i-1}, R_{i-1}) \cap B(x_i, R_i) \neq 0$ for $i = 1, \dots, m$, $y \in B(x_m, R_m)$. We already know that $u \equiv M$ on $B(x_0, 2R_0)$. Because of $B(x_0, R_0) \cap B(x_1, R_1) \neq 0$, this implies

$$\sup_{B(x_1,R_1)} u = M;$$

hence by our preceding reasoning

$$u \equiv M \quad \text{on } B(x_1, 2R_1).$$

Iteratively, we obtain

$$u \equiv M \quad \text{on } B(x_m, 2R_m),$$

and because of $y \in B(x_m, R_m)$,

$$u(y) = M.$$

Since y was arbitrary, it follows that

$$u \equiv M \quad \text{in } \Omega. \qquad \square$$

As another application of the Harnack inequality, we shall now demonstrate a result of Liouville type:

Theorem 14.2.3. *Any bounded (weak) solution of* $Lu = 0$ *that is defined on all of* $\mathbb{R}^d$*, where* L *has measurable bounded coefficients* $a^{ij}(x)$ *satisfying*

$$\lambda\,|\xi| \le \sum_{i,j} a^{ij}(x)\xi_i\xi_j, \quad \left|a^{ij}(x)\right| \le \Lambda$$

for fixed constants $0 < \lambda \le \Lambda < \infty$ *and all* $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$*, is constant.*

Proof. Since u is bounded, $\inf_{\mathbb{R}^d} u$ and $\sup_{\mathbb{R}^d} u$ are finite. Thus, for any

$$\mu < \inf_{\mathbb{R}^d} u,$$

$u - \mu$ is a positive solution of $Lu = 0$ on $\mathbb{R}^d$. Therefore, by Corollary 14.1.1,

$$0 \le \sup_{B(0,R)} u - \mu \le c_3 \left(\inf_{B(0,R)} u - \mu \right)$$

for any $R > 0$ and any $\mu < \inf_{\mathbb{R}^d} u$, and passing to the limit, then this also holds for

$$\mu = \inf_{\mathbb{R}^d} u.$$

Since c_3 does not depend on R, it follows that

$$0 \le \sup_{\mathbb{R}^d} u - \mu \le c_3 \left(\inf_{\mathbb{R}^d} u - \mu \right) = 0,$$

and hence

$$u \equiv \text{const.} \qquad \square$$

14.3 An Example: Regularity of Bounded Solutions of Semilinear Elliptic Equations

In this section, we shall show how the Harnack inequality naturally applies for the regularity of solutions of nonlinear equations. We take up once more the semilinear equation (12.2.7)

$$\Delta u + \Gamma(u)|Du|^2 = 0 \tag{14.3.1}$$

with a smooth function $\Gamma(u)$, on an open and bounded $\Omega \subset \mathbb{R}^d$. In Sect. 12.2, we have shown that a weak solution $u \in W^{1,p_1}(\Omega)$ for some $p_1 > d$ is smooth. We recall that this condition implies that u is bounded, by the Sobolev embedding Theorem 11.1.1 (see also Morrey's Theorem 11.1.5). In this section, we wish to show that all bounded solutions are smooth, as an application of the Harnack inequality. The crucial point will be to find auxiliary functions constructed from a solution that are subharmonic and to which therefore a Harnack inequality can be applied.

We start with the following computation for a smooth solution u. Let $x_0 \in \Omega$, and $C > 0$, and p some constant.

$$\begin{aligned}
\Delta e^{C(u(x)-p)^2} &= \sum_i \frac{\partial}{\partial x^i}(2C(u-p)u_{x^i}\, e^{C(u(x)-p)^2}) \\
&= 2C(u-p)\Delta u e^{C(u(x)-p)^2} \\
&\quad +2C|Du|^2 e^{C(u(x)-p)^2} \\
&\quad +4C^2(u-p)^2|Du|^2 e^{C(u(x)-p)^2} \\
&= -2C\Gamma(u)(u-p)|Du|^2 e^{C(u(x)-p)^2} \\
&\quad +2C|Du|^2 e^{C(u(x)-p)^2} \\
&\quad +4C^2(u-p)^2|Du|^2 e^{C(u(x)-p)^2}.
\end{aligned}$$

If we now assume

$$\Gamma(u) \le a, \tag{14.3.2}$$

and choose C with

$$a^2 \le C, \tag{14.3.3}$$

then

$$\Delta e^{C(u(x)-p)^2} \ge 0, \tag{14.3.4}$$

i.e., we have constructed a subharmonic function from a solution of (14.3.1).

Since we wish to prove a regularity result, we cannot yet assume that u is a classical solution of (14.3.1). We need to consider weak solutions; $u \in W^{1,2}(\Omega)$ with $\Gamma(u)$ bounded is called a weak solution of (14.3.1) if

$$\int \left(\sum_i D_i u D_i \varphi - \Gamma(u)|Du|^2 \varphi \right) \mathrm{d}x = 0 \text{ for all } \varphi \in H_0^{1,2} \cap L^\infty(\Omega); \quad (14.3.5)$$

here, we need to require that the test function φ be bounded in order to ensure that the integral $\int |Du|^2 \varphi$ be finite. For a weak solution of (14.3.4), (14.3.5) then is also satisfied in the weak sense when the conditions (14.3.2) and (14.3.3) hold (see Sect. 14.1 for weakly subharmonic functions), i.e.,

$$\int_\Omega \sum_i D_i (\mathrm{e}^{C(u(x)-p)^2}) D_i \eta(x) \mathrm{d}x \leq 0 \text{ for all } \eta \in H_0^{1,2}(\Omega), \eta \geq 0; \quad (14.3.6)$$

here, we need to require u to be bounded in order to ensure that, computed by the chain rule, $D_i(\mathrm{e}^{C(u(x)-p)^2})$ is in $L^2(\Omega)$. For the details, so that you can see how a computation in the smooth case is translated into one in the weak case via an integration by parts (the reasoning is the same as in the proof of Lemma 14.1.2): The inequality (14.3.6) then is obtained via

$$\begin{aligned}
\int \sum_i D_i (\mathrm{e}^{C(u-p)^2} D_i \eta &= \int \sum_i 2C(u-p) D_i u \mathrm{e}^{C(u-p)^2} D_i \eta \\
&= \int \sum_i D_i u D_i (2C(u-p)\mathrm{e}^{C(u-p)^2} \eta) \\
&\quad -2 \int \sum_i D_i u D_i ((u-p)\mathrm{e}^{C(u-p)^2}) \eta \\
&= \int |Du|^2 \Gamma(u) 2C(u-p)\mathrm{e}^{C(u-p)^2} \eta \\
&\quad - \int 2C|Du|^2 \mathrm{e}^{C(u-p)^2} \eta \\
&\quad - \int 4C^2 |Du|^2 (u-p)^2 \mathrm{e}^{C(u-p)^2} \eta \\
&\leq 0
\end{aligned}$$

for $\eta \geq 0$, using (14.3.2), (14.3.3) as before.

