Chapter 13
The Regularity Theory of Schauder and the Continuity Method (Existence Techniques IV)

13.1 C^α-Regularity Theory for the Poisson Equation

In this chapter we shall need the fundamental concept of Hölder continuity, which we now recall from Sect. 11.1:

Definition 13.1.1. Let $f : \Omega \to \mathbb{R}$, $x_0 \in \Omega$, $0 < \alpha < 1$. The function f is called Hölder continuous at x_0 with exponent α if

$$\sup_{x\in\Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty. \tag{13.1.1}$$

Moreover, f is called Hölder continuous in Ω if it is Hölder continuous at each $x_0 \in \Omega$ (with exponent α); we write $f \in C^\alpha(\Omega)$. If (13.1.1) holds for $\alpha = 1$, then f is called Lipschitz continuous at x_0. Similarly, $C^{k,\alpha}(\Omega)$ is the space of those $f \in C^k(\Omega)$ whose kth derivative is Hölder continuous with exponent α.

We define a seminorm by

$$|f|_{C^\alpha(\Omega)} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{13.1.2}$$

We define

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)}$$

and

$$\|f\|_{C^{k,\alpha}(\Omega)}$$

as the sum of $\|f\|_{C^k(\Omega)}$ and the Hölder seminorms of all kth partial derivatives of f. As in Definition 13.1.1, in place of $C^{0,\alpha}$, we usually write C^α. The following result is elementary:

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Lemma 13.1.1. *If $f_1, f_2 \in C^\alpha(G)$ on $G \subset \mathbb{R}^d$, then $f_1 f_2 \in C^\alpha(G)$, and*

$$|f_1 f_2|_{C^\alpha(G)} \le \left(\sup_G |f_1|\right) |f_2|_{C^\alpha(G)} + \left(\sup_G |f_2|\right) |f_1|_{C^\alpha(G)} .$$

Proof.

$$\frac{|f_1(x)f_2(x) - f_1(y)f_2(y)|}{|x-y|^\alpha} \le \frac{|f_1(x) - f_1(y)|}{|x-y|^\alpha} |f_2(x)| + \frac{|f_2(x) - f_2(y)|}{|x-y|^\alpha} |f_1(y)|,$$

which directly implies the claim. □

Theorem 13.1.1. *As always, let $\Omega \subset \mathbb{R}^d$ be open and bounded,*

$$u(x) := \int_\Omega \Gamma(x,y) f(y) \mathrm{d}y, \tag{13.1.3}$$

where Γ is the fundamental solution defined in Sect. 2.1.

(a) If $f \in L^\infty(\Omega)$ (i.e., $\sup_{x\in\Omega} |f(x)| < \infty$),[1] *then $u \in C^{1,\alpha}(\Omega)$, and*

$$\|u\|_{C^{1,\alpha}(\Omega)} \le c_1 \sup |f| \quad \textit{for } \alpha \in (0,1). \tag{13.1.4}$$

(b) If $f \in C_0^\alpha(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$, and

$$\|u\|_{C^{2,\alpha}(\Omega)} \le c_2 \|f\|_{C^\alpha(\Omega)} \quad \textit{for } 0 < \alpha < 1. \tag{13.1.5}$$

The constants in (13.1.4) and (13.1.5) depend on α, d, and on Ω (on its volume $|\Omega|$ and its diameter).

Proof. (a) Up to a constant factor, the first derivatives of u are given by

$$v^i(x) := \int_\Omega \frac{x^i - y^i}{|x-y|^d} f(y) \mathrm{d}y \quad (i = 1, \dots, d).$$

From this formula,

$$\left|v^i(x_1) - v^i(x_2)\right| \le \sup_\Omega |f| \cdot \int_\Omega \left| \frac{x_1^i - y^i}{|x_1 - y|^d} - \frac{x_2^i - y^i}{|x_2 - y|^d} \right| \mathrm{d}y. \tag{13.1.6}$$

By the intermediate value theorem, on the line from x_1 to x_2, there exists some x_3 with

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[1] "sup" here is the essential supremum, as explained in the appendix.

$$\left|\frac{x_1^i - y^i}{|x_1 - y|^d} - \frac{x_2^i - y^i}{|x_2 - y|^d}\right| \leq \frac{c_3\,|x_1 - x_2|}{|x_3 - y|^d}. \tag{13.1.7}$$

We put $\delta := 2\,|x_1 - x_2|$. Since Ω is bounded, we can find $R > 0$ with $\Omega \subset B(x_3, R)$, and we replace the integral on Ω in (13.1.6) by the integral on $B(x_3, R)$, and we decompose the latter as

$$\int_{B(x_3,R)} = \int_{B(x_3,\delta)} + \int_{B(x_3,R)\setminus B(x_3,\delta)} = I_1 + I_2, \tag{13.1.8}$$

where without loss of generality, we may take $\delta < R$. We have

$$I_1 \leq 2\int_{B(x_3,\delta)} \frac{1}{|x_3 - y|^{d-1}}\mathrm{d}y = 2d\omega_d\delta \tag{13.1.9}$$

and by (13.1.7)

$$I_2 \leq c_4\delta(\log R - \log\delta), \tag{13.1.10}$$

and hence

$$I_1 + I_2 \leq c_5\,|x_1 - x_2|^\alpha \quad \text{for any } \alpha \in (0, 1).$$

This proves (a) because obviously we also have

$$\left|v^i(x)\right| \leq c_6 \sup_\Omega |f|\,. \tag{13.1.11}$$

(b) Up to a constant factor, the second derivatives of u are given by

$$w^{ij}(x) = \int \left(|x-y|^2\,\delta_{ij} - d\left(x^i - y^i\right)\left(x^j - y^j\right)\right)\frac{1}{|x-y|^{d+2}}f(y)\,\mathrm{d}y;$$

however, we still need to show that this integral is finite if our assumption $f \in C_0^\alpha(\Omega)$ holds. This will also follow from our subsequent considerations.

We first put $f(x) = 0$ for $x \in \mathbb{R}^d \setminus \Omega$; this does not affect the Hölder continuity of f. We write

$$\begin{aligned} K(x-y) &:= \left(|x-y|^2\,\delta_{ij} - d\left(x^i - y^i\right)\left(x^j - y^j\right)\right)\frac{1}{|x-y|^{d+2}} \\ &= \frac{\partial}{\partial x^j}\left(\frac{x^i - y^i}{|x-y|^d}\right). \end{aligned}$$

