5 Radiation Mechanisms

In the previous chapters we have studied the physical properties and detection of electromagnetic radiation. Next we shall briefly discuss concepts related to emission and absorption of radiation. Since we can give here only a summary of some essential results without delving into quantum mechanical explanations, the reader interested in the details is advised to consult any good physics textbook.

5.1 Radiation of Atoms and Molecules

Electromagnetic radiation is emitted or absorbed when an atom or a molecule moves from one energy level to another. If the energy of the atom decreases by an amount ΔE, the atom emits or radiates a quantum of electromagnetic radiation, called a *photon*, whose frequency ν is given by the equation

$$\Delta E = h\nu, \tag{5.1}$$

where h is the *Planck constant*, $h = 6.6256 \times 10^{-34}\ \mathrm{J\,s}$. Similarly, if the atom receives or absorbs a photon of a frequency ν, its energy increases by $\Delta E = h\nu$.

The classical model describes an atom as a nucleus surrounded by a swarm of electrons. The nucleus consists of Z protons, each having a charge $+e$ and N electrically neutral neutrons; Z is the *charge number* of the atom and $A = Z + N$ is its *mass number*. A neutral atom has as many electrons (charge $-e$) as protons.

An energy level of an atom usually refers to an energy level of its electrons. The energy E of an electron cannot take arbitrary values; only certain energies are allowed: the energy levels are *quantised*. An atom can emit or absorb radiation only at certain frequencies ν_{if} corresponding to energy differences between some initial and final states i and f: $|E_{\mathrm{i}} - E_{\mathrm{f}}| = h\nu_{\mathrm{if}}$. This gives rise to the *line spectrum*, specific for each element (Fig. 5.1). Hot gas under low pressure produces an *emission spectrum* consisting of such discrete lines. If the same gas is cooled down and observed against a source of white light (which has a continuous spectrum), the same lines are seen as dark *absorption lines*.

At low temperatures most atoms are in their lowest energy state, the *ground state*. Higher energy levels are *excitation states*; a transition from lower to higher state is called *excitation*. Usually the excited atom will return to the lower state very rapidly, radiating a photon (*spontaneous emission*); a typical lifetime of an excited state might be 10^{-8} seconds. The frequency of the emitted photon is given by (5.1). The atom may return to the lower state directly or through some intermediate states, emitting one photon in each transition.

Downward transitions can also be induced by radiation. Suppose our atom has swallowed a photon and become excited. Another photon, whose frequency ν corresponds to some possible downward transition from the excited state, can now irritate the atom, causing it to jump to a lower state, emitting a photon with the same

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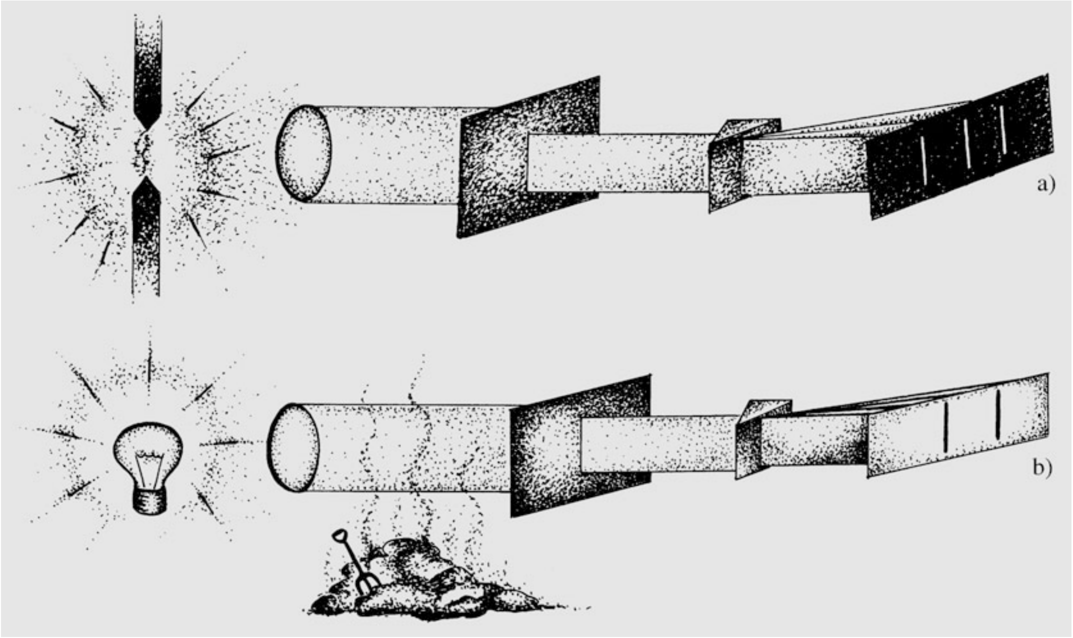

Fig. 5.1 Origin of line spectra. (**a**) Emission spectrum. Atoms of glowing gas returning from excited states to lower states emit photons with frequencies corresponding to the energy difference of the states. Each element emits its own characteristic wavelengths, which can be measured by spreading the light into a spectrum with a prism or diffraction grating. (**b**) Absorption spectrum. When white light containing all wavelengths travels through gas, the wavelengths characteristic of the gas are absorbed

frequency ν. This is called *induced* or *stimulated emission*. Photons emitted spontaneously leave the atom randomly in all directions with random phases: the radiation is isotropic and incoherent. Induced radiation, on the other hand, is *coherent*; it propagates in the same direction as and in phase with the inducing radiation.

Also collisions of atoms cause transitions up and down. In these transitions photons are not emitted, instead the kinetic energies of the atoms are changed. (An upward transition can, of course, cause e.g. a spontaneous emission.) The higher the density of the gas the more dominant the collisional transitions are.

The zero level of the energy states is usually chosen so that a bound electron has negative energy and a free electron positive energy (cf. the energy integral of planetary orbits, Chap. 6). If an electron with energy $E < 0$ receives more energy than $|E|$, it will leave the atom, which becomes an ion. In astrophysics ionisation is often called a *bound-free* transition (Fig. 5.2). Unlike in excitation all values of energy ($E > 0$) are now possible. The extraneous part of the absorbed energy goes to the kinetic energy of the liberated electron. The inverse process, in which an atom captures a free electron, is the *recombination* or free–bound transition.

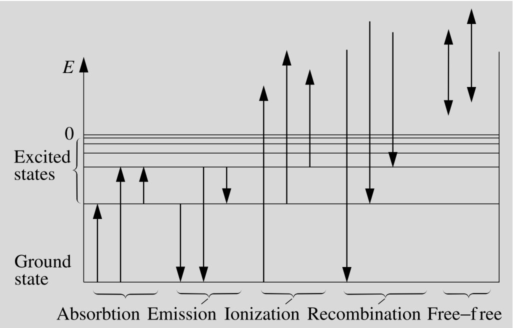

Fig. 5.2 Different kinds of transitions between energy levels. Absorption and emission occur between two bound states, whereas ionisation and recombination occur between a bound and a free state. Interaction of an atom with an free electron can result in a free–free transition

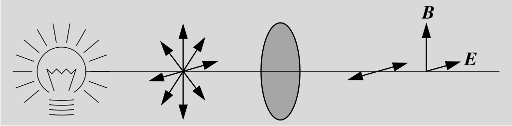

Fig. 5.3 Polarisation of light. The light of an incandescent bulb contains all possible directions of vibration and is therefore unpolarised. Some crystals, for example, pass electric fields oscillating only in certain directions, and the transmitted part of the light becomes linearly polarised. ***E*** is the electric field and ***B*** the magnetic field

When an electron scatters from a nucleus or an ion without being captured, the electromagnetic interaction can change the kinetic energy of the electron producing *free–free* radiation. In a very hot gas ($T > 10^6$ K) hydrogen is fully ionised, and the free–free radiation is the most important source of emission. It is then usually called *thermal bremsstrahlung*. The latter part of the name derives from the fact that decelerating electrons hitting the anode of an X-ray tube emit similar radiation. In an analogous way the absorption process can be called a bound–bound transition.

Electromagnetic radiation is transverse wave motion; the electric and magnetic fields oscillate perpendicular to each other and also perpendicular to the direction of propagation. The light of an ordinary incandescent lamp has a random distribution of electric fields vibrating in all directions. If the directions of electric fields in the plane perpendicular to the direction of propagation are not evenly distributed, the radiation is *polarised* (Fig. 5.3). The direction of polarisation of *linearly polarised* light means the plane determined by the electric vector and the direction of the light ray. If the electric vector describes a circle, the radiation is *circularly polarised*. If the amplitude of the electric field varies at the same time, the polarisation is *elliptic*.