Lemma 14.3.1. *Let $u : B(x_0, 4R) \to \mathbb{R}$ ($B(x_0, 4R)$ a ball in $\mathbb{R}^d$) be bounded, with*

$$\sup_{y_1, y_2 \in B(x_0, 2R)} |u(y_1) - u(y_2)| = M, \quad (14.3.7)$$

and satisfy

$$\Delta \mathrm{e}^{C(u(x)-p)^2} \geq 0 \quad (14.3.8)$$

in the weak sense for every $p \in \mathbb{R}$. Then there exists some

$$M' < M$$

with

$$\sup_{z_1, z_2 \in B(x_0, R)} |u(z_1) - u(z_2)| = M'. \tag{14.3.9}$$

Proof. By (14.3.7), we can find some $x_1 \in B(x_0, 2R)$ with

$$\text{meas}\left(\left\{x \in B(x_0, R) : |u(x) - u(x_1)| \le \frac{M}{4}\right\}\right) \ge \frac{1}{4}\text{meas}(B(x_0, R) = \frac{\omega_d}{4} R^d. \tag{14.3.10}$$

We consider the auxiliary function

$$g(x) := \frac{1}{e^{CM^2}} e^{C(u(x) - u(x_1))^2}.$$

We have

$$\mu := \sup_{x \in B(x_0, 2R)} g(x) \le 1. \tag{14.3.11}$$

On the other hand, by (14.3.7), there exists some $y \in B(x_0, 2R)$ with

$$|u(y) - u(x_1)| \ge \frac{M}{2};$$

hence

$$\mu \ge e^{-\frac{3}{4}CM^2}. \tag{14.3.12}$$

On $\{x \in B(x_0, R) : |u(x) - u(x_1)| \le \frac{M}{4}\}$ [as in (14.3.10)], we have

$$g(x) \le e^{-\frac{15}{16}CM^2}. \tag{14.3.13}$$

We then consider the auxiliary function

$$h(x) := \mu - g(x) \ge 0 \text{ on } B(x_0, 2R). \tag{14.3.14}$$

From (14.3.12)and (14.3.13), we have

$$h(x) \ge e^{-\frac{3}{4}CM^2} - e^{-\frac{15}{16}CM^2} \text{ on } \left\{x \in B(x_0, R) : |u(x) - u(x_1)| \le \frac{M}{4}\right\}. \tag{14.3.15}$$

By (14.3.8) and the definitions of g, h,

$$\Delta h(x) \le 0 \text{ weakly in } B(x_0, 2R), \tag{14.3.16}$$

so that, with (14.3.14), we can apply the Harnack inequality for positive superharmonic functions, Theorem 14.1.2, to obtain

$$\begin{aligned}\inf_{x\in B(x_0,R)} h(x) &\geq \frac{c}{R^d}\int_{B(x_0,R)} h(x)\mathrm{d}x \text{ for some constant } c>0\\ &\geq c'\left(\mathrm{e}^{-\frac{3}{4}CM^2}-\mathrm{e}^{-\frac{15}{16}CM^2}\right)\end{aligned} \tag{14.3.17}$$

for some constant c' that is independent of u, by (14.3.15) and (14.3.10). □

The key of the proof was, of course, the Harnack inequality. The principle is that when we can control a supersolution h on some sufficiently large part of the ball $B(x_0, 2R)$ from below, then we can control h everywhere from below on the smaller ball $B(x_0, R)$.

Clearly, we can iterate the proof of this lemma, to show that, given $\epsilon > 0$, we find some $\delta > 0$ with

$$\sup_{\xi_1,\xi_2\in B(x_0,\delta)} |u(\xi_1)-u(\xi_2)| < \epsilon.$$

The iteration works because, by (14.3.15), whenever $\sup_{\xi_1,\xi_2\in B(x_0,2R)} |u(\xi_1) - u(\xi_2)| \geq \epsilon$, then we can decrease that supremum on $B(x_0, R)$ by at least $c'(\mathrm{e}^{-\frac{3}{4}C\epsilon^2} - \mathrm{e}^{-\frac{15}{16}C\epsilon^2})$ for some constant c' that does not depend on u.

Thus, we have

Theorem 14.3.1. *Let u be a bounded solution of*

$$\int_\Omega \left(\sum_i D_i u D_i\varphi - \Gamma(u)|Du|^2\varphi\right)\mathrm{d}x = 0 \textit{ for all } \varphi \in H_0^{1,2}\cap L^\infty(\Omega) \tag{14.3.18}$$

with a smooth and bounded function Γ. Then u is continuous in Ω.

Once we know that u is continuous, we can derive further regularity properties of u. As in Sect. 12.2, one shows in the end that u is smooth. In the special case where u is a solution of the variational problem (12.2.8), this is particularly easy. We simply take a function f with $f'(u) = \sqrt{g(u)}$ which is possible since u, hence $g(u)$ is continuous. Then the variational problem $\int g(u)|Du|^2 \to \min$ becomes the variational problem $\int |Dv|^2 \to \min$ for $v = f\circ u$, i.e., the Dirichlet integral, that we have already treated in Sects. 10.1 and 11.2. Since f is differentiable with positive derivative, the regularity of v then translates into the regularity of u, indeed. Actually, inspired by this argument, we may also want to treat (14.3.1) in a similar manner. We simply solve

$$\Phi'(u) = \Gamma(u) \text{ and } f'(u) = \mathrm{e}^{\Phi(u)} \tag{14.3.19}$$

and then have

$$\begin{aligned}\Delta(f \circ u) &= f'(u)\Delta u + f''(u)|D(u)|^2 \\ &= e^{\Phi(u)}(\Delta u + \Phi'(u)|D(u)|^2) \\ &= e^{\Phi(u)}(\Delta u + \Gamma(u)|D(u)|^2),\end{aligned}$$

and so, $f \circ u$ is harmonic, hence regular, when u solves (14.3.1). There are technical issues involved, like the solvability of (14.3.19) for a continuous u, the weak formulation of the preceding formula, and the necessary iteration to get from continuity to smoothness, however, that we do not address here.