We have

$$\int_{R_1<|y|<R_2} K(y)\mathrm{d}y = \int_{|y|=R_2} \frac{y^j}{R_2} \cdot \frac{y^i}{|y|^d} - \int_{|y|=R_1} \frac{y^j}{R_1} \cdot \frac{y^i}{|y|^d}$$
$$= 0, \tag{13.1.12}$$

since $\frac{y^i}{|y|^d}$ is homogeneous of degree $1-d$. Thus also

$$\int_{\mathbb{R}^d} K(y)\mathrm{d}y = 0. \tag{13.1.13}$$

We now write

$$w^{ij}(x) = \int_{\mathbb{R}^d} K(x-y)f(y)\mathrm{d}y$$
$$= \int_{\mathbb{R}^d} (f(y)-f(x))\, K(x-y)\mathrm{d}y \tag{13.1.14}$$

by (13.1.13). As before, on the line from x_1 to x_2, there is some x_3 with

$$|K(x_1-y)-K(x_2-y)| \le \frac{c_7\,|x_1-x_2|}{|x_3-y|^{d+1}}. \tag{13.1.15}$$

We again put

$$\delta := 2\,|x_1-x_2|$$

and write [cf. (13.1.14)]

$$w^{ij}(x_1) - w^{ij}(x_2)$$
$$= \int_{\mathbb{R}^d} \{(f(y)-f(x_1))\, K(x_1-y) - (f(y)-f(x_2))\, K(x_2-y)\}\,\mathrm{d}y$$
$$= I_1 + I_2, \tag{13.1.16}$$

where I_1 denotes the integral on $B(x_1,\delta)$ and I_2 that on $\mathbb{R}^d \setminus B(x_1,\delta)$. Since $|f(y)-f(x)| \le \|f\|_{C^\alpha} \cdot |x-y|^\alpha$, it follows that

$$|I_1| \le \|f\|_{C^\alpha} \int_{B(x_1,\delta)} \{|K(x_1-y)|\,|x_1-y|^\alpha + |K(x_2-y)|\,|x_2-y|^\alpha\}\,\mathrm{d}y$$
$$\le c_8\,\|f\|_{C^\alpha} \cdot \delta^\alpha. \tag{13.1.17}$$

Moreover,

$$I_2 = \int_{\mathbb{R}^d \setminus B(x_1,\delta)} (f(x_2) - f(x_1))\, K(x_1 - y)\, \mathrm{d}y$$
$$+ \int_{\mathbb{R}^d \setminus B(x_1,\delta)} (f(y) - f(x_2))\, (K(x_1 - y) - K(x_2 - y))\, \mathrm{d}y, \qquad (13.1.18)$$

and the first integral vanishes because of (13.1.12). Employing (13.1.15), and since for $y \in \mathbb{R}^d \setminus B(x_1, \delta)$,

$$\frac{1}{|x_3 - y|^{d+1}} \leq \frac{c_9}{|x_1 - y|^{d+1}},$$

it follows that

$$|I_2| \leq c_{10}\delta \, \|f\|_{C^\alpha} \int_{\mathbb{R}^d \setminus B(x_1,\delta)} |x_1 - y|^{\alpha-d-1} \leq c_{11}\delta^\alpha \, \|f\|_{C^\alpha} . \qquad (13.1.19)$$

Inequality (13.1.5) then follows from (13.1.16), (13.1.17), and (13.1.19). ☐

Theorem 13.1.2. *As always, let $\Omega \subset \mathbb{R}^d$ be open and bounded, and $\Omega_0 \subset\subset \Omega$. Let u be a weak solution of $\Delta u = f$ in Ω.*

(a) If $f \in C^0(\Omega)$, then $u \in C^{1,\alpha}(\Omega)$, and

$$\|u\|_{C^{1,\alpha}(\Omega_0)} \leq c_{12} \left(\|f\|_{C^0(\Omega)} + \|u\|_{L^2(\Omega)} \right). \qquad (13.1.20)$$

(b) If $f \in C^\alpha(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$, and

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leq c_{13} \left(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^2(\Omega)} \right). \qquad (13.1.21)$$

Remark. The restriction $0 < \alpha < 1$ is essential for Theorem 13.1.2, as well as for the subsequent results. For example, in some neighborhood of 0, the function

$$u\left(x^1, x^2\right) = \left|x^1\right| \left|x^2\right| \log\left(\left|x^1\right| + \left|x^2\right|\right)$$

satisfies the inequality

$$|u| + |\Delta u| \leq \text{const},$$

while the mixed second derivative $\frac{\partial^2 u}{\partial x^1 \partial x^2}$ behaves like

$$\log\left(\left|x^1\right| + \left|x^2\right|\right).$$

Consequently, the $C^{1,1}$-norm of u cannot be controlled by pointwise bounds for $f := \Delta u$ and u.

Proof. We demonstrate the estimates (13.1.20) and (13.1.21) first under the assumption $u \in C^{2,\alpha}(\Omega)$. We may cover Ω_0 by finitely many balls that are contained in Ω. Therefore, it suffices to verify the estimates for the case

$$\Omega_0 = B(0, r),$$
$$\Omega = B(0, R), \quad 0 < r < R < \infty.$$

Let $0 < R_1 < R_2 < R$. We choose some $\eta \in C_0^\infty(B(0, R_2))$ with $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le R_1$, and

$$\|\eta\|_{C^{k,\alpha}(B(0,R_2))} \le c_{14}(R_2 - R_1)^{-k-\alpha}. \tag{13.1.22}$$

We put

$$\phi := \eta u. \tag{13.1.23}$$

Then ϕ vanishes outside of $B(0, R_2)$, and by (2.1.12),

$$\phi(x) = \int_\Omega \Gamma(x, y)\Delta\phi(y)\mathrm{d}y. \tag{13.1.24}$$

Here,

$$\Delta\phi = \eta\Delta u + 2Du \cdot D\eta + u\Delta\eta, \tag{13.1.25}$$

and so

$$\|\Delta\phi\|_{C^0} \le \|\Delta u\|_{C^0} + c_{15}\,\|\eta\|_{C^2} \cdot \|u\|_{C^1}\,, \tag{13.1.26}$$

and by Lemma 13.1.1

$$\|\Delta\phi\|_{C^\alpha} \le c_{16}\,\|\eta\|_{C^{2,\alpha}}\left(\|\Delta u\|_{C^\alpha} + \|u\|_{C^{1,\alpha}}\right), \tag{13.1.27}$$

where all norms are computed on $B(0, R_2)$. From Theorem 13.1.1 and (13.1.26) and (13.1.27), we obtain

$$\|\phi\|_{C^{1,\alpha}} \le c_{17}\left(\|\Delta u\|_{C^0} + \|\eta\|_{C^2}\,\|u\|_{C^1}\right) \tag{13.1.28}$$

and

$$\|\phi\|_{C^{2,\alpha}} \le c_{18}\,\|\eta\|_{C^{2,\alpha}}\left(\|\Delta u\|_{C^\alpha} + \|u\|_{C^{1,\alpha}}\right), \tag{13.1.29}$$

respectively. Since $u(x) = \phi(x)$ for $|x| \le R_1$, and recalling (13.1.22), we obtain

$$\|u\|_{C^{1,\alpha}(B(0,R_1))} \le c_{19}\left(\|\Delta u\|_{C^0(B(0,R_2))} + \frac{1}{(R_2 - R_1)^2}\,\|u\|_{C^1(B(0,R_2))}\,,\right) \tag{13.1.30}$$

and

$$\|u\|_{C^{2,\alpha}(B(0,R_1))} \le c_{20}\frac{1}{(R_2-R_1)^{2+\alpha}}\left(\|\Delta u\|_{C^\alpha(B(0,R_2))}+\|u\|_{C^{1,\alpha}(B(0,R_2))}\right) \tag{13.1.31}$$

respectively.

We now interrupt the proof for some auxiliary results:

Lemma 13.1.2.