If polarised radiation travels through a magnetic field, the direction of the polarisation will rotate. The amount of such *Faraday rotation* is proportional to the component of the magnetic field parallel to the line of sight, number of electrons along the line of sight, distance travelled, and square of the wavelength of the radiation.

Scattering is an absorption followed by an instantaneous emission at the same wavelength but usually in a new direction. On the macroscopic scale, radiation seems to be reflected by the medium. The light coming from the sky is sunlight scattered from atmospheric molecules. Scattered light is always polarised, the degree of polarisation being highest in the direction perpendicular to the direction of the original radiation.

5.2 The Hydrogen Atom

The hydrogen atom is the simplest atom, consisting of a proton and an electron. According to the Bohr model the electron orbits the proton in a circular orbit. (In spite of the fact that this model has very little to do with reality, it can be successfully used to predict some properties of the hydrogen atom.) Bohr's first postulate says that the angular momentum of the electron must be a multiple of $\hbar$:

$$mvr = n\hbar, \tag{5.2}$$

where

$m =$ mass of the electron,
$v =$ speed of the electron,
$r =$ radius of the orbit,
$n =$ the principal quantum number,
$n = 1, 2, 3, \ldots,$
$\hbar = h/2\pi,$
$h =$ the Planck constant.

The quantum mechanical interpretation of Bohr's first postulate is obvious: the electron is represented as a standing wave, and the "length of the orbit" must be a multiple of the de Broglie wavelength, $\lambda = \hbar/p = \hbar/mv$.

A charged particle in a circular orbit (and thus in accelerated motion) should emit electromagnetic radiation, losing energy, were it to obey the rules of classical electrodynamics. Therefore our electron should spiral down towards the nucleus. But obviously, Nature does not behave this way, and we have to accept Bohr's second postulate, which says that an electron moving in an allowed orbit around a nucleus does not radiate. Radia-

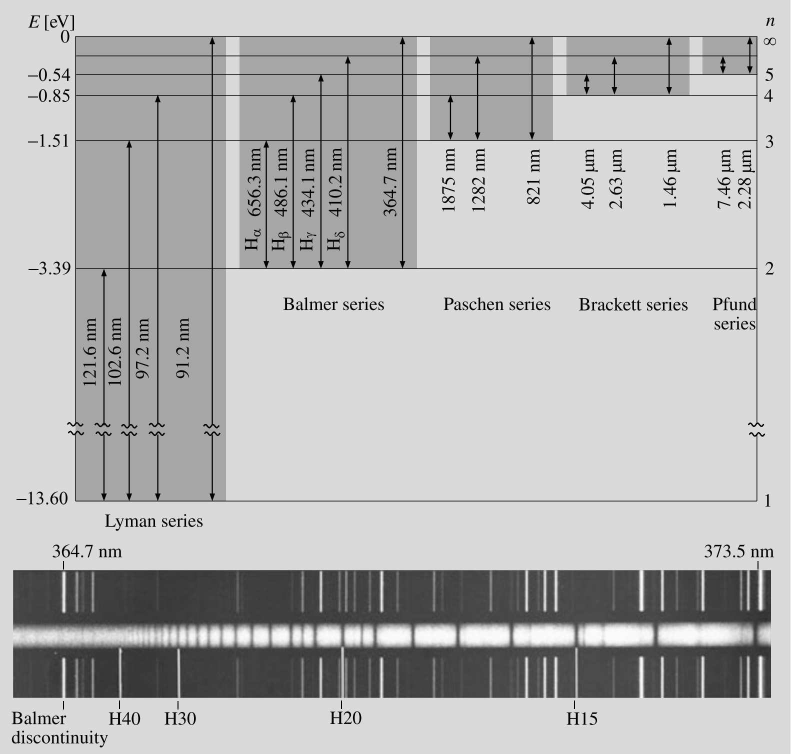

Fig. 5.4 Transitions of a hydrogen atom. *The lower picture* shows a part of the spectrum of the star HD193182. On both sides of the stellar spectrum we see an emission spectrum of iron. The wavelengths of the reference lines are known, and they can be used to calibrate the wavelengths of the observed stellar spectrum. The hydrogen Balmer lines are seen as dark absorption lines converging towards the Balmer ionisation limit (also called the Balmer discontinuity) at $\lambda = 364.7$ nm to the left. The numbers $(15, \ldots, 40)$ refer to the quantum number n of the higher energy level. (Photo by Mt. Wilson Observatory)

tion is emitted only when the electron jumps from a higher energy state to a lower one. The emitted quantum has an energy $h\nu$, equal to the energy difference of these states:

$$h\nu = E_{n_2} - E_{n_1}. \tag{5.3}$$

We shall now try to find the energy of an electron in the state E_n. Coulomb's law gives the force pulling the electron towards the proton:

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2}, \tag{5.4}$$

where

$\epsilon_0 =$ the vacuum permittivity

$= 8.85 \times 10^{-12}\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}\,\mathrm{C}^2$,

$e =$ the charge of the electron $= 1.6 \times 10^{-19}$ C,

$r_n =$ the distance between the electron and the proton.

The acceleration of a particle moving in a circular orbit of radius r_n is

$$a = \frac{v_n^2}{r_n},$$

and applying Newton's second law ($F = ma$), we get

$$\frac{mv_n^2}{r_n} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2}. \tag{5.5}$$

From (5.2) and (5.5) it follows that

$$v_n = \frac{e^2}{4\pi\epsilon_0\hbar}\frac{1}{n}, \qquad r_n = \frac{4\pi\epsilon_0\hbar^2}{me^2}n^2.$$

The total energy of an electron in the orbit n is now

$$\begin{aligned} E_n = T + V &= \frac{1}{2}mv_n^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n} \\ &= -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} \equiv -C\frac{1}{n^2}, \end{aligned} \tag{5.6}$$

where C is a constant. For the ground state ($n = 1$), we get from (5.6)

$$E_1 = -2.18 \times 10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}.$$

From (5.3) and (5.6) we get the energy of the quantum emitted in the transition $E_{n_2} \to E_{n_1}$:

$$h\nu = E_{n_2} - E_{n_1} = C\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right). \tag{5.7}$$

In terms of the wavelength λ this can be expressed as

$$\frac{1}{\lambda} = \frac{\nu}{c} = \frac{C}{hc}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \equiv R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \tag{5.8}$$

where R is the *Rydberg constant*, $R = 1.097 \times 10^7\,\mathrm{m}^{-1}$.

Equation (5.8) was derived experimentally for $n_1 = 2$ by Johann Jakob Balmer as early as 1885. That is why we call the set of lines produced by transitions $E_n \to E_2$ the *Balmer series*. These lines are in the visible part of the spectrum. For historical reasons the Balmer lines are often denoted by symbols H_α, H_β, H_γ etc. If the electron returns to its ground state ($E_n \to E_1$), we get the *Lyman series*, which is in the ultraviolet. The other series with specific names are the *Paschen series* ($n_1 = 3$), *Bracket series* ($n_1 = 4$) and *Pfund series* ($n_1 = 5$) (see Fig. 5.4).

5.3 Line Profiles

The previous discussion suggests that spectral lines would be infinitely narrow and sharp. In reality, however, they are somewhat broadened. We will now consider briefly the factors affecting the shape of a spectral line, called a *line profile*. An exact treatment would take us too deep into quantum mechanics, so we cannot go into the details here.

According to quantum mechanics everything cannot be measured accurately at the same time. For example, even in principle, there is no way to determine the x coordinate and the momentum p_x in the direction of the x axis with arbitrary precision simultaneously. These quantities have small uncertainties Δx and Δp_x, such that

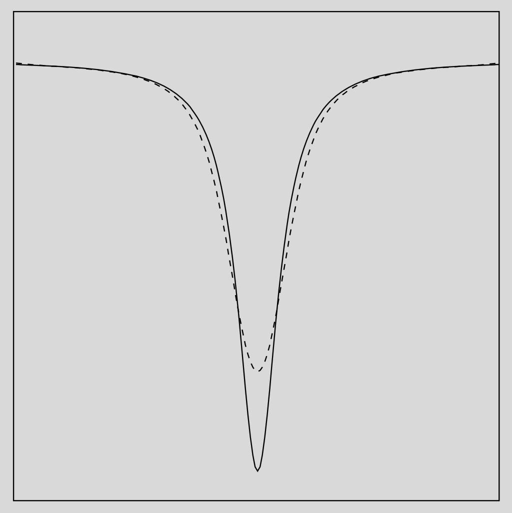

Fig. 5.5 Each spectral line has its characteristic natural width (*solid line*). Motions of particles broaden the line further due to the Doppler effect, resulting in the Voigt profile (*dashed line*). Both profiles have the same area

$$\Delta x \Delta p_x \approx \hbar.$$

A similar relation holds for other directions, too. Time and energy are also connected by an uncertainty relation,

$$\Delta E \Delta t \approx \hbar.$$

The natural width of spectral lines is a consequence of this *Heisenberg uncertainty principle*.