When we want to proceed in a more analytical manner, we can obtain the following Caccioppoli inequality:

Lemma 14.3.2. *Let $u \in W^{1,2}(\Omega)$ be a bounded and continuous weak solution of (14.3.18) in Ω. Assume $|\Gamma(u)| \leq a$. For all $x_0 \in \Omega$, there then exists a radius $R_0 < \operatorname{dist}(x_0, \partial\Omega)$, depending only on the modulus of continuity of u and the bound a in (14.3.2), such that for all radii $0 < r < R \leq R_0$, with $u_R := u_{B(x_0,R)}$ (the mean value of u on the ball $B(x_0, R)$), we have*

$$\int_{B(x_0,r)} |Du|^2 \leq \frac{32}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u - u_R|^2 . \tag{14.3.20}$$

Proof. We choose $\eta \in H_0^{1,2}(B(x_0, R))$ with

$$\begin{aligned}&0 \leq \eta \leq 1, \\ &\eta \equiv 1 \quad \text{on } B(x_0, r);\ \text{hence } D\eta \equiv 0 \quad \text{on } B(x_0, r), \\ &|D\eta| \leq \frac{2}{R-r}.\end{aligned}$$

As in Sect. 11.2, we employ the test function

$$\varphi = (u - u_R)\eta^2$$

and obtain

$$\begin{aligned}\int_{B(x_0,R)} |Du|^2\eta^2 &= -\int_{B(x_0,R)} 2D_i u\eta D_i\eta(u - u_R) + \int_{B(x_0,R)} \Gamma(u)|Du|^2(u - u_R)\eta^2 \\ &\leq \frac{1}{4}\int_{B(x_0,R)} |Du|^2\eta^2 + 4\int_{B(x_0,R)} (u - u_R)^2|D\eta|^2 \\ &\quad + a \sup_{B(x_0,R)} |u - u_R| \int_{B(x_0,R)} |Du|^2\eta^2.\end{aligned}$$

By continuity of u, we may choose R so small that $a \sup_{B(x_0,R)} |u - u_R| \le \frac{1}{4}$. We then obtain

$$\int_{B(x_0,R)} |Du|^2\eta^2 \le 8 \int_{B(x_0,R)} (u-u_R)^2 |D\eta|^2$$
$$\le \frac{32}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u-u_R|^2 . \qquad (14.3.21)$$

This yields (14.3.20). □

The key point here is that we can use a test function like $u(x) - u_R$ or $u(x) - u(x_0)$ that, because of the continuity of u, on a sufficiently small ball $B(x_0, R)$ leads to an arbitrarily small factor for the nonlinear term $\Gamma(u)|D(u)|^2$ so that it can be dominated by the linear term.

We have seen the use of such an inequality already in Sect. 11.2, and we shall see in Sect. 14.4 below how the Caccioppoli inequality can be used to show higher results.

References for continuity results via Moser's Harnack inequality for equations and systems of the type (14.3.1) are [15,26]. As the example at the end of Sect. 12.2 shows, weak solutions of elliptic systems of the type considered here need not be continuous, even if they are bounded. However, continuous weak solutions are smooth; see [24].

14.4 Regularity of Minimizers of Variational Problems

The aim of this section is the proof of (a special case of) the fundamental result of de Giorgi on the regularity of minima of variational problems with elliptic Euler–Lagrange equations:

Theorem 14.4.1. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a function of class C^∞ satisfying the following conditions: For some constants $K, \Lambda < \infty$, $\lambda > 0$ and for all $p = (p_1, \dots, p_d) \in \mathbb{R}^d$:*

(i) $\left|\frac{\partial F}{\partial p_i}(p)\right| \le K\,|p| \quad (i = 1, \dots, d)$.

(ii) $\lambda\,|\xi|^2 \le \sum \frac{\partial^2 F(p)}{\partial p_i \partial p_j}\xi_i\xi_j \le \Lambda\,|\xi|^2$ *for all* $\xi \in \mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $u \in W^{1,2}(\Omega)$ be a minimizer of the variational problem

$$I(v) := \int_\Omega F(Dv(x))\mathrm{d}x,$$

i.e.,

$$I(u) \le I(u+\varphi) \quad \text{for all } \varphi \in H_0^{1,2}(\Omega). \qquad (14.4.1)$$

Then $u \in C^\infty(\Omega)$.

Remark. Because of (i), there exist constants c_1, c_2 with

$$|F(p)| \le c_1 + c_2 |p|^2 . \tag{14.4.2}$$

Since Ω is assumed to be bounded, this implies

$$I(v) = \int_\Omega F(Dv) < \infty$$

for all $v \in W^{1,2}(\Omega)$. Therefore, our variational problem, namely, to minimize I in $W^{1,2}(\Omega)$, is meaningful.

We shall first derive the Euler–Lagrange equations for a minimizer of I:

Lemma 14.4.1. *Suppose that the assumptions of Theorem 14.4.1 hold. We then have for all $\varphi \in H_0^{1,2}(\Omega)$,*

$$\int_\Omega \sum_{i=1}^d F_{p_i}(Du) D_i \varphi = 0 \tag{14.4.3}$$

(using the abbreviation $F_{p_i} = \frac{\partial F}{\partial p_i}$).

Proof. By (i),

$$\int_\Omega \sum_{i=1}^d F_{p_i}(Dv) D_i \varphi \le d\ K \int_\Omega |Dv|\,|D\varphi| \le d\ K \,\|Dv\|_{L^2(\Omega)} \|D\varphi\|_{L^2(\Omega)} ,$$

and this is finite for $\varphi, v \in W^{1,2}(\Omega)$. By a standard result of Lebesgue integration theory, on the basis of this inequality, we may compute

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u + t\varphi)$$

by differentiation under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u + t\varphi) = \int_\Omega \sum F_{p_i}(Du + tD\varphi) D_i \varphi. \tag{14.4.4}$$

In particular, $I(u + t\varphi)$ is a differentiable function of $t \in \mathbb{R}$, and since u is a minimizer,

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u + t\varphi)|_{t=0} = 0. \tag{14.4.5}$$

Equation (14.4.4) for $t = 0$ then implies (14.4.3). □

Lemma 14.4.1 reduces Theorem 14.4.1 to the following:

Theorem 14.4.2. *Let $A^i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \dots, d$, be C^∞-functions satisfying the following conditions: There exist constants $K, \Lambda < \infty$, $\lambda > 0$ such that for all $p \in \mathbb{R}^d$:*

(i) $\left|A^i(p)\right| \le K\,|p| \quad (i = 1, \dots, d)$.
(ii) $\lambda\,|\xi|^2 \le \sum_{i,j=1}^d \frac{\partial A^i(p)}{\partial p_j}\xi_i\xi_j$ *for all* $\xi \in \mathbb{R}^d$.
(iii) $\left|\frac{\partial A^i(p)}{\partial p_j}\right| \le \Lambda$.

Let $u \in W^{1,2}(\Omega)$ be a weak solution of

$$\sum_{i=1}^d \frac{\partial}{\partial x^i} A^i(Du) = 0 \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{14.4.6}$$

i.e., for all $\varphi \in H_0^{1,2}(\Omega)$, let

$$\int_\Omega \sum_{i=1}^d A^i(Du) D_i\varphi = 0. \tag{14.4.7}$$

Then $u \in C^\infty(\Omega)$.