(a) There exists a constant c_a such that for every $\rho > 0$ and any function $v \in C^1(B(0,\rho))$:

$$\|v\|_{C^0(B(0,\rho))} \le \|Dv\|_{C^0(B(0,\rho))} + c_a\|v\|_{L^2(B(0,\rho))} : \tag{13.1.32}$$

(b) There exists a constant c_b such that for every $\rho > 0$ and any function $v \in C^{1,\alpha}(B(0,\rho))$:

$$\|v\|_{C^1(B(0,\rho))} \le |Dv|_{C^\alpha(B(0,\rho))} + c_b\|v\|_{L^2(B(0,\rho))} \tag{13.1.33}$$

[here, $|Dv|_{C^\alpha}$ is the Hölder seminorm defined in (13.1.2)].

Proof. If (a) did not hold, for every $n \in \mathbb{N}$, we could find a radius ρ_n and a function $v_n \in C^1(B(0,\rho_n))$ with

$$1 = \|v_n\|_{C^0(B(0,\rho_n))} \ge \|Dv_n\|_{C^0(B(0,\rho_n))} + n\|v_n\|_{L^2(B(0,\rho_n))}. \tag{13.1.34}$$

We first consider the case where the radii ρ_n stay bounded for $n \to \infty$ in which case we may assume that they converge towards some radius ρ_0 and we can consider everything on the fixed ball $B(0,\rho_0)$.

Thus, in that situation, we have a sequence $v_n \in C^1(B(0,\rho_0))$ for which $\|v_n\|_{C^1(B(0,\rho_0))}$ is bounded. This implies that the v_n are equicontinuous. By the theorem of Arzela–Ascoli, after passing to a subsequence, we can assume that the v_n converge uniformly towards some $v_0 \in C^0(B(0,\rho))$ with $\|v_0\|_{C^0(B(0,\rho_0))} = 1$. But (13.1.34) would imply $\|v_0\|_{L^2(B(0,\rho_0))} = 0$; hence $v \equiv 0$, a contradiction.

It remains to consider the case where the ρ_n tend to ∞. In that case, we use (13.1.34) to choose points $x_n \in B(0,\rho_n)$ with

$$|v_n(x_n)| \ge \frac{1}{2}\|v_n\|_{C^0(B(0,\rho_n))} = \frac{1}{2}. \tag{13.1.35}$$

We then consider $w_n(x) := v_n(x+x_n)$ so that $w_n(0) \ge \frac{1}{2}$ while (13.1.34) holds for w_n on some fixed neighborhood of 0. We then apply the Arzela–Ascoli argument to the w_n to get a contradiction as before.

(b) is proved in the same manner. The crucial point now is that for a sequence v_n for which the norms $\|v_n\|_{C^{1,\alpha}}$ are uniformly bounded both the v_n and their first derivatives are equicontinuous. □

Lemma 13.1.3.

(a) For $\varepsilon > 0$, there exists $M(\varepsilon)\,(< \infty)$ such that for all $u \in C^1(B(0,1))$

$$\|u\|_{C^0(B(0,1))} \leq \varepsilon \|u\|_{C^1(B(0,1))} + M(\varepsilon) \|u\|_{L^2(B(0,1))} \tag{13.1.36}$$

for all $u \in C^{1,\alpha}$. For $\varepsilon \to 0$,

$$M(\varepsilon) \leq \text{const.}\, \varepsilon^{-d}. \tag{13.1.37}$$

(b) For every $\alpha \in (0,1)$ and $\varepsilon > 0$, there exists $N(\varepsilon)\,(< \infty)$ such that for all $u \in C^{1,\alpha}(B(0,1))$

$$\|u\|_{C^1(B(0,1))} \leq \varepsilon \|u\|_{C^{1,\alpha}(B(0,1))} + N(\varepsilon) \|u\|_{L^2(B(0,1))} \tag{13.1.38}$$

for all $u \in C^{1,\alpha}$. For $\varepsilon \to 0$,

$$N(\varepsilon) \leq \text{const.}\, \varepsilon^{-\frac{d+1}{\alpha}}. \tag{13.1.39}$$

(c) For every $\alpha \in (0,1)$ and $\varepsilon > 0$, there exists $Q(\varepsilon)\,(< \infty)$ such that for all $u \in C^{2,\alpha}(B(0,1))$

$$\|u\|_{C^{1,\alpha}(B(0,1))} \leq \varepsilon \|u\|_{C^{2,\alpha}(B(0,1))} + Q(\varepsilon) \|u\|_{L^2(B(0,1))} \tag{13.1.40}$$

for all $u \in C^{1,\alpha}$. For $\varepsilon \to 0$,

$$Q(\varepsilon) \leq \text{const.}\, \varepsilon^{-d-1-\alpha}. \tag{13.1.41}$$

Proof. We rescale:

$$u_\rho(x) := u\left(\frac{x}{\rho}\right), \ u_\rho : B(0,\rho) \to \mathbb{R}. \tag{13.1.42}$$

Equation (13.1.36) then is equivalent to

$$\|u_\rho\|_{C^0(B(0,\rho))} \leq \varepsilon\rho \|u_\rho\|_{C^1(B(0,\rho))} + M(\varepsilon)\rho^{-d} \|u_\rho\|_{L^2(B(0,\rho))}. \tag{13.1.43}$$

We choose ρ such that $\varepsilon\rho = 1$, i.e., $\rho = \varepsilon^{-1}$ and apply (a) of Lemma 13.1.2. This shows (13.1.43), and (a) follows.

For (b), we shall show

$$\|Du\|_{C^0(B(0,1))} \leq \varepsilon |Du|_{C^\alpha(B(0,1))} + N(\varepsilon) \|u\|_{L^2(B(0,1))}. \tag{13.1.44}$$

Combining this with (a) then shows the claim. We again rescale by (13.1.42). This transforms (13.1.44) into

$$\|Du_\rho\|_{C^0(B(0,\rho))} \leq \varepsilon\rho^\alpha |Du_\rho|_{C^\alpha(B(0,\rho))} + N(\varepsilon)\rho^{-d-1} \|u_\rho\|_{L^2(B(0,\rho))}. \tag{13.1.45}$$

We choose ρ such that $\varepsilon\rho^\alpha = 1$, i.e., $\rho = \varepsilon^{-\frac{1}{\alpha}}$ and apply (b) of Lemma 13.1.2. This shows (13.1.45) and completes the proof of (b).