If the average lifetime of an excitation state is T, the energy corresponding to the transition can only be determined with an accuracy of $\Delta E = \hbar/T = h/(2\pi T)$. From (5.1) it follows that $\Delta\nu = \Delta E/h$. In fact, the uncertainty of the energy depends on the lifetimes of both the initial and final states. The *natural width* of a line is defined as

$$\gamma = \frac{\Delta E_\mathrm{i} + \Delta E_\mathrm{f}}{\hbar} = \frac{1}{T_\mathrm{i}} + \frac{1}{T_\mathrm{f}}. \tag{5.9}$$

It can be shown that the corresponding line profile is

$$I_\nu = \frac{\gamma}{2\pi}\frac{I_0}{(\nu - \nu_0)^2 + \gamma^2/4}, \tag{5.10}$$

where ν_0 is the frequency at the centre of the line and I_0 the total intensity of the line. At the centre of the line the intensity per frequency unit is

$$I_{\nu_0} = \frac{2}{\pi\gamma} I_0,$$

and at the frequency $\nu = \nu_0 + \gamma/2$,

$$I_{\nu_0+\gamma/2} = \frac{1}{\pi\gamma} I_0 = \frac{1}{2} I_{\nu_0}.$$

Thus the width γ is the width of the line profile at a depth where the intensity is half of the maximum. This is called the *full width at half maximum* (FWHM).

Doppler Broadening Atoms of a gas are moving the faster the higher the temperature of the gas. Thus spectral lines arising from individual atoms are shifted by the Doppler effect. The observed line consists of a collection of lines with different Doppler shifts, and the shape of the line depends on the number of atoms with different velocities.

Each Doppler shifted line has its characteristic natural width. The resulting line profile is obtained by giving each Doppler shifted line a weight proportional to the number of atoms given by the velocity distribution and integrating over all velocities. This gives rise to the *Voigt profile* (Fig. 5.5), which already describes most spectral lines quite well. The shapes of different profiles don't seem very different; the most obvious consequence of the broadening is that the maximum depth decreases.

One way to describe the width of a line is to give its full width at half maximum (Fig. 5.6). Due to Doppler broadening this is usually greater than the natural width. The *equivalent width* is another measure of a line strength. It is the area of a rectangular line that has the same area as the line profile and that emits no light at all. The equivalent width can be used to describe the energy corresponding to a line independently of the shape of the line profile.

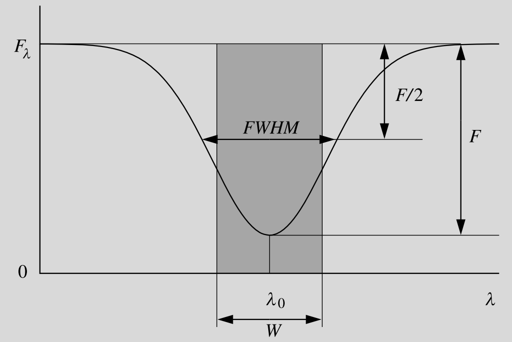

Fig. 5.6 The full width at half maximum (FWHM) of a spectral line is the width at the depth where the intensity is half of the maximum. The equivalent width W is defined so that *the line* and *the shaded rectangle* have the same area in the picture. The two measures are not generally the same, although they usually are close to each other

5.4 Quantum Numbers, Selection Rules, Population Numbers

Quantum Numbers The Bohr model needs only one quantum number, n, to describe all the energy levels of the electron. This can explain only the coarse features of an atom with a single electron.

Quantum mechanics describes the electron as a three dimensional wave, which only gives the probability of finding the electron in a certain place. Quantum mechanics has accurately predicted all the energy levels of hydrogen atoms. The energy levels of heavier atoms and molecules can also be computed; however, such calculations are very complicated. Also the existence of quantum numbers can be understood from the quantum mechanical point of view.

The quantum mechanical description involves four quantum numbers, one of which is our n, the *principal quantum number*. The principal quantum number describes the quantised energy levels of the electron. The classical interpretation of discrete energy levels allows only certain orbits given by (5.6). The orbital angular momentum of the electron is also quantised. This is described by the *angular momentum quantum number* l. The angular momentum corresponding to a quantum number l is

$$L = \sqrt{l(l+1)}\hbar.$$

The classical analogy would be to allow some elliptic orbits. The quantum number l can take only the values

$$l = 0, 1, \ldots, n - 1.$$

For historical reasons, these are often denoted by the letters s, p, d, f, g, h, i, j.

Although l determines the magnitude of the angular momentum, it does not give its direction. In a magnetic field this direction is important, since the orbiting electron also generates a tiny magnetic field. In any experiment, only one component of the angular momentum can be measured at a time. In a given direction z (e.g. in the direction of the applied magnetic field), the projection of the angular momentum can have only the values

$$L_z = m_l \hbar,$$

where m_l is the *magnetic quantum number*

$$m_l = 0, \pm 1, \pm 2, \ldots, \pm l.$$

The magnetic quantum number is responsible for the splitting of spectral lines in strong magnetic fields, known as the *Zeeman effect*. For example, if $l = 1$, m_l can have $2l + 1 = 3$ different values. Thus, the line arising from the transition $l = 1 \rightarrow l = 0$ will split into three components in a magnetic field (Fig. 5.7).

The fourth quantum number is the *spin* describing the intrinsic angular momentum of the electron. The spin of the electron is

$$S = \sqrt{s(s+1)}\hbar,$$

where the spin quantum number is $s = \frac{1}{2}$. In a given direction z, the spin is

$$S_z = m_s \hbar,$$

where m_s can have one of the two values:

$$m_s = \pm \frac{1}{2}.$$

All particles have a spin quantum number. Particles with an integral spin are called *bosons* (photon, mesons); particles with a half-integral spin are *fermions* (proton, neutron, electron, neutrino etc.).

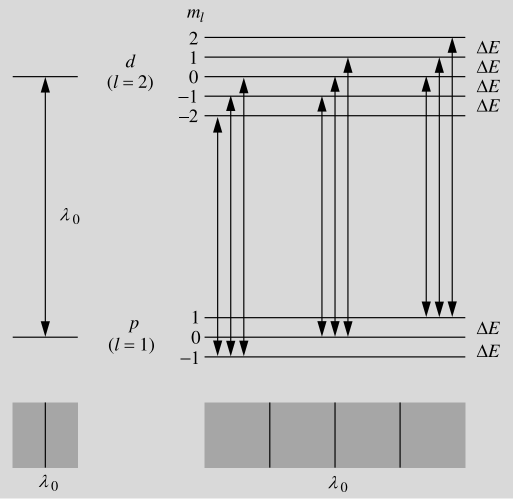

Fig. 5.7 The Zeeman effect. In strong magnetic fields each energy level of a hydrogen atom splits into $(2l + 1)$ separate levels, which correspond to different values of the magnetic quantum number $m_l = l, l - 1, \ldots, -l$. The energy differences of the successive levels have the same constant value ΔE. For example the p state $(l = 1)$ splits into three and the d state $(l = 2)$ into five sublevels. The selection rules require that in electric dipole transitions Δm_l equals 0 or ± 1, and only nine different transitions between p and d states are possible. Moreover, the transitions with the same Δm_l have the same energy difference. Thus the spectrum has only three separate lines

Classically, spin can be interpreted as the rotation of a particle; this analogy, however, should not be taken too literally.

The total angular momentum $\boldsymbol{J}$ of an electron is the sum of its orbital and spin angular momentum:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}.$$

Depending on the mutual orientation of the vectors $\boldsymbol{L}$ and $\boldsymbol{S}$ the quantum number j of total angular momentum can have one of two possible values,

$$j = l \pm \frac{1}{2}$$

(except if $l = 0$, when $j = \frac{1}{2}$). The z component of the total angular momentum can have the values

$$m_j = 0, \pm 1, \pm 2, \ldots \pm j.$$

Spin also gives rise to the fine structure of spectral lines. Lines appear as close pairs or doublets.