The crucial step in the proof will be Theorem 14.2.1, of de Giorgi and Nash. Important steps towards Theorem 14.4.2 had been obtained earlier by S. Bernstein, L. Lichtenstein, E. Hopf, C. Morrey, and others.

We shall start with a lemma.

Lemma 14.4.2. *Under the assumptions of Theorem 14.4.2, for any $\Omega' \subset\subset \Omega$ we have $u \in W^{2,2}(\Omega')$, and moreover, $\|u\|_{W^{2,2}(\Omega')} \le c\,\|u\|_{W^{1,2}(\Omega)}$, where $c = c(\lambda, \Lambda, \operatorname{dist}(\Omega', \partial\Omega))$.*

Proof. We shall proceed as in the proof of Theorem 11.2.1. For

$$|h| < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega),$$

$\varphi_{k,-h}(x) := \varphi(x - he_k)$ (e_k being the kth unit vector) is of class $H_0^{1,2}(\Omega)$ as well. Therefore,

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^d A^i(Du(x)) D_i\varphi_{k,-h}(x)\mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^d A^i(Du(x)) D_i\varphi(x - he_k)\mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^d A^i(Du(y + he_k)) D_i\varphi(y)\mathrm{d}y \\ &= \int_\Omega \sum_{i=1}^d A^i\left((Du)_{k,h}\right) D_i\varphi. \end{aligned}$$

Subtracting (14.4.7), we obtain

$$\int \sum_i \left(A^i(Du(x+he_k)) - A^i(Du(x))\right) D_i\varphi(x) = 0. \tag{14.4.8}$$

For almost all $x \in \Omega$

$$\begin{aligned} &A^i\,(Du(x+he_k)) - A^i\,(Du(x)) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} A^i\,(tDu(x+he_k) + (1-t)Du(x))\,\mathrm{d}t \\ &= \int_0^1 \left(\sum_{j=1}^d A^i_{p_j}\,(tDu(x+he_k) + (1-t)Du(x))\,D_j\,(u(x+he_k) - u(x))\right)\mathrm{d}t. \end{aligned} \tag{14.4.9}$$

We thus put

$$a_h^{ij}(x) := \int_0^1 A^i_{p_j}\,(tDu(x+he_k) + (1-t)Du(x))\,\mathrm{d}t,$$

and using (14.4.9), we rewrite (14.4.8) as

$$\int_\Omega \sum_{i,j} a_h^{ij}(x) D_j \left(\frac{u(x+he_k) - u(x)}{h}\right) D_i\varphi(x)\mathrm{d}x = 0. \tag{14.4.10}$$

Here, because of (ii) and (iii),

$$\lambda\,|\xi|^2 \le \sum_{i,j} a_h^{ij}(x)\xi_i\xi_j \le \mathrm{d}\Lambda\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

We may thus proceed as in Sect. 11.2 and put

$$\varphi = \frac{1}{h}\,(u(x+he_k) - u(x))\,\eta^2$$

with $\eta \in C_0^1(\Omega'')$, where we choose Ω'' satisfying

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega,$$

$\operatorname{dist}(\Omega'', \partial\Omega), \operatorname{dist}(\Omega', \partial\Omega'') \ge \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega)$, and require

$$\begin{aligned} &0 \le \eta \le 1, \\ &\eta(x) = 1 \quad \text{for } x \in \Omega', \\ &|D\eta| \le \frac{8}{\operatorname{dist}(\Omega', \partial\Omega)}, \end{aligned}$$

as well as

$$|2h| < \operatorname{dist}(\Omega'', \partial\Omega).$$

Using the notation

$$\Delta_k^h u(x) = \frac{u(x + he_k) - u(x)}{h},$$

(14.4.10) then implies

$$\begin{aligned}
\lambda \int_\Omega \left|D\Delta_k^h u\right|^2 \eta^2 &\leq \int_\Omega \sum_{i,j} a_h^{ij} \left(D_j \Delta_k^h u\right)\left(D_i \Delta_k^h u\right) \eta^2 \\
&= -\int_\Omega \sum_{i,j} a_h^{ij} D_j \Delta_k^h u \, 2\eta (D_i \eta) \Delta_h^k u \quad \text{by (14.4.10)} \\
&\leq \varepsilon \mathrm{d}\Lambda \int_\Omega \left|D\Delta_k^h u\right|^2 + \frac{\mathrm{d}\Lambda}{\varepsilon} \int_\Omega \left|\Delta_k^h u\right|^2 |D\eta|^2 \quad \text{for all } \varepsilon > 0,
\end{aligned}$$

and with $\varepsilon = \frac{\lambda}{2\mathrm{d}\Lambda}$,

$$\int_\Omega \left|D\Delta_k^h u\right|^2 \eta^2 \leq c_1 \int_{\Omega''} \left|\Delta_k^h u\right|^2 \leq c_1 \int_\Omega |Du|^2$$

by Lemma 11.2.1, with c_1 independent of h. Hence

$$\left\|D\Delta_k^h u\right\|_{L^2(\Omega')} \leq c_1 \|Du\|_{L^2(\Omega)}. \tag{14.4.11}$$

Since the right-hand side of (14.4.11) does not depend on h, from Lemma 11.2.2 we obtain $D^2 u \in L^2(\Omega')$ and the inequality

$$\left\|D^2 u\right\|_{L^2(\Omega')} \leq c_1 \|Du\|_{L^2(\Omega)}. \tag{14.4.12}$$

Consequently, $u \in W^{2,2}(\Omega')$. □

Performing the limit $h \to 0$ in (14.4.10), with

$$\begin{aligned}
a^{ij}(x) &:= A^i_{p_j}(Du(x)), \\
v &:= D_k u,
\end{aligned} \tag{14.4.13}$$

we also obtain

$$\int_\Omega \sum_{i,j} a^{ij}(x) D_j v D_i \varphi = 0 \quad \text{for all } \varphi \in H_0^{1,2}(\Omega).$$

By (ii), (iii), $(a^{ij}(x))_{i,j=1,\dots,d}$ satisfies the assumptions of Theorem 14.2.1. Applying that result to $v = D_k u$ then yields the following result:

Lemma 14.4.3. *Under the assumptions of Theorem 14.2.1,*

$$Du \in C^{\alpha}(\Omega)$$

for some $\alpha \in (0,1)$, *i.e.,*

$$u \in C^{1,\alpha}(\Omega).$$

Thus $v = D_k u$, $k = 1, \dots, d$, is a weak solution of

$$\sum_{i,j=r}^{d} D_i \left(a^{ij}(x) D_j v\right) = 0. \tag{14.4.14}$$

Here, the coefficients $a^{ij}(x)$ satisfy not only the ellipticity condition

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j, \quad \left|a^{ij}(x)\right| \le \Lambda$$

for all $\xi \in \mathbb{R}^d$, $x \in \Omega$, $i, j = 1, \dots, d$, but by (14.4.13), they are also Hölder continuous, since A^i is smooth and Du is Hölder continuous by Lemma 14.4.3. For the proof of Theorem 14.4.2, we thus need a regularity theory for such equations. Equation (14.4.14) is of divergence type, in contrast to those treated in Chap. 13, and therefore, we cannot apply the results of Schauder directly. However, one can develop similar methods. For the sake of variety, here, we shall present the method of Campanato as an alternative approach. As a preparation, we shall now prove some auxiliary results for equations of type (14.4.14) with constant coefficients. (Of course, these results are already essentially known from Chap. 11.)