(c) is proved in the same manner. □

We now continue the proof of Theorem 13.1.2:
For homogeneous polynomials $p(t), q(t)$, we define

$$A_1 := \sup_{0\le r\le R} p(R-r)\, \|u\|_{C^{1,\alpha}(B(0,r))}\,,$$

$$A_2 := \sup_{0\le r\le R} q(R-r)\, \|u\|_{C^{2,\alpha}(B(0,r))}\,.$$

For the proof of (a), we choose R_1 such that

$$A_1 \le 2p(R-R_1)\, \|u\|_{C^{1,\alpha}(B(0,R_1))}\,, \tag{13.1.46}$$

and for (b), such that

$$A_2 \le 2q(R-R_1)\, \|u\|_{C^{2,\alpha}(B(0,R_1))}\,. \tag{13.1.47}$$

(In general, the R_1 of (13.1.46) will not be the same as that of (13.1.47).) Then (13.1.30) and (13.1.38) imply

$$\begin{aligned} A_1 \le c_{21}\, p(R-R_1)\Big(&\|\Delta u\|_{C^0(B(0,R_2))} + \frac{\varepsilon}{(R_2-R_1)^2}\|u\|_{C^{1,\alpha}(B(0,R_2))} \\ &+ \frac{1}{(R_2-R_1)^2} N(\varepsilon)\, \|u\|_{L^2(B(0,R_2))}\Big) \\ \le c_{22}\, &\frac{p(R-R_1)}{p(R-R_2)} \cdot \frac{\varepsilon}{(R_2-R_1)^2} \cdot A_1 \\ &+ c_{23}\, p(R-R_1)\, \|\Delta u\|_{C^0(B(0,R_2))} + c_{24}\, N(\varepsilon) \frac{p(R-R_1)}{(R_2-R_1)^2}\, \|u\|_{L^2(B(0,R_2))}\,. \end{aligned} \tag{13.1.48}$$

We choose $R_2 = \frac{R+R_1}{2} \in (R_1, R)$. Then, because the polynomial p is homogeneous,

$$\frac{p(R-R_1)}{p(R-R_2)} = \frac{p(R-R_1)}{p(\frac{R-R_1}{2})}$$

is independent of R and R_1. Therefore,

$$\varepsilon = \frac{(R_2-R_1)^2}{2c_{22}} \frac{p(R-R_2)}{p(R-R_1)} \sim (R-R_1)^2$$

and

$$N(\varepsilon) \sim (R - R_1)^{-\frac{2(d+1)}{\alpha}}$$

by Lemma 13.1.2(b). Thus, when we choose

$$p(t) = t^{\frac{2(d+1)}{\alpha}+2},$$

the coefficient of $\|u\|_{L^2(B(0,R_2))}$ in (13.1.48) is controlled.

Thus, finally

$$\begin{aligned} \|u\|_{C^{1,\alpha}(B(0,r))} &\leq \frac{1}{p(R-r)} A_1 \\ &\leq c_{25} \left(\|\Delta u\|_{C^0(B(0,R))} + \|u\|_{L^2(B(0,R))} \right), \end{aligned} \tag{13.1.49}$$

with a constant that now also depends on the radii occurring.

In the same manner, from (13.1.31) and (13.1.40), we obtain

$$\|u\|_{C^{2,\alpha}(B(0,r))} \leq c_{26} \left(\|\Delta u\|_{C^\alpha(B(0,R))} + \|u\|_{L^2(B(0,R))} \right) \tag{13.1.50}$$

for $0 < r < R$. Since $\Delta u = f$, we have thus proved (13.1.20) and (13.1.21) for $u \in C^{2,\alpha}(\Omega)$.

For $u \in W^{1,2}(\Omega)$ we consider the mollifications u_h as in Lemma A.2 of the appendix. Let $0 < h < \operatorname{dist}(\Omega_0, \partial\Omega)$. Then

$$\int_\Omega Du_h \cdot Dv = -\int_\Omega f_h v \quad \text{for all } v \in H_0^{1,2}(\Omega),$$

and since $u_h \in C^\infty$, also

$$\Delta u_h = f_h.$$

Moreover, by Lemma A.2,

$$\|f_h - f\|_{C^0} \to 0,$$

and for $h \to 0$, the f_h therefore constitute a Cauchy sequence in $C^0(\Omega)$. Applying (13.1.20) to $u_{h_1} - u_{h_2}$, we obtain

$$\|u_{h_1} - u_{h_2}\|_{C^{1,\alpha}(\Omega_0)} \leq c_{27} \left(\|f_{h_1} - f_{h_2}\|_{C^0(\Omega)} + \|u_{h_1} - u_{h_2}\|_{L^2(\Omega)} \right). \tag{13.1.51}$$

The limit function u thus is contained in $C^{1,\alpha}(\Omega_0)$ and satisfies (13.1.20).

We also easily check that

$$\|f_h\|_{C^\alpha} \leq \|f\|_{C^\alpha}.$$

Therefore, by using the Arzela-Ascoli Theorem, the f_h converge to f in C^β for every $\beta < \alpha$ (see Section 5 in [19]). Hence

$$\|u_{h_1} - u_{h_2}\|_{C^{2,\beta}(\Omega_0)} \leq c_{28} \left(\|f_{h_1} - f_{h_2}\|_{C^\beta(\Omega)} + \|u_{h_1} - u_{h_2}\|_{L^2(\Omega)} \right). \quad (13.1.52)$$

The limit function u thus is contained in $C^{2,\beta}(\Omega_0)$ and satisfies (13.1.21) for every $\beta < \alpha$. Since the constant c_{28} in (13.1.52) and hence also the constant c_{13} in (13.1.21) can be taken to be independent of $\beta < \alpha$, we obtain (13.1.21) also for the exponent α, and hence u is contained in $C^{2,\alpha}(\Omega_0)$ and satisfies the required estimate. □

Part (a) of the preceding theorem can be sharpened as follows:

Theorem 13.1.3. *Let u be a weak solution of $\Delta u = f$ in Ω (Ω a bounded domain in $\mathbb{R}^d$), $f \in L^p(\Omega)$ for some $p > d$, $\Omega_0 \subset\subset \Omega$. Then $u \in C^{1,\alpha}(\Omega)$ for some α that depends on p and d, and*

$$\|u\|_{C^{1,\alpha}(\Omega_0)} \leq \text{const} \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Proof. Again, we consider the Newton potential

$$w(x) := \int_\Omega \Gamma(x, y) f(y) \mathrm{d}y,$$

and

$$v^i(x) := \int_\Omega \frac{x^i - y^i}{(x - y)^d} f(y) \mathrm{d}y.$$

Using Hölder's inequality, we obtain

$$\left| v^i(x) \right| \leq \|f\|_{L^p(\Omega)} \left(\int \frac{\mathrm{d}y}{|x - y|^{(d-1)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}},$$

and this expression is finite because of $p > d$. In this manner, one also verifies that $\frac{\partial}{\partial x^i} w = \text{const} v^i$ and obtains the Hölder estimate as in the proof of Theorem 13.1.1(a) and Theorem 13.1.2(a). □

Corollary 13.1.1. *If $u \in W^{1,2}(\Omega)$ is a weak solution of $\Delta u = f$ with $f \in C^{k,\alpha}(\Omega)$, $k \in \mathbb{N}$, $0 < \alpha < 1$, then $u \in C^{k+2,\alpha}(\Omega)$, and for $\Omega_0 \subset\subset \Omega$,*

$$\|u\|_{C^{k+2,\alpha}(\Omega_0)} \leq \text{const} \left(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

If $f \in C^\infty(\Omega)$, so is u.