Selection Rules The state of an electron cannot change arbitrarily; transitions are restricted by selection rules, which follow from certain conservation laws. The selection rules express how the quantum numbers must change in a transition. Most probable are the *electric dipole transitions*, which make the atom behave like an oscillating dipole. The conservation laws require that in a transition we have

$$\begin{aligned}\Delta l &= \pm 1,\\ \Delta m_l &= 0, \pm 1.\end{aligned}$$

In terms of the total angular momentum the selection rules are

$$\begin{aligned}\Delta l &= \pm 1,\\ \Delta j &= 0, \pm 1,\\ \Delta m_j &= 0, \pm 1.\end{aligned}$$

The probabilities of all other transitions are much smaller, and they are called *forbidden transitions*; examples are magnetic dipole transitions and all quadrupole and higher multipole transitions.

Spectral lines originating in forbidden transitions are called *forbidden lines*. The probability of such a transition is so low that under normal circumstances, the transition cannot take place before collisions force the electron to change state. Forbidden lines are possible only if the gas is extremely rarefied (like in auroras and planetary nebulae).

The spins of an electron and nucleus of a hydrogen atom can be either parallel or antiparallel (Fig. 5.8). The energy of the former state is 0.0000059 eV higher. But the selection rules make an electric dipole transition between these states impossible. The transition, which is a magnetic dipole transition, has a very low probability, $A = 2.8 \times 10^{-15}\ \mathrm{s}^{-1}$. This means that the average lifetime of the higher state is $T = 1/A = 11 \times 10^6$ years. Usually collisions change the state of the electron well before this period of time has elapsed. But in interstellar space the density of hydrogen is so low and the total amount of hydrogen so great that a considerable number of these transitions can take place.

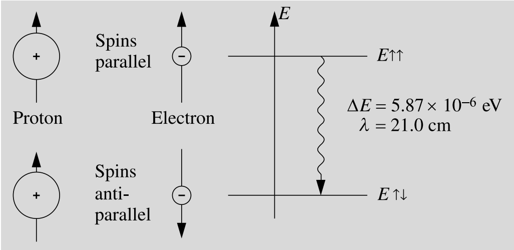

Fig. 5.8 The origin of the hydrogen 21 cm line. The spins of the electron and the proton may be either parallel or opposite. The energy of the former state is slightly larger. The wavelength of a photon corresponding to a transition between these states is 21 cm

The wavelength of the radiation emitted by this transition is 21 cm, which is in the radio band of the spectrum. Extinction at radio wavelengths is very small, and we can observe more distant objects than by using optical wavelengths. The 21 cm radiation has been of crucial importance for surveys of interstellar hydrogen.

Population Numbers The population number n_i of an energy state i means the number of atoms in that state per unit volume. In thermal equilibrium, the population numbers obey the *Boltzmann distribution*:

$$\frac{n_i}{n_0} = \frac{g_i}{g_0} \mathrm{e}^{-\Delta E/(kT)}, \tag{5.11}$$

where T is the temperature, k is the Boltzmann constant, $\Delta E = E_i - E_0 = h\nu$ is the energy difference between the excited and ground state, and g_i is the statistical weight of the level i (it is the number of different states with the same energy E_i). The subscript 0 always refers to the ground state. Often the population numbers differ from the values given by (5.11), but still we can define an *excitation temperature* T_{exc} in such a way that (5.11) gives correct population numbers, when T is replaced by T_{exc}. The excitation temperature may be different for different energy levels.

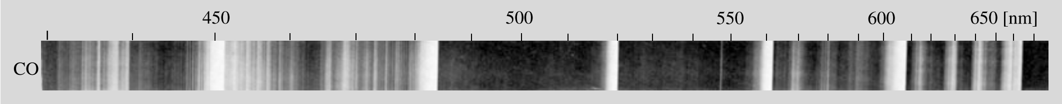

Fig. 5.9 Spectrum of carbon monoxide CO from 430 nm to 670 nm. The various bands correspond to different vibrational transitions. Each band is composed of numerous rotational lines. Near the right edge of each band the lines are so closely packed that they overlap and at this resolution, the spectrum looks continuous. (R.W.B. Pearse, A.G. Gaydon: *The Identification of Molecular Spectra* (Chapman & Hall Ltd., London 1976) p. 394)

5.5 Molecular Spectra

The energy levels of an atom are determined by its electrons. In the case of a molecule, there are many more possibilities: atoms can vibrate around their equilibria and the molecule can rotate around some axis. Both vibrational and rotational states are quantised. Transitions between successive vibrational states typically involve photons in the infrared band, while transitions between rotational states involve photons in the microwave band. These combined with transitions of electrons produce a band spectrum, characteristic for molecules (Fig. 5.9). The spectrum has several narrow bands composed of a great number of lines.

5.6 Continuous Spectra

We have already mentioned some processes that produce continuous spectra. Continuous emission spectra can originate in recombinations and free–free transitions. In recombination, an atom captures a free electron whose energy is not quantised; in free–free transitions, both initial and final states are unquantised. Thus the emission line can have any frequency whatsoever. Similarly, ionisations and free–free transitions can give rise to a continuous absorption spectrum.

Each spectrum contains a continuous component, or *continuum*, and spectral lines. Sometimes, however, the lines are so closely packed and so broad that they seem to form a nearly continuous spectrum.

When the pressure of hot gas is increased, the spectral lines begin to broaden. At high pressure, atoms bump into each other more frequently, and the close neighbours disturb the energy levels. When the pressure is high enough, the lines begin to overlap. Thus the spectrum of hot gas at high pressure is continuous. Electric fields also broaden spectral lines (the Stark effect).

In liquids and solids the atoms are more densely packed than in gaseous substances. Their mutual perturbations broaden the energy levels, producing a continuous spectrum.

5.7 Blackbody Radiation

A *blackbody* is defined as an object that does not reflect or scatter radiation shining upon it, but absorbs and re-emits the radiation completely. A blackbody is a kind of an ideal radiator, which cannot exist in the real world. Yet many objects behave very much as if they were blackbodies.

The radiation of a blackbody depends only on its temperature, being perfectly independent of its shape, material and internal constitution. The wavelength distribution of the radiation follows *Planck's law*, which is a function of temperature only. The intensity at a frequency ν of a blackbody at temperature T is

$$B_\nu(T) = B(\nu; T) = \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/(kT)} - 1}, \quad (5.12)$$

where

$$h = \text{the Planck constant} = 6.63 \times 10^{-34}\ \mathrm{J\,s},$$

$$c = \text{the speed of light} \approx 3 \times 10^{8}\ \mathrm{m\,s^{-1}},$$

$$k = \text{the Boltzmann constant}$$

$$= 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}.$$

By definition of the intensity, the dimension of B_ν is $\mathrm{W\,m^{-2}\,Hz^{-1}\,sterad^{-1}}$.

Blackbody radiation can be produced in a closed cavity whose walls absorb all radiation incident upon them (and coming from inside the cavity). The walls and the radiation in the cavity are in equilibrium; both are at the same temperature, and the walls emit all the energy they receive. Since radiation energy is constantly transformed into thermal energy of the atoms of the walls and back to radiation, the blackbody radiation is also called *thermal radiation*.

The spectrum of a blackbody given by Planck's law (5.12) is continuous. This is true if the size of the radiator is very large compared with the dominant wavelengths. In the case of the cavity, this can be understood by considering the radiation as standing waves trapped in the cavity. The number of different wavelengths is larger, the shorter the wavelengths are compared with the size of the cavity. We already mentioned that spectra of solid bodies are continuous; very often such spectra can be quite well approximated by Planck's law.