The first result is the Caccioppoli inequality:

Lemma 14.4.4. *Let* $(A^{ij})_{i,j=1,\dots,d}$ *be a matrix with* $\left|A^{ij}\right| \le \Lambda$ *for all* i, j, *and*

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} A^{ij}\xi_i\xi_j \quad \text{for all } \xi \in \mathbb{R}^d$$

with $\lambda > 0$. *Let* $u \in W^{1,2}(\Omega)$ *be a weak solution of*

$$\sum_{i,j=1}^{d} D_j \left(A^{ij} D_i u\right) = 0 \quad \text{in } \Omega. \tag{14.4.15}$$

We then have for all $x_0 \in \Omega$ *and* $0 < r < R < \mathrm{dist}(x_0, \partial\Omega)$ *and all* $\mu \in \mathbb{R}$,

$$\int_{B(x_0,r)} |Du|^2 \le \frac{c_2}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u-\mu|^2 . \tag{14.4.16}$$

Proof. We choose $\eta \in H_0^{1,2}(B(x_0, R))$ with

$$\begin{aligned} 0 \le \eta \le 1,& \\ \eta \equiv 1 \quad \text{on } B(x_0,r),& \text{ hence } D\eta \equiv 0 \quad \text{on } B(x_0,r), \\ |D\eta| \le \frac{2}{R-r}.& \end{aligned}$$

As in Sect. 11.2, we employ the test function

$$\varphi = (u-\mu)\eta^2$$

and obtain

$$\begin{aligned} 0 &= \int \sum_{i,j} A^{ij} D_i u D_j \left((u-\mu)\eta^2\right) \\ &= \int \sum_{i,j} A^{ij} D_i u D_j u\, \eta^2 + \int 2 \sum_{i,j} A^{ij} D_i u(u-\mu)\eta D_j \eta. \end{aligned}$$

Using the ellipticity conditions, we deduce the inequality

$$\begin{aligned} \lambda \int_{B(x_0,R)} |Du|^2\eta^2 \le& \int_{B(x_0,R)} \sum A^{ij} D_i u D_j u\, \eta^2 \\ &\le \varepsilon \Lambda\, d \int_{B(x_0,R)} |Du|^2\, \eta^2 \\ &\quad + \frac{\Lambda}{\varepsilon} d \int_{B(x_0,R)\setminus B(x_0,r)} |D\eta|^2\, |u-\mu|^2 , \end{aligned}$$

since $D\eta = 0$ on $B(x_0, r)$. Hence, with $\varepsilon = \frac{1}{2}\frac{\lambda}{\Lambda d}$,

$$\int_{B(x_0,R)} |Du|^2\, \eta^2 \le \frac{c_2}{(R-r)^2} \int_{B(x_0,R)\setminus B(x_0,r)} |u-\mu|^2 ,$$

and because of

$$\int_{B(x_0,r)} |Du|^2 \le \int_{B(x_0,R)} |Du|^2\, \eta^2,$$

the claim results. □

The next lemma contains the Campanato estimates:

Lemma 14.4.5. *Under the assumptions of Lemma 14.4.4, we have*

$$\int_{B(x_0,r)} |u|^2 \le c_3 \left(\frac{r}{R}\right)^d \int_{B(x_0,R)} |u|^2 \tag{14.4.17}$$

as well as

$$\int_{B(x_0,r)} \left|u - u_{B(x_0,r)}\right|^2 \le c_4 \left(\frac{r}{R}\right)^{d+2} \int_{B(x_0,R)} \left|u - u_{B(x_0,R)}\right|^2. \tag{14.4.18}$$

Proof. Without loss of generality $r < \frac{R}{2}$. We choose $k > d$. By the Sobolev embedding theorem (Theorem 11.1.1) or an extension of this result analogous to Corollary 11.1.3,

$$W^{k,2}(B(x_0,R)) \subset C^0(B(x_0,R)).$$

By Theorem 11.3.1, now $u \in W^{k,2}\left(B\left(x_0, \frac{R}{2}\right)\right)$, with an estimate analogous to Theorem 11.2.2. Therefore,

$$\begin{aligned}\int_{B(x_0,r)} |u|^2 &\le c_5 r^d \sup_{B(x_0,r)} |u|^2 \le c_6 \frac{r^d}{R^{d-2k}} \|u\|_{W^{k,2}\left(B\left(x_0,\frac{R}{2}\right)\right)} \\ &\le c_3 \frac{r^d}{R^d} \int_{B(x_0,R)} |u|^2 .\end{aligned}$$

(Concerning the dependence on the radius: The power r^d is obvious. The power R^d can easily be derived from a scaling argument, instead of carefully going through all the intermediate estimates). This yields (14.4.17). Since we are dealing with an equation with constant coefficients, Du is a solution along with u. For $r < \frac{R}{2}$, we thus obtain

$$\int_{B(x_0,r)} |Du|^2 \le c_7 \frac{r^d}{R^d} \int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 . \tag{14.4.19}$$

By the Poincaré inequality (Corollary 11.1.4),

$$\int_{B(x_0,r)} \left|u - u_{B(x_0,r)}\right|^2 \le c_8 r^2 \int_{B(x_0,r)} |Du|^2 . \tag{14.4.20}$$

By the Caccioppoli inequality (Lemma 14.4.4)

$$\int_{B\left(x_0,\frac{R}{2}\right)} |Du|^2 \le \frac{c_9}{R^2} \int_{B(x_0,R)} \left|u - u_{B(x_0,R)}\right|^2 . \tag{14.4.21}$$

Then (14.4.19)–(14.4.21) imply (14.4.18). □

We may now use Campanato's method to derive the following regularity result:

Theorem 14.4.3. *Let $a^{ij}(x)$, $i, j = 1, \dots, d$, be functions of class C^α, $0 < \alpha < 1$, on $\Omega \subset \mathbb{R}^d$, satisfying the ellipticity condition*

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \quad \textit{for all } \xi \in \mathbb{R}^d, x \in \Omega \tag{14.4.22}$$

and

$$\left|a^{ij}(x)\right| \le \Lambda \quad \textit{for all } x \in \Omega, i, j = 1, \dots, d, \tag{14.4.23}$$

with fixed constants $0 < \lambda \le \Lambda < \infty$. Then any weak solution v of

$$\sum_{i,j=1}^{d} D_j\left(a^{ij}(x)D_i v\right) = 0 \tag{14.4.24}$$

is of class $C^{1,\alpha'}(\Omega)$ for any α' with $0 < \alpha' < \alpha$.