Proof. Since $u \in C^{2,\alpha}(\Omega)$ by Theorem 13.1.2, we know that it weakly solves

$$\Delta \frac{\partial}{\partial x^i} u = \frac{\partial}{\partial x^i} f.$$

Theorem 13.1.2 then implies

$$\frac{\partial}{\partial x^i} u \in C^{2,\alpha}(\Omega) \quad (i \in \{1, \dots, d\}),$$

and thus $u \in C^{3,\alpha}(\Omega)$. The proof is concluded by induction. □

13.2 The Schauder Estimates

In this section, we study differential equations of the type

$$Lu(x) := \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2 u(x)}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \frac{\partial u(x)}{\partial x^i} + c(x)u(x) = f(x) \quad (13.2.1)$$

in some domain $\Omega \subset \mathbb{R}^d$. We make the following assumptions:

(A) Ellipticity: There exists $\lambda > 0$ such that for all $x \in \Omega$, $\xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 .$$

Moreover, $a^{ij}(x) = a^{ji}(x)$ for all i, j, x.

(B) Hölder continuous coefficients: There exists $K < \infty$ such that

$$\left\| a^{ij} \right\|_{C^\alpha(\Omega)}, \left\| b^i \right\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)} \leq K$$

for all i, j.

The fundamental estimates of J. Schauder are the following:

Theorem 13.2.1. *Let $f \in C^\alpha(\Omega)$, and suppose $u \in C^{2,\alpha}(\Omega)$ satisfies*

$$Lu = f \quad (13.2.2)$$

in Ω ($0 < \alpha < 1$). For any $\Omega_0 \subset\subset \Omega$, we then have

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leq c_1 \left(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^2(\Omega)} \right), \quad (13.2.3)$$

with a constant c_1 depending on $\Omega, \Omega_0, \alpha, d, \lambda, K$.

For the proof, we shall need the following lemma:

Lemma 13.2.1. *Let the symmetric matrix* $(A^{ij})_{i,j=1,\dots,d}$ *satisfy*

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} A^{ij}\xi_i\xi_j \le \Lambda\,|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^d \tag{13.2.4}$$

with

$$0 < \lambda < \Lambda < \infty.$$

Let u *satisfy*

$$\sum_{i,j=1}^{d} A^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} = f \tag{13.2.5}$$

with $f \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$. *For any* $\Omega_0 \subset\subset \Omega$, *we then have*

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \le c_2 \left(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{13.2.6}$$

Proof. We shall employ the following notation:

$$A := (A^{ij})_{i,j=1,\dots,d}, \quad D^2u := \left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right)_{i,j=1,\dots,d}.$$

If B is a nonsingular $d \times d$-matrix and if $y := Bx$, $v := u \circ B^{-1}$, i.e., $v(y) = u(x)$, we have

$$AD^2u(x) = AB^t D^2v(y)B,$$

and hence

$$\mathrm{Tr}(AD^2u(x)) = \mathrm{Tr}(BAB^t D^2v(y)). \tag{13.2.7}$$

Since A is symmetric, we may choose B such that $B^t A B$ is the unit matrix. In fact, B can be chosen as the product of the diagonal matrix

$$D = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & & \\ & \ddots & \\ & & \lambda_d^{-\frac{1}{2}} \end{pmatrix}$$

$(\lambda_1, \dots, \lambda_d$ being the eigenvalues of $A)$ with some orthogonal matrix R. In this way we obtain the transformed equation

$$\Delta v(y) = f\left(B^{-1}y\right). \tag{13.2.8}$$

Theorem 13.1.2 then yields $C^{2,\alpha}$-estimates for v, and these can be transformed back into estimates for $u = v \circ B$. The resulting constants will also depend on the bounds

λ, Λ for the eigenvalues of A, since these determine the eigenvalues of D and hence of B. □

Proof of Theorem 13.2.1: We shall show that for every $x_0 \in \bar{\Omega}_0$ there exists some ball $B(x_0, r)$ on which the desired estimate holds. The radius r of this ball will depend only on $\operatorname{dist}(\Omega_0, \partial\Omega)$ and the Hölder norms of the coefficients a^{ij}, b^i, c. Since $\bar{\Omega}_0$ is compact, it can be covered by finitely many such balls, and this yields the estimate in Ω_0.

Thus, let $x_0 \in \bar{\Omega}_0$. We rewrite the differential equation $Lu = f$ as

$$\begin{aligned}\sum_{i,j} a^{ij}(x_0) \frac{\partial^2 u(x)}{\partial x^i \partial x^j} &= \sum_{i,j} \left(a^{ij}(x_0) - a^{ij}(x)\right) \frac{\partial^2 u(x)}{\partial x^i \partial x^j} \\ &\quad - \sum_i b^i(x) \frac{\partial u(x)}{\partial x^i} - c(x)u(x) + f(x) \\ &=: \varphi(x). \end{aligned} \tag{13.2.9}$$

If we are able to estimate the C^α-norm of φ, putting $A^{ij} := a^{ij}(x_0)$ and applying Lemma 13.2.1 will yield the estimate of the $C^{2,\alpha}$-norm of u. The crucial term for the estimate of φ is $\sum (a^{ij}(x_0) - a^{ij}(x)) \frac{\partial^2 u}{\partial x^i \partial x^j}$. Let $B(x_0, R) \subset \Omega$. By Lemma 13.1.1

$$\begin{aligned}&\left| \sum_{i,j} \left(a^{ij}(x_0) - a^{ij}(x)\right) \frac{\partial^2 u(x)}{\partial x^i \partial x^j} \right|_{C^\alpha(B(x_0,R))} \\ &\quad \leq \sup_{i,j,x \in B(x_0,R)} \left| a^{ij}(x_0) - a^{ij}(x) \right| \left| D^2 u \right|_{C^\alpha(B(x_0,R))} \\ &\qquad + \sum_{i,j} \left| a^{ij} \right|_{C^\alpha(B(x_0,R))} \sup_{B(x_0,R)} \left| D^2 u \right|. \end{aligned} \tag{13.2.10}$$

Thus, also

$$\begin{aligned}&\left\| \sum \left(a^{ij}(x_0) - a^{ij}(x)\right) \frac{\partial^2 u}{\partial x^i \partial x^j} \right\|_{C^\alpha(B(x_0,R))} \\ &\quad \leq \sup \left| a^{ij}(x_0) - a^{ij}(x) \right| \|u\|_{C^{2,\alpha}(B(x_0,R))} + c_3 \|u\|_{C^2(B(x_0,R))}, \end{aligned} \tag{13.2.11}$$

where c_3 in particular depends on the C^α-norm of the a^{ij}.