We can also write Planck's law as a function of the wavelength. We require that $B_\nu \,\mathrm{d}\nu = -B_\lambda \,\mathrm{d}\lambda$. The wavelength decreases with increasing frequency; hence the minus sign. Since $\nu = c/\lambda$, we have

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = -\frac{c}{\lambda^2}, \tag{5.13}$$

whence

$$B_\lambda = -B_\nu \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = B_\nu \frac{c}{\lambda^2}, \tag{5.14}$$

or

$$\begin{aligned} B_\lambda(T) &= \frac{2hc^2}{\lambda^5} \frac{1}{\mathrm{e}^{hc/(\lambda kT)} - 1}, \\ [B_\lambda] &= \mathrm{W\,m^{-2}\,m^{-1}\,sterad^{-1}}. \end{aligned} \tag{5.15}$$

The functions B_ν and B_λ are defined in such a way that the total intensity can be obtained in the same way using either of them:

$$B(T) = \int_0^\infty B_\nu \,\mathrm{d}\nu = \int_0^\infty B_\lambda \,\mathrm{d}\lambda .$$

Let us now try to find the total intensity using the first of these integrals:

$$B(T) = \int_0^\infty B_\nu(T) \,\mathrm{d}\nu = \frac{2h}{c^2} \int_0^\infty \frac{\nu^3 \,\mathrm{d}\nu}{\mathrm{e}^{h\nu/(kT)} - 1}.$$

We now change the integration variable to $x = h\nu/(kT)$, whence $\mathrm{d}\nu = (kT/h)\,\mathrm{d}x$:

$$B(T) = \frac{2h}{c^2}\frac{k^4}{h^4} T^4 \int_0^\infty \frac{x^3 \,\mathrm{d}x}{\mathrm{e}^x - 1}.$$

The definite integral in this expression is just a real number, independent of the temperature. Thus we find that

$$B(T) = AT^4, \tag{5.16}$$

where the constant A has the value

$$A = \frac{2k^4}{c^2h^3}\frac{\pi^4}{15}. \tag{5.17}$$

(In order to get the value of A we have to evaluate the integral. There is no elementary way to do that. We can tell those who are familiar with all the exotic functions so beloved by theoretical physicists, that the integral can rather easily be expressed as $\Gamma(4)\zeta(4)$, where ζ is the Riemann zeta function and Γ is the gamma function. For integral values, $\Gamma(n)$ is simply the factorial $(n-1)!$. The difficult part is showing that $\zeta(4) = \pi^4/90$. This can be done by expanding $x^4 - x^2$ as a Fourier-series and evaluating the series at $x = \pi$.)

The flux density F for isotropic radiation of intensity B is (Sect. 4.1):

$$F = \pi B$$

or

$$F = \sigma T^4. \tag{5.18}$$

This is the *Stefan-Boltzmann law*, and the constant σ $(= \pi A)$ is the *Stefan-Boltzmann constant*,

$$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}.$$

From the Stefan-Boltzmann law we get a relation between the luminosity and temperature of a star. If the radius of the star is R, its surface area is $4\pi R^2$, and if the flux density on the surface is F, we have

$$L = 4\pi R^2 F.$$

If the star is assumed to radiate like a blackbody, we have $F = \sigma T^4$, which gives

$$L = 4\pi\sigma R^2 T^4. \tag{5.19}$$

In fact this defines the *effective temperature* of the star, discussed in more detail in the next section.

The luminosity, radius and temperature of a star are interdependent quantities, as we can see from (5.19). They are also related to the absolute bolometric magnitude of the star. Equation (4.13) gives the difference of the absolute bolometric magnitude of the star and the Sun:

$$M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} = -2.5 \lg \frac{L}{L_\odot}. \tag{5.20}$$

But we can now use (5.19) to express the luminosities in terms of the radii and temperatures:

$$M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} = -5 \lg \frac{R}{R_\odot} - 10 \lg \frac{T}{T_\odot}. \tag{5.21}$$

As we can see in Fig. 5.10, the wavelength of the maximum intensity decreases with increasing total intensity (equal to the area below the curve). We can find the wavelength $\lambda_{\max}$ corresponding to the maximum intensity by differentiating Planck's function $B_\lambda(T)$ with respect to λ and finding zero of the derivative. The result is the *Wien displacement law*:

$$\lambda_{\max} T = b = \mathrm{const}, \tag{5.22}$$

where the *Wien displacement constant b* is

$$b = 0.0028978\ \mathrm{K\,m}.$$

We can use the same procedure to find the maximum of B_ν. But the frequency $\nu_{\max}$ thus obtained is different from $\nu_{\max} = c/\lambda_{\max}$ given by (5.22). The reason for this is the fact that the intensities are given per unit frequency or unit wavelength, and the dependence of frequency on wavelength is nonlinear.

When the wavelength is near the maximum or much longer than $\lambda_{\max}$ Planck's function can be approximated by simpler expressions. When $\lambda \approx \lambda_{\max}$ (or $hc/(\lambda kT) \gg 1$), we have

$$e^{hc/(\lambda kT)} \gg 1.$$

In this case we get the *Wien approximation*

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda kT)}. \tag{5.23}$$

When $hc/(\lambda kT) \ll 1$ ($\lambda \gg \lambda_{\max}$), we have

$$e^{hc/\lambda kT} \approx 1 + hc/(\lambda kT),$$

which gives the *Rayleigh–Jeans approximation*

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} \frac{\lambda kT}{hc} = \frac{2ckT}{\lambda^4}. \tag{5.24}$$

This is particularly useful in radio astronomy.

Classical physics predicted only the Rayleigh–Jeans approximation. Were (5.24) true for all wavelengths, the intensity would grow beyond all limits when the wavelength approaches zero, contrary to observations. This contradiction was known as the ultraviolet catastrophe.

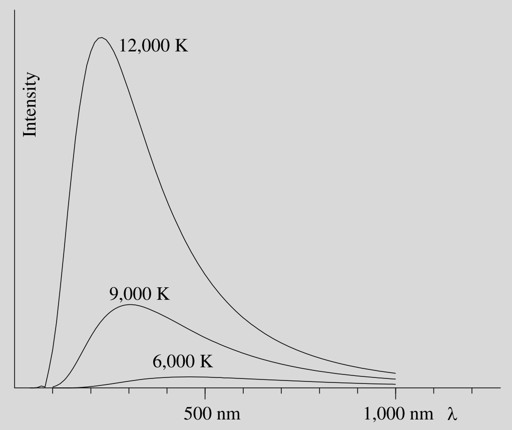

Fig. 5.10 Intensity distributions of blackbodies at temperature 12,000 K, 9000 K and 6000 K. Since the ratios of the temperatures are 4:3:2, the wavelengths of intensity maxima given by the Wien displacement law are in the proportions 1:4, 1:3 and 1:2, or 3, 4 and 6. The actual wavelengths of the maxima are 241.5 nm, 322 nm and 483 nm. The total intensities or the areas below the curves are proportional to 4^4, 3^4 and 2^4

5.8 Temperatures

Temperatures of astronomical objects range from almost absolute zero to millions of degrees. Temperature can be defined in a variety of ways, and its numerical value depends on the specific definition used. All these different temperatures are needed to describe different physical phenomena, and often there is no unique 'true' temperature.

Often the temperature is determined by comparing the object, a star for instance, with a blackbody. Although real stars do not radiate exactly like blackbodies, their spectra can usually be approximated by blackbody spectra after the effect of spectral lines has been eliminated. The resulting temperature depends on the exact criterion used to fit Planck's function to observations.

The most important quantity describing the surface temperature of a star is the *effective temperature* T_e. It is defined as the temperature of a blackbody which radiates with the same total flux density as the star. Since the effective temperature depends only on the total radiation power integrated over all frequencies, it is well defined for all energy distributions even if they deviate far from Planck's law.

In the previous section we derived the Stefan-Boltzmann law, which gives the total flux density as a function of the temperature. If we now find a value T_e of the temperature such that the Stefan-Boltzmann law gives the correct flux density F on the surface of the star, we have found the effective temperature. The flux density on the surface is

$$F = \sigma T_e^4. \tag{5.25}$$

The total flux is $L = 4\pi R^2 F$, where R is the radius of the star, and the flux density at a distance r is

$$F' = \frac{L}{4\pi r^2} = \frac{R^2}{r^2} F = \left(\frac{\alpha}{2}\right)^2 \sigma T_e^4, \tag{5.26}$$

where $\alpha = 2R/r$ is the observed angular diameter of the star. For direct determination of the effective temperature, we have to measure the total flux density and the angular diameter of the star. This is possible only in the few cases in which the diameter has been found by interferometry.

If we assume that at some wavelength λ the flux density F_λ on the surface of the star is obtained from Planck's law, we get the *brightness temperature* T_b. In the isotropic case we have then $F_\lambda = \pi B_\lambda(T_b)$. If the radius of the star is R and distance from the Earth r, the observed flux density is

$$F'_\lambda = \frac{R^2}{r^2} F_\lambda.$$

Again F_λ can be determined only if the angular diameter α is known. The brightness temperature T_b can then be solved from

$$F'_\lambda = \left(\frac{\alpha}{2}\right)^2 \pi B_\lambda(T_b). \tag{5.27}$$

Since the star does not radiate like a blackbody, its brightness temperature depends on the particular wavelength used in (5.27).