Proof. For $x_0 \in \Omega$, we write

$$a^{ij} = a^{ij}(x_0) + \left(a^{ij}(x) - a^{ij}(x_0)\right).$$

Letting

$$A^{ij} := a^{ij}(x_0),$$

(14.4.24) becomes

$$\sum_{i,j=1}^{d} D_j\left(A^{ij}D_i v\right) = \sum_{i,j=1}^{d} D_j\left((a^{ij}(x_0) - a^{ij}(x))D_i v\right) = \sum_{j=1}^{d} D_j\left(f^j(x)\right)$$

with

$$f^j(x) := \sum_{i=1}^{d}\left((a^{ij}(x_0) - a^{ij}(x))D_i v\right). \tag{14.4.25}$$

This means that

$$\int_\Omega \sum_{i,j=1}^{d} A^{ij}D_i v D_j\varphi = \int_\Omega \sum_{j=1}^{d} f^j D_j\varphi \quad \text{for all } \varphi \in H_0^{1,2}(\Omega). \tag{14.4.26}$$

For some ball $B(x_0, R) \subset \Omega$, let

$$w \in H^{1,2}(B(x_0, R))$$

be a weak solution of

$$\sum_{i,j=1}^{d} D_j \left(A^{ij} D_i w\right) = 0 \quad \text{in } B(x_0, R), \qquad w = v \quad \text{on } \partial B(x_0, R). \tag{14.4.27}$$

Thus w is a solution of

$$\int_{B(x_0,R)} \sum_{i,j=1}^{d} A^{ij} D_i w D_j \varphi = 0 \quad \text{for all } \varphi \in H_0^{1,2}(B(x_0, R)). \tag{14.4.28}$$

Such a w exists by the Lax–Milgram theorem (see appendix). Note that we seek $z = w - v$ with

$$\begin{aligned} B(\varphi, z) &:= \int \sum A^{ij} D_i z D_j \varphi \\ &= -\int \sum A^{ij} D_i v D_j \varphi \\ &=: F(\varphi) \quad \text{for all } \varphi \in H_0^{1,2}(B(x_0, R)). \end{aligned}$$

Since (14.4.27) is a linear equation with constant coefficients, then if w is a solution, so is $D_k w$, $k = 1, \dots, d$ (with different boundary conditions, of course). We may thus apply (14.4.17) from Lemma 14.4.5 to $u = D_k w$ and obtain

$$\int_{B(x_0,r)} |Dw|^2 \le c_{10} \left(\frac{r}{R}\right)^d \int_{B(x_0,R)} |Dw|^2 . \tag{14.4.29}$$

(Here, Dw stands for the vector $(D_1 w, \dots, D_d w)$.) Since $w = v$ on $\partial B(x_0, R)$, $\varphi = v - w$ is an admissible test function in (14.4.28), and we obtain

$$\int_{B(x_0,R)} \sum_{i,j=1}^{d} A^{ij} D_i w D_j w = \int_{B(x_0,R)} \sum_{i,j=1}^{d} A^{ij} D_i w D_j v. \tag{14.4.30}$$

Using (14.4.27), (14.4.23) and the Cauchy–Schwarz inequality, this implies

$$\int_{B(x_0,R)} |Dw|^2 \le \left(\frac{\Lambda d}{\lambda}\right)^2 \int_{B(x_0,R)} |Dv|^2 . \tag{14.4.31}$$

Equations (14.4.26) and (14.4.28) imply

$$\int_{B(x_0,R)} \sum_{i,j=1}^{d} A^{ij} D_i (v - w) D_j \varphi = \int_{B(x_0,R)} \sum_{i,j=1}^{d} f^j D_j \varphi$$

for all $\varphi \in H_0^{1,2}(B(x_0, R))$. We utilize once more the test function $\varphi = v - w$ to obtain

$$\begin{aligned}\int_{B(x_0,R)} |D(v-w)|^2 &\le \frac{1}{\lambda}\int_{B(x_0,R)} \sum_{i,j} A^{ij} D_i(v-w) D_j(v-w)\\ &= \frac{1}{\lambda}\int_{B(x_0,R)} \sum_j f^j D_j(v-w)\\ &\le \frac{1}{\lambda}\left(\int_{B(x_0,R)} |D(v-w)|^2\right)^{\frac{1}{2}} \left(\int_{B(x_0,R)} \sum_j \left|f^j\right|^2\right)^{\frac{1}{2}}\end{aligned}$$

by the Cauchy–Schwarz inequality, i.e.,

$$\int_{B(x_0,R)} |D(v-w)|^2 \le \frac{1}{\lambda^2}\int_{B(x_0,R)} \sum_j \left|f^j\right|^2. \tag{14.4.32}$$

We now put the preceding estimates together. For $0 < r \le R$, we have

$$\begin{aligned}\int_{B(x_0,r)} |Dv|^2 &\le 2\int_{B(x_0,r)} |Dw|^2 + 2\int_{B(x_0,r)} |D(v-w)|^2\\ &\le c_{11}\left(\frac{r}{R}\right)^d \int_{B(x_0,R)} |Dv|^2 + 2\int_{B(x_0,r)} |D(v-w)|^2\end{aligned}$$

by (14.4.29) and (14.4.31). Now

$$\begin{aligned}\int_{B(x_0,r)} |D(v-w)|^2 &\le \int_{B(x_0,R)} |D(v-w)|^2, \quad \text{since } r \le R\\ &\le \frac{1}{\lambda^2}\int_{B(x_0,R)} \sum_j \left|f^j\right|^2 \quad \text{by (14.4.32)}\\ &\le \frac{1}{\lambda^2} \sup_{\substack{i,j\\ x\in B(x_0,R)}} \left|a^{ij}(x_0) - a^{ij}(x)\right|^2 \int_{B(x_0,R)} |Dv|^2\\ &\qquad \text{by (14.4.25)}\\ &\le c_{12} R^{2\alpha} \int_{B(x_0,R)} |Dv|^2, \end{aligned} \tag{14.4.33}$$

since the a^{ij} are of class C^α. Altogether, we obtain

$$\int_{B(x_0,r)} |Dv|^2 \le \gamma\left(\left(\frac{r}{R}\right)^d + R^{2\alpha}\right)\int_{B(x_0,R)} |Dv|^2 \tag{14.4.34}$$

with some constant γ. If (14.4.34) did not contain the term $R^{2\alpha}$ (which is present solely for the reason that the $a^{ij}(x)$, while Hölder continuous, are not necessarily constant), we would have a useful inequality. That term, however, can be made to disappear by a simple trick. For later purposes, we formulate a somewhat more general result:

Lemma 14.4.6. *Let $\sigma(r)$ be a nonnegative, monotonically increasing function satisfying*

$$\sigma(r) \le \gamma\left(\left(\frac{r}{R}\right)^{\mu} + \delta\right)\sigma(R) + \kappa R^{\nu}$$

for all $0 < r \le R \le R_0$, with $\mu > \nu$ and $\delta \le \delta_0(\gamma, \mu, \nu)$. If δ_0 is sufficiently small, for $0 < r \le R \le R_0$, we then have

$$\sigma(r) \le \gamma_1\left(\frac{r}{R}\right)^{\nu}\sigma(R) + \kappa_1 r^{\nu},$$

with γ_1 depending on γ, μ, ν, and κ_1 depending in addition on κ ($\kappa_1 = 0$ if $\kappa = 0$).

Proof. Let $0 < \tau < 1$, $R < R_0$. Then by assumption

$$\sigma(\tau R) \le \gamma\tau^{\mu}(1 + \delta\tau^{-\mu})\sigma(R) + \kappa R^{\nu}.$$

We choose $0 < \tau < 1$ such that

$$2\gamma\tau^{\mu} = \tau^{\lambda}$$

with $\nu < \lambda < \mu$ (without loss of generality $2\gamma > 1$), and assume that

$$\delta_0\tau^{-\mu} \le 1.$$

It follows that

$$\sigma(\tau R) \le \tau^{\lambda}\sigma(R) + \kappa R^{\nu}$$

and thus iteratively for $k \in \mathbb{N}$

$$\begin{aligned}\sigma(\tau^{k+1}R) &\le \tau^{\lambda}\sigma(\tau^k R) + \kappa\tau^{k\nu}R^{\nu}\\ &\le \tau^{(k+1)\lambda}\sigma(R) + \kappa\tau^{k\nu}R^{\nu}\sum_{j=0}^{k}\tau^{j(\lambda-\nu)}\\ &\le \gamma_0\tau^{(k+1)\nu}(\sigma(R) + \kappa R^{\nu})\end{aligned}$$

(where γ_0, as well as the subsequent γ_1, contains a factor $\frac{1}{\tau}$). We now choose $k \in \mathbb{N}$ such that

$$\tau^{k+2}R < r \le \tau^{k+1}R,$$

and obtain

$$\sigma(r) \le \sigma(\tau^{k+1}R) \le \gamma_1 \left(\frac{r}{R}\right)^{\nu} \sigma(R) + \kappa_1 r^{\nu}. \qquad \square$$

Continuing with the proof of Theorem 14.4.3, applying Lemma 14.4.6 to (14.4.34), where we have to require $0 < r \le R \le R_0$ with $R_0^{2\alpha} \le \delta_0$, we obtain the inequality

$$\int_{B(x_0,r)} |Dv|^2 \le c_{13} \left(\frac{r}{R}\right)^{d-\varepsilon} \int_{B(x_0,R)} |Dv|^2 \tag{14.4.35}$$

for each $\varepsilon > 0$, where c_{13} and R_0 depend on ε. We repeat this procedure, but this time applying (14.4.18) from Lemma 14.4.5 in place of (14.4.17). Analogously to (14.4.29), we obtain

$$\int_{B(x_0,r)} \left|Dw - (Dw)_{B(x_0,r)}\right|^2 \le c_{14} \left(\frac{r}{R}\right)^{d+2} \int_{B(x_0,R)} \left|Dw - (Dw)_{B(x_0,R)}\right|^2. \tag{14.4.36}$$

We also have

$$\int_{B(x_0,R)} \left|Dw - (Dw)_{B(x_0,R)}\right|^2 \le \int_{B(x_0,R)} \left|Dw - (Dv)_{B(x_0,R)}\right|^2,$$

because for any L^2-function g, the following relation holds:

$$\int_{B(x_0,R)} \left|g - g_{B(x_0,R)}\right|^2 = \inf_{\kappa \in \mathbb{R}} \int_{B(x_0,R)} |g - \kappa|^2. \tag{14.4.37}$$

(*Proof:* For $g \in L^2(\Omega)$, $F(\kappa) := \int_\Omega |g-\kappa|^2$ is convex and differentiable with respect to κ, and

$$F'(\kappa) = \int_\Omega 2(\kappa - g);$$

hence $F'(\kappa) = 0$ precisely for

$$\kappa = \frac{1}{|\Omega|} \int_\Omega g,$$

and since F is convex, a critical point has to be a minimizer.)

Moreover,

$$\begin{aligned}
&\int_{B(x_0,R)} \left|Dw - (Dv)_{B(x_0,R)}\right|^2 \\
&\qquad \le \frac{1}{\lambda} \int_{B(x_0,R)} \sum_{i,j} A^{ij} \left(D_i w - (D_i v)_{B(x_0,R)}\right) \left(D_j w - (D_j v)_{B(x_0,R)}\right) \\
&\qquad = \frac{1}{\lambda} \int_{B(x_0,R)} \sum_{i,j} A^{ij} \left(D_i w - (D_i v)_{B(x_0,R)}\right) \left(D_j v - (D_j v)_{B(x_0,R)}\right) \\
&\qquad\quad + \frac{1}{\lambda} \int_{B(x_0,R)} \sum_{i,j} A^{ij} \left(D_i v\right)_{B(x_0,R)} \left(D_j v - D_j w\right)
\end{aligned}$$

by (14.4.30). The last integral vanishes, since $A^{ij} (D_i v)_{B(x_0,R)}$ is constant and $v-w \in H_0^{1,2}(B(x_0, R))$. Applying the Cauchy–Schwarz inequality as usual, we altogether obtain

$$\int_{B(x_0,R)} \left|Dw - (Dw)_{B(x_0,R)}\right|^2 \le \frac{\Lambda^2}{\lambda^2} d^2 \int_{B(x_0,R)} \left|Dv - (Dv)_{B(x_0,R)}\right|^2 . \quad (14.4.38)$$