Analogously,

$$\left\| \sum_i b^i(x) \frac{\partial u}{\partial x^i}(x) \right\|_{C^\alpha(B(x_0,R))} \leq c_4 \|u\|_{C^{1,\alpha}(B(x_0,R))}, \tag{13.2.12}$$

$$\|c(x)u(x)\|_{C^\alpha(B(x_0,R))} \leq c_5 \|u\|_{C^\alpha(B(x_0,R))}. \tag{13.2.13}$$

Altogether, we obtain

$$\|\varphi\|_{C^\alpha(B(x_0,R))} \leq \sup_{i,j,x\in B(x_0,R)} \left|a^{ij}(x_0) - a^{ij}(x)\right| \|u\|_{C^{2,\alpha}(B(x_0,R))} + c_6 \|u\|_{C^2(B(x_0,R))} + \|f\|_{C^\alpha(B(x_0,R))}. \tag{13.2.14}$$

By Lemma 13.2.1, from (13.2.9) and (13.2.14) for $0 < r < R$, we obtain

$$\|u\|_{C^{2,\alpha}(B(x_0,r))} \leq c_7 \sup_{i,j,x\in B(x_0,R)} \left|a^{ij}(x_0) - a^{ij}(x)\right| \|u\|_{C^{2,\alpha}(B(x_0,R))} + c_8 \|u\|_{C^2(B(x_0,R))} + c_9 \|f\|_{C^\alpha(B(x_0,R))}. \tag{13.2.15}$$

Since the a^{ij} are continuous on Ω, we may choose $R > 0$ so small that

$$c_7 \sup_{i,j,x\in B(x_0,R)} \left|a^{ij}(x_0) - a^{ij}(x)\right| \leq \frac{1}{2}. \tag{13.2.16}$$

With the same method as in the proof of Theorem 13.1.2, the corresponding term can be absorbed in the left-hand side. We then obtain from (13.2.15)

$$\|u\|_{C^{2,\alpha}(B(x_0,R))} \leq 2c_8 \|u\|_{C^2(B(x_0,R))} + 2c_9 \|f\|_{C^\alpha(B(x_0,R))}. \tag{13.2.17}$$

By (13.1.40), for every $\varepsilon > 0$, there exists some $Q(\varepsilon)$ with

$$\|u\|_{C^2(B(x_0,R))} \leq \varepsilon \|u\|_{C^{2,\alpha}(B(x_0,R))} + Q(\varepsilon) \|u\|_{L^2(B(x_0,R))}. \tag{13.2.18}$$

With the same method as in the proof of Theorem 13.1.2, from (13.2.18) and (13.2.17), we deduce the desired estimate

$$\|u\|_{C^{2,\alpha}(B(x_0,R))} \leq c_{10} \left(\|f\|_{C^\alpha(B(x_0,R))} + \|u\|_{L^2(B(x_0,R))}\right). \tag{13.2.19}$$

We may now state the global estimate of J. Schauder for the solution of the Dirichlet problem for L:

Theorem 13.2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^{2,\alpha}$ (analogously to Definition 11.3.1, we require the same properties as there, except that (iii) is replaced by the condition that ϕ and ϕ^{-1} are of class $C^{2,\alpha}$). Let $f \in C^\alpha(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$ (as in Definition 11.3.2), and let $u \in C^{2,\alpha}(\bar{\Omega})$ satisfy*

$$\begin{aligned} Lu(x) &= f(x) && \text{for } x \in \Omega, \\ u(x) &= g(x) && \text{for } x \in \partial\Omega. \end{aligned} \tag{13.2.20}$$

Then

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq c_{11} \left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}\right), \tag{13.2.21}$$

with a constant c_{11} depending on $\Omega, \alpha, d, \lambda$, and K.

The Proof essentially is a modification of that of Theorem 13.2.1, with modifications that are similar to those employed in the proof of Theorem 11.3.3. We shall therefore provide only a sketch of the proof. We start with a simplified model situation, namely, the Poisson equation in a half-ball, from which we shall derive the general case.

As in Sect. 11.3, let

$$B^+(0,R) = \left\{x = (x^1,\dots,x^d) \in \mathbb{R}^d; |x| < R, x^d > 0\right\}.$$

Moreover, let

$$\partial^0 B^+(0,R) := \partial B^+(0,R) \cap \left\{x^d = 0\right\},$$
$$\partial^+ B^+(0,R) := \partial B^+(0,R) \setminus \partial^0 B^+(0,R).$$

We consider $f \in C^\alpha\left(\overline{B^+(0,R)}\right)$ with

$$f = 0 \quad \text{on } \partial^+ B^+(0,R).$$

In contrast to the situation considered in Theorem 13.1.1(b), f no longer must vanish on all of the boundary of our domain $\Omega = B^+(0,R)$, but only on a certain portion of it. Again, we consider the corresponding Newton potential

$$u(x) := \int_{B^+(0,R)} \Gamma(x,y) f(y)\mathrm{d}y. \tag{13.2.22}$$

Up to a constant factor, the first derivatives of u are given by

$$v^i(x) = \int_{B^+(0,R)} \frac{x^i - y^i}{|x-y|^d} f(y)\mathrm{d}y \quad (i = 1,\dots,d), \tag{13.2.23}$$

and they can be estimated as in the proof of Theorem 13.1.1(a), since there, we did not need any assumption on the boundary values.

Up to a constant factor, the second derivatives are given by

$$w^{ij}(x) = \int_{B^+(0,R)} \frac{\partial}{\partial x^j}\left(\frac{x^i - y^i}{|x-y|^d}\right) f(y)\mathrm{d}y \quad \left(= w^{ji}(x)\right). \tag{13.2.24}$$

For $K(x-y) = \frac{\partial}{\partial x^j}\left(\frac{x^i-y^i}{|x-y|^d}\right)$, and $i \neq d$ or $j \neq d$,

$$\int_{\substack{R_1<|y|<R_2 \\ y^d>0}} K(y)\mathrm{d}y = 0 \tag{13.2.25}$$

by homogeneity as in (13.1.12). Thus, for $i \neq d$ or $j \neq d$, the α-Hölder norm of the second derivative $\frac{\partial^2}{\partial x^i \partial x^j} u$ can be estimated as in the proof of Theorem 13.1.1(b). The differential equation $\Delta u = f$ implies

$$\frac{\partial^2}{(\partial x^d)^2} u = f - \sum_{i=1}^{d-1} \frac{\partial^2}{(\partial x^i)^2} u, \tag{13.2.26}$$

and so we obtain estimates for the α-Hölder norm of $\frac{\partial^2}{(\partial x^d)^2} u$ as well. We can thus estimate all second derivatives of u.

As in the proof of Theorem 13.1.2, we then obtain $C^{2,\alpha}$-estimates in $B^+(0, R)$ for solutions of

$$\begin{aligned} \Delta u &= f \qquad \text{in } B^+(0, R) \quad \text{with } f \in C^\alpha\left(\overline{B^+(0, R)}\right), \\ u &= 0 \qquad \text{on } \partial^0 B^+(0, R), \end{aligned} \tag{13.2.27}$$

for $0 < r < R$:

$$\|u\|_{C^{2,\alpha}(B^+(0,r))} \leq c_{12} \left(\|f\|_{C^\alpha(B^+(0,R))} + \|u\|_{L^2(B^+(0,R))}\right). \tag{13.2.28}$$

Namely, putting

$$\varphi := \eta u$$

as in (13.1.23) with the same cutoff function as in (13.1.22), we have $\varphi = 0$ on $\partial B^+(0, R_2)$ $(0 < R_1 < R_2 < R)$, since η vanishes on $\partial^+ B^+(0, R_2)$, and u on $\partial^0 B^+(0, R_2)$. Thus, again

$$\varphi(x) = \int_{B^+(0,R)} \Gamma(x, y) \Delta\varphi(y) \mathrm{d}y$$

is a Newton potential, and the preceding estimates can be used to deduce the same result as in Theorem 13.1.2: For $0 < r < R$,

$$\|u\|_{C^{2,\alpha}(B^+(0,r))} \leq c_{13} \left(\|f\|_{C^\alpha(B^+(0,R))} + \|u\|_{L^2(B^+(0,R))}\right). \tag{13.2.29}$$