In radio astronomy, brightness temperature is used to express the intensity (or surface brightness) of the source. If the intensity at frequency ν is I_ν, the brightness temperature is obtained from

$$I_\nu = B_\nu(T_b).$$

T_b gives the temperature of a blackbody with the same surface brightness as the observed source.

Since radio wavelengths are very long, the condition $h\nu \ll kT$ of the Rayleigh–Jeans approximation is usually satisfied (except for millimetre and submillimetre bands), and we can write Planck's law as

$$\begin{aligned} B_\nu(T_b) &= \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT_b)} - 1} \\ &= \frac{2h\nu^3}{c^2} \frac{1}{1 + h\nu/(kT_b) + \cdots - 1} \\ &\approx \frac{2k\nu^2}{c^2} T_b. \end{aligned}$$

Thus we get the following expression for the radio astronomical brightness temperature:

$$T_b = \frac{c^2}{2k\nu^2} I_\nu = \frac{\lambda^2}{2k} I_\nu. \tag{5.28}$$

A measure of the signal registered by a radio telescope is the *antenna temperature* T_A. After

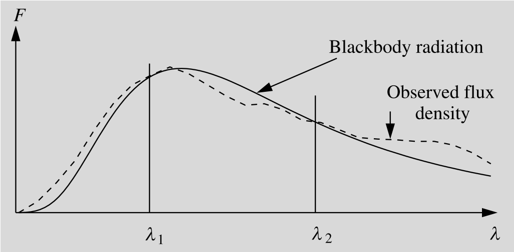

Fig. 5.11 Determination of the colour temperature. The ratio of the flux densities at wavelengths λ_1 and λ_2 gives the temperature of a blackbody with the same ratio. In general the result depends on the wavelengths chosen

the antenna temperature is measured, we get the brightness temperature from

$$T_{\mathrm{A}} = \eta T_{\mathrm{b}}, \tag{5.29}$$

where η is the *beam efficiency* of the antenna (typically $0.4 \lesssim \eta \lesssim 0.8$). Equation (5.29) holds if the source is wide enough to cover the whole beam, i.e. the solid angle Ω_{A} from which the antenna receives radiation. If the solid angle subtended by the source, Ω_{S}, is smaller than Ω_{A}, the observed antenna temperature is

$$T_{\mathrm{A}} = \eta \frac{\Omega_{\mathrm{S}}}{\Omega_{\mathrm{A}}} T_{\mathrm{b}} \quad (\Omega_{\mathrm{S}} < \Omega_{\mathrm{A}}). \tag{5.30}$$

The *colour temperature* T_{c} can be determined even if the angular diameter of the source is unknown (Fig. 5.11). We only have to know the relative energy distribution in some wavelength range $[\lambda_1, \lambda_2]$; the absolute value of the flux is not needed. The observed flux density as a function of wavelength is compared with Planck's function at different temperatures. The temperature giving the best fit is the colour temperature in the interval $[\lambda_1, \lambda_2]$. The colour temperature is usually different for different wavelength intervals, since the shape of the observed energy distribution may be quite different from the blackbody spectrum.

A simple method for finding a colour temperature is the following. We measure the flux density F'_λ at two wavelengths λ_1 and λ_2. If we assume that the intensity distribution follows Planck's law, the ratio of these flux densities must be the same as the ratio obtained from Planck's law:

$$\frac{F'_{\lambda_1}(T)}{F'_{\lambda_2}(T)} = \frac{B_{\lambda_1}(T)}{B_{\lambda_2}(T)} = \frac{\lambda_2^5}{\lambda_1^5} \frac{\mathrm{e}^{hc/(\lambda_2 kT)} - 1}{\mathrm{e}^{hc/(\lambda_1 kT)} - 1}. \tag{5.31}$$

The temperature T solved from this equation is a colour temperature.

The observed flux densities correspond to certain magnitudes m_{λ_1} and m_{λ_2}. The definition of magnitudes gives

$$m_{\lambda_1} - m_{\lambda_2} = -2.5 \lg \frac{F'_{\lambda_1}}{F'_{\lambda_2}} + \text{const},$$

where the constant term is a consequence of the different zero points of the magnitude scales. If the temperature is not too high, we can use the Wien approximation in the optical part of the spectrum:

$$\begin{aligned} m_{\lambda_1} - m_{\lambda_2} &= -2.5 \lg \frac{B_{\lambda_1}}{B_{\lambda_2}} + \text{const} \\ &= -2.5 \lg \left(\frac{\lambda_2}{\lambda_1} \right)^5 \\ &\quad + 2.5 \frac{hc}{kT} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \lg \mathrm{e} + \text{const}. \end{aligned}$$

This can be written as

$$m_{\lambda_1} - m_{\lambda_2} = a + b/T_{\mathrm{c}}, \tag{5.32}$$

where a and b are constants. This shows that there is a simple relationship between the difference of two magnitudes and the colour temperature.

Strictly speaking, the magnitudes in (5.32) are monochromatic, but the same relation can be also used with broadband magnitudes like B and V. In that case, the two wavelengths are essentially the effective wavelengths of the B and V bands. The constant is chosen so that $B - V = 0$ for stars of the spectral type A0 (see Chap. 9). Thus the colour index $B - V$ also gives a colour temperature.

The *kinetic temperature* T_{k}, is related to the average speed of gas molecules. The kinetic energy of an ideal gas molecule as a function of temperature follows from the kinetic gas theory:

$$\text{Kinetic energy} = \frac{1}{2}mv^2 = \frac{3}{2}kT_\mathrm{k}.$$

Solving for T_k we get

$$T_\mathrm{k} = \frac{mv^2}{3k}, \tag{5.33}$$

where m is the mass of the molecule, v its average velocity (or rather its r.m.s velocity, which means that v^2 is the average of the squared velocities), and k, the Boltzmann constant. For ideal gases the pressure is directly proportional to the kinetic temperature (cf. Box 11.1):

$$P = nkT_\mathrm{k}, \tag{5.34}$$

where n is the number density of the molecules (molecules per unit volume). We previously defined the excitation temperature T_exc as a temperature which, if substituted into the Boltzmann distribution (5.11), gives the observed population numbers. If the distribution of atoms in different levels is a result of mutual collisions of the atoms only, the excitation temperature equals the kinetic temperature, $T_\mathrm{exc} = T_\mathrm{k}$.

The *ionisation temperature* T_i is found by comparing the number of atoms in different states of ionisation. Since stars are not exactly blackbodies, the values of excitation and ionisation temperatures usually vary, depending on the element whose spectral lines were used for temperature determination.

In *thermodynamic equilibrium* all these various temperatures are equal.

5.9 Other Radiation Mechanisms

The radiation of a gas in thermodynamic equilibrium depends on the temperature and density only. In astrophysical objects deviations from thermodynamic equilibrium are, however, quite common. Some examples of *non-thermal radiation* arising under such conditions are mentioned in the following.

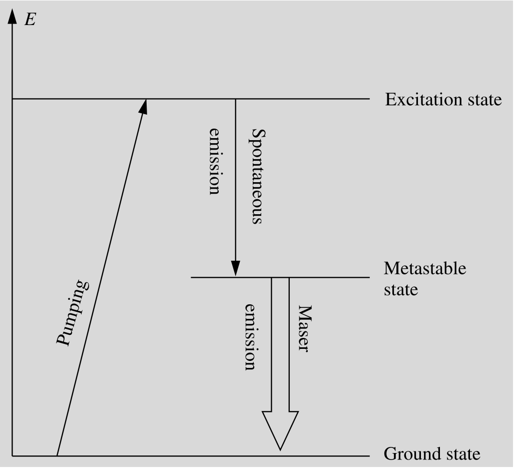

Fig. 5.12 The operational principle of the maser and the laser. A metastable state (a state with a relatively long average lifetime) stores atoms where they accumulate; there are more atoms in the metastable state than in the ground state. This population inversion is maintained by radiatively exciting atoms to a higher excitation state ("pumping"), from which they spontaneously jump down to the metastable state. When the atoms are illuminated by photons with energies equal to the excitation energy of the metastable state, the photons will induce more radiation of the same wavelength, and the radiation is amplified in geometric progression

Maser and Laser (See Fig. 5.12.) The Boltzmann distribution (5.11) shows that usually there are fewer atoms in excited states than in the ground state. There are, however, means to produce a *population inversion*, an excited state containing more atoms than the ground state. This inversion is essential for both the *maser* and the *laser* (Microwave/Light Amplification by Stimulated Emission of Radiation). If the excited atoms are now illuminated with photons having energies equal to the excitation energy, the radiation will induce downward transitions. The number of photons emitted greatly exceeds the number of absorbed photons, and radiation is amplified. Typically the excited state is a *metastable state*, a state with a very long average lifetime, which means that the contribution of spontaneous emission is negligible. Therefore the resulting radiation is coherent and monochromatic. Several maser sources have been found in interstellar molecular clouds and dust envelopes around stars.