Finally,

$$\begin{aligned}
\int_{B(x_0,r)} \left|Dv - (Dv)_{B(x_0,r)}\right|^2 \le 3 &\int_{B(x_0,r)} \left|Dw - (Dw)_{B(x_0,r)}\right|^2 \\
+ 3 &\int_{B(x_0,r)} |Dv - Dw|^2 \\
+ 3 &\int_{B(x_0,r)} \left((Dv)_{B(x_0,r)} - (Dw)_{B(x_0,r)}\right)^2 .
\end{aligned}$$

The last expression can be estimated by Hölder's inequality

$$\int_{B(x_0,r)} \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} (Dv - Dw)\right)^2 \le \int_{B(x_0,r)} (Dv - Dw)^2.$$

Thus

$$\begin{aligned}
&\int_{B(x_0,r)} \left|Dv - (Dv)_{B(x_0,r)}\right|^2 \\
&\qquad \le 3 \int_{B(x_0,r)} \left|Dw - (Dw)_{B(x_0,r)}\right|^2 + 6 \int_{B(x_0,r)} |Dv - Dw|^2 \\
&\qquad \le 3 \int_{B(x_0,r)} \left|Dw - (Dw)_{B(x_0,r)}\right|^2 + c_{15} R^{2\alpha} \int_{B(x_0,R)} |Dv|^2 \quad (14.4.39)
\end{aligned}$$

by (14.4.33). From (14.4.36), (14.4.38), and (14.4.39), we obtain

$$\begin{aligned}
&\int_{B(x_0,r)} \left|Dv - (Dv)_{B(x_0,r)}\right|^2 \\
&\quad \le c_{16}\left(\frac{r}{R}\right)^{d+2} \int_{B(x_0,R)} \left|Dv - (Dv)_{B(x_0,R)}\right|^2 + c_{17}R^{2\alpha}\int_{B(x_0,R)} |Dv|^2 \\
&\quad \le c_{16}\left(\frac{r}{R}\right)^{d+2} \int_{B(x_0,R)} \left|Dv - (Dv)_{B(x_0,R)}\right|^2 + c_{18}R^{d-\varepsilon+2\alpha}, \qquad (14.4.40)
\end{aligned}$$

applying (14.4.35) for $0 < R \le R_0$ in place of $0 < r \le R$. Lemma 14.4.6 implies

$$\begin{aligned}
&\int_{B(x_0,r)} \left|Dv - (Dv)_{B(x_0,r)}\right|^2 \\
&\qquad \le c_{19}\left(\frac{r}{R}\right)^{d+2\alpha-\varepsilon} \int_{B(x_0,R)} \left|Dv - (Dv)_{B(x_0,R)}\right|^2 + c_{20}r^{d+2\alpha-\varepsilon}.
\end{aligned}$$

The claim now follows from Campanato's theorem (Corollary 11.1.7). □

It is now easy to prove Theorem 14.4.2:

Proof of Theorem 14.4.2: We apply Theorem 14.4.3 to $v = Du$ and obtain $v \in C^{1,\alpha'}$; hence $u \in C^{2,\alpha'}$. We may then differentiate the equation with respect to x^k and observe that the second derivatives $D_{jk}u$, $j,k = 1,\dots,d$, again satisfy equations of the same type. By Theorem 14.4.3, then $D^2u \in C^{1,\alpha''}$; hence $u \in C^{3,\alpha''}$. Iteratively, we obtain $u \in C^{m,\alpha_m}$ for all $m \in \mathbb{N}$ with $0 < \alpha_m < 1$. Therefore, $u \in C^\infty$.

Remark. The regularity Theorem 14.4.1 of de Giorgi more generally applies to minimizers of variational problems of the form

$$I(v) := \int_\Omega F(x, v(x), Dv(x))\mathrm{d}x,$$

where $F \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)$ again satisfies conditions like (i), (ii) of Theorem 14.4.1 with respect to p, and $\frac{1}{|p|^2}F(x,v,p)$ satisfies smoothness conditions with respect to the variables x and v uniformly in p.

References for this section are Giaquinta [10, 11].

Summary

Moser's Harnack inequality says that positive weak solutions u of

$$Lu = \sum_{i,j} \frac{\partial}{\partial x^j} \left(a^{ij}(x) \frac{\partial}{\partial x^i} u(x) \right) = 0$$

satisfy an estimate of the form

$$\sup_{B(x_0,R)} u \leq \text{ const } \inf_{B(x_0,R)} u$$

in each ball $B(x_0, R)$ in the interior of their domain of definition Ω. Here, the coefficients a^{ij} need to satisfy only an ellipticity condition, and have to be measurable and bounded, but they need not satisfy any further conditions like continuity. Moser's inequality yields a proof of the fundamental result of de Giorgi and Nash about the Hölder continuity of weak solutions of linear elliptic differential equations of second order with measurable and bounded coefficients. These assumptions are appropriate and useful for applications to *nonlinear* elliptic equations of the type

$$\sum_{i,j} \frac{\partial}{\partial x^j} \left(A^{ij}(u(x)) \frac{\partial}{\partial x^i} u(x) \right) = 0.$$

Namely, if one does not yet know any detailed properties of the solution u, then, even if the A^{ij} themselves are smooth, one can work only with the boundedness of the coefficients

$$a^{ij}(x) := A^{ij}(u(x)).$$

Here, a nonlinear equation is treated as a linear equation with not necessarily regular coefficients.

An application is de Giorgi's theorem on the regularity of minimizers of variational problems of the form

$$\int F(Du(x)) \text{ (14.4.38)} x \to \min$$

under the structural conditions

(i) $|\frac{\partial F}{\partial p_i}(p)| \leq K|p|$,

(ii) $\lambda|\xi|^2 \leq \sum \frac{\partial^2 F(p)}{\partial p_i \partial p_j} \xi_i \xi_j \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$,

with constants $K, \Lambda < \infty, \lambda > 0$.

Exercises

14.1. Formulate conditions on the coefficients of a differential operator of the form

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j} \left(a^{ij}(x) \frac{\partial}{\partial x^i} u(x) \right) + \sum_{i=1}^{d} \frac{\partial}{\partial x^i} (b^i(x)u(x)) + c(x)u(x)$$

that imply a Harnack inequality of the type of Corollary 14.1.1. Carry out the detailed proof.

14.2. As in Lemma 14.1.4, let

$$\phi(p, R) = \left(⨍_{B(x_0,R} u^p \, \mathrm{d}x \right)^{1/p}$$

for a fixed positive $u : B(x_0, R) \to \mathbb{R}$.

Show that

$$\lim_{p \to 0} \phi(p, R) = \exp \left(⨍_{B(x_0,R)} \log u(x) \, \mathrm{d}x \right).$$

14.3. Show the regularity of bounded minimizers of

$$\int_\Omega g(x, u(x)) |Du(x)|^2 \mathrm{d}x$$

where g is smooth, bounded, and $\geq \lambda$ for some constant $\lambda > 0$.