We next consider a solution of

$$\Delta u = f \quad \text{in } B^+(0, R) \quad \text{with } f \in C^\alpha\left(\overline{B^+(0, R)}\right), \tag{13.2.30}$$

$$u = g \quad \text{on } \partial^0 B^+(0, R) \quad \text{with } g \in C^{2,\alpha}\left(\overline{B^+(0, R)}\right). \tag{13.2.31}$$

As in Sect. 11.3, we put $\bar{u} := u - g$. We see that $\bar{u}$ satisfies

$$\begin{aligned} \Delta \bar{u} &= f - \Delta g =: \bar{f} \in C^\alpha\left(\overline{B^+(0, R)}\right) \text{ in } B^+(0, R), \\ \bar{u} &= 0 \text{ on } \partial^0 B^+(0, R). \end{aligned}$$

We have thus reduced our considerations to the above case (13.2.27), and so, from (13.2.29), we obtain

$$\begin{aligned}\|u\|_{C^{2,\alpha}(B^+(0,r))} &\leq \|\bar{u}\|_{C^{2,\alpha}(B^+(0,r))} + \|g\|_{C^{2,\alpha}(B^+(0,r))} \\ &\leq c_{14}\left[\|\bar{f}\|_{C^\alpha(B^+(0,R))} + \|\bar{u}\|_{L^2(B^+(0,R))} + \|g\|_{C^{2,\alpha}(B^+(0,R))}\right] \\ &\leq c_{15}\left[\|f\|_{C^\alpha(B^+(0,R))} + \|g\|_{C^{2,\alpha}(B^+(0,R))} + \|u\|_{L^2(B^+(0,R))}\right].\end{aligned} \tag{13.2.32}$$

In order to finally treat the situation of Theorem 13.2.2, as in Sect. 11.3, we transform a neighborhood U of a boundary point $x_0 \in \partial\Omega$ with a $C^{2,\alpha}$-diffeomorphism ϕ to the ball $\mathring{B}(0,R)$, such that the portion of u that is contained in Ω is mapped to $B^+(0,R)$, and the intersection of U with $\partial\Omega$ is mapped to $\partial^0 B^+(0,R)$. Again, $\tilde{u} := u \circ \phi^{-1}$ on $B^+(0,R)$ satisfies a differential equation of the same type as $Lu = f$, $\tilde{L}\tilde{u} = \tilde{f}$, again with different constants λ, K in (A) and (B). By the preceding considerations, we obtain a $C^{2,\alpha}$-estimate for $\tilde{u}$ in $B^+(0,R/2)$. Again ϕ transforms this estimate into one for u on a subset U' of U. Since Ω is bounded, $\partial\Omega$ is compact and can thus be covered by finitely many such neighborhoods U'. The resulting estimates, together with the interior estimate of Theorem 13.2.1, applied to the complement Ω_0 of those neighborhoods in Ω, yield the claim of Theorem 13.2.2.

Corollary 13.2.1. *In addition to the assumptions of Theorem 13.2.2, suppose that $c(x) \leq 0$ in Ω. Then*

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq c_{16}\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)}\right). \tag{13.2.33}$$

Proof. Because of $c \leq 0$, the maximum principle (see, e.g., Theorem 3.3.2) implies

$$\sup_\Omega |u| \leq \max_{\partial\Omega} |u| + c_{17} \sup_\Omega |f| = \max_{\partial\Omega} |g| + c_{17} \sup_\Omega |f|.$$

Therefore, the L^2-norm of u can be estimated in terms of the C^0-norms of f and g, and the claim follows from (13.2.21). □

13.3 Existence Techniques IV: The Continuity Method

In this section, we wish to study the existence problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

in a $C^{2,\alpha}$-region Ω with $f \in C^\alpha(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$. The starting point for our considerations will be the corresponding result for the Poisson equation, Kellogg's theorem:

Theorem 13.3.1. *Let Ω be a bounded domain of class C^∞ in $\mathbb{R}^d$, $f \in C^\alpha(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$. The Dirichlet problem*

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{13.3.1}$$

then possesses a unique solution u of class $C^{2,\alpha}(\bar{\Omega})$.

Proof. Uniqueness follows from the maximum principle (see Corollary 3.1.1). For the existence proof, we first assume that f and g are of class C^∞. The variational methods of Sect. 10.3 yield a weak solution, which then is of class $C^\infty(\Omega)$ by Theorem 11.3.1. Moreover, by Corollary 13.2.1,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq c_1 \left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} \right). \tag{13.3.2}$$

We now return to the $C^{2,\alpha}$-case. We approximate f and g by C^∞-functions f_n and g_n that are defined on Ω. Let u_n be the solution of the corresponding Dirichlet problem

$$\begin{aligned} \Delta u_n &= f_n \quad \text{in } \Omega, \\ u_n &= g_n \quad \text{on } \partial\Omega. \end{aligned}$$

For $n \geq m$, $u_n - u_m$ then satisfies (13.3.2) on Ω, i.e.,

$$\|u_n - u_m\|_{C^{2,\alpha}(\Omega)} \leq c_1 \left(\|f_n - f_m\|_{C^\alpha(\Omega)} + \|g_n - g_m\|_{C^{2,\alpha}(\Omega)} \right). \tag{13.3.3}$$

Here, the constant c_1 does not depend on the solutions; it depends only on the $C^{2,\alpha}$-geometry of the domain. We assume that f_n converges to f in $C^\alpha(\Omega)$, and g_n to g in $C^{2,\alpha}(\Omega)$, and so the u_n constitute a Cauchy sequence in $C^{2,\alpha}(\Omega)$ and therefore converge towards some $u \in C^{2,\alpha}(\Omega)$ that satisfies

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

and the estimate (13.3.2). □

We now state the main existence result of this chapter:

Theorem 13.3.2. *Let Ω be a bounded domain of class C^∞ in $\mathbb{R}^d$. Let the differential operator*

$$L = \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x^i} + c(x) \tag{13.3.4}$$

satisfy (A) and (B) from Sect. 13.2, and in addition,

$$c(x) \le 0 \quad \text{in } \Omega. \tag{13.3.5}$$

For any $f \in C^{\alpha}(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$ *there then exists a unique solution* $u \in C^{2,\alpha}(\bar{\Omega})$ *of the Dirichlet problem*

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{13.3.6}$$

Remark. It is quite instructive to compare this result and its assumptions with Theorem 11.4.4.

Proof. Considering, as usual, $\bar{u} = u - g$ in place of u, we may assume $g = 0$, as our problem is equivalent to

$$\begin{aligned} L\bar{u} &= \bar{f} := f - Lg \in C^{\alpha}(\Omega), \\ \bar{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We thus assume $g = 0$ (and write u in place of $\bar{u}$). We consider the family of equations

$$\begin{aligned} L_t u &= f \quad \text{for } 0 \le t \le 1, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{13.3.7}$$

with

$$L_t = tL + (1-t)\Delta. \tag{13.3.8}$$

The differential operators L_t satisfy the structural conditions (A) and (B) with

$$\lambda_t = \min(1, \lambda), \quad K_t = \max(1, K). \tag{13.3.9}$$

We have $L_0 = \Delta$, $L_1 = L$. By Theorem 13.3.1, we can solve (13.3.7) for $t = 0$. We intend to show that we may then also solve this equation for all $t \in [0, 1]$, in particular for $t = 1$. The latter is what is claimed in the theorem.