Synchrotron Radiation A free charge in accelerated motion will emit electromagnetic radiation. Charged particles moving in a magnetic field follow helices around the field lines. As seen from the direction of the field, the motion is circular and therefore accelerated. The moving charge will radiate in the direction of its velocity vector. Such radiation is called *synchrotron radiation*. It will be further discussed in Chap. 16.

5.10 Radiative Transfer

Propagation of radiation in a medium, also called radiative transfer, is one of the basic problems of astrophysics. The subject is too complicated to be discussed here in any detail. The fundamental equation of radiative transfer is, however, easily derived.

Assume we have a small cylinder, the bottom of which has an area $\mathrm{d}A$ and the length of which is $\mathrm{d}r$. Let I_ν be the intensity of radiation perpendicular to the bottom surface going into a solid angle $\mathrm{d}\omega$ ($[I_\nu] = \mathrm{W\,m^{-2}\,Hz^{-1}\,sterad^{-1}}$). If the intensity changes by an amount $\mathrm{d}I_\nu$ in the distance $\mathrm{d}r$, the energy changes by

$$\mathrm{d}E = \mathrm{d}I_\nu\,\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t$$

in the cylinder in time $\mathrm{d}t$. This equals the emission minus absorption in the cylinder. The absorbed energy is (cf. (4.14))

$$\mathrm{d}E_{\mathrm{abs}} = \alpha_\nu I_\nu\,\mathrm{d}r\,\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t, \tag{5.35}$$

where α_ν is the opacity of the medium at frequency ν. Let the amount of energy emitted per hertz at frequency ν into unit solid angle from unit volume and per unit time be j_ν ($[j_\nu] = \mathrm{W\,m^{-3}\,Hz^{-1}\,sterad^{-1}}$). This is called the *emission coefficient* of the medium. The energy emitted into solid angle $\mathrm{d}\omega$ from the cylinder is then

$$\mathrm{d}E_{\mathrm{em}} = j_\nu\,\mathrm{d}r\,\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t. \tag{5.36}$$

The equation

$$\mathrm{d}E = -\mathrm{d}E_{\mathrm{abs}} + \mathrm{d}E_{\mathrm{em}}$$

gives then

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \mathrm{d}r + j_\nu \mathrm{d}r$$

or

$$\frac{\mathrm{d}I_\nu}{\alpha_\nu\,\mathrm{d}r} = -I_\nu + \frac{j_\nu}{\alpha_\nu}. \tag{5.37}$$

We shall denote the ratio of the emission coefficient j_ν to the absorption coefficient or opacity α_ν by S_ν:

$$S_\nu = \frac{j_\nu}{\alpha_\nu}. \tag{5.38}$$

S_ν is called the *source function*. Because $\alpha_\nu\,\mathrm{d}r = \mathrm{d}\tau_\nu$, where τ_ν is the optical thickness at frequency ν, (5.37) can be written as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu. \tag{5.39}$$

Equation (5.39) is the basic equation of radiative transfer. Without solving the equation, we see that if $I_\nu < S_\nu$, then $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu > 0$, and the intensity tends to increase in the direction of propagation. And, if $I_\nu > S_\nu$, then $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu < 0$, and I_ν will decrease. In an equilibrium the emitted and absorbed energies are equal, in which case we find from (5.35) and (5.36)

$$I_\nu = j_\nu/\alpha_\nu = S_\nu. \tag{5.40}$$

Substituting this into (5.39), we see that $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu = 0$. In thermodynamic equilibrium the radiation of the medium is blackbody radiation, and the source function is given by Planck's law:

$$S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{h\nu/(kT)} - 1}.$$

Even if the system is not in thermodynamic equilibrium, it may be possible to find an excitation temperature T_{exc} such that $B_\nu(T_{\mathrm{exc}}) = S_\nu$. This temperature may depend on frequency.

A formal solution of (5.39) is

$$I_\nu(\tau_\nu) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} \mathrm{e}^{-(\tau_\nu - t)}\,S_\nu(t)\,\mathrm{d}t. \tag{5.41}$$

Here $I_\nu(0)$ is the intensity of the background radiation, coming through the medium (e.g. an interstellar cloud) and decaying exponentially in the medium. The second term gives the emission in the medium. The solution is only formal, since in general, the source function S_ν is unknown and must be solved simultaneously with the intensity. If $S_\nu(\tau_\nu)$ is constant in the cloud and the background radiation is ignored, we get

$$I_\nu(\tau_\nu) = S_\nu \int_0^{\tau_\nu} \mathrm{e}^{-(\tau_\nu - t)}\,\mathrm{d}t = S_\nu\bigl(1 - \mathrm{e}^{-\tau_\nu}\bigr). \tag{5.42}$$

If the cloud is optically thick ($\tau_\nu \gg 1$), we have

$$I_\nu = S_\nu, \tag{5.43}$$

i.e. the intensity equals the source function, and the emission and absorption processes are in equilibrium.

An important field of application of the theory of radiative transfer is in the study of planetary and stellar atmospheres. In this case, to a good approximation, the properties of the medium only vary in one direction, say along the z axis. The intensity will then depend only on z and θ, where θ is the angle between the z axis and the direction of propagation of the radiation.

In applications to atmospheres it is customary to define the optical depth τ_ν in the vertical direction as

$$\mathrm{d}\tau_\nu = -\alpha_\nu\,\mathrm{d}z.$$

Conventionally z increases upwards and the optical depth inwards in the atmosphere. The vertical line element $\mathrm{d}z$ is related to that along the light ray, $\mathrm{d}r$, according to

$$\mathrm{d}z = \mathrm{d}r\cos\theta.$$

With these notational conventions, (5.39) now yields

$$\cos\theta\,\frac{\mathrm{d}I_\nu(z,\theta)}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu. \tag{5.44}$$

This is the form of the equation of radiative transfer usually encountered in the study of stellar and planetary atmospheres.

A formal expression for the intensity emerging from an atmosphere can be obtained by integrating (5.44) from $\tau_\nu = \infty$ (we assume that the bottom of the atmosphere is at infinite optical depth) to $\tau_\nu = 0$ (corresponding to the top of the atmosphere). This yields

$$I_\nu(0,\theta) = \int_0^\infty S_\nu\,\mathrm{e}^{-\tau_\nu\sec\theta}\sec\theta\,\mathrm{d}\tau_\nu. \tag{5.45}$$

This expression will be used later in Chap. 8 on the interpretation of stellar spectra.

5.11 Examples

Example 5.1 Find the wavelength of the photon emitted in the transition of a hydrogen atom from $n_2 = 110$ to $n_1 = 109$.

Equation (5.8) gives

$$\begin{aligned}\frac{1}{\lambda} &= R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\\ &= 1.097\times 10^7\ \mathrm{m}^{-1}\left(\frac{1}{109^2} - \frac{1}{110^2}\right)\\ &= 16.71\ \mathrm{m}^{-1},\end{aligned}$$

whence

$$\lambda = 0.060\ \mathrm{m}.$$

This is in the radio band. Such radiation was observed for the first time in 1965 by an NRAO radio telescope.

Example 5.2 The effective temperature of a star is 12,000 K and the absolute bolometric magnitude 0.0. Find the radius of the star, when the effective temperature of the Sun is 5000 K and the absolute bolometric magnitude 4.7.

We can apply (5.21):

$$\begin{aligned}M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} &= -5\lg\frac{R}{R_\odot} - 10\lg\frac{T}{T_\odot}\\ \Rightarrow\quad \frac{R}{R_\odot} &= \left(\frac{T_{\mathrm{e}\odot}}{T_\mathrm{e}}\right)^2 10^{-0.2(M_{\mathrm{bol}} - M_{\mathrm{bol},\odot})}\\ &= \left(\frac{5800}{12{,}000}\right)^2 10^{-0.2(0.0-4.7)}\\ &= 2.0.\end{aligned}$$

Thus the radius is twice the Solar radius.

Example 5.3 Derive the Wien displacement laws.

Let us denote $x = hc/(\lambda kT)$. Planck's law then becomes

$$B_\lambda(T) = \frac{2k^5T^5}{h^4c^3}\,\frac{x^5}{\mathrm{e}^x - 1}.$$

For a given temperature, the first factor is constant. Thus, it is sufficient to find the maximum of the function $f(x) = x^5/(e^x - 1)$.

First we must evaluate the derivative of f:

$$f'(x) = \frac{5x^4(e^x - 1) - x^5 e^x}{(e^x - 1)^2} = \frac{x^4 e^x}{(e^x - 1)^2}\left(5 - 5e^{-x} - x\right).$$

By definition, x is always strictly positive. Hence $f'(x)$ can be zero only if the factor $5 - 5e^{-x} - x$ is zero. This equation cannot be solved analytically. Instead we write the equation as $x = 5 - 5e^{-x}$ and solve it by iteration:

$$\begin{aligned} x_0 &= 5 \quad \text{(this is just a guess)},\\ x_1 &= 5 - 5e^{-x_0} = 4.96631,\\ &\vdots\\ x_5 &= 4.96511. \end{aligned}$$

Thus the result is $x = 4.965$. The Wien displacement law is then

$$\lambda_{\max} T = \frac{hc}{xk} = b = 2.898 \times 10^{-3}\ \mathrm{K\,m}.$$

In terms of frequency Planck's law is

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}.$$

Substituting $x = h\nu/(kT)$ we get

$$B_\nu(T) = \frac{2k^3T^3}{h^2c^2}\frac{x^3}{e^x - 1}.$$

Now we study the function $f(x) = x^3/(e^x - 1)$:

$$f'(x) = \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} = \frac{x^2 e^x}{(e^x - 1)^2}\left(3 - 3e^{-x} - x\right).$$

This vanishes, when $3 - 3e^{-x} - x = 0$. The solution of this equation is $x = 2.821$. Hence

$$\frac{cT}{\nu_{\max}} = \frac{hc}{kx} = b' = 5.100 \times 10^{-3}\ \mathrm{K\,m}$$

or

$$\frac{T}{\nu_{\max}} = 1.701 \times 10^{-11}\ \mathrm{K\,s}.$$

Note that the wavelength corresponding to $\nu_{\max}$ is different from $\lambda_{\max}$. The reason is that we have used two different forms of Planck's function, one giving the intensity per unit wavelength, the other per unit frequency.

Example 5.4 (a) Find the fraction of radiation that a blackbody emits in the range $[\lambda_1, \lambda_2]$, where λ_1 and $\lambda_2 \gg \lambda_{\max}$. (b) How much energy does a 100 W incandescent light bulb radiate in the radio wavelengths, $\lambda \geq 1$ cm? Assume the temperature is 2500 K.

Since the wavelengths are much longer than $\lambda_{\max}$ we can use the Rayleigh–Jeans approximation $B_\lambda(T) \approx 2ckT/\lambda^4$. Then

$$B' = \int_{\lambda_1}^{\lambda_2} B_\lambda(T)\,d\lambda \approx 2ckT\int_{\lambda_1}^{\lambda_2}\frac{d\lambda}{\lambda^4} = \frac{2ckT}{3}\left(\frac{1}{\lambda_1^3} - \frac{1}{\lambda_2^3}\right),$$

and hence

$$\frac{B'}{B_{\text{tot}}} = \frac{5c^3h^3}{k^3\pi^4}\frac{1}{T^3}\left(\frac{1}{\lambda_1^3} - \frac{1}{\lambda_2^3}\right).$$

Now the temperature is $T = 2500$ K and the wavelength range $[0.01\ \text{m}, \infty)$, and so

$$B' = 100\ \mathrm{W} \times 1.529 \times 10^{-7}\frac{1}{2500^3}\frac{1}{0.01^3} = 9.8 \times 10^{-10}\ \mathrm{W}.$$

It is quite difficult to listen to the radio emission of a light bulb with an ordinary radio receiver.

Example 5.5 (Determination of Effective Temperature) The observed flux density of Arcturus is

$$F' = 4.5 \times 10^{-8}\ \mathrm{W\,m^{-2}}.$$

Interferometric measurements give an angular diameter of $\alpha = 0.020''$. Thus, $\alpha/2 = 4.85 \times 10^{-8}$

radians. From (5.26) we get

$$T_e = \left(\frac{4.5 \times 10^{-8}}{(4.85 \times 10^{-8})^2 \times 5.669 \times 10^{-8}} \right)^{1/4} \mathrm{K}$$

$$= 4300 \mathrm{~K}.$$

Example 5.6 Flux densities at the wavelengths 440 nm and 550 nm are 1.30 and 1.00 $\mathrm{W\,m^{-2}\,m^{-1}}$, respectively. Find the colour temperature.

If the flux densities at the wavelengths λ_1 and λ_2 are F_1 and F_2, respectively, the colour temperature can be solved from the equation

$$\frac{F_1}{F_2} = \frac{B_{\lambda_1}(T_c)}{B_{\lambda_2}(T_c)} = \left(\frac{\lambda_2}{\lambda_1} \right)^5 \frac{e^{hc/(\lambda_2 k T_c)} - 1}{e^{hc/(\lambda_1 k T_c)} - 1}.$$

If we denote

$$A = \frac{F_1}{F_2} \left(\frac{\lambda_1}{\lambda_2} \right)^5,$$

$$B_1 = \frac{hc}{\lambda_1 k},$$

$$B_2 = \frac{hc}{\lambda_2 k},$$

we get the equation

$$A = \frac{e^{B_2/T_c} - 1}{e^{B_1/T_c} - 1}$$

for the colour temperature T_c. This equation must be solved numerically.

In our example the constants have the following values:

$$A = \frac{1.00}{1.30} \left(\frac{550}{440} \right)^5 = 2.348,$$

$$B_1 = 32{,}700 \mathrm{~K}, \qquad B_2 = 26{,}160 \mathrm{~K}.$$

By substituting different values for T_c, we find that $T_c = 7545$ K satisfies our equation.

5.12 Exercises

Exercise 5.1 Show that in the Wien approximation the relative error of B_λ is

$$\frac{\Delta B_\lambda}{B_\lambda} = -e^{-hc/(\lambda k T)}.$$

Exercise 5.2 If the transition of the hydrogen atom $n + 1 \rightarrow n$ were to correspond to the wavelength 21.05 cm, what would the quantum number n be? The interstellar medium emits strong radiation at this wavelength. Can this radiation be due to such transitions?

Exercise 5.3 The space is filled with background radiation, remnant of the early age of the universe. Currently the distribution of this radiation is similar to the radiation of a blackbody at the temperature of 2.7 K. What is $\lambda_{\max}$ corresponding to this radiation? What is its total intensity? Compare the intensity of the background radiation to the intensity of the Sun at the visual wavelengths.

Exercise 5.4 The temperature of a red giant is $T = 2500$ K and radius 100 times the solar radius.

(a) Find the total luminosity of the star, and the luminosity in the visual band 400 nm $\leq \lambda \leq$ 700 nm.
(b) Compare the star with a 100 W lamp that radiates 5 % of its energy in the visual band. What is the distance of the lamp if it looks as bright as the star?

Exercise 5.5 The effective temperature of Sirius is 10,000 K, apparent visual magnitude -1.5, distance 2.67 kpc and bolometric correction 0.5. What is the radius of Sirius?

Exercise 5.6 The observed flux density of the Sun at $\lambda = 300$ nm is 0.59 $\mathrm{W\,m^{-2}\,nm^{-1}}$. Find the brightness temperature of the Sun at this wavelength.

Exercise 5.7 The colour temperature can be determined from two magnitudes corresponding to two different wavelengths. Show that

$$T_c = \frac{7000 \mathrm{~K}}{(B - V) + 0.47}.$$

The wavelengths of the B and V bands are 440 nm and 548 nm, respectively, and we assume that $B = V$ for stars of the spectral class A0, the colour temperature of which is about 15,000 K.

Exercise 5.8 The kinetic temperature of the plasma in the solar corona can reach 10^6 K. Find the average speed of the electrons in such a plasma.