The operator

$$L_t : B_1 := C^{2,\alpha}(\bar{\Omega}) \cap \{u : u = 0 \quad \text{on } \partial\Omega\} \to C^{\alpha}(\bar{\Omega}) =: B_2$$

is a bounded linear operator between the Banach spaces B_1 and B_2. Let u_t be a solution of $L_t u_t = f$, $u_t = 0$ on $\partial\Omega$. By Corollary 13.2.1,

$$\|u_t\|_{C^{2,\alpha}(\Omega)} \le c_2 \|f\|_{C^{\alpha}(\Omega)},$$

i.e.,

$$\|u\|_{B_1} \le c_2 \|L_t u\|_{B_2}, \tag{13.3.10}$$

for all $u \in B_1$. Here, the constant c_2 does not depend on t, because by (13.3.9), the structure constants λ_t, K_t of the operators L_t can be controlled independently of t.

We want to show that for any $f \in B_2$ there exists a solution u_t of (13.3.7), i.e., of $L_t u_t = f$, in B_1. In other words, we want to show that the operators $L_t : B_1 \to B_2$ are surjective for $0 \le t \le 1$. This, however, follows from the general result stated as the next theorem. With that result, we then conclude the proof of Theorem 13.3.2.

Theorem 13.3.3. *Let $L_0, L_1 : B_1 \to B_2$ be bounded linear operators between the Banach spaces B_1, B_2. We put*

$$L_t := (1-t)L_0 + tL_1 \quad \text{for } 0 \le t \le 1.$$

We assume that there exists a constant c that does not depend on t, with

$$\|u\|_{B_1} \le c \|L_t u\|_{B_2} \quad \text{for all } u \in B_1. \tag{13.3.11}$$

If then L_0 is surjective, so is L_1.

Proof. Let L_τ be surjective for some $\tau \in [0, 1]$. By (13.3.11), L_τ then is injective as well, and thus bijective. We therefore have an inverse operator

$$L_\tau^{-1} : B_2 \to B_1.$$

For $t \in [0, 1]$, we rewrite the equation

$$L_t u = f \quad \text{for } f \in B_2 \tag{13.3.12}$$

as

$$L_\tau u = f + (L_\tau - L_t)u = f + (t-\tau)(L_0 u - L_1 u),$$

or

$$u = L_\tau^{-1} f + (t-\tau) L_\tau^{-1}(L_0 - L_1)u =: \Lambda u.$$

Thus, for solving (13.3.12), we need to find a fixed point of the operator $\Lambda : B_1 \to B_2$. By the Banach fixed point theorem, such a fixed-point exists if we can find some $q < 1$ with

$$\|\Lambda u - \Lambda v\|_{B_1} \le q \|u - v\|_{B_1}.$$

We have

$$\|\Lambda u - \Lambda v\| \le \left\| L_\tau^{-1} \right\| (\|L_0\| + \|L_1\|) \, |t-\tau| \, \|u - v\|.$$

By (13.3.11), $\left\| L_\tau^{-1} \right\| \leq c$. Therefore, it suffices to choose

$$|t - \tau| \leq \frac{1}{2} \left(c \left(\|L_0\| + \|L_1\| \right) \right)^{-1} =: \eta$$

for obtaining the desired fixed point. This means that if $L_\tau u = f$ is solvable, so is $L_t u = f$ for all t with $|t - \tau| \leq \eta$. Since L_0 is surjective by assumption, L_t then is surjective for $0 \leq t \leq \eta$. Repeating the preceding argument, this time for $\tau = \eta$, we obtain surjectivity for $\eta \leq t \leq 2\eta$. Iteratively, all L_t for $t \in [0, 1]$, and in particular L_1, are surjective. $\square$

Basic references about Schauder's approach are [2, 12]. Our treatment of the fundamental C^α-estimate for the Poisson equation uses scaling relations in place of the usual weighted Hölder spaces and is hopefully a little simpler.

Summary

A solution of the Poisson equation

$$\Delta u = f$$

with α-Hölder continuous f is contained in the space $C^{2,\alpha}$; i.e., it possesses α-Hölder continuous second derivatives for $0 < \alpha < 1$. (This is no longer true for $\alpha = 0$ or $\alpha = 1$. For example, if f is only continuous, a solution need not be twice continuously differentiable.) By linear coordinate transformations this result can be easily extended to linear elliptic differential equations with constant coefficients. Schauder then succeeded in extending these results to solutions of elliptic equations

$$Lu(x) := \sum_{i,j} a^{ij}(x) \frac{\partial^2 u(x)}{\partial x^i \partial x^j} + \sum_i b^i(x) \frac{\partial u}{\partial x^i} + c(x)u(x) = f(x)$$

with α-Hölder continuous coefficients, by considering such an operator L as a local perturbation of an operator with constant coefficients a^{ij}, b^i, c.

The continuity method reduces the solution of

$$Lu = f$$

to that of the Poisson equation

$$\Delta u = f$$

by considering the operators

$$L_t := tL + (1 - t)\Delta$$

for $0 \leq t \leq 1$, and showing that the set of $t \in [0,1]$ for which

$$L_t u = f$$

can be solved is open and closed (and nonempty, because the Poisson equation can be solved). The proof of closedness rests on Schauder's estimates.

Exercises

13.1. Let $K \subset \mathbb{R}^d$ be bounded, $f_n : K \to \mathbb{R}$ ($n \in \mathbb{N}$) a sequence of functions with

$$\|f_n\|_{C^\alpha(K)} \leq \text{const} \quad \text{(independent of } n\text{)},$$

for some $0 < \alpha \leq 1$. (Here and in the next exercise, in the case $\alpha = 1$, we consider the space $C^{0,1}$ of Lipschitz continuous functions.) Show that $(f_n)_{n\in\mathbb{N}}$ has to contain a uniformly convergent subsequence.

13.2. Is it true that for all domains $\Omega \subset \mathbb{R}^d$, $0 < \alpha < \beta \leq 1$,

$$C^\beta(\Omega) \subset C^\alpha(\Omega)?$$

13.3. Let $u \in C^{k,\alpha}(\Omega)$ satisfy

$$Lu = f$$

for some $f \in C^{k,\alpha}(\Omega)$ ($k \in \mathbb{N}, 0 < \alpha < 1$). Here, we assume that the operator L from (13.2.1) satisfies the ellipticity condition (A) as well as

$$\left\|a^{ij}\right\|_{C^{k,\alpha}(\Omega)}, \left\|b^i\right\|_{C^{k,\alpha}(\Omega)}, \|c\|_{C^{k,\alpha}(\Omega)} \leq K$$

for all i, j. Show that $u \in C^{k+2,\alpha}(\Omega_0)$ for any $\Omega_0 \subset\subset \Omega$, and

$$\|u\|_{C^{k+2,\alpha}(\Omega)} \leq c(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}),$$

with a constant c depending on K and the quantities of Theorem 13.2.